Steps towards a ‘directed homotopy hypothesis’. $(\infty, 1)$-categories, directed spaces and perhaps rewriting

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June 12, 2015

Some history

$(\infty, 0)$-**categories, spaces and rewriting.**

- Letters from Grothendieck to Larry Breen (1975).
- Letter from AG to Quillen, [4], in 1983, forming the very first part of ‘Pursuing Stacks’, [5], pages 13 to 17 of the original scanned file.
- Letter from TP to AG (16/06/1983).

Grothendieck on ∞-groupoids (from PS)

At first sight, it seemed to me that the Bangor group had indeed come to work out (quite independently) one basic intuition of the program I had envisaged in those letters to Larry Breen – namely the study of n-truncated homotopy types (of semi-simplicial sets, or of topological spaces) was essentially equivalent to the study of so-called n-groupoids (where n is a natural integer). This is expected to be achieved by associating to any space (say) X its 'fundamental n-groupoid' $\Pi_n(X)$, generalizing the familiar Poincaré fundamental groupoid for $n = 1$. The obvious idea is that 0-objects of $\Pi_n(X)$ should be points of X, 1-objects should be 'homotopies' or paths between points, 2-objects should be homotopies between 1-objects, etc. This $\Pi_n(X)$ should embody the n-truncated homotopy type of X in much the same way as for $n = 1$ the usual fundamental groupoid embodies the 1-truncated homotopy type.

In a letter to AG, (16/06/1983), I suggested that Kan complexes gave a solution to what ∞-groupoids were. Grothendieck did not like this solution for several reasons.

- Simplicial sets are not globular like the intuition of higher categories.
- Composition is not defined precisely, only up to homotopy.

Grothendieck’s points can be countered to some extent but that is not our main purpose here.

Homotopy Hypothesis

Kan complexes *do* form *one* model for weak ∞-groupoids and so do satisfy what has become known as Grothendieck's 'Homotopy Hypothesis' which can be interpreted as saying

- there is an equivalence of (weak) $(n+1)$-categories

$$\begin{array}{ccc} \text{spaces} & \longleftrightarrow & n\text{-groupoids} \\ \text{up to } n\text{-homotopy} & & \text{up to } (n+1)\text{-equivalence} \end{array}$$

  for all $n \leq \infty$.

- The challenge is to make definitions of 'n-category' and 'n-groupoid' (and probably also of 'spaces'), so that this works.
  This serves as a test for any notion of n-groupoid put forward, (and as always, here, n can be ∞).

Summary of some classical results of simplicial homotopy theory

Classical case of a weak form of the 'HH':

- $Sing : Spaces \to Kan$ gives a Kan complex for each space.
- Geometric realisation, $| - | : \mathcal{S} \to Spaces$, gives an adjoint to $Sing$,
  and
- the two homotopy categories are equivalent by the induced functors.

  (Here $\mathcal{S}$ is the category of simplicial sets.)

- *Are Kan complexes algebraic enough to be a good / useful model of some notion of ∞-groupoids?*
- Possible solution: add composites in to make them more algebraic, e.g., generate a free simplicially enriched groupoid on each simplicial set.

$\mathcal{S}-Grpds$ = simplicially enriched groupoids, adding in formal composites of 'horns'.

- $\mathcal{G} : sSet \to \mathcal{S}-Grpds$.
- The functor $\mathcal{G}$ has a left adjoint, $\overline{W}$.
- For any $\mathcal{S}$-groupoid, $\mathbb{G}$, $\overline{W}\mathbb{G}$ is a Kan complex.
- These functors give an equivalence of homotopy categories, and thus
- $\mathcal{S}$-groupoids 'satisfy the HH'.

Towards a ‘dHH’ for directed homotopy?

Assume some idea of ∞-category, (to be returned to shortly). Let r be a non-negative integer

Idea:

- An ∞-category is an (∞, r)-category if all n-cells are weakly invertible for all $n \geq r$.
- ..., so an ∞-groupoid is an $(\infty, 0)$-category; similarly for (n, r)-categories.

- Perhaps a form of dHH would be:

$$\begin{array}{ccc} \text{Dir.Spaces} & \longleftrightarrow & (n,1)\text{-categories} \\ \text{up to } n\text{-homotopy} & & \text{up to } (n+1)\text{-equivalence} \end{array}$$

... and it is this idea that we want to test.

