BASIC CONCEPTS IN HOMOTOPY THEORY

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Abstract. These are notes on basic concepts in the modern approach to homotopy theories also known as (∞,1)-categories assembled from many good references and aimed at students of computer science, physics or mathematics. They are supposed to cohesively fill the gap of knowledge (and in the current literature) for students who had first courses in category theory, algebraic topology and homological algebra and want to be able to seriously start to think about and understand concepts in higher categories. For this purpose, the present notes try to organically combine a review of these courses from a more advanced categorical perspective with a treatment of basics concepts in model categories, quasicategories and also (∞, n)-categories.

0. Summary and outline

Summary. In these notes we present basic ideas of the modern perspective on homotopy theory which (in its 1-categorical incarnation) can be regarded as the study of categories with weak equivalences, or (in its higher categorical incarnation) as the study of (∞,1)-categories. We will also review the basic tools from category theory to explain these ideas concisely and include a short account on classical homotopy theory and cohomology to provide motivation for why they were developed. All of the ideas presented in this essay are well-established (though some have been developed only in the past decade) and in particular there is no claim of originality by the author. We still hope that the selection of topics and the discussion of how they relate to each other proves useful to other students and the interested reader. These notes are intended to be expanded and improved[1] gradually with time. The main references for the work presented so far are [Hat02], [May99], [Rie14], [DS95], [Gro15] and [SP14].

Outline. After a very brief discussion of what constitutes category theory and how to possibly generalize this to higher category theory, the first part of section 1 will review some of its important tools. We start with giving various examples to a central concept in category theory: representing mathematical structure within a category by universal properties. This includes notions of free generation, duality, weighted and conical limits, coends, cographs and further tools for their description like the Grothendieck construction and profunctors. An item of the same list but so much universally applicable that it deserves its own section is the concept of Kan extensions. We will use them to find easy access to many ideas like Yoneda structures, Yoneda extensions and simplicial sets. The latter will be explained more in detail from different perspectives including a first digression to quasicategories

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[1]There will be necessarily loads of errors which are all due to the author, and will gratefully be fixed if emailed to christoph.dorn@cs.ox.ac.uk

and we will try to argue for why simplicial (and semi-simplicial) sets are so useful for many applications in mathematics. A brief introduction to accessibility and locally presentable categories will finish our review of category theory.

In the second part of section 1 we will discuss classical topics in algebraic topology with a rather categorical flavour. This is the perspective on algebraic topology that we need to understand the usefulness of all following chapters. We start by introducing the notion of (co)fiber sequences which will lay the groundwork to easily state many elementary results which would have otherwise required quite a lot of "topological intuition". For our discussion of cohomology, we do not choose the usual path of constructing a classical homology theory, but instead deduce their existence from a homotopical characterisation of (co)homology via spectra and their uniqueness from a result based on their axiomatisation. A very short discussion on classifying spaces is included at the end of this part which has relevance for some further topics discussed in [SP14].

Finally we will very briefly broaden our view to discuss cohomology in a more general setting than topological spaces. For this we introduce the notion of generalized spaces and quantities. Based on some examples that we provide, we will find the need for higher categories is ubiquitous in order to have a complete categorical description of some well-known mathematical structures. This higher structure makes the study of generalized spaces into the study of $(\infty,1)$-toposes.

In section 2 we will mostly discuss the basics concepts of unenriched model categories. This mainly consists of reiterating constructions which we have already seen for topological spaces in a more formal context. After giving the construction of the small object argument, the use of the recognition theorem (as presented e.g. in [Hir03]) will facilitate our discussion of the most famous examples $\mathbf{Ch}_{\geq 0}(R)$ and **Top**. Note that the enriched case is the more general approach to the subject and has the advantage that the hom objects already inherit homotopical structure from the category $\mathcal{V}$ we are enriching in. It is discussed e.g. in [Rie14] (which is partly based on [Shu09]). This is in contrast to the ordinary treatment where we have to introduce some concepts from mid-air inspired by our discussion of topological spaces. We will give a discussion of derived functors in the unenriched framework, and remark on how they provide a notion of global homotopy (co)limits. Only in the last part of this section we will briefly generalize this and give an overview of the more modern approach following Riehl's book (since this will be compared to constructions on quasicategories) - in this last section some knowledge about enriched category theory is required for which we refer the reader to the standard reference [Kel82] (or for a much more leisure introduction [Dor14]).

We will reach a much younger theory in section 3, when we will start our discussion of the theory of quasicategories, a certain model of $(\infty,1)$-categories most prominently developed by Lurie, Joyal and others. Apart from presenting the very basic constructions on quasicategories this section has two purposes: Firstly, we want to show how the concept of a (symmetric) monoidal structure can be transferred to the setting of $(\infty,1)$-categories as it will be used in the final part of this essay. Secondly, based on the first part we want to finish off our short discussion of the category of spectra and its stable homotopy category which we mentioned in our review of classical cohomology. This will generalize to a process of stabilisation of certain infinity categories, and also yield a notion of highly structured ring spectrum describing special cohomology theories. Our presentation will mostly follow [Gro15].

In the final section 4 of this essay we will try to bring together the topics discussed so far. We start with remarks of how the reconcile the view on homotopy coherent functors and homotopy (co)limits in (∞,1)-categories and model categories. This will be a key step in understanding how the translation between the (∞,1)-categorical picture and the 1-categorical perspective works. We will then broaden our view and discuss the simplest model of (∞, n)-categories, namely Segal n-categories. For this we will need to introduce the notion of weak enrichment and we will see how it relates in many ways to the classical concept of enrichment. With a model of (∞, n)-categories at hand we can then speak about the cobordism hypothesis in a formal way leaving out most of the geometry except for a short remark at the end of this section. Instead we will focus on understanding how the "category of dualizibility data" looks like. Finally, we give a very brief account of a possible axiomatisation of the theory of (∞, n)-categories following [SP14] and discuss some of its strong and useful consequences.

I would like to thank my supervisors Samson Abramsky and Christopher Douglas for always answering my questions and letting me spend an unreasonable amount of time on writing this unnecessarily long document. I would like to apologise that this document is still not in a polished state, and unfortunately crossing the boundary between mathematics and philosophy in a few places.

Contents

1. Preliminaries and Motivation

The purpose of this introduction is to present preliminaries beyond basic category theory (categories, functors, adjoints, monads, monoidal categories) and algebraic topology (singular, cellular and simplicial (co)homology, fundamental group, fiber bundles). The focus will be on the motivation of these concepts and how they relate to each other, *not* comprehensive definitions with extensive lists of examples. The latter can be found e.g. in standard textbooks or on the nLab.

1.1. **Category theory.**

We start by recalling some philosophy and basic concepts of ordinary category theory which will be useful for this essay.

1.1.1. *Meeting higher categories.* In category theory we study blackboxed processes called *morphisms* with finite inputs and outputs of certain types called *objects* together with confluent and terminating *reduction rules* for all compatible composites. Usually we further expand the collection of morphisms (and accordingly our rules) to include identity morphisms. More commonly, one would just say composition has identities and is associative. But this does not emphasize the process-like nature of composition itself which will play a role here.

Ordinary category theory is powerful enough to describe composition itself if we add a notion of "monoidal composition" at the right place: Composition in a category $\mathcal{C}$ is described by processes with inputs $\mathcal{C}(a,b)$, $\mathcal{C}(b,c)$ and output $\mathcal{C}(a,c)$. Here, $\mathcal{C}(a,c)$ stands for morphisms from a to c, but importantly we will just blackbox this into an object of some category $\mathcal{V}$. Composition should become a morphism in $\mathcal{V}$. To make this happen we need a way to think of the input tuple $\mathcal{C}(a,b)$, $\mathcal{C}(b,c)$ as one object. This is achieved via a *monoidal structure* allowing for an single object $\mathcal{C}(a,b) \otimes \mathcal{C}(b,c) \in \mathcal{V}$. Building the rewriting rules from the first paragraph into this monoidal structure we obtain the theory of ordinary $\mathcal{V}$-enriched categories. Details on monoidal structures and enrichment can be found in [Kel82]. This "monoidal" composition however is secretly making $\mathcal{V}$ into a higher category.

Remark 1.1. In fact, the classic idea of enrichment leads to higher categories in at least two ways

(i) (Delooping). A monoidal product on $\mathcal{V}$ allows for objects A, B to be composed $A \otimes B$ just as morphisms. In fact we can naturally interpret objects of $\mathcal{V}$ to actually be morphisms, which then live in the delooping of $\mathcal{V}$, $B\mathcal{V}$, a so-called bicategory. A bicategory, has objects and morphism but also 2-morphisms. Morphisms of $\mathcal{V}$ become 2-morphisms of $B\mathcal{V}$, and their domain and codomain (which where objects of $\mathcal{V}$) are now morphisms of $B\mathcal{V}$. But importantly, the associativity law for the monoidal product is not formulated by an equality (as it is the case for morphisms), but using an isomorphism in $\mathcal{V}$. This translates to morphisms in $B\mathcal{V}$ composing associatively "up to 2-isomorphism", but not up to equality.

(ii) (Inductive enrichment). A special case is the enrichment over the category of categories and functors **Cat**, or n-categories, inductively yielding strict $(n+1)$-categories and a category $(n+1)$-**Cat** $=$ (n-**Cat**)-**Cat**. This is however makes the associativity law in 2-categories hold as an equality. Clearly this is not in line with the previous item. Instead we have discovered the notion of "strict" higher categories. Non-strict (usually called weak) higher categories, have isomorphism in place of equalities in their strict analogues, as well as possibly higher isomorphisms relating these isomorphisms. These isomorphism are usually collectively referred to as "coherence data". Coherence data is reason for weak higher categories being substantially more difficult to describe than strict higher categories.

In this work we will work with coherence data from a "topological" perspective, which is the most well-accepted way to uniformly describe all higher coherence, and based on a deep intuition between the algebraic and topological world: The *homotopy hypothesis* (cf. Lemma 3.10). Further note that the fact that we had to add a monoidal product in order to describe categorical composition abstractly is also part of a more general phenomenon: The *microcosm principle*.

1.1.2. *Universality and free generation.* We will specialize to the ordinary case $\mathcal{V} = \mathbf{Set}$ from now on, but many of the ideas presented below don't require Hom objects to be sets. Details and generalistions of the next section can be found in [Rie14], which we often follow in our presentation.

An object $c \in \mathcal{A}$ is said to have the ***universal property*** with respect to "information" $I_a \in \mathbf{Set}$ *functorially* associated to each $a \in \mathcal{A}$ if this information is *naturally* represented by the Hom object of c and a. In other words, an object having universal property I means it is *representing* the functor $I : \mathcal{A} \to \mathcal{V}$, i.e. there is a natural isomorphism

$$\mathcal{A}(c, a) \cong I_a$$

We consider special cases of this idea

Example 1.2. First, let I_a take itself the form of a Hom functor of a category $\mathcal{B}$ precomposed with a functor $U : \mathcal{A} \to \mathcal{B}$. Then we say $c \in \mathcal{A}$ is ***freely generated*** by $b \in B$ with respect to U if for all $a \in \mathcal{A}$

$$\mathcal{A}(c, a) \cong \mathcal{B}(b, Ua)$$

If for every b we can generate a c_b this defines a (left) adjoint functor $F : \mathcal{B} \to \mathcal{A}$ of U with $Fb = c_b$.

Hom functors $\mathcal{A}(a, -) =: Ya$ are themselves freely generated. By abuse of notation denote constant functors $b : * \to \mathcal{B}$ by their value from now on. Then $\mathcal{A}(a, -)$ is generated by $* : * \to \mathbf{Set}$ for $U = a^*$ (precomposition with $a : * \to \mathcal{A}$)

$$\mathbf{Set}^{\mathcal{A}}(Ya, F) \cong \mathbf{Set}^{*}(*, UF) \cong \mathbf{Set}(*, \mathrm{ev}_a F) \cong \mathrm{ev}_a F \tag{1.3}$$

where in the last expression applies to both the enriched and unenriched setting. Noting that this statement is also natural in a gives the full ordinary ***Yoneda Lemma***. Passing to $\mathcal{A}^{\mathrm{op}}$ in the above discussion, we define the ***Yoneda embedding*** $\mathcal{A}(-, a) =: ya$ into the category of presheafs $\mathrm{PSh}(\mathcal{A}) := \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$. Just as a group homomorphism $\mathbb{Z} \to G$ can be described by a single element (namely the image of the free generator of $\mathbb{Z}$), the Yoneda lemma let's us encode natural isomorphisms h, k of a universal property in set elements η, ϵ

$$\mathbf{Set}^{\mathcal{A}}(Yc, I) \ni h \mapsto \eta \in \mathrm{ev}_c\, I$$
$$\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}(yc, I) \ni k \mapsto \epsilon \in \mathrm{ev}_c\, I$$

η resp. ϵ are called units resp. counits. In the case of adjunctions $\mathcal{A}(Fb, a) \cong \mathcal{B}(b, Ga)$ (writing G for U), we have an isomorphism of representables. So the above principle can be applied in both directions: The bijection is determined by both units and counits. Describing the same bijection, they have to be related by the so-called triangular identities. These are exhibiting F and G as (the prototype of) ***duals*** which means in a general 2-category, and in particular in **Cat**, that

$$\Big(F \xrightarrow{\sim} 1 \circ F \xrightarrow{\epsilon_F \,=\, \epsilon.1} F \circ G \circ F \xrightarrow{F\eta \,=\, 1.\eta} F \circ 1 \xrightarrow{\sim} F\Big) = F \xrightarrow{1_F} F \tag{1.4}$$
$$\Big(G \xrightarrow{\sim} G \circ 1 \xrightarrow{1.\epsilon} G \circ F \circ G \xrightarrow{\eta.1} G \circ 1 \xrightarrow{\sim} G\Big) = G \xrightarrow{1_G} G$$

Here, "." denotes *whiskering* of 2-cells (natural transformations) with 1-cells (functors). Finally note that Hom functors themselves also have adjoints called ***copower*** and ***power***, and denoted as:

$$\mathcal{A}(S \cdot a, b) \cong \mathbf{Set}(S, \mathcal{A}(a,b)) \tag{1.5}$$
$$\mathcal{A}(b, a^S) \cong \mathbf{Set}^{\mathrm{op}}(\mathcal{A}(b,a), S)$$

This is a particular instance of (co)tensoredness over the category we are enriching in for the ordinary case of **Set**-enriched categories.

Example 1.6. Specialising the previous case, given $J : \mathcal{D} \to \mathcal{A}$ we consider

$$\mathcal{A}(c,a) \cong \mathbf{Set}^{\mathcal{D}^{op}}(*, \mathcal{A}(J-, a))$$

and call such a c a ***colimit*** of the diagram J if it exists. Here $*$ denotes the constant functor on the singleton set. Passing to opposite categories $\mathcal{D}^{\mathrm{op}}, \mathcal{A}^{\mathrm{op}}$ we obtain the dual concept of ***limits***.

$$\mathcal{A}(a,c) \cong \mathbf{Set}^{\mathcal{D}}(*, \mathcal{A}(a, J-))$$

Construction 1.7 (Coends and ends). We consider a special colimit called ***coend*** whose significance will become more apparent in the next example. Given a bifunctor $H : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{A}$ the coend is defined as the following coequalizer (and denoted by an integral)

$$\textstyle\coprod_{f:c\to c'} H(c',c) \overset{H(f,1)}{\underset{H(1,f)}{\rightrightarrows}} \coprod_{c\in\mathcal{C}} H(c,c) \dashrightarrow \int^{c\in\mathcal{C}} H(c,c)$$

Passing to opposite categories, we obtain a limit called ***end*** of H and denote it by $\int_{c\in\mathcal{C}} H(c,c)$. Alternatively, we can regard a coend as a colimit of the diagram obtained from H by erasing all images of morphisms that are not of the form $(1_c, f : c \to c')$ or $(f : c \to c', 1_{c'})$. Thus, if H is constant in $\mathcal{C}^{\mathrm{op}}$ the above coend is just the usual colimit of H. From the above formula we can also easily calculate the special case of $H = \mathcal{B}(F-, G-)$, for $F, G : \mathcal{C} \to \mathcal{B}$, in which case the end of H is just the set of natural transformations between F and G.