The challenge is, thus, to make definitions of '*n*-category' (and also of 'Dir.Spaces'), so that this works.

We will collect up some oldish ideas and constructions and add in some new thoughts.

Some reminders, terminology, notation, etc.

- Pospace, $X = (X, \leq)$, a space with a closed partial order.
- A d-space (Grandis) is a space, X, with a set, dX, of *distinguished paths*, or *dipaths*, closed under existence of constant paths, 'subpaths' and concatenation; ref. Grandis, [3]. (NB. Pospaces give d-spaces.)
- $\vec{I}$, ordered interval of length 1.
- $p : \vec{I} \to X$, a dipath from a to b, so $p(0) = a$ and $p(1) = b$.

- $\vec{P}(X)(a, b)$ = the set of dipaths from a to b in X.
- $\vec{T}(X)(a, b) = \vec{P}(X)(a, b)/\sim$, (with $\sim$ being equivalence by increasing parametrisation), that is,

  $\vec{T}(X)(a, b)$ is the set of *traces* in X from a to b.

  $\vec{T}(X)$ is the *trace category* of X.

More reminders, etc.

- Δ^n= standard n-simplex (trivial partial order)
- $\vec{\Delta}^n$= standard n-simplex induced order from ordered n-cube, so
- $\underline{x} \in \vec{\Delta}^n \Leftrightarrow \underline{x} = (x_1, \ldots, x_n)$ with $0 \leq x_1 \leq x_2 \leq \ldots \leq x_n \leq 1$ as a subobject of $(\vec{I})^n$.
- Intermediate models allowing simplices with *some* direction, i.e. as subspaces of $I^k \times (\vec{I})^{n-k}$. (We will concentrate on the simplest case, with the Δ^n.)

Singular simplicial traces

- $\vec{\mathbb{P}}(X)_n(a, b) =$ the set of singular simplicial dipaths of dimension n, from a to b in X, more precisely,
- it consists of dimaps: $\sigma : \vec{I} \times \Delta^n \to X$, such that $\sigma|_{0\times\Delta^n}$ is constant at a, whilst $\sigma|_{1\times\Delta^n}$ is constant at b.

- $\vec{\mathbb{T}}(X)_n(a, b) =$

  the set of singular traces of dipaths of dimension n, from a to b in X

- i.e., $\vec{\mathbb{T}}(X)_n(a, b) = \vec{\mathbb{P}}(X)_n(a, b)/\sim$, (with $\sim$ being equivalence by increasing parametrisation on the $\vec{I}$-variable).

We note:

- varying n gives a Kan complex, $\vec{\mathbb{T}}(X)(a,b)$;
- there is a composition

$$\vec{\mathbb{T}}(X)(a,b)\times\vec{\mathbb{T}}(X)(b,c)\to\vec{\mathbb{T}}(X)(a,c),$$

  which is associative, and
- there are identity traces at each vertex:
  in other words,
- $\vec{\mathbb{T}}(X)$ is a (fibrant / locally Kan) $\mathcal{S}$-category.

Suggestions on how to use $\vec{\mathbb{T}}(X)$:

- Take chain complex of each $\vec{\mathbb{T}}(X)(a, b)$ (over some field, $\Bbbk$). This gives a differential graded category, which includes the information on the 'Natural Homology' of Dubut-Goubault and, with Goubault-Larrecq, of 'Directed Homology', (see Jeremy's presentation).
- Take the fundamental groupoid of each Hom-set $\vec{\mathbb{T}}(X)(a, b)$ to get a groupoid enriched category, $\Pi_1\vec{\mathbb{T}}(X)$.
- For given (simple) d-spaces, look for small models of $\vec{\mathbb{T}}(X)$, i.e., with a finite set of objects and 'manageable' simplicial sets, yet weakly equivalent to $\vec{\mathbb{T}}(X)$, (perhaps some sort of 'minimal model theory'?)
- ... following that up, is there a theory of 'Sullivan forms' on such objects?