Example 1.8. Generalising the previous case, we replace the constant functor $* : \mathcal{D} \to \mathbf{Set}$ with an arbitrary *weight* $W : \mathcal{D}^{\mathrm{op}} \to \mathbf{Set}$ yielding a definition of ***weighted colimits*** via

$$\mathcal{A}(c,a) \cong \mathbf{Set}^{\mathcal{D}^{op}}(W, \mathcal{A}(J-, a))$$
$$c \equiv \mathrm{colim}^W J$$

Similarly, with opposite categories and using a weight $W : \mathcal{D} \to \mathbf{Set}$ we obtain ***weighted limits*** $\lim^W J$.

Construction 1.9 (Grothendieck construction). For such copresheafs W we can construct a ***category of elements*** $\mathbf{el}\,\boldsymbol{W}$, by letting objects $(a, x), x \in W(a)$ be elements in the union of all fibers $W(a)$ and morphisms $(a, x) \to (a', x')$ lie over some $f : a \to a'$ with $W(f)x = x'$ (resp. $W(f)x' = x$ in the presheaf case). We have a canonical discrete ***opfibration*** $\Pi : \mathbf{el}\,\boldsymbol{W} \to \mathcal{A}$ (resp. a ***fibration*** in the presheaf case). Here, opfibration means that fibers depend covariantly functorially on $a \in \mathcal{A}$ and contravariantly for fibrations: In the discrete case of opfibrations, this just means for $\Pi : \mathbf{el}\,\boldsymbol{W} \to \mathcal{A}$, given $f : \Pi(e) = a \to a'$ we have a unique $\phi : e \to e'$ such that $\Pi(\phi) = f$, and so e' is (covariantly) associated to e over f. In the non-discrete case, we will introduce the notion of *cartesian morphism* for this purpose in section 3. The above construction is called the ***Grothendieck construction***. Using the uniqueness property in the case of discrete fibrations it is straight forward to establish an equality of categories

$$\mathrm{DiscreteFib}(\mathcal{A}) = [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$$

As we will see in section 3 this can be categorified to a (strict) 2-equivalence of 2-categories

$$\mathrm{Fib}(\mathcal{A}) \simeq [\mathcal{A}^{\mathrm{op}}, \mathbf{Cat}] \tag{1.10}$$

The (discrete) Grothendieck construction illustrates our previous claim that representables $\mathcal{D}(d, -)$ are freely generated by $\{ 1_d \}$ for ev_d: In the category $\mathbf{el}\mathcal{D}(d, -)$ 1_d is initial.

The Grothendieck construction also allows us to reformulate weighted limits. Expressing natural transformations as ends by Construction 1.7 and using the adjunction (1.5) we first note,

$$\begin{aligned}
\mathcal{A}(a, c) &\cong \mathbf{Set}^{\mathcal{D}}(W, \mathcal{A}(a, J-)) \\
&\cong \int_{d \in \mathcal{D}} \mathbf{Set}(Wd, \mathcal{A}(a, Jd)) \\
&\cong \int_{d \in \mathcal{D}} \mathcal{A}(a, (Jd)^{Wd}) \\
&\cong \mathcal{A}\left(a, \int_{d \in \mathcal{D}} (Jd)^{Wd}\right)
\end{aligned}$$

In the final step we used that hom functors preserve limits. The above (and it's dual) establishes that

Lemma 1.11. *For a diagram J with weight W we have*

$$\begin{aligned}
\mathrm{colim}^W J &= \int^{d \in \mathcal{D}} (Wd) \cdot (Jd) \\
\lim^W J &= \int_{d \in \mathcal{D}} (Jd)^{Wd}
\end{aligned}$$

□

In our specific case of $\mathcal{V} = \mathbf{Set}$, after plugging in the defining limit formula for ends, we can employ our Grothendieck construction to describe the diagram as follows:

$$\begin{aligned}\int_{d\in\mathcal{D}}(Jd)^{Wd} &= \mathrm{eq}\left(\prod_{d\in\mathcal{D}}(Jd)^{Wd} \rightrightarrows \prod_{f:d\to d'}(Jd')^{Wd} \right)\\ &= \mathrm{eq}\left(\prod_{d\in\mathcal{D}}\prod_{x\in Wd}(Jd) \rightrightarrows \prod_{f:d\to d'}\prod_{x\in Wd}(Jd') \right)\\ &= \mathrm{eq}\left(\prod_{w\in \mathbf{el}\,\boldsymbol{W}} J\Pi w \rightrightarrows \prod_{g\in \mathrm{mor}\,\mathbf{el}\,\boldsymbol{W}} J\Pi\,\mathrm{cod}\,g \right)\\ &= \lim_{\mathbf{el}\,\boldsymbol{W}} J\Pi\end{aligned}$$

In the last step we used the standard result that ordinary limits can be expressed as equalisers of products. Thus in the case $\mathcal{V} = \mathbf{Set}$ limits weighted by W reduce to usual limits over the category of elements of W. Still, weights can yield more concise formulations of universal properties even in the **Set** case.

Example 1.12 (Profunctors)*.* In Remark 1.1 (ii) we described the process of introducing higher morphisms in order to recover previously left out mathematical structure, which is often called ***categorification***.

In the broad sense of this word, categorifying functions yields functors, categorifying a notion of homotopy of functions yields natural transformations, and relations become ***profunctors***. A profunctor $H : \mathcal{A} \nrightarrow \mathcal{B}$ is given by a functor $H : \mathcal{B}^{\mathrm{op}} \times \mathcal{A} \to \mathbf{Set}$, and composition of profunctors H, K is defined via

$$(a, c) \mapsto \int^{b\in\mathcal{B}} K(c, b) \cdot H(b, a)$$

Hom functors can also be regarded as a categorified relations. Even better, as shown in the next section about Kan extensions, they provide the identities to the 2-category **Prof** of categories, profunctors and natural transformations. Note that since composition had to be defined as a colimit, this category is our first example of a bicategory (weak 2-category). It provides a natural setting for Hom profunctors, allowing a *proarrow equipment* of **Cat** in **Prof** (namely, given by the 2-functor $P : \mathbf{Cat} \to \mathbf{Prof}$ mapping $(F : \mathcal{A} \to \mathcal{B}) \mapsto \mathcal{B}(-, F-)$). This facilitates the formulation and generalisation of universal properties of concepts like adjunctions ($\mathcal{A}(F-, -) \cong \mathcal{B}(-, G-)$ is really an arrow in **Prof**), full and faithfulness ($\mathcal{A}(-, -) \cong \mathcal{B}(F-, F-)$ is really an arrow in **Prof**), cographs (see next construction) or pointwise Kan extensions (as weighted limits with generalized objects) to theories of more general categories which come with a proarrow equipment - as it is the case for example for $\mathcal{V}$-**Cat**.

Remark 1.13 (more on proarrow equipments)*.* Fully spelled out, the definition of a proarrow equipment does not only require an objectwise bijective and locally full and faithful functor P mapping $F : \mathcal{A} \to \mathcal{B}$ to $\mathcal{B}(-, F-)$, but also that duals $\mathcal{B}(F-, -)$ to each $\mathcal{B}(-, F-)$ exist in the proarrow category. In the explicit case of **Prof** this can be seen using tools from the next section. It is worth mentioning that a proarrow equipement is naturally organized in a double category with profunctors being horizontal arrows, and functors vertical arrows. From this perspective, it is a double category with *companions* and *cojoints*, and in particular the definition of *weighted limits with generalized objects*, which we just alluded to for the purpose of defining pointwise Kan extensions, can then be stated using adjoints to horizontal composition. Other encodings of “categorified relations” (a.k.a. correspondences a.k.a. profunctors) can be given as an element of the slice category over the 1-cell

$\mathbf{Cat} \downarrow C_1$ (used again in section 4) or as a functor into presheafs $K : \mathcal{A} \to \mathrm{PSh}(\mathcal{B})$ if a *Yoneda structure* is present. The latter will be mentioned again in Example 1.17.

Construction 1.14 (Cographs and joins). We finish this section with an example of a colimit in higher category theory. With the correct weak notions of 2-functors, 2-natural transformations and replacing isomorphism by equivalence our discussion about (co)limits also applies to bicategories, i.e. for a 2-category $\mathcal{A}$, 2-functors $W : \mathcal{D} \to \mathbf{Cat}$, $J : \mathcal{D} \to \mathcal{A}$ a W-weighted *2-colimit* of J is an $a \in \mathcal{A}$ s.t. for all $c \in \mathcal{A}$

$$\mathcal{A}(c, a) \simeq \mathbf{Cat}^{\mathcal{D}^{op}}(W, \mathcal{A}(J-, a))$$

Now considering the diagram $J : C_1 \to \mathbf{Prof}$ consisting of a single arrow $R : \mathcal{A} \nrightarrow \mathcal{B}$ in **Prof**, the above $*$-weighted colimit of R will be a category $\mathcal{C}$ called the ***cograph*** such that $\operatorname{obj} \mathcal{C} = \operatorname{obj} \mathcal{A} \cup \operatorname{obj} \mathcal{B}$ and $\mathcal{C}(c, c')$ will be one of the sets $\mathcal{A}(c, c')$ or $\mathcal{B}(c, c')$ or $R(c, c')$ or empty, obviously depending on where c, c' live. In the case that R is terminal, i.e. constant equal to $\{ * \}$, we denote $\mathcal{C}$ by $\mathcal{A} \star \mathcal{B}$ called the ***join*** of $\mathcal{A}$ and $\mathcal{B}$. Unwinding the previous definitions of cographs this means $\operatorname{obj} \mathcal{A} \star \mathcal{B} = \operatorname{obj} \mathcal{A} \cup \operatorname{obj} \mathcal{B}$ and

$$(\mathcal{A} \star \mathcal{B})(c, c') = \begin{cases} \mathcal{A}(c, c') & c, c' \in \mathcal{A} \\ \mathcal{B}(c, c') & c, c' \in \mathcal{B} \\ * & c \in \mathcal{A}, c' \in \mathcal{B} \\ \emptyset & \text{otherwise} \end{cases}$$

The reader should convince himself/herself that this leads to the following universal property of the comma category Δ/J over a diagram $J : \mathcal{A} \to \mathcal{C}$ (where $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{A}}$ is the constant diagram functor):

$$\mathbf{Cat}(\mathcal{B}, \Delta/J) \cong \mathbf{Cat}_J(\mathcal{B} \star \mathcal{A}, \mathcal{C}) \tag{1.15}$$

The right-hand side denotes the full subcategory of functors which are constant and equal to J on $\mathcal{A} \hookrightarrow \mathcal{B} \star \mathcal{A}$. Now, in the ordinary case $\Delta/J =: \mathcal{C}_{/J}$ is nothing but the category of cones over J by the following observation

$$\mathrm{Cones}(J) \cong \mathbf{Cat}_J(* \star \mathcal{A}, \mathcal{C}) \cong \mathbf{Cat}(*, \mathcal{C}_{/J}) \cong \mathcal{C}_{/J} \tag{1.16}$$

This reformulation and the above universal property will be the starting point for defining limits in a certain model of $(\infty, 1)$-categories, and demonstrates some of the usefulness of proarrow equipment, as comma categories are often not easily available.

1.1.3. *Kan extensions.* The ***left Kan extension*** of $F : \mathcal{A} \to \mathcal{C}$ along $K : \mathcal{A} \to \mathcal{B}$ is the functor $\mathrm{Lan}_K F : \mathcal{B} \to \mathcal{C}$ freely generated by F with respect to precomposition by K, i.e. it has the universal property

$$\mathcal{C}^{\mathcal{B}}(\mathrm{Lan}_K F, G) \cong \mathcal{C}^{\mathcal{A}}(F, GK)$$

for all $G : \mathcal{B} \to \mathcal{C}$. Dually we obtain the concept of a *left lifting*. And by passing to opposite categories we obtain right Kan extensions

$$\mathcal{C}^{\mathcal{B}}(G, \mathrm{Ran}_K F) \cong \mathcal{C}^{\mathcal{A}}(GK, F)$$

Example 1.17 (Yoneda structures). A natural example for these universal properties is again provided by the Yoneda Lemma. Our previous statement of the Yoneda Lemma in (1.3) was

$$\mathbf{Set}^{\mathcal{A}}(Ya, F) \cong \mathbf{Set}^{*}(*, Fa)$$

for all F, which establishes that $Ya = \mathrm{Lan}_a *$. Less prominently, but equally important (since often implicitly used) we have

$$\mathcal{A}^{*}(a, b) \cong \mathbf{Set}^{*}(*, (Ya)b)$$

for all $b : * \to \mathcal{A}$, and this establishes a as the left lifting of $*$ along Ya. With these two observations we already found 2 out of 3 axioms defining a ***Yoneda structure***PSh : $\mathcal{D} \to \mathcal{D}$ on a bicategory $\mathcal{D}$. The third is giving a condition for "universal elements" to correspond to representable functors and can details be found in [SW78].

The above definition of Kan extensions is concise, but does not have the same easily verifiable properties (e.g. regarding existence) as for instance weighted limits. We say a right Kan extension as defined above is ***pointwise***, if it is preserved by representables $\mathcal{C}(c, -)$. That is, for all $G : \mathcal{B} \to \mathbf{Set}$ and for all $c \in \mathcal{C}$

$$\mathbf{Set}^{\mathcal{B}}(G, \mathcal{C}(c, \mathrm{Ran}_K F)) \cong \mathbf{Set}^{\mathcal{A}}(GK, \mathcal{C}(c, F))$$

Similarly a left Kan extension is pointwise if it is preserved by $(\mathcal{C}^{\mathrm{op}}(c, -))^{\mathrm{op}} = \mathcal{C}(-, c) : \mathcal{C} \to \mathbf{Set}^{\mathrm{op}}$. Denote $\mathrm{Lan}_K F = (\mathrm{Ran}_{K^{\mathrm{op}}} F^{\mathrm{op}})^{\mathrm{op}}$ by L. Let L be pointwise. By the Yoneda Lemma we then obtain

$$\begin{aligned} \mathcal{C}(Lb, c) = \mathcal{C}^{\mathrm{op}}(c, Lb) &\cong \mathbf{Set}^{\mathcal{B}^{\mathrm{op}}}(\mathcal{B}^{\mathrm{op}}(b, -), \mathcal{C}^{\mathrm{op}}(c, L-)) \\ &\cong \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}(\mathcal{B}^{\mathrm{op}}(b, K-), \mathcal{C}^{\mathrm{op}}(c, F-)) \\ &= \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}(\mathcal{B}(K-, b), \mathcal{C}(F-, c)) \end{aligned} \tag{1.18}$$

which exhibits pointwise Kan extensions as weighted (co)limits. Since representables preserve limits we thus obtain the following

Lemma 1.19. *A left (or right) Kan extension is pointwise if and only if* $\mathrm{Lan}_K Fb$ *is a weighted colimit (and* $\mathrm{Ran}_K Fb$ *a weighted limit resp.) of* F*, with weights* $\mathcal{B}(K-, b)$ *(and* $\mathcal{B}(b, K-)$ *resp.)*

Now by Lemma 1.11 this let's us express pointwise right and left extensions as coends and ends respectively as follows (alternatively, we could write out the (co)limit over the category of elements)

$$\begin{aligned} \mathrm{Lan}_K Fb &= \int^{a \in \mathcal{A}} \mathcal{B}(Ka, b) \cdot Fa \\ \mathrm{Ran}_K Fb &= \int_{a \in \mathcal{A}} (Fa)^{\mathcal{B}(b, Ka)} \end{aligned} \tag{1.20}$$

Example 1.21 ((co)Yoneda lemma). The usefulness of these formulas can be demonstrated by the following derivations for the case $\mathcal{C} = \mathbf{Set}$ and $K = 1_{\mathcal{A}}$. Using Construction 1.7 and (1.20) we find

$$Fb \cong \mathrm{Ran}_1 Fb \cong \int_{a \in \mathcal{A}} (Fa)^{\mathcal{A}(b,a)} = \int_{a \in \mathcal{A}} \mathbf{Set}(\mathcal{A}(b, a), Fa) = \mathbf{Set}^{\mathcal{A}}(\mathcal{A}(b, -), F)$$

$$Fb \cong \mathrm{Lan}_1 Fb \cong \int^{a \in \mathcal{A}} \mathcal{A}(a, b) \cdot Fa = \int^{a \in \mathcal{A}} Fa \cdot \mathcal{A}(a, b) = \mathrm{colim}^F \mathcal{A}(-, b)$$

The first equation is just the Yoneda Lemma, the second is called the ***coYoneda Lemma*** (and can be regarded as the Yoneda Lemma for the Yoneda structure based on copresheafs instead of presheafs).

Similarly, we obtain the following

Lemma 1.22. *If K is fully faithful, we have isomorphisms* $(\mathrm{Ran}_K F)Ka \cong Fa$ *for all* $a \in \mathcal{A}$.

Proof. Following the above proof of the Yoneda Lemma we find

$$(\mathrm{Ran}_K F)Ka \cong \int_{a' \in \mathcal{A}} (Fa')^{\mathcal{B}(Ka,Ka')} = \int_{a' \in \mathcal{A}} (Fa')^{\mathcal{A}(a,a')} \cong Fa$$

□

Example 1.23 (PSh($\mathcal{A}$) is the free colimit completion)*.* Consider the important special case where $K = y$ and $\mathcal{B} = \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$, in which case Kan extensions are called ***Yoneda extensions***. First note that in this case (1.18) takes the form

$$\begin{aligned}\mathcal{C}(\mathrm{Lan}_y Fb, c) &\cong \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}\left(\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}(y-,b), \mathcal{C}(F-,c)\right) \\ &\cong \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}(b, \mathcal{C}(F-,c))\end{aligned}$$

and this establishes $\mathcal{C}(F-,c)$ (the ***nerve***) as right adjoint to $\mathrm{Lan}_y F$ (the ***realisation*** based on F). The coYoneda lemma now obtains the following meaning: $\mathrm{PSh}(\mathcal{A}) := \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$ is the ***free colimit completion of*** $\mathcal{A}$, in the sense that it is freely generated by $\mathcal{A}$ as follows

$$\mathbf{Cat}^L(\mathrm{PSh}(\mathcal{A}), \mathcal{C}) \simeq \mathbf{Cat}(\mathcal{A}, \mathcal{C})$$

where $\mathbf{Cat}^L$ denotes the subcategory of cocomplete categories and colimit preserving functors: Indeed, for every $F \in \mathbf{Cat}(\mathcal{A}, \mathcal{C})$ the extension $\mathrm{Lan}_y F$ is a left adjoint realisation, thus preserves colimits, i.e. it lives in $\mathbf{Cat}^L(\mathrm{PSh}(\mathcal{A}), \mathcal{C})$. By y being full and faithful and Lemma 1.22 we know $\mathrm{Lan}_y F$ actually does extend F. Further, by the coYoneda lemma clearly there is a unique colimit preserving extension (up to isomorphism).

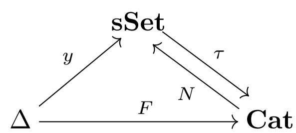
Example 1.24 (Yoneda extensions)*.* Next we consider any functor $G : \mathrm{PSh}(\mathcal{A}) \to \mathcal{C}$ preserving colimits, with $\mathcal{C}$ cocomplete (so that we can form pointwise left Kan extensions). Set $F = Gy$. Again, by Lemma 1.22 since y is fully faithful we have $\mathrm{Lan}_y Fa \cong Fa = (Gy)a$. But we just established that PSh($\mathcal{A}$) is the free colimit completion of $\mathcal{A}$. Since G preserves colimits by assumption and $\mathrm{Lan}_y F$ by Example 1.21, we deduce $\mathrm{Lan}_y F \cong G$ and by previous example G has right adjoint $G \dashv \mathcal{C}(F-,-)$.

Example 1.25 (Representables on presheafs)*.* Now specialize to a colimit preserving $G : \mathrm{PSh}(\mathcal{A}) \to \mathbf{Set}^{\mathrm{op}}$ and denote by $X^{\mathrm{op}} = Gy : \mathcal{A} \to \mathbf{Set}^{\mathrm{op}}$ the corresponding presheaf. Clearly, both G and $\mathrm{PSh}(\mathcal{A})(-,X) : \mathrm{PSh}(\mathcal{A}) \to \mathbf{Set}^{\mathrm{op}}$ preserve colimits. They coincide on the image of y and are therefore isomorphic to the same left Kan extension. We thus proved that $G : \mathrm{PSh}(\mathcal{A}) \to \mathbf{Set}^{\mathrm{op}}$ is representable if and only if it preserves colimits.

Construction 1.26 (Simplicial sets)*.* Based on our discussion of Kan extensions we recall basics about simplicial sets. Let $[n]$ denote the poset category of $n+1$ ordered elements. Let Δ denote the full subcategory of $\mathbf{Cat}$ of all $[n], n \in \mathbb{N}_0$. It is not hard to show that all functors in Δ can be build from "co-deleting" and "co-duplicating" operations $d_i : [n-1] \to [n]$ and $s_i : [n+1] \to [n]$ where d_i^{-1} deletes i, and s_i^{-1} duplicates i. These operations are called ***face*** and ***degeneracy*** maps respectively. The category of simplicial sets **sSet** is given by $\mathrm{PSh}(\Delta)$. We just established the latter to be the free colimit completion of $y : \Delta \hookrightarrow \mathrm{PSh}(\Delta)$.

Define $\Delta^n := y[n]$ which is called an n-simplex. Given a simplicial set X by Yoneda we have $X_n \equiv X[n] \cong \mathbf{sSet}(\Delta^n, X)$. Thus we can think of "$n$-simplices in X" (defined to be elements of X_n) as natural transformations from Δ^n. In particular, if x is and n-simplex in X then it's i-th face is the $(n-1)$-simplex denoted by the natural transformation $xd_i \equiv xy(d_i) : \Delta^{n-1} \to X$ (as opposed to the usual notation $d_i(x) \equiv X(d_i)(x)$). This notation and further details can be found in [Rie11a]. Define $\Delta^T = \mathbf{Cat}(-, T)$ for a poset T. Containment of simplicial sets is inherited from **Set** (monos are pointwise monos): for instance $yd_1 : \Delta^1 \hookrightarrow \Delta^2$ is denoted by $\Delta^{\{0,2\}} \subset \Delta^2$. A ***horn*** Λ^n_i is the simplicial subset of Δ^n with the only nondegenerate n-simplex and the i-th face removed: for instance $\Lambda^2_1 = \Delta^{\{0,1\}} \cup \Delta^{\{1,2\}} \subset \Delta^2$. We say a horn $\Lambda^n_i \hookrightarrow X$ can be ***filled*** if there is $\Delta^n \hookrightarrow X$ such that Λ^n_i is the ith horn of this Δ^n. We say a horn Λ^n_i is inner if $0 < i < n$.

We can transfer Example 1.24 to our current situation. First let $F : \Delta \hookrightarrow \mathbf{Cat}$ be the inclusion. This yields an adjunction $\tau := \mathrm{Lan}_y F \dashv N := \mathbf{Cat}(F-, -)$, where N is called the ***nerve of a category***, $\tau(X)$ realizes X as a category build from X_0, X_1 and X_2 by the rules described in the next remark.



A simplicial set lies in the essential image of N if it satisfies e.g. the Segal condition (as used in section 4) or the inner horn filling condition (as used in section 3). Both will be explained further in the next construction. This is a first (and most basic example) how the nerve functor translates between the "algebraic" and "non-algebraic" perspective on category theory: Algebraically **Cat** is just the category of algebras $\mathbf{Alg}(T)$ of the free category monad T on ordinary graphs, which (full and faithfully) translates via N into certain presheaves satisfying e.g. the Segal condition (in particular, "non-algebraic" approaches subsume "algebraic approaches"). The same distinction holds for approaches to higher categories, replacing graphs by n-graphs (n-globular sets), see [Lei04] for more details.

Next we let $F : \Delta \hookrightarrow \mathbf{Top}$ be the realisation of the poset $[n]$ as topological n-simplex with usual face and degeneracy maps. This in turn yields an adjunction $|-| := \mathrm{Lan}_y F \dashv S_{\mathrm{top}} := \mathbf{Top}(F-, -)$, where $|-|$ is called ***geometric realisation***, and S_{top} just gives the familiar simplicial sets consisting of singular chains. We call $S_{\mathrm{top}} = \mathbf{Top}(F-, -)$ the ***singular nerve***.

Finally, given a simplicial set Y let $G = - \times Y : \mathbf{sSet} \hookrightarrow \mathbf{sSet}$. This preserves colimits and thus (again employing Example 1.24 we obtain an adjunction $G \dashv \mathbf{sSet}((y - \times Y), -) = (-)^Y$ showing that **sSet** is *cartesiain closed*.

The following examples, constructions and remarks will all try to connect the definitions that we just gave to theory that will be developed only later on in this essay, drawing the bigger picture of how we are going to use them. The reader might want to skip directly to section 1.1.4 and come back to this part later, or not care too much about the use of notions that have not yet been defined.

Example 1.27 (Dold-Kan correspondence). Let us shift to the enriched setting for a moment, namely enriching in abelian groups (this assumes we have found an appropriate notion of enriched Kan extension - Actually we have done so already: pointwise Kan extension where exhibited as weighted limits which carry over to the enriched setting straight-forwardly. Also note that we can use the free group functor $F : \mathbf{Set} \to \mathbf{Ab}$ as a *change of base functor* in the following). Then the enriched Yoneda extension of a certain $i : \Delta^{\mathrm{op}} \to \mathrm{Ch}(\mathbb{Z})$ along $y : \Delta^{\mathrm{op}} \to \mathbf{Ab}^{\Delta^{\mathrm{op}}} = \mathbf{sAb}$ yields as before an adjunction $N \dashv |-|$: Here the precise definition of i (it maps $[n]$ to the chain complex associated to $Fy[n]$ by taking alternating sums of faces and modding out images of degeneracies) is not so important to us, but rather we want to note that this adjunction consists of fully faithful essentially surjective functors and thus gives rise to an enriched equivalence of categories.

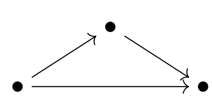
Conceptually, what we need to take away from this is the following: the homotopy theory on $\mathbf{Ch}(\mathbb{Z})$ which we will define in section 2 can be equivalently regarded as a homotopy theory on $\mathbf{sAb}$. The latter gives a generalizing and unifying perspective for comparing “ordinary” homotopy theory on $\mathbf{sSet}$ and enriched homotopy theories $\mathcal{C}^{\Delta^{\mathrm{op}}}$ like the one on $\mathbf{sAb}$. Indeed, categories of simplicial objects $\mathcal{C}^{\Delta^{\mathrm{op}}}$ yield a natural homotopy theory for general categories $\mathcal{C}$ (with coproducts): They are naturally enriched in $\mathbf{sSet}$, actually naturally *tensored* over $\mathbf{sSet}$, by setting $(K \otimes X)_n = K_n \cdot X_n, X \in \mathcal{C}^{\Delta^{\mathrm{op}}}, K \in \mathbf{sSet}$. This action $\otimes : \mathbf{sSet} \times \mathcal{C}^{\Delta^{\mathrm{op}}} \to \mathcal{C}^{\Delta^{\mathrm{op}}}$ then gives rise to simplicial enrichment by an adjunction

$$\mathcal{C}^{\Delta^{\mathrm{op}}}(K \otimes X, Y) = \mathbf{sSet}(K, \underline{\mathcal{C}^{\Delta^{\mathrm{op}}}}(X, Y))$$

where $\underline{\mathcal{C}^{\Delta^{\mathrm{op}}}}$ denotes the enriched hom (details in [Rie14] §3.7). But $\mathbf{sSet}$-enriched categories carry a notion of homotopy (1-simplices) between morphisms (0-simplices) in ther hom spaces. So this also applies to $\mathcal{C}^{\Delta^{\mathrm{op}}}$. In particular, in section 2.5 we use the *simplicial* bar construction to compute a “resolution” of diagram functor regarded as constant *simplicial* object in a diagram category $\mathcal{C} = \mathcal{M}^{\mathcal{D}}$.

Construction 1.28 (quasicategories). As described above, the importance of simplicial sets arises from them giving “non-algebraic” structures naturally subsuming and generalizing the “algebraic” approach to monoids and categories via the nerve functor. To see how this happens morally consider the following: The description of a category in a ZFC-like[2] style was quite long relative to the simple idea of building it from identities, morphisms and their equations. Once $\mathbf{Cat}$ is defined a description of this procedure becomes more natural: A (small) category is obtained from gluing together and taking unions of 0-cells $[0] = \bullet$, 1-cells $[1] = \bullet \longrightarrow \bullet$ and the free walking composition[3]

$[2] \quad =$



So instead of Δ we first consider the full subcategory Δ_2 of [0], [1] and [2] in $\mathbf{Cat}$, similarly one uses Δ_1 for graphs. We have just established that $\mathrm{PSh}(\Delta_2)$ is the free

[2] or preferably for this essay ZFC + (Grothendieck universes exist).

[3] We could also include a (-1)-cell $[-1] = \emptyset$ to restrict ourselves to pushouts. This corresponds to using the category Δ^+ discussed in the next example.

colimit completion of Δ_2 and it is in this way that $\mathrm{PSh}(\Delta_2)$ generalizes **Cat**, the latter being build itself from Δ_2 by colimits. To now take the leap from $\mathrm{PSh}(\Delta_2)$ to $\mathrm{PSh}(\Delta)$ we first look at 1-object categories. A monoid is given by maps

$$m : M \times M \to M \ , \ e : * \to M$$

satisfying associativity. Everyone will be able to compute $m_1 m_2 m_3$, $m_i \in M$ from this by iterated use of m. But instead we could ask for additional maps $m_n : M^{\times n} \to M$ that fully describe multiplication on arbitrary length composites on M. This is the step that takes us from $\mathrm{PSh}(\Delta_2)$ to $\mathrm{PSh}(\Delta)$ - but it only leads to a well-defined composition with additional conditions like the following

Lemma 1.29. *$X \cong N(\mathcal{C})$, $\mathcal{C} \in$ **Cat**, if and only if there exists a unique filler for every inner horn in X.*

An central interesting observation is that a category of convenient topological spaces (e.g. CW complexes as discussed in section 1.2.2) is also built by colimits from Δ interpreted as topological n-simplices via the inclusion above - in other words Δ again is a ***dense*** subcategory. This means that **sSet** also generalizes **Top**! Here, for a simplicial set to lie in the essential image of the topological nerve functor a different condition is in place: there is a (possibly non-unique) filler for *every* horn. This is called a ***Kan complex***.

Lemma 1.30. *$X \cong S_{\mathrm{top}}(Y)$, $Y \in$ **Top**, if and only if it is a Kan complex.*

What happens if we "merge" these two conditions in **sSet**?

Definition 1.31. A ***quasicategory*** is a $X \in$ **sSet** such that there exists a filler for every inner horn.

This will be further discussed in section 3.

Remark 1.32 (Computational view on Δ^n). Let us first describe informally what's going on. The slogan is: An n-simplex encodes the free witnesses of coherence in the binary rewriting of a single string of n arrows $f_1 f_2 ... f_n$. This is really just saying that the categorical nerve of $[n]$ is actually $\Delta^n = y[n]$ as we will soon realize. Disallowing rewrites with identities (degeneracies) for the a moment, i.e. effectively reducing to the case of ***semi-simplicial sets***, we can make the following trivial observations

- **$n = 1$:** A single arrow f_1 is in "normal form", i.e. it cannot be rewritten to a different string without using identites/degeneracies.
- **$n = 2$:** $f_1 f_2$ admits one binay rewrite. We denote the unique "witness" of this rewrite by $g_{12} : f_1 f_2 \to f_{12}$ and call it a 2-simplex.
- **$n = 3$:** Consider $f_1 f_2 f_3$. Apply g_{12} to get $f_{12} f_3$, and then apply $g_{(12)3}$ to get $f_{(12)3}$. But $f_1 f_2 f_3$ actually admits to start with another binary rewrite: Apply g_{23} and then $g_{1(23)}$. Coherence requires a witness of these rules being equivalent which we denote by $h_{123} : g_{(12)3} g_{12} \to g_{1(23)} g_{23}$ and call it a 3-simplex (h_{123} is "directed" in the same way g_{12} is naturally directed based on the ordering induced by the string).
- **$n = 4$:** Consider $f_1 f_2 f_3 f_4$. There are five ways to insert 2 meaningful pairs of brackets in a string of length 4. They correspond to the order of application of our 2-simplex witnesses and can be ordered in a well known pentagramm as they are related by five 3-simplices. They thus give us two witnesses

of coherence, namely $h_{234}h_{1(23)4}h_{123}$ and $h_{12(34)}h_{(12)34}$. So we introduce a 4-simplex j_{1234} witnessing coherence of these witnesses of coherence.