- ‘Future’ and ‘past’: representable and corepresentable functors
  For a point $a \in X$, consider the *simplicial functor*,

  $$\vec{\mathbb{T}}(X)(a, -) : \vec{\mathbb{T}}(X) \to \mathcal{S}.$$

  This encodes the possible future from a onwards. Similarly, for $b \in X$, $\vec{\mathbb{T}}(X)(-, b)$ encodes the possible past of b. (Note these are functors on $\vec{\mathbb{T}}(X)$, so their invariants should probably also be functors on $\vec{\mathbb{T}}(X)$.)
- Use bar construction, (free linear cocategory construction), to obtain twisted cochains, classifying varying fibre-bundle-like constructions, see [6].

Some models for $(\infty, 1)$-categories:

- simplicially enriched categories (like our $\vec{\mathbb{T}}(X)$)
- quasi-categories (= weak Kan complexes)
- Segal categories
- A_∞-categories (non-linear form)
- ...

  (Look at Bergner, [2], for details.)

We will look only at the first two.

Simplicially enriched categories or, more briefly, $\mathcal{S}$-categories.

- Any 'category with weak equivalences' gives an $\mathcal{S}$-category (Dwyer-Kan Hammock localisation);
- (Bergner, [1], 2007) $\mathcal{S}-Cat$ has a cofibrantly generated model category structure with
  - weak equivalences : - $f : \mathcal{C} \to \mathcal{D}$ such that each $f(a_1, a_2) : \mathcal{C}(a_1, a_2) \to \mathcal{D}(f(a_1), f(a_2))$ is a weak equivalence in $\mathcal{S}$ and $\pi_0(f)$ is an equivalence of categories;
  - fibrations : each $f(a_1, a_2)$ is a fibration in $\mathcal{S}$ and f is essentially epi modulo homotopy,
  - cofibrations : LLP w.r.t acyclic fibrations,

so ...

- fibrant $\mathcal{S}$-categories are ‘locally Kan’, like our $\vec{\mathbb{T}}(X)$,

  thus fibrant $\mathcal{S}$-categories are enriched over ‘∞-groupoids’ (not a bad start) and
- a cofibrant $\mathcal{S}$-category is a retract of a free $\mathcal{S}$-category.

- Idea: starting with a ‘real-life’ d-space, X, find a finite ‘simplicial polygraph/simplicial computad’ presenting it, i.e., generating a fibrant simplicially enriched category weakly equivalent to it. (This is to catch and present lots of higher dimensional information, not just ‘connectivity’ or similar.)

Quasi-categories aka *weak Kan complexes*: these are

- simplicial sets having fillers for all (n, k)-horns with $n > 0$ and $0 < k < n$. (N.B. Kan complexes have fillers for all (n, k)-horns with $0 \leq k \leq n$.)

  Notion due to Boardman-Vogt, (1973), exploited by Jean-Marc Cordier (and with TP), (1980s), then popularised and their theory expanded by Joyal, (2002), Lurie, (2009), and others.
- **Examples:** If C is a category, $Ner(\mathrm{C})$ is a quasi-category. (It will be a Kan complex if, and only if, C is a groupoid.)
- QCat will denote the category of quasi-categories (NB. This is not that nice as a category, just like Kan is not!)

Two useful model category structures on $\mathcal{S}$:

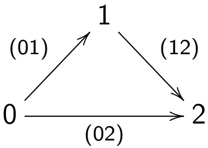
- The category $\mathcal{S}$ has a model category structure in which the fibrant objects are the Kan complexes (namely the classical one).
- The category $\mathcal{S}$ has a model category structure in which the fibrant objects are the quasi-categories (namely Joyal's).

From $\mathcal{S}-Cat$ to $QCat$ and back:

The homotopy coherent nerve of a fibrant $\mathcal{S}$-category is a quasi-category. (Cordier-Porter, (1986))

- **Explanation:** For each $n > 0$, let $[n] = \{0 < \ldots < n\}$, thought of as a small category. There is a functorial comonadic simplicial resolution, $S[n] \to [n]$, which is the identity on objects

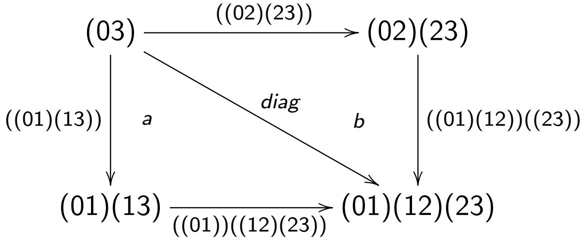
Examples: [2] looks like



so $[2](0,2)$ is a singleton, but the ‘hom’ from 0 to 2 in $S[2]$,

$$S[2](0,2) = \left((02) \xrightarrow{((01)(12))} (01)(12) \right) \cong \Delta[1]$$

As for $S[3](0,3)$, this is a square, $\Delta[1]^2$, as follows:



where the diagonal $diag = ((01)(12)(23))$,
$a = (((01))((12)(23)))$ and $b = (((01)(12))((23)))$, ..., and so on.