...

$n = k$ **:** There are $\binom{k}{2}$ ways to insert $k-2$ meaningful pairs of brackets. We get 2 witnesses witnessing coherence of witnesses witnessing coherence, and their coherence requires introduction of a k-simplex. The calculations are straight forward though we might run out of letters of our alphabet.

0-simplices are implicit in the above construction when writing down the composite string of arrows in a certain order - they can be recovered from the position of an arrow. Degenerate n-simplices can be included by re-allowing degeneracies (which as previously indicated were disallowed in the above). This construction is just one of many ways to describe an n-simplex and not a spectacular one, but it elucidates how to obtain a n-(semi-)simplex by *freely adding witnesses of coherence.*

Motivated by the above, we define the ***spine*** S_n of an n-simplex $\Delta^n = \Delta^{\{0,1,..,n\}}$ to be the subsimplicial set given by the union $\cup_{0\leq k\leq n-1}\Delta^{\{k,k+1\}}$. The inclusion induces a map $s_n : \mathbf{sSet}(\Delta^n, X) \to \mathbf{sSet}(S_n, X)$. In the view of the above, we can think of a simplicial sets as a *set of composable arrows* with identities such that if n of them are composable, say $f_1...f_n$, we can associate to them a possibly empty set of n-simplices $s_n^{-1}(f_1, ..., f_n)$, each thought of as a coherent set of rules for their composition in the sense of the previous remark.

The map given by s_n will be later called a ***Segal map***. Our informal description of simplicial sets above takes the more concise form of a functor $X : \Delta_{X_0} \to \mathbf{Set}$: Here, Δ_{X_0} is the categories of "spine points" and $X(x_0, ..., x_n)$ should be thought of as the set of n-simplices with these points along their spine. Such a functor is also called a ***precategory***, to which we will come back in section 4. The Segal maps

$$s_n : X(x_0, ..., x_n) \to X(x_0, x_1) \times X(x_1, x_2) \times ... \times X(x_{n-1}, x_n) \tag{1.33}$$

are induced by the obvious iterated face maps in $(\Delta \downarrow X_0)$. If s_n is an isomorphism, then $s_n^{-1}(f_1, ..., f_n)$ is a singleton and thus rules for composition are well-defined, making X a category.

Remark 1.34 (Universality of **sSet**). It would be nice to have some characterisation of **sSet** in terms of a universal property. We will sketch two approaches. Firstly, let $\Delta^+ := * \star \Delta$. Then Δ^+ is the free strict symmetric monoidal category containing a monoid M

$$* \xrightarrow{1} M \equiv [0] \xleftarrow{m} M \otimes M \equiv [1]$$

Δ itself can then be universally characterised by $\mathrm{Fun}(\Delta, \mathcal{C}) \cong \mathrm{Fun}^{colax}(\Delta^+, \mathcal{C})$ for cartesian monoidal categories $\mathcal{C}$.

Simplicial sets also arise universally using the theory of *derivators.* The following is due to Denis-Charles Cisinski ([Cis10]) and actually seeks to answer two much more general questions: "*What is the mathematical structure over which everything is canonically enriched? And [...] how can we formulate correctly such a question?*" Details to the following rough sketch can be found in the original post.

In the **Cat**-enriched setting a ***prederivator*** is a 2-presheaf on **Cat**, and we denote the corresponding 2-category by **PDer**. In particular, for all $\mathcal{C} \in \mathbf{Cat}$ we get a prederivator $\mathbf{Y}\mathcal{C} : \mathcal{X} \mapsto \mathcal{C}^{\mathcal{X}}, \mathcal{X} \in \mathbf{Cat}$ via the Yoneda embedding. For a category $\mathcal{C}$ with weak equivalences (a class of morphisms with 2-out-of-3 property, as explained in section 2) we get a prederivator $\mathrm{Ho}(\mathcal{C})$ mapping $\mathcal{X}$ to the localisation of $\mathcal{C}^{\mathcal{X}}$ at objectwise weak equivalences. If $\mathcal{C}$ is a model category $\mathrm{Ho}(\mathcal{C})$ has the properties of a ***derivator***, a nice prederivator with left duals $u_!$ and right duals u_* for functors in its image $\mathrm{Ho}(\mathcal{C})(F : \mathcal{X} \to \mathcal{Y})$, and these encode for instance the theory of homotopy colimits and resp. limits in $\mathcal{C}$. We can set up a category of derivators **Der** (not a full subcategory of **PDer** since we now want morphisms to commute with left duals $u_!$, i.e. homotopy colimits, but different definitions are possible, e.g. requiring commutation with homotopy limits). The structure that a derivator captures is suggested by Cisinki as an answer to the second question.

It turns out that representable prederivators freely generate derivators with respect to the forgetful functor $U : \mathbf{Der} \to \mathbf{PDer}$, leading to a "derivator Yoneda embedding" as follows:

$$\mathbf{Der}(\widehat{\mathbf{Y}\mathcal{C}}, \mathcal{D}) \simeq \mathbf{PDer}(\mathbf{Y}\mathcal{C}, \mathcal{D}) \cong \mathrm{ev}_{\mathcal{C}}\, \mathcal{D}.$$

Moreover $\widehat{\mathbf{Y}\mathcal{C}}$ can be explicitely described by $\mathrm{Ho}(\mathbf{sSet}^{\mathcal{C}})$, where $\mathbf{sSet}^{\mathcal{C}}$ carries projective model structure as explained in section 2. In particular $\mathrm{Ho}(\mathbf{sSet})$, *the homotopy theory of simplicial sets*, is freely generated by a point $\mathbf{Y}(*)$. Cisinki concludes that derivators are canonically "enriched" over $\mathrm{Ho}(\mathbf{sSet})$ and suggests this as an answer to his first question. For more details on derivators we refer the reader e.g. to [Gro13].

We will see very similar statements later in this essay: We will note that the $(\infty, 1)$-*category of spaces* is similarly freely generated under homotopy colimits by a point $*$: Once having established a notion of (co)limits in section 3 and presheaves in $(\infty, 1)$-categories this statement is just another instance of the presheaf categories being "free colimit completions" by the coYoneda Lemma in Example 1.23. It is thus the $(\infty, 1)$-categorical analogue of saying **Set** is freely generated under colimits by a point. Regarding the last statement, we will also explain and use in section 4 that every model category is "almost" enriched in simplicial sets.

1.1.4. *Accessibility.* So far we didn't speak about size questions for set theory based category theory. In this setting, size and existence issues can be adressed for instance by using (Grothendieck) universes. The notion of accessibility which we will sketch in the following short paragraph is a classical tool for keeping track of sizes in this stratified setting. It will play a role for the definiton of presentable $(\infty, 1)$-categories in section 3, and Quillen's "small object argument" in section 2. Note that a constructive approach like homotopy type theory can circumvent the use of these arguments to handle size issues.

Fix a cardinal κ. Historically, colimits of diagrams with shape of a ***directed sets*** (a preorder with upper bounds for finite subsets) were useful in many places as they commute with taking finite limits. The categorification of directed sets are ***filtered categories*** $\mathcal{C}$, in which every functor $F : \mathcal{D} \to \mathcal{C}$ with small domain can be extended to an "upper bound" functor $F^+ : \mathcal{D} \star * \to \mathcal{C}$. If this holds for domains $\mathcal{D}$ with less then κ morphism, $\mathcal{C}$ is called a κ***-filtered category***. Similarly, κ***-filtered colimits*** are colimits of diagrams with domain $\mathcal{C}$ a κ-filtered category. Note that a κ-filtered colimits are in particular λ-filtered colimits, for $\lambda > \kappa$. An object $A \in \mathcal{A}$

is called κ-***compact*** if $\mathrm{Hom}(A, -)$ preserves κ-filtered colimits, i.e. colimits of some $F : \mathcal{C} \to \mathcal{A}$ with $\mathcal{C}$ κ-filtered. Less obscurely this means that the induced map

$$\alpha : \mathrm{colim}_{\mathcal{C}} \mathrm{Hom}(X, F) \to \mathrm{Hom}(X, \mathrm{colim}_{\mathcal{C}} F)$$

is an isomorphism. But in **Set** a colimit is just the union of all objects in the diagram, modulo an equivalence relation induced by it's morphisms. Then, an equivalence class $[f] \in \mathrm{colim}_{\mathcal{C}} \mathrm{Hom}(X, F-)$ with representative $f \in \mathrm{Hom}(X, Fc)$ is mapped to $\lambda_c f \in \mathrm{Hom}(X, \mathrm{colim}_{\mathcal{C}} F)$ by α, where $\lambda : F \to \Delta_{\mathrm{colim}_{\mathcal{C}} F}$ denotes the cocone under F. Thus, being compact guarantees that every map in $\mathrm{Hom}(X, \mathrm{colim}_{\mathcal{C}} F)$ can be factored through some Fc as $\lambda_c f$. An object is referred to as just ***compact*** (or small) if it is κ-compact for some κ (and thus for all $\lambda > \kappa$).

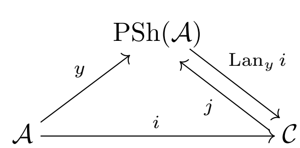
Given a category $\mathcal{B}$ there is a canonical supercategory κ-ind-$\mathcal{B}$ called ***ind-category*** in which all objects of $\mathcal{B}$ are κ-compact: ind-$\mathcal{B}$ is the free κ-filtered colimit completion of $\mathcal{B}$. In other words, it is the full subcategory of $\mathrm{PSh}(\mathcal{B})$ (which we described as free cocompletion in Example 1.23) of functors which can be written as κ-filtered colimits of representables. Finally, an κ-***accessible category*** $\mathcal{C}$ is a category of the form κ-ind-$\mathcal{A}$ for some small category $\mathcal{A}$. A functor between κ-accessible categories is called an ***accessible functor*** if it preserves κ-filtered colimits. If $\mathcal{C}$ happens to be cocomplete it is called a ***locally*** κ-***presentable category***. We refer to it as just locally presentable if some κ makes it locally κ-presentable. It can be shown without to much effort that all objects in a locally presentable category are compact. In addition we have

Proposition 1.35. *An functor $G : \mathcal{A} \to \mathcal{B}$ between locally presentable categories has left adjoint if and only if it preserves small limits and is accessible.*

Proof Sketch. (following [ARJ94]) Assume, G preserves small limits and is accessible. We show it satisfies the *solution set condition* and thus the General Adjoint Functor Theorem applies: Indeed for $B \in \mathcal{B}$ choose $\lambda > \kappa$ such that B is λ-compact and $\mathcal{A}, \mathcal{B}$ locally κ-presentable (recall all objects are compact). Given $A \in \mathcal{A}$ write $A = \mathrm{colim}\, K_i \xleftarrow{h_i} K_i$ by accessibility and $GA = \mathrm{colim}\, GK_i \xleftarrow{Gh_i} GK_i$ by assumption on G. By compactness of B every $f : B \to GA$ now factors through some Gh_i as required.

Conversely, assume G has left adjoint F. We need to show its λ-accesibility for some λ. Let λ be a cardinal bounding from above the set κ_B where B is κ-compact ($\mathcal{A}, \mathcal{B}$ locally κ-presentable) and FB is κ_B compact. Then for a λ-filtered colimit $C_i \xrightarrow{c_i} \mathrm{colim}_{\mathcal{C}} C_i$ we need to show that $GC_i \xrightarrow{Gc_i} G\, \mathrm{colim}_{\mathcal{C}} C_i$ is a colimit. But by κ-accessibility it is enough to verify universality for κ-compact objects B and this then follows from the adjunction (details in Thm. 1.66 of [ARJ94]). $\square$

Remark 1.36 (Presentations of locally presentable categories). For a locally presentable category $\mathcal{C} = \kappa$-ind-$\mathcal{A}$ cocompleteness implies that we can left Kan extend the inclusion $i : \mathcal{A} \hookrightarrow \mathcal{C}$ along the composite inclusion $y : \mathcal{A} \hookrightarrow \mathcal{C} \hookrightarrow \mathrm{PSh}(\mathcal{A})$ which is just the Yoneda embedding:



From $\mathcal{C}$ being a full subcategory and Example 1.24 we deduce an adjunction $\mathrm{PSh}(\mathcal{A})(y-,-) \dashv \mathrm{Lan}_y\, i$, which by the Yoneda lemma immediately implies $j \dashv \mathrm{Lan}_y\, i$ where $j : \mathcal{C} \hookrightarrow \mathrm{PSh}(\mathcal{A})$. But j is full and faithful and so this establishes $\mathcal{C}$ as a reflective localisation (see next remark) of $\mathrm{PSh}(\mathcal{A})$. This explains the terminology "presentable": A presentation of an algebraic structure morally consists of a freely generated structure localized by some relations. The converse also holds (reflective subcategories inherit colimits via the left adjoint "reflector"), we obtain

Theorem 1.37 (Classification theorem)**.** *A category $\mathcal{C}$ is locally representable if and only if it is a reflective localisation of* $\mathrm{PSh}(\mathcal{A})$ *for some small $\mathcal{A}$.*

Remark 1.38 (Localisations and S-locality). We recall the notion of reflective localisation and give a slight reformulation. Given a relative $(\mathcal{C}, S)$, that is a category $\mathcal{C}$ and a collection of morphisms S, the localisation $\gamma : \mathcal{C} \to \mathcal{C}[S^{-1}]$ at S satisfies the universal property

$$\mathrm{Fun}(\mathcal{C}[S^{-1}], \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{rel}}((\mathcal{C}, S), (\mathcal{D}, Isom))$$

where the right hand side means functors $\mathcal{C} \to \mathcal{D}$ mapping S to isomorphisms. Given an adjunction $F \dashv G : \mathcal{B} \to \mathcal{C}$ we say it is a ***reflective localisation*** if G is full and faithful, which is the case if and only if the counits ϵ_B are isomorphisms. By the triangle equalities this implies that images $F\eta_C$ of the units η_C become isomorphisms as well. It is then straight forward to verify that $F : \mathcal{C} \to \mathcal{B} \equiv \mathcal{C}[S^{-1}]$ is a localisation in the sense above, where we set $S = F^{-1}(Isom)$. We want to reformulate this description:

Definition 1.39. Given a relative $(\mathcal{C}, S)$ an object $c \in \mathcal{C}$ is called ***S-local*** if for all $(f : c_1 \to c_2) \in S$ the induced map

$$f^* : \mathrm{Hom}(c_2, c) \to \mathrm{Hom}(c_1, c)$$

is a bijection. A map $(g : d_1 \to d_2) \in \mathrm{mor}\, \mathcal{C}$ is called a ***S-equivalence*** if for all S-local objects $c \in \mathcal{C}$ the induced map

$$g^* : \mathrm{Hom}(d_2, c) \to \mathrm{Hom}(d_1, c)$$

is a bijection

We can then make the following observation

Lemma 1.40. *Given a reflective localisation $F \dashv G$ with notation fixed above we have that c is S-local iff it is in the essential image of G and f is a S-equivalence iff f is in S.*

Proof sketch. Clearly every object in the image of G is S-local by the naturality of the adjunction. Conversely, we need to show that c being S-local implies that η_c is iso (indeed, assume c is in the essential image of G - by naturality of η and η_G being iso this implies η_c is iso.) By $\eta_c \in S$ and c being S local we can find a left inverse for η_c. By naturality of η and the triangle equalities this is also a right inverse. The second statement follows similarly. □

This reformulation in terms of S-locality is useful once we are interested in a homotopical weakening of localisation. For instance it is the terminology which is used for *Bousfield localisations.*

1.2. **Algebraic topology.** Homotopy theory describes the situation when we are interested in the information presented by a category only up to a class of morphisms called weak equivalences. Such a situation is encountered naturally when Hom sets $\mathrm{Hom}(X,Y)$ carry a notion of homotopy between elements $f,g \in \mathrm{Hom}(X,Y)$: This yields the notion of ***homotopy equivalences*** defined as maps with inverse up to homotopy and it allows for a set $[X,Y]$ of homotopy classes of maps. In the unenriched case we will need to encode homotopy data in hom spaces: Data of right homotopies into Y is encoded in the the functor $\mathrm{Hom}(-,Y^I)$ for a ***path object*** Y^I of Y, or dually for left homotopies from X in the functor $\mathrm{Hom}(X\times I,-)$ for a ***cylinder object*** $X\times I$ of X. Both objects should "retract" onto Y and X respectively, and details of the formalisation will be given in section 2. We give a rough overview of this section from this general perspective

(i) The functor $[X,-]$ probes other objects by the test space X. We call it abstract homotopy with coefficients in X. It is convenient to restrict our attention to only a manageable collection of test spaces. In particular, if these come equipped for instance with a comultiplication $\psi : X \to X \sqcup X$ our probe functor $[X,-]$ will factor over categories with richer structure. In the setting of based topological spaces (see section 1.2.1), we recover ***ordinary homotopy theory*** when considering *spheres* as test spaces (here S is the 1-sphere)

$$\pi_k(X) = [\Sigma^n S, X]$$

Based on the information that we obtain from these test spaces about some Z we can introduce a new notion of equivalence which we call ***weak equivalence***. We have to ask when weak equivalence coincides with our old notion of homotopy equivalence: The answer is the subcategory of **CW-complexes**. This subcategory is a *deformation of* $\mathbf{Top}_*$ in the sense that all other spaces can be weakly approximated by CW-complexes - see section 1.2.2 and general deformations in 2.5.

(ii) Dually, we can consider the functor $[-,Y]$ which quantifies all other spaces by how they map into Y. We call it abstract cohomology with coefficients in Y.

$$H(X;Y) = [X,Y] \tag{1.41}$$

Again, our functor factors over a richer category if Y has e.g. a multiplication $\phi : Y\times Y \to Y$. In the setting of based topological spaces we have a notion of homotopy kernel and cokernel allowing a discussion of exactness as long as our functor factors over an abelian category like **AbGp**. One shortcoming of ordinary homotopy theory is that $[S^1, Z]$ is in general not abelian. If we contravariantly mimick the axiomatic properties of ordinary homotopy theory *and* require $[-,Y]$ to take values in **AbGp** we obtain what is called a ***reduced generalised cohomology***. Generalised cohomology theories are classified by ***spectra***, i.e. they correspond to Y taking values in the category of spectra "stabilizing" the category of spaces (as defined with more detail in section 1.2.3). This yields cohomology functors H^k as follows

$$H^k X = \pi_k \,\mathrm{Maps}(X,Y)$$

The category of spectra is enriched in spaces and with further conditions becomes a closed symmetric monoidal category. Some technicalities in these statements have to be discussed section 1.2.3.

(iii) Finally, if we invert the variance of generalized cohomology we obtain functors H_k constituting a ***generalized homology*** theory. This also has a description in the category of spectra, namely $H_k X$ it is the k-th homotopy group in the category of spectra as follows[4]

$$H_k X = \pi_k X \wedge Y$$

which in more technical detail will be again discussed in 1.2.3.

Of course, our initial notation for paths and cylinder objects is suggestive for the case of the category of topological spaces **Top**, i.e. from now on let I be the unit intervall and take path and cylinder spaces Y^I, $X \times I$ as usual. Note that we will work in a convenient category of "constructive" spaces (e.g. compactly generated) and topologise sets of maps by the usual compact-open topology. Then **Top** becomes cartesian closed, i.e.

$$\mathbf{Top}(X \times Y, Z) \cong \mathbf{Top}(X, Z^Y) \tag{1.42}$$

Before we can speak about the concepts of homotopy, CW complexes, cohomology and spectra on **Top** we will lay some technical foundations based on the homotopical structure of **Top** in the next section which will facilitate the description of their properties.

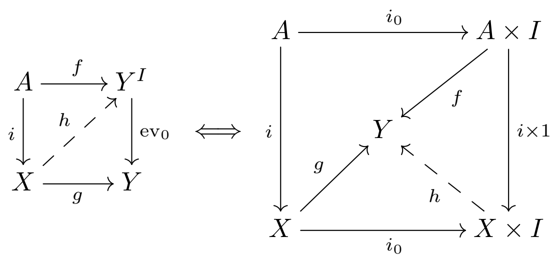
For most parts in this section we will follow [May99] in our presentation and take some inspiration for proofs from [Hat02] from time to time.

1.2.1. *Fiber and cofiber sequences.* We will see in section 2 that a model structure, i.e. notions of cofibrations and fibrations, is a tool to construct the localisation of a category at its weak equivalences yielding its **homotopy category** and the theory of **homotopy (co)limits**. Clearly, this localisation is a posteriori independent of the chosen notion of cofibration and fibration. Further, with view towards section 3 we note that not every homotopy theory a.k.a **(∞,1)-category** is described on a 1-categorical level by a category with this model structure, but in general only by a category with weak equivalences. But since **Top** is a category with weak equivalences that allows a model structure, we will now introduce this structure and then use it to calculate homotopy (co)limits as explained in section 2 - precisely we will need homotopy pushouts and pullbacks along the zero morphism which we will call homotopy cokernels and kernels (see Definition 1.48) and denote by hcoker and hker respectively.

We define ***(Hurewicz) cofibrations*** $i : A \to X$ to be the maps along which we can "extend" homotopy data with given initial conditions, i.e. for all f, g, Y below

[4] The nice dual looking formulation of homology and cohomology was taken from a comment by Tom Goodwillie http://mathoverflow.net/a/63975/74792 and we will attempt to explain it in (1.87).

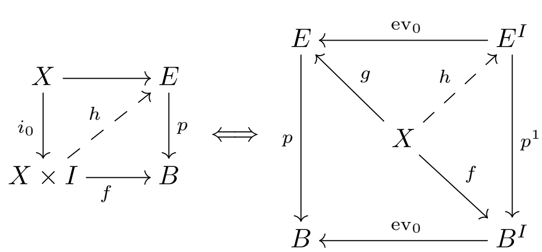
we have an extending homotopy h



for all Y. The mutual implication is obtained by cartesian closedness (1.42) of **Top**. The diagrams have a universal form (evident in the diagram on the right), namely when Y is the pushout of the outer arrows i and $i_0 : A \to A \times I$ which is called ***mapping cylinder*** and denoted by $M_i = A \times I \sqcup_A X$. For every other Y the diagram (and a solution h) will factor over M_f and its solution $r : X \times I \to M_i$. Considering the canonical map $j : M_i \to X \times I$ this shows that $i : A \to X$ is a cofibration if and only if M_i is a retract of $X \times I$. By direct construction of deformation data this can be equivalently characterised by

Lemma 1.43. *$i : A \to X$ is a cofibration $\iff$ $X \times I$ deformation retracts to M_i $\iff$ A is a neighbourhood deformation retract of X*

Dually, we define ***(Hurewicz) fibrations*** to be maps $p : E \to B$ along which we can "lift" homotopy data with given initial conditions, i.e. for all f, g, X we have a lifting homotopy h



for all X. The universal diagram now gives rise to the ***mapping path space*** N_p as the pullback of p and ev_0. It can thus be described as the subspace of $E \times B^I$ of tuples (e, β) such that $\beta(0) = p(e)$. Note that (by Lemma 1.43 and a direct verification) for a map $f : X \to Y$ the following holds: $i_f := i_1 : X \to M_f$ is a cofibration, $e_f := ev_1 : N_f \to Y$ is a fibration and $r : M_f \to Y$ (retract onto Y), $c : E \to N_f$ (embed constant paths) are weak equivalences. These factor f via $f = ri_f$ and via $f = e_f c$, and i_f and e_f are called the ***cofibrant replacement*** and the ***fibrant replacement*** of f, respectively.

Having defined our notion of fibrations and cofibrations we can now calculate our first homotopy (co)limits as promised in the beginning of this section. Recall that in a pointed category the kernel and cokernel (more often called quotient) of a map f are defined by

(1.44)
$$\begin{array}{ccc} F_f & \xrightarrow{\ker f} & X \\ \downarrow & \lrcorner & \downarrow f \\ * & \longrightarrow & Y \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \operatorname{coker} f \\ * & \longrightarrow & \ulcorner C_f \end{array}$$

Construction 1.45 (based spaces). **Top** has a terminal object $*$ and quotient spaces (a.k.a. cokernels, cofibers), but it does not allow kernels using the initial object $\emptyset$. To obtain a fully dual notion of kernels and fibers we thus "initialize" $*$ by passing to the slice category $\mathbf{Top}_* := */\mathbf{Top}$ of ***based spaces***. We will use $*$ to denote basepoints generically, or specify them explicitely in a tuple (X, x_0). The hom sets $\mathbf{Top}_*(X, Y)$ can be naturally topologized as subspaces of the hom spaces Y^X in **Top** and have the constant map $c_* : x \mapsto * \in Y$ as their basepoint. Then we obtain a new monoidal product $\wedge$ called the ***smash product*** in the adjunction

$$\mathbf{Top}_*(X \wedge Y, Z) \cong \mathbf{Top}_*(X, Z^Y) \tag{1.46}$$

Explicitely, $X \wedge Y = X \times Y/\left((X \times *) \sqcup (* \times Y)\right)$. We write $[Y, Z] = \pi_0 Y^Z$ for the path components of the mapping space. To translate our notion of homotopy from **Top** to $\mathbf{Top}_*$we note that we have an adjunction $(-)_+ \dashv U : \mathbf{Top}_* \to \mathbf{Top}$ where U is the forgetful functor and $(-)_+$ adds a discrete basepoint. Thus a path $\gamma \in \mathbf{Top}(I, X)$ is represented by an element of $\mathbf{Top}_*(I_+, X) = X^{I_+}$ yielding our new paths objects. Dually, using our monoidal product, we thus define a cylinder object to be $X \wedge I_+$. Making these replacements in our defining diagrams for fibrations and cofibrations, and taking pullbacks and pushouts in $\mathbf{Top}_*$, we obtain the new notions of ***reduced mapping paths space*** $\overline{N_f}$ and ***mapping cylinder*** $\overline{M_f}$. $\overline{N_f} \subset E \times B^{I_+}$ is effectively N_f with base point $(*, c_*)$. $\overline{M_f} = (A \wedge I_+) \sqcup_A X$ can be obtained from M_f by quotienting out $\{\, * \,\} \times I$ (note: the new coproduct is called ***wedge product*** and denoted by $\vee$ instead of *sqcup*). Cofibrant and fibrant replacements for $f : X \to Y$ factoring f through $\overline{M_f}$ and $\overline{N_f}$ resp. can be found exactly as before and we will keep the notation i_f and e_f resp. for them.

Our new notion of homotopy $X \wedge I_+ \to Y$ is a based one and thus stricter than the unbased case. But we would like our notion of homotopy equivalent spaces to be consistent. This is the case if we consider only ***nondegenerately basepointed spaces***, i.e. spaces X such that the inclusion $* \hookrightarrow X$ is a cofibration. For these spaces, any unbased homotopy equivalence yields a based homotopy equivalence by the lifting property of cofibrations.

We define the ***cone*** functor C by $CX = X \wedge (I, 1)$ ($(I, 1)$ denotes I with basepoint 1). We further define the ***loop*** functor Ω and the ***suspension*** functor Σ as a special case of (1.46), namely for $Y = S^1$ we obtain the mutual adjoint functors

$$\Sigma X := X \wedge S^1 \ , \ \Omega Z = Z^{S^1}$$

They have canonical unbased and unreduced analogues which will be denoted by the same letters when their use is clear from context.

Remark 1.47. By (1.46) and symmetry of $\wedge$ the functor $- \wedge Z$ preserves cofibrations, and thus so does Σ. They are also preserved by usual products $- \times Z$ (since it commutes with $- \wedge I_+$ and the coproduct $\vee$, and thus we have can extend our retract to "$r \times 1$": $(X \times Z) \wedge I_+ \to \overline{M_{f \times 1}}$).

Finally, we can define the homotopy (co)limit analogues of (1.44)

Definition 1.48. We define the ***homotopy cokernel*** $\overline{M}_f \xrightarrow{\text{hcoker } f} C_f$ of a map $X \xrightarrow{f} Y$ to be the cokernel of its cofibrant replacement i_f. Analogously, the ***homotopy kernel*** $F_f \xrightarrow{\text{hker } f} N_f$ of f is the kernel of its fibrant replacement e_f. We define the ***homotopy cofiber*** to be the composition $\text{hcof } f : Y \xrightarrow{i} \overline{M}_f \xrightarrow{\text{hcoker } f} C_f$, and the ***homotopy fiber*** to be the composition $\text{hfib } f : F_f \xrightarrow{\text{hker } f} \overline{N}_f \xrightarrow{p} X$, for canonical inclusion i and projection p. Note that by the universal properties of $\overline{M}_f$, $\overline{N}_f$, coker and ker we can regard F and C as functors from the arrow category Arr(**Top**) to **Top**.

We have $\text{hcoker}^n f = *, \text{hker}^n f = *$ (up to homotopy) as usual for $n \geq 2$. But this is not the case for homotopy (co)fibers since in each step we need to pass to cofibrant replacements M_f of Y. We thus define

Definition 1.49. The ***cofiber sequence*** associated to a map $f : X \to Y$ is the sequence

$$X \xrightarrow{f} Y \xrightarrow{\text{hcof } f} C_f \xrightarrow{\text{hcof}^2 f} C_{\text{hcof } f} \xrightarrow{\text{hcof}^3 f} C_{\text{hcof}^2 f} \xrightarrow{\text{hcof}^4 f} \cdots$$

Lemma 1.50. *The cofiber sequence is termise homotopy equivalent to the sequence*

$$X \xrightarrow{f} Y \xrightarrow{\text{hcof } f} C_f \xrightarrow{p_f} \Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma \text{hcof } f} \Sigma C_f \xrightarrow{\Sigma p_f} \Sigma^2 X \xrightarrow{\Sigma^2 f} \cdots$$

where p_f is the quotient map onto $M_f/Y = \Sigma X$. These homotopy equivalences commute with the sequence maps up to homotopy.

Proof Sketch. First, note that $\text{hcof } f$ is a cofibration being a pushout of the cofibration $i_0 : X \hookrightarrow CX$ along $f : X \to Y$: Indeed, i_0 is a cofibration by Lemma 1.43 and pushouts preserve cofibrations as is shown generally in section 2. The statement then follows by the following claim:

Claim 1.51. *Given a cofibration $i : A \to B$ we have*

(i) a homotopy inverse $\phi : B/A \to C_i$ to the projection $\pi : C_i \to B/A$
(ii) and in the case $i = \text{hcof } f$ this makes

$$\begin{array}{ccccccc} Y & \xrightarrow{\text{hcof } f} & C_f & \xrightarrow{p_f} & \Sigma X & \xrightarrow{\Sigma f} & \Sigma Y \\ & & & {\scriptstyle \text{hcof}^2 f} \searrow & \pi \uparrow \downarrow \phi & \nearrow {\scriptstyle p_{\text{hcof } f}} & \\ & & & & C_{\text{hcof } f} & & \end{array}$$

commute at least up to homotopy.

For *(i)* we construct a map ϕ to C_i explicitely (there is no canonical choice) using the lifting property of cofibrations. Statement of *(ii)* can be easily verified (we just need to consider π of course). The lemma now follows by inductively "replacing" suspension objects by homotopy cofibers. ☐

Remark 1.52. Claim 1.51 shows that for our definition of cofibrations the notion of homotopy cofiber and cokernel coincide up to homotopy.