- These $S[n]$ form the basic building blocks for $\mathcal{S}$-categories.

The homotopy coherent nerve:

$$Ner_{h.c.}(\mathcal{C})_n = \mathcal{S}-Cat(S[n], \mathcal{C}).$$

$$Ner_{h.c.} : \mathcal{S}-Cat \to \mathcal{S}$$

with left adjoint *Rel*, given by gluing copies of the $S[n]$ together. (See Emily Riehl's [7] for some relevant results on this.)

This gives a Quillen equivalence between the $\mathcal{S}$-category model structure and the structure of Joyal on $\mathcal{S}$.

If $\mathcal{C}$ is an $\mathcal{S}$-groupoid, then $Ner_{h.c.}(\mathcal{C})$ is homotopy equivalent to $\overline{W}(\mathcal{C})$, the classifying space of $\mathcal{C}$. (The equivalence can be made explicit.)

Back to 'd-spaces':

- Write $\vec{\mathbf{T}}(X) = Ner_{h.c}(\vec{\mathbb{T}}(X))$.
- This is a quasi-category, and varies functorially with X.
- As a simplicial set, $\vec{\mathbf{T}}(X)_0 = \mathcal{S}-Cat(S[0], \vec{\mathbb{T}}(X))$, so it has the points of X as its zero simplices;
- $\vec{\mathbf{T}}(X)_1 = \mathcal{S}-Cat(S[1], \vec{\mathbb{T}}(X))$, so consists, just, of the traces between points,
- $\vec{\mathbf{T}}(X)_2 = \mathcal{S}-Cat(S[2], \vec{\mathbb{T}}(X))$, so gives undirected homotopies between traces,

  and so on.

- (N.B. for *directed* homotopies, we would need to use the directed $\vec{\Delta}^n$ instead of the undirected ones, and we would have a structure which was not a quasi-category.)

This gives

Dir.Spaces	$\rightarrow$	$(n,1)$-categories
up to n-homotopy		up to $(n+1)$-equivalence

at least, for $n = \infty$, and a realisation functor in the other direction:

Using the directed simplices, $\vec{\Delta}^n$, as d-spaces, in the coend description of $|K|$ gives a d-space. (I suspect, but have not proved, that this gives an ∞-equivalence of some type.)

Questions and ‘things to do’:

1. Check if the proposed ‘dHH’ works? If it does, now what? If it does not, what is the subtlety?
2. Look at n-types, for low values of n, and apply to analysis of d-spaces.
3. Look at variants of $(\infty, 1)$-categories, occurring via polygraphs in rewriting theory, and apply the techniques of directed homotopy to that context. (This would involve examining [7] from a rewriting perspective.)
4. Try to develop ‘minimal model’ theory for $(n, 1)$-categories, so as to aid our understanding of applications in concurrency.

[1] J. E. Bergner, *A model category structure on the category of simplicial categories*, Trans. Amer. Math. Soc, 359, (2007), 2043 – 2058.

[2] J. E. Bergner, 2010, *A survey of $(\infty, 1)$-categories*, in *Towards higher categories*, volume 152 of *The IMA Volumes in Mathematics and its Applications,*, 69 – 83, Springer.

[3] M. Grandis, 2009, *Directed Algebraic Topology: Models of non-reversible worlds*, Cambridge University Press.

[4] A. Grothendieck, 1983, *Letter from Grothendieck to Dan Quillen, (19/2/1983)*, Grothendieck circle.

[5] A. Grothendieck, 1983, *Pursuing Stacks*, 'tapuscript', 600 + pages.

[6] T. Porter, *Enriched categories and models for spaces of evolving states*, Theoretical Computer Science, 405, (2008), 88–100.

[7] E. Riehl, *On the structure of simplicial categories associated to quasi-categories*, Math. Proc. Camb. Phil. Soc., 150, (2011), 489–504.