Proving the dual lemma and claim leads to the following two representations of a fiber sequence

$$\cdots \xrightarrow{\Omega^2 f} \Omega^2 Y \xrightarrow{\Omega j_f} \Omega F_f \xrightarrow{\Omega\,\mathrm{hfib}\,f} \Omega X \xrightarrow{\Omega f} \Omega Y \xrightarrow{j_f} F_f \xrightarrow{\mathrm{hfib}\,f} X \xrightarrow{f} Y \tag{1.53}$$

which is homotopy equivalent (as stated precisely in Lemma 1.50 in dual form) to the more concise sequence

$$\cdots \xrightarrow{\mathrm{hfib}^4 f} F_{\mathrm{hfib}^2 f} \xrightarrow{\mathrm{hfib}^3 f} F_{\mathrm{hfib}\,f} \xrightarrow{\mathrm{hfib}^2 f} F_f \xrightarrow{\mathrm{hfib}\,f} X \xrightarrow{f} Y$$

From the concise forms of our sequences (and the remark that $\mathrm{hcof}\,f$ and $\mathrm{hfib}\,f$ are cofibrations and fibration respectively) we immediately deduce the following lemma

Lemma 1.54. *Given $Z \in \mathbf{Top}_*$ the sequences obtained by applying $[-, Z]$ to the cofiber sequence and $[Z, -]$ are fiber sequences yield exact sequences in $\mathbf{Set}_*$.* □

1.2.2. *Homotopy and CW complexes.* We keep working the context of $\mathbf{Top}_*$. As sketched in the beginning of this section in ordinary homotopy theory we are probing spaces by spheres and these allow a canonical comultiplication $p : S^n = \Sigma S^{n-1} \to \Sigma S^{n-1} \wedge \Sigma S^{n-1} = S^n \wedge S^n$ (up to our identification $S^n \cong \Sigma S^{n-1}$) which we call ***pinch map***. More, precisely we define

Definition 1.55. For $n \geq 0$ the ***n-th homotopy group*** of a space X is defined as

$$\pi_n(X) = [S^n, X] = [\Sigma^n S^0, X] = [S^0, \Omega^n X] = \pi_0(\Omega^n X)$$

and the ***n-th relative homotopy group*** for a pair (X, A) with inclusion $i : A \hookrightarrow X$ by

$$\pi_n(X, A) = \pi_{n-1}(F_i)$$

which more explicitely can be described as the group of homotopy classes of maps of triples $(I^n, \partial I^n, (\partial I^{n-1} \times I) \cup (I^{n-1} \times 1) =: J) \to (X, A, *)$. In particular, $\pi_n(X, *) = \pi_n(X)$.

We can now apply the dual of Lemma 1.54 to our fiber sequence (1.53) for $Z = S^0$ and $f = i : A \to X$ which yields the following exact sequence

$$\cdots \xrightarrow{(i)_*} \pi_2 X \xrightarrow{(j_i)_*} \pi_2(X, A) \xrightarrow{(\mathrm{hfib}\,i)_*} \pi_1(A) \xrightarrow{(i)_*} \pi_1(Y) \xrightarrow{(j_i)_*} \pi_1(X, A) \xrightarrow{(\mathrm{hfib}\,i)_*} \pi_0(A) \xrightarrow{(i)_*} \pi_0(X)$$

which we define to be the ***long exact sequence (LES) of a pair*** (X, A). It is straightforward to work out this sequence explicitely (based on our explicit description of $\pi_k(X, A)$) which can be found e.g. in [Hat02]. More generally, we obtain such a ***LES for a map*** $g : X \to Y$ with $\pi_{n-1}(F_g)$ in place of $\pi_n(X, A)$. We extend the above sequence to the right by $\pi_0(X, A)$ resp. $\pi_{-1}(F_g)$ given by the cokernel of i_* resp. g_* in $\mathbf{Set}_*$. With these induced maps on homotopy $f_* = [S^n, f]$ we now define a new notion of weak equivalence based on the information given about a space from its homotopy groups

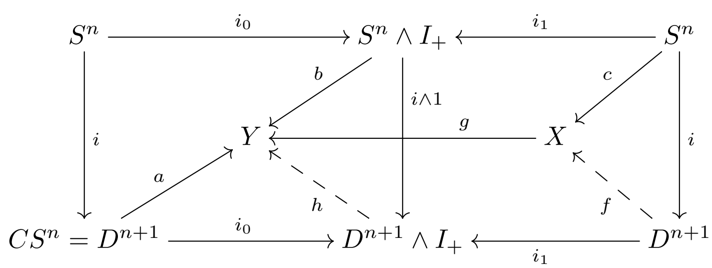
Definition 1.56. A map $f : X \to Y$ is called a ***weak equivalence*** if induces $f_* : \pi_n X \to \pi_n Y$ induces an isomorphism for all n. f is called a ***weak m-equivalence*** if it induces isomorphisms for $n \leq m$. We will refer to it just as a

k-equivalence if it induces isomorphisms for $n < k$ and a surjection for $n = k$: Using our sequence extension this is the precisely the case if $\pi_n(F_f) = 0$ for $-1 \leq n \leq k-1$. Finally, a space X is said to be ***k-connected*** if $\pi_n X = 0, n \leq k$. Using our extension to $\pi_0(X, A) = \operatorname{coker}(i)_*$, we can define ***k-equivalences of pairs*** and ***k-connected pairs*** completely analogously.

We emphasize once more for clarity that our notion of a k-equivalence and a weak m-equivalence are indeed two *slightly different things* by the previous definition.

k-equivalences can be characterised concisely by the following observation. A map $f : X \to Y$ is nullhomotopic if and only if it extends to a map $f : CX \to Y$. A map $g : Y \to Z$ induces a surjection $g_* = [X, g] : [X, Y] \to [X, Z]$ if and only of for all $f' \in [X, Z]$ we have a homotopy $h : f' \simeq f''$ such that $g_*(f) = f''$ for some f, i.e. g_* has images in all path components of Z^X. Setting $X = S^n$, with the convention $S^{-1} = \emptyset$, this motivates the following lemma:

Lemma 1.57. *$g : Y \to Z$ is a k-equivalence if and only if for all a, b, c and $-1 \leq n < k$ in the following diagram*



we have solutions h and f making the diagram commute.

Proof. Assume solutions to the diagram can always be found for $n < k$. Taking b and c to be the constant maps to $*$ makes a effectively into a map from S^{n+1} to Y, and forces h to be a based homotopy of spheres $a \simeq a'$. Then $g_*(f) = a'$ showing surjectivity on the groups π_{n+1}. Injectivity on π_n follows by taking a to be a nullhomotopy of $g_*(c)$ (take b to be constant) so that the solution f shows c is nullhomotopic as well.

Conversely, let g be an k-equivalence and let $n < k$. Define $\alpha_x(t) = a(x, t)$, $\beta_x(t) = b(x, t)$, and a map $s : S^n \to F_g$ by $s(x) = (c(x), \beta_x \alpha_x^{-1})$. By the LES of g we know that F_g is $(k-1)$-connected and thus s is nullhomotopic via $h' : S^n \wedge I_+ \to F_g$, $h'(x, t) = (f(x, t), \gamma(x, t))$. The first component yields f and h is obtained from $\tilde{h}(x, t, s) = \gamma(x, s)(t)$ after reparametrizing the unit square appropriately. $\square$

Remark 1.58. The lemma also holds for the unbased analogue of the diagram with $- \wedge I_+$ replaced by $- \times I$. The proof is essentially the same: we just regard c and g as based maps, and then s can be shown to be homotopic to a based map $S^n \to F_g$ by the lifting property of the cofibration $* \hookrightarrow S^n$.

With our new definiton of weak equivalence we already posed the question for which spaces it coincides with our old notion of homotopy equivalence. The answer was given the name "CW-complexes". We now give their definition, and the Theorem 1.62 shows that they are indeed the answer.

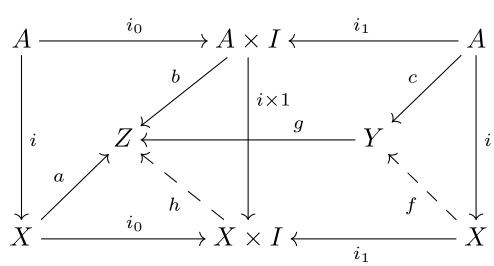
Construction 1.59 (CW complexes). A ***CW complex*** X is a space obtained as a colimit of a sequence of inclusions $X^0 \hookrightarrow X^1 \hookrightarrow X^2 \hookrightarrow \cdots$ where X^0 is a discrete space of points indexed by the set I_0 and X^n is obtained from X^{n-1} by attaching ***n-cells*** D^n via attaching maps $s_j : S^n \to X^{n-1}$ indexed by I_n as follows:

$$\begin{array}{ccc} \bigsqcup_{j\in I_n} S^n & \xrightarrow{\sqcup_j s_j} & X^{n-1} \\ {\scriptstyle \sqcup_j i_j}\downarrow & & \downarrow{\scriptstyle i} \\ \bigsqcup_{j\in I_n} D^n & \longrightarrow & X^n \end{array}$$

We set $\dim X \in \mathbb{N} \cup \{ \infty \}$ to be the highest n such that X has at least one n-cell. A ***subcomplex*** A of a CW-complex X is a subspace of X and a CW complex, such that all n-cells (and their attaching maps) also occur in X. If $I^0 = A \in \mathbf{Top}$ rather than a discrete space the construction yields a ***relative CW-complex*** denoted by (X, A). A map $f : X \to Y$ between CW complexes is called ***cellular*** if $f(X^n) \subset Y^n$ for all n. Subject to certain conditions many operations evidently preserve CW complexes (quotients by subcomplexes, wedges of based complexes with $* \in X^0$, products, pushouts of cellular maps along inclusions of subcomplexes, etc.). A (ordinary or relative) CW complex is said to be ***n-connected*** if it does not have k-cells for $k \leq n$ apart from a single 0-cell.

By virtue of their construction, CW-complexes interact with weak n-equivalences as follows

Corollary 1.60. *If $g : Y \to Z$ is a k-equivalence, (X, A) a relative CW complex with $\dim X \leq k \in \mathbb{N} \cup \{ \infty \}$, then for all a, b, c in the following diagram we have solutions f, h:*



Proof. Note that by Remark 1.58 we have a unbased version of Lemma 1.57. First, we show the statement holds in finite dimensions, i.e. for $k < \infty$. Argue inductively assuming the claim was shown for (X^n, A) (with the convention $X^{-1} = \emptyset$). The claim follows for finitely many attached n-cells $(X^n \cup D^n \cup \cdots \cup D^n, X^n)$ from applying (the unbased version of) Lemma 1.57 for each n-cell and stacking diagrams together. Passing to colimits we obtain the claim for (X^{n+1}, X^n) and thus by inductive hypothesis for (X^{n+1}, A). Finally, from the finite dimensional case by passing to colimits again the full statement follows. □

Proposition 1.61. *For a CW complex X, a k-equivalence $g : Y \to Z$ induces a bijection $g_* : [X, Y] \to [X, Z]$ if $\dim(X) < k$ and a surjection if $\dim(X) = k$.*

Proof. For surjectivity apply Corollary 1.60 for $(X, \emptyset)$. For injectivity, assume $h : gd \simeq ge$ for $e, d \in [X, Y]$ and apply Corollary 1.60 for $(X \times I, X \times \partial I)$ with

$a(x,t) = gd(x)$, $b(x,1,t) = h(x,t), b(x,0,t) = a(x,t)$ and $c(x,0) = d, c(x,1) = e$. Also, note $\dim X \times 1 = \dim X + 1$. □

Theorem 1.62 (Whitehead)**.** *A k-equivalence $g : Y \to Z$ between CW complexes, where $k \in \mathbb{N} \cup \{ \infty \}$ is a homotopy equivalence if* $\dim Y, \dim Z < k$.

Proof. Apply Proposition 1.61 for $X = Z$ to find $g \circ f \simeq 1$. Then $g \circ f \circ g \simeq g$ and applying Proposition 1.61 for $X = Y$ this implies $f \circ g \simeq 1$. □

In the beginning of this section we noted that the category of CW complexes **CW** was a deformation of **Top** where we meant that every space could be weakly approximated by a CW complex. This will be a consequence of Theorem 1.65. The following tells us that on top of that, up to homotopy, we can restrict our attention in **CW** to cellular maps.

Theorem 1.63 (Cellular approximation)**.** *A map $f : X \to Y$ between CW-complexes is homotopic to a cellular map $g : X \to Y$.*

Proof. First note that for a n-connected relative complex (X, A) we have $\pi_k(X, A) = 0, k \leq n$: Indeed, arguing inductively assume $X = A \cup D^m, m > n \geq k$ and let $f : I^k \to X$. Identify the interior of D^m with $\mathbb{R}^m$, take $x \in \mathbb{R}^m$and set $K_1 = f^{-1}(B_1(x)), K_2 = f^{-1}(B_2(x))$. Approximate f by some f' such that $f = f'$ outside K_2 and f' is smooth on K_1 (where parameters of the smooth approximation should be chosen depending on the distance of $I - K_2$ and $\overline{K_1}$). Clearly $f \simeq f'$. Choosing the approximation sufficiently close, f' misses some $x' \in B_1(x) \in D^m$. Since $A \cup D^m - x'$ deformation retracts to A, f corresponds to 0 in $\pi_k(X, A)$ (cf. Definition 1.55). Thus an n-connected relative complex is n-connected homotopically.

In particular we record that (Y, Y^n) is n-connected and the inclusion $i : Y^n \hookrightarrow Y$ is an n-equivalence. To find a cellular $g \simeq f$ we argue inductively. Suppose we have found $g_n : X^n \to Y^n$ such that $h : ig_n \simeq f|_{X^n}$. Given an $(n+1)$-cell $d : D^{n+1} \to X$ with attaching map $s : S^n \to X^n$ we set $j : S^n \xrightarrow{g_n s} Y^n \hookrightarrow Y^{n+1}$. We can apply Corollary 1.60 to $i : Y^{n+1} \hookrightarrow Y$ with $a = fd$, $b = h(ij \times 1)$ and $c = j$ to find an extension $\tilde{g}_n$ of g_n to $X^n \sqcup_{S^n} D^{n+1}$ by the solution map f. The statement follows (cf. proof of Corollary 1.60). □

Before we show have CW complexes weakly approximate all other spaces, we make the following definition:

Definition 1.64. A ***triad*** of $(X; A, B)$ consists of spaces X, A, B such that $A, B \subset X$ and $A \cup B = X$. A CW triad $(X; A, B)$ consists of CW complexes X, A, B such that A, B are subcomplexes of X and their union is X. A ***triple*** of spaces (X, A, B) means spaces such that $A \subset B \subset X$.

Theorem 1.65 (CW approximation)**.** *In each of the following cases we have CW-approximation functor Γ and a (up to homotopy) natural transformation $\gamma : i \circ \Gamma \to \mathrm{Id}$ (i being the canonical inclusion in each case) such that $\Gamma X \xrightarrow{\gamma_X} X$ are a weak equivalences:*

(i) Γ : $\mathbf{Top}_*$ $\to$ $\mathbf{CW}_*$
(ii) Γ : $\mathbf{Top}_*$**-pairs** $\to$ $\mathbf{CW}_*$**-pairs**
(iii) Γ : $\mathbf{Top}_*$**-triads** $\to$ $\mathbf{CW}_*$**-triads**

Remark 1.66. (i) We didn't fully define all of these categories. The slogan is that maps between pairs (and triads) need to induce compatible maps on their subspaces or subcomplexes, including $C = A \cap B$ in the case of triads.

(ii) Each case generalizes its preceeding case in the above. So we can fix *one* CW-approximation functor Γ for "all purposes".

Proof. For (i) we first note that by Proposition 1.61 $(\gamma_Y)_* : [\Gamma X, \Gamma Y] \to [\Gamma X, Y]$ is a bijection and so naturality determines the functor Γ once we found γ

$$\begin{array}{ccc} \Gamma X & \xrightarrow{\Gamma f} & \Gamma Y \\ {\scriptstyle \gamma_X}\downarrow & & \downarrow{\scriptstyle \gamma_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

To find γ_X and we inductively build $\gamma_n : (\Gamma X)^n \to X$ by firstly attaching n-cells $d_i : D^n \to (\Gamma X)^n$ with attaching maps $s_i : S^{n-1} \to (\Gamma X)^{n-1}$ for each generator of the kernel of γ_{n-1}. Since $\gamma_{n-1} s_i \simeq c_*$ (the constant map), we can obtain $\tilde{\gamma}_n$ by extending γ_n to all cells d_i. Secondly, for each generator of $t_j : D^n/S^{n-1} \cong S^n \to X$ of $\pi_n X$ we attach an n-cell $d_j : D^n \to (\Gamma X)^n$ with $s_j = c_*$ and extend $\tilde{\gamma}_n$ to γ_n such that $\gamma_n d_j = t_j \pi$ for $\pi : D^n \to D^n/S^{n-1}$. Passing to the colimit of these inductively constructed spaces, the verification of the statement (using compactness of spheres and their homotopies) is straightforward.

The natural approaches to proving (ii) and (iii) do not bring essentially new ideas, apart from noting that a map of triads $(X, A, B) \to (X', A', B')$ which restricts to give weak equivalences on A, B and $C = A \cap B$ also is a weak equivalence for the full spaces as one might expect. □

Defining a triad $(X; A, B)$ on X automatically yields a map of pairs $i : (A, A \cap B) \to (X, B)$. As noted in Definition 1.48 F is functorial on Arr(**Top**$_*$). Let $Fi : F_j \to F_l$ where $j : B \hookrightarrow X, l : A \cap B \hookrightarrow A$. Defining the ***triad homotopy group*** $\pi_k(X; A, B) = \pi_{k-1}(F_{Fi})$ this yields a ***LES of triads***

$$\cdots \longrightarrow \pi_2(A, A \cap B) \xrightarrow{i_*} \pi_2(X, B) \longrightarrow \pi_2(X; A, B) \longrightarrow \pi_1(A, A \cap B) \xrightarrow{i_*} \pi_1(X, B)$$

In the case $B \subset A \Rightarrow A \cap B = B$, necessarily dropping the condition $A \cup B = X$ in the definition of triads for a moment, this recovers the notion of LES of pairs as defined e.g. in [Hat02]. More interestingly, recalling Definition 1.56, one can show that if $(A, A \cap B)$ is n-connected for $n > 0$ and $(B, A \cap B)$ is m-connected then $\pi_k(X; A, B) = 0$ for $k \leq m + n$. A possible approach to this statement is to use Theorem 1.65 and argue inductively, proofs based on this idea can be found e.g. in [May99] and [Hat02]. By the LES of triads we immediately deduce

Theorem 1.67 (Excision)**.** *For a triad $(X; A, B)$, if $(A, A \cap B)$ is n-connected for $n > 0$ and $(B, A \cap B)$ is m-connected then i_* is an $(m + n)$-equivalence.*

Corollary 1.68. *Let $f : Y \to Z$ be a based map and a n-equivalence between $n - 1$ connected spaces (for $n > 0$). Then*

(i) $\pi : (M_f, Y) \to (M_f/Y, *)$ is a $2n$-equivalence

(ii) if X and Y are non-degenerately based then $p_f : (\overline{M}_f, Y) \to (C_f, *)$ is a $2n$-equivalence

(iii) There is a canonical map $\eta : F_f \to \Omega C_f$ (as defined below), and it induces an isomorphism $\eta_* : \pi_{n-1}(F_f) \to \pi_n(C_f)$

Proof. (i) follows from Theorem 1.67 if we set $(X; A, B) = (C_f, Z \cup (Y \times [0, 2/3]), Y \times [1/3, 1])$, and use the LES of (M_f, Y) to show n-connectedness of $(A, A \cap B) \simeq (M_f, Y)$ and the LES of (CY, Y) to show n-connectedness of $(B, A \cap B) \simeq (CY, Y)$. (ii) follows from (i) after taking note that Remark 1.47 and Claim 1.51 imply that the reduced constructions are homotopy equivalent their non-reduced analogues for non-degenerately based spaces (More generally Remark 1.47 implies that nondegenerately based cofibrations are unbased cofibrations[5]). For (iii) we first define $\eta : F_f \to \Omega C_f$ to map (y, ζ) to a loop γ which is $\zeta(2t)$ for $t \in [0, 1/2]$ and then goes straight up I to reach $*$. We have a evident homotopy $\eta F r \simeq F(p_f) : F_{i_1 : Y \to \overline{M}_f} \to \Omega C_f$, where $r : \overline{M}_f \to Z$ is the usual retraction and to be precise we wrote $Fr = F(1, r)$ by abuse of notation (also p_f is the map from (ii)). But $(Fr)_* : \pi_k(M_f, Y) \to \pi_{k-1}(F_f)$ is a isomorphism by the five lemma in the LES for (M_f, Y) and f. Using part (ii) we deduce the statement. □

Theorem 1.69 (Exactness)**.** *If $i : A \hookrightarrow X$ is a cofibration, (X, A) is n-connected and both X and A are $n-1$ connected (for $n > 0$) then $\pi : (X, A) \to (X/A, *)$ is a $2n$-equivalence.*

Proof. Using Corollary 1.68 and Claim 1.51 the statement follows from

$$\begin{array}{ccc} \pi_k(M_f, A) & \xrightarrow{\pi_*} & \pi_k(C_f, *) \\ \downarrow r_* & & \downarrow \pi_* \\ (X, A) & \xrightarrow[\pi_*]{} & (X/A, *) \end{array}$$

where r denotes the retract of M_f to X. □

Note that $\wedge$ is commutative and thus

Definition 1.70. Σ commutes with homotopies, inducing a ***"homotopy" suspension map*** on homotopy groups $\Sigma : \pi_k(X) \to \pi_{k+1}(\Sigma X) = \pi_{k+1}(\Sigma X, *)$ given by $[f] \mapsto [f \wedge S^1] = [f \wedge 1_{S^1}]$.

Theorem 1.71 (Freudenthal)**.** *For an $(n-1)$-connected space X the homotopy suspension map $\Sigma : \pi_k(X) \to \pi_{k+1}(\Sigma X)$ is iso for $k < 2n$ and epi for $k = 2n$.*

Proof. The statement follows from

$$\begin{array}{ccccc} & & \pi_k(CX, X) & & \\ & \swarrow^{\partial} & & \searrow^{\pi_*} & \\ \pi_{k-1}(X) & & \xrightarrow{\quad\Sigma\quad} & & \pi_k(\Sigma X, *) \end{array}$$

[5]If we have an unbased lifing problem $(h : A \to Y^I, g : X \to Y)$ for a based cofibration $i : A \hookrightarrow X$ we first use nondegeneracy to lift $\big((*, s) \mapsto (t \mapsto h(*, (1-t)s)) : * \times I \to Y^I, h : A \times I \to Y\big)$ to a map $\tilde{k} : A \times I \to Y^I$. Then use the based lifiting problem $\Big((a, t) \mapsto \tilde{k}(-, t), (x, t) \mapsto \tilde{g}(x, t) = g(x)\Big)$ for $i : A \times I \hookrightarrow X \times I$ which is a cofibration by Remark 1.47, and whose solution $\tilde{h}$ solves the initial problem when restricted to $t = 0$

Indeed, since CX is contractible, the LES of (CX, X) implies that ∂ is iso in all dimensions and so (CX, X) is n-connected. For $[f] \in \pi_{k-1}X$ we have $\partial[f \wedge (I,1)] = [f]$ and $\pi_*([f \wedge (I,1)]) = \Sigma[f]$ so the above commutes. But π is a $2n$ equivalence by Theorem 1.69 (as previously noted $X \hookrightarrow CX$ is a cofibration) and therefore the statement follows. □

We preliminarily define the ***homotopy category*** of **Top** to be Ho(**Top**) with objects from **Top** and morphisms the homotopy classes $[X, Y]$ of morphisms in **Top**. This comes equipped with a (localisation) functor

$$\gamma : \mathbf{Top} \to \mathrm{Ho}(\mathbf{Top})$$

sending morphisms to their equivalence classes. We can then summarise the properties we distilled about ordinary homotopy theory as follows

- **functoriality**: We have functors $\pi_0 : \mathrm{Ho}(\mathbf{Top}) \to \mathbf{Set}_*$, $\pi_1 : \mathrm{Ho}(\mathbf{Top}) \to \mathbf{Gp}$ and $\pi_k : \mathrm{Ho}(\mathbf{Top}) \to \mathbf{AbGp}$ for $k \geq 1$. The last statement follows from the usual ***Eckmann-Hilton argument***:

$$\begin{aligned}\alpha \circ \beta &\simeq (\alpha \cdot 1) \circ (1 \cdot \beta) \simeq (\alpha \circ 1) \cdot (1 \circ \beta) \\ &\simeq (1 \circ \alpha) \cdot (\beta \circ 1) \simeq (1 \cdot \beta) \circ (\alpha \cdot 1) \simeq \beta \circ \alpha\end{aligned}$$

  here "∘" denotes usual composition (i.e. pasting $S^n \cong I^n/\partial I^n \cong \Sigma\left(I^{n-1}/\partial I^{n-1}\right)$ along the coordinate of suspension), "·" denotes pasting along some other fixed coordinate ($n \geq 2$!) and $\alpha, \beta, 1$ are maps in $\pi_n(X)$ where 1 is the constant map.
- **weak equivalences**: By virtue of their Definition 1.56, weak equivalences induce isomorphisms on homotopy groups.
- (conditional) **exactness**: By Theorem 1.69, for a cofibration and n-equivalence $i : A \hookrightarrow X$ of $(n-1)$-connected spaces the following is exact for $k < 2n$:

$$\pi_k(A) \longrightarrow \pi_k(X) \longrightarrow \pi_k(X/A)$$

- (conditional) **suspension**: From Definition 1.70 and Theorem 1.71 we have have a natural isomorphism $\Sigma : \pi_k X \to \pi_k \Sigma X$ for $(n-1)$-connected spaces X and $k < 2n$.
- **additivity** It is straight forward to show from Definition 1.55 that π_k is "additive" on products, i.e.

$$\pi_k\left(\prod_i X_i\right) = \prod_i \pi_k(X_i)$$

As we will now learn there are more easily computable functors on Ho(**Top**) satisfying axioms similar to the above ones "unconditionally".

1.2.3. *Spectra and (co)homology.* Even though we saw in the beginning of this section that we can regard cohomology as dual to homotopy and homology as a covariant cohomology, we will approach homology in the traditional axiomatic way based on the properties we just distilled for homotopy. We want homology to behave "like" homotopy does in the category of $(n-1)$-connected spaces and n-equivalences:

Definition 1.72. A ***homology theory*** (H, Σ) consists of functors H_k from $\mathrm{Ho}(\mathbf{Top}_*)$ to **AbGp** satisfying the following axioms

- **weak equivalences** are mapped to isomorphisms by H_k. We denote $H_k(f) = f_*$ as before.
- **exactness** is preserved by H_k, i.e. for a cofibration $i : A \hookrightarrow X$ the following is exact

$$H_k(A) \xrightarrow{i_*} H_k(X) \xrightarrow{\pi_*} H_k(X/A)$$

- **suspension**: Σ is the natural suspension isomorphism $H_k(X) \cong H_{k+1}(\Sigma X)$.
- **additivity**: H_k preserve coproducts

$$H_k\left(\bigvee_i X_i\right) \cong \bigoplus_i H_k(X_i)$$

(spelling out "preserving" this implies that the isomorphism is given by $\oplus_k(i_k)_*$ for $i_k : X_k \hookrightarrow \vee_l X_l$. As a consequence the projections $\pi_k : \vee_l X_l \to X_k$ become projections $(\pi_k)_*$ of the biproduct.)

We can give a exactly analoguos definition for a ***homology theory on CW complexes*** H' (more precisely, based CW complexes), but note that by Theorem 1.62 (Whitehead's Theorem) we don't require the first axiom anymore. Even without the first axiom we still have the following equivalence

Theorem 1.73. *Fixing a CW-approximation* Γ *from Theorem 1.65 then* H *and* H' *mutually determine each other.*

Proof. Clearly, given H it restricts to a homology theory on CW complexes. Conversely, given H' we set $H(X) = H'(\Gamma X)$. This defines a homology theory by functoriality of Γ, naturality of the weak equivalences γ_X and Whitehead's Theorem 1.62. Finally, we clearly recover our original theories when doing the round trip. □

Remark 1.74 (Ordinary homology)*.* How do we recover ordinary homology on CW complexes from these axioms? We claim the above axioms are equivalent to the following more standard axiomatisation as proved in Proposition 1.76

Definition 1.75*.* An ***unreduced homology theory*** (H, ∂) on CW complexes is given by functor H_k from the homotopy category of unbased CW pairs to **AbGp** satisfying the following more familiar axioms

- **exactness**: For an inclusion $A \hookrightarrow X$, the natural map $\partial : H_k(X, A) \to H_{k-1}(A, \emptyset)$ makes the following exact

$$\cdots \to H_k(A, \emptyset) \to H_k(X, \emptyset) \to H_k(X, A) \xrightarrow{\partial} H_{k-1}(A, \emptyset) \to H_{k-1}(X, \emptyset) \to \cdots$$

- **excision**: For a CW triad $(X; A, B)$, inclusion induces an isomorphism

$$H_k(A, A \cap B) \xrightarrow{\sim} H_k(X, B)$$

- **additivity**: H_k preserve coproducts.

To see that these are equivalent we use our cofiber sequences in $\mathbf{Top}_*$ and make the following definition based on Claim 1.51: The ***topological boundary map*** $\delta : X/A \to \Sigma A$ of a cofibration $i : A \to X$ is given by the composition

$$X/A \xrightarrow{\phi} C_i \xrightarrow{\pi} \Sigma A$$

where ϕ is a homotopy inverse to $\pi : C_i \to X/A$.

Proposition 1.76. *An unreduced homology theory H and a homology theory H' on CW complexes mutually determine each other subject to the conditions*

$$H'_\bullet(X) = H_\bullet(X, *) \qquad H_\bullet(X, A) = H'_\bullet(C_{i_+ : A_+ \to X_+})$$

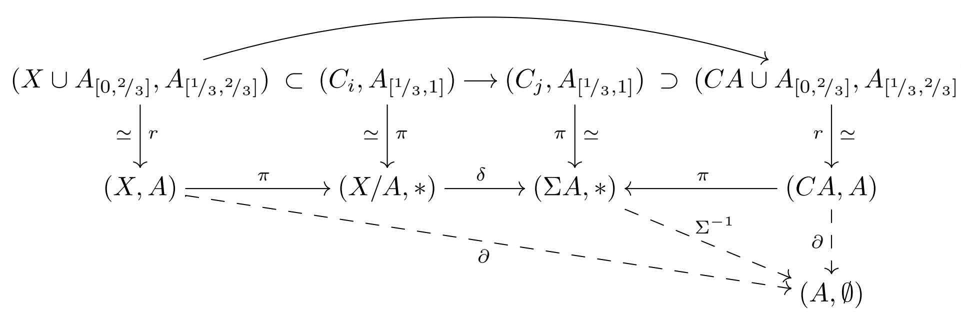
Proof. First let H be an unreduced theory and set $H'_\bullet(X) = H_\bullet(X, *)$. For a cofibration $i : A \to X$ we reiterate the proof of Theorem 1.69 (exactness for homotopy) using the excision axiom of H. We immediately find that $H_\bullet(X, A) \xrightarrow{\pi_*} H_\bullet(X/A, *)$ is an isomorphism. Plugging this into the LES of (X, A) we find the **exactness** axiom. Additionally, the **suspension** isomorphism can now be obtained by setting $\Sigma^{-1} : H_\bullet(\Sigma A, *) \xleftarrow{\pi_*} H_\bullet(CA, A) \xrightarrow{\partial} H_\bullet(A, *)$. **Additivity** also follows, since for $*_i \to X_i$ (which is always a cofibration for CW complexes) we have

$$\bigoplus_i H_\bullet(X_i, *_i) = H_\bullet\left(\bigsqcup_i X_i, \bigsqcup_i *_i\right) = H_\bullet\left(\bigsqcup_i X_i / \bigsqcup_i *_i, *\right) = H_\bullet\left(\bigvee_i X_i, *\right)$$

Note that more generally for any inclusion $i : A \to X$ we can use its cofibration replacement to find that $H_\bullet(X, A) \cong H_\bullet(M_i, A) \xrightarrow{\pi_*} H_\bullet(C_{i_+}, *)$ is an isomorphism, where for any inclusion $i : A \to X$ we obtain $i_+ : A_+ \to X_+$ by applying $(-)_+$ - so C_{i_+} is just the unreduced cone construction.

Conversely, given H' we set $H_\bullet(X, A) = H'_\bullet(C_{i_+ : A_+ \to X_+})$. **Additivity** can be verified straightforwardly. **Excision** follows by noting that for a CW triad $(X; A, B)$ we have $A/(A \cap B) = X/B$. Finally, we need to check **exactness**. Define $\partial = \Sigma^{-1}\delta_*\pi_*$. Note that for a CW subcomplex (X, A) we have $C_{i_+ : A_+ \to X_+} \simeq X_+/A_+ = X/A$ and also $H_\bullet(X, \emptyset) = H'_\bullet(C_{i_+ : * \to X_+}) = H'_\bullet(X_+)$. With this choice of ∂ exactness of H follows from naturality of Σ and exactness of H' when applied to the cofiber sequence of $i_+ : A_+ \to X_+$ (together with Claim 1.51 (ii)).

Finally, we need to check that by doing a round-trip we actually recover the notion of homology that we started with (since we claimed that H and H' *"mutually determine"* each other). This is straightforward for the functors H_k, H'_k and for Σ. We will only check it for ∂. Indeed, this can be seen again analogously to the proof of Theorem 1.69: For a cofibration $i : A \hookrightarrow X$ and $j : A \hookrightarrow CA$ the following commutes when passing to (unreduced) homology H_k



the dashed arrows make sense after applying H_k to the diagram and $A_{[a,b]}$ denotes $A \times [a, b]$ up to quotients in C_i, C_j. Then the lower triangle gives

$$\partial = \Sigma^{-1}\delta_*\pi_*$$

as required: Its commutativity follows using naturality of ∂ along the upper map and outer side retractions (inducing the identity on $(A, \emptyset$ in homology). Finally, for general $i : A \to X$ we can just use its cofibrant replacement $(M_i, A) \simeq (X, A)$ in the above argument. $\square$

Note that we have a correspondence between *unreduced* homology theories on **Top** and CW complexes by arguing exactly analogous to Theorem 1.73. These correspondences together with the previous Proposition 1.76 imply a correspondence between (reduced) homology and unreduced homology on **Top**.

To fully single out classical (reduced) homology theory (i.e. cellular/singular/simplicial homology) and making it "dual" to classical homotopy theory, we need one more axiom which we will assume from now on:

- **dimension** : $H_0(S^0) = \mathbb{Z}$ and $H_k(S^0) = 0$ for $k > 0$.

We fix a generator e_0 of $H_0(S^0)$ and using the suspension isomorphism we fix generators e_n for $H_n(S^n)$ as well.

Lemma 1.77. *For a CW complex X we have $H_k(X/X^n) = 0$ for all $k \leq n$.*

Proof. (Following [Hat02]) Assume X is n-connected, i.e. $X^n = *$. Then we need to show $H_k(X) = 0$. Note that X is homotopic to a "telescope space"

$$X \times [0, \infty) \simeq T = (X^0 \times [0, 1]) \sqcup_{X^0 \times \{ 1 \}} (X^1 \times [1, 2]) \sqcup_{X^1 \times \{ 2 \}} (X^2 \times [2, 3]) \sqcup \cdots$$

by inductively retracting $X \times [n, \infty) \cup T$ to $X \times [n + 1, \infty) \cup T$ using Lemma 1.43 (sticking all homotopies together gives a continuous map as usual, since they are continuous on all i-skeleta of $X \times [0, \infty)$, which itself is the colimit of them). Note also that $X \simeq X \times [0, \infty) \simeq T$. Let $R = X^0 \times [0, \infty)$ and $Z = R \cup (\bigcup_i X_i \times \{ i \})$. Then considering the LES associated to $Z/R = \bigvee_i X^i$ (n-connected) noting R is contractible yields by additivity that $H_k(Z) = 0$, $k \leq n$, and by the LES for $T/Z = \bigvee_i \Sigma X^i$ (($n+1$)-connected) we deduce $H_k(T) = 0$ as required. $\square$

Corollary 1.78 (approximation for homology)**.** *$i_* : H_k(X^n) \cong H_k(X)$ for $k < n$ and $i_* : H_n(X^n) \to H_n(X)$ is surjective.*

Proof. Using the LES for (X, X^n) this follows directly from Lemma 1.77. $\square$

We are now interested in the connections of homotopy and homology. Define the ***Hurewicz map***

$$h : \pi_n(X) \to H_n(X) \qquad h(f : S^n \to X) = f_*(e_n)$$

Representing $[f + g]$ by $S^n \xrightarrow{p} S^n \vee S^n \xrightarrow{f \vee g} X \vee X \xrightarrow{(1_X, 1_X)} X$ we see by spelling out the additivity axiom in Definition 1.72 that h is a homomorphism. Further, by definition of h and naturality of the (homological) suspension Σ the following diagrams commute

$$\begin{array}{ccc} \pi_n(X) & \xrightarrow{h} & H_n(X) \\ \downarrow g_* & & \downarrow g_* \\ \pi_n(Y) & \xrightarrow{h} & H_n(Y) \end{array} \qquad \begin{array}{ccc} \pi_n(X) & \xrightarrow{h} & H_n(X) \\ \Sigma \downarrow & & \downarrow \Sigma \\ \pi_{n+1}(\Sigma X) & \xrightarrow{h} & H_{n+1}(\Sigma X) \end{array} \tag{1.79}$$

Theorem 1.80 (Hurewicz)**.** *Let X be a wedge of spheres $\bigvee_\alpha S^n_\alpha$ and let Y be a $(n-1)$-connected CW complex for $n > 0$. Then*

(i) $h : \pi_n(X) \to H_n(X)$ is iso if $n > 1$ and the abelianisation if $n = 1$
(ii) $h : \pi_n(Y) \to H_n(Y)$ is iso if $n > 1$ and the abelianisation if $n = 1$

Proof. (i) follows by noting $\pi_n(X)$ is the free group on generators e_α for $n = 1$, and the free abelian group on generators e_α for $n > 1$ (the latter follows from considering additivity of π_* for the CW pair $(\prod_\alpha S^n_\alpha, \bigvee_\alpha S^n_\alpha)$). Clearly, h maps generators to generators yielding the statement.

For (ii) we first replace Y by Y^{n+1} by Corollary 1.78. Then $Y = Y^{n+1}$ is the cofiber of a map s given by $K := \bigvee_{\beta \in I_{n+1}} S^n_\beta \xrightarrow{s=(\dots,s_\beta,\dots)} \bigvee_{\alpha \in I_n} S^n_\alpha = Y^n$ (where I is indexing cells of Y and s_β are attaching maps). Since h is natural the following commutes:

$$\begin{array}{ccccccc}
\pi_n(K) & \xrightarrow{s_*} & \pi_n(Y^n) & \xrightarrow{(\mathrm{hcof}\, s)_*} & \pi_n(Y) & \xrightarrow{\pi_*} & 0 \\
\downarrow h & & \downarrow h & & \downarrow h & & \downarrow \\
H_n(K) & \xrightarrow{s_*} & H_n(Y^n) & \xrightarrow{(\mathrm{hcof}\, s)_*} & H_n(Y) & \xrightarrow{\pi_*} & H_n(Y/Y^n)
\end{array}$$

Clearly the bottom row is exact (recall the proof of Proposition 1.76). Note $H_n(Y/Y^n) = 0$ by Corollary 1.78. Since K and Y^n are wedges of n-spheres part (i) applies. The statement now follows from the five lemma: In the case $n = 1$ we pass to the abelianisation of the top row which is exact by Van Kampen's theorem. In the case $n > 1$ the top row is exact by comparing it to the LES of the pair (M_s, K). □

From here it is not very far to show that actually classical homology (if it exists) is the **unique** homology theory satisfying the dimension axiom by exhibiting that $H_n(X)$ can be calculated as chain homology of a chain $C_k = H_k(X^k/X^{k-1})$ and using the right diagram in (1.79) to characterise the chain maps in terms of the *degrees* of attaching maps. This resembles the proof of equivalence of simplicial and cellular homology given in most courses on Algebraic Topology. Thus we just note:

Theorem 1.81. *Up to natural isomorphism there is at most one homology theory H^{cell} satisfying the dimension axiom which we call cellular homology.*

Remark 1.82.
(i) The discussion generalizes when we replace $\mathbb{Z}$ by some other abelian group G in the dimension axiom. The unique cellular homology theory is then just $H^{cell}_*(-) \otimes G$.
(ii) Singular homology is cellular homology for a certain approximation functor Γ. Namely ΓX is the geometric realiziation (recall Construction 1.26) of the singular nerve $\mathbf{Top}(F-, X)$ of X for $F : \Delta \hookrightarrow \mathbf{Top}$ as previously defined. This is a common choice of approximation functor.

We will now concern ourselves with the existence of such a homology theory, by finally giving the *homotopical* characterisation of homology theories. We first note that there is completely analogous discussion of the *axiomatic* characterisation of cohomology theories as contravariant homology theories (and their axioms now look fully dual to the characterisation we gave for homotopy). Explicitely, we have the following axioms

Definition 1.83. A ***cohomology theory*** (H, Σ) consists of functors H^k from $\mathrm{Ho}(\mathbf{Top}_*)$ to $\mathbf{AbGp}^{\mathrm{op}}$, for which we write $f^* := H^k(f)^{\mathrm{op}}$, satisfying the following axioms

- **exactness** : Given a cofibration $A \hookrightarrow X$ we have an exact sequence

$$H^k(X/A) \xrightarrow{\pi^*} H^k(X) \xrightarrow{i^*} H^k A$$

- **suspension** : Σ is the natural suspension isomorphism $H^k(\Sigma X) \cong H^k(X)$
- **additivity** : H^k preserve coproducts (mapping into $\mathbf{AbGp}^{\mathrm{op}}$) which spelled out in **AbGp** means that i_j^* induce
$$\prod_j H^k(X_j) \cong H^k(\bigvee_j X_j)$$
where $i_j : X_j \hookrightarrow \vee_j X_j$ denote the canonical inclusions.
- **weak equivalences** H^k turns weak equivalences into isomorphisms.

The discussion about reduced, unreduced and CW-restricted theories applies completely analogously.

Our main tool for the homotopical characterisation of (co)homology will be the following construction. We restrict the notion of "space" to mean "based CW-complex" from now on.

Construction 1.84 (Spectra). A ***prespectrum*** T is a sequence of spaces T_n with maps $g_n : \Sigma T_n \to T_{n+1}$. For instance, for any space X we have a associated ***suspension prespectrum*** $\Sigma^\infty X$ given via $(\Sigma^\infty X)_{n\geq 0} = \Sigma^n X$ and $\Sigma^\infty X_{n<0} = *$. A morphism between prespectra T and T' is given by $f_n : T_n \to T'_n$ such that $g'_n \Sigma f_n = f_{n+1} g_n$ yielding a category **PSp**. For prespectrum T and every space X we obtain a prespectrum $T \wedge X$ with $(T \wedge X)_n = T_n \wedge X$. This defines a functor
$$T \wedge - : \mathbf{CW}_* \to \mathbf{PSp}$$
from spaces to prespectra. $\mathbf{PSp}(T \wedge -, T')$ preserves colimits so by Example 1.25 we obtain an adjunction
$$\mathbf{CW}_*(X, \mathrm{Map}(T, T')) \cong \mathbf{PSp}(T \wedge X, T') \tag{1.85}$$
making **PSp** enriched in spaces and giving us a natural notion of homotopy on the Hom sets:
$$\mathbf{CW}_*(I_+, \mathrm{Map}(T, T')) \cong \mathbf{PSp}(T \wedge I_+, T')$$
A ***spectrum*** E (also called a CW prespectrum) is prespectrum such that $h_n : \Sigma E_n \to E_{n+1}$ are inclusions of subcomplexes. This allows us to weaken our notion of morphism: It suffices that maps $f_n : E_n \to E'_n$ are defined on a subcomplex of E_n as long as we have a k such that f_{n+k} is defined on all of $\Sigma^k E_n \hookrightarrow E_{n+k}$. For instance, consider the sphere spectrum $S = \Sigma^\infty S^0$. Then $f^k : S \to S$ with $f^k_{n>k} = 1_{S^n}$ (and trivial otherwise) gives an example of such a morphism which is "eventually defined everywhere". This notion of morphism yields a category **Sp** of spectra. As before a homotopy is given by a map in $\mathbf{Sp}(T \wedge I_+, T')$. If we analogously to Ho(**Top**) only consider homotopy classes of morphisms we obtain the ***stable homotopy category of spectra***. We also have a natural suspension functor Σ and this is now *invertible*:
$$(\Sigma E)_n = E_{n+1} \ , \ (\Sigma^{-1} E)_n = E_{n-1}$$
Following Definition 1.55 this yields a natural notion of homotopy on suspensions of *the* sphere:
$$\pi_k(E) = [\Sigma^k S, E] \ , \ k \in \mathbb{Z}$$
This can be equivalently written as $\mathrm{colim}_n\, \pi_{n+k} E_n$, with the maps in the diagram being the following composition
$$\ldots \to \pi_{k+n-1} E_{n-1} \xrightarrow{\Sigma} \pi_{k+n} \Sigma E_{n-1} \xrightarrow{h_n} \pi_{k+n} E_n \to \ldots \tag{1.86}$$

This colimit can serve as a definition of homotopy groups on **PSp** as well. We define ***generalized homology of a (pre)spectrum*** $\boldsymbol{T}$ on spaces $X \in \mathbf{CW}$ by setting

$$T_k(X) = \pi_k(X \wedge T)\ ,\ k \in \mathbb{Z}$$

and ***generalized cohomology of*** $\boldsymbol{T}$ by setting

$$T^k(X) = [\Sigma^{-k}(\Sigma^\infty X), T]\ ,\ k \in \mathbb{Z}$$

Recall that $\Sigma^\infty X$ denotes the suspension spectrum of X. In particular T^0 equals our expression $H(X;T)$ from (1.41). To exhibit the duality between the two we consider only negative values for k, then using our adjunction (1.85) we obtain (1.87)

$$[\Sigma^n(\Sigma^\infty X), T] = [\Sigma^\infty X \wedge S^n, T] = [S^n, \mathrm{Maps}(\Sigma^\infty X, T)] = \pi_n \,\mathrm{Maps}(\Sigma^\infty X, T)$$

where π_n on the right denotes homotopy groups of spaces, and $n = -k \in \mathbb{N}$.

Remark 1.88. Morally, our category of spectra is a model (category) of " stabilised spaces" with a stabilised notion homotopy (take e.g. the suspension spectrum of a space in the above definition of π_k). There are many different such models. The definition of spectra which we presented above is due to Adams. However, they all lead to essentially the same **stable homotopy category**. The latter is symmetric monoidal closed - which is a structure already encoded in **Sp** but only present up to homotopy in our presentation. There are different models which allow a direct formulation of the monoidal structure (e.g. symmetric spectra $\mathbf{Sp}^{\mathbf{\Sigma}}$ where E_n come equipped with a Σ_n-action[6]). But we will not concern ourselves with these approaches now, but instead find an elegant solution in the setting of $(\infty, 1)$-categories for defining the monoidal product $\wedge$ (still called ***smash product***) in section 3. We note that having a smash product on spectra would allow the above definition of homology to generalised to the category of spectra.

We are now ready to characterise axiomatic (co)homology theories by spectra.

Definition 1.89. A spectrum (or presprectrum) T is ***connected*** if T_n is $(n-1)$-connected.

Theorem 1.90. *Let T be a connected positive prespectrum. This determines an (axiomatic) homology theory H_k:*

$$H_k(X) = \mathrm{colim}_n\, \pi_{n+k} X \wedge T_n = \pi_k(T \wedge X)$$

where the colimit is taken over the sequence (cf. (1.86)*)*

$$X \wedge T_0 \to \cdots \to X \wedge T_n \xrightarrow{(1\wedge g_n)\Sigma} X \wedge T_{n+1} \to \cdots$$

Proof (sketch). Note that $X \wedge T_n$ is $(n-1)$ connected. Then restrict the sequences to $n + k < 2n \iff k < n$ without affecting the colimit such that we can apply exactness and suspension results from homotopy. Exactness follows from Theorem 1.69 (exactness in homotopy) noting that taking filtered colimits (e.g. sequences) is an exact functor in **AbGp**. The suspension axiom follows from Theorem 1.71 (Freudenthal suspension theorem) and $\Sigma(X \wedge T_n) \cong (\Sigma X) \wedge T_n$. Additivity follows from cellular approximation when noting that $(\bigvee_i X_i) \wedge T_n =$

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[6] For a short basic introduction to spectra we refer reader to the webpage of the Geometric Langlands seminar 2013 at University of Chicago

$\bigvee_i (X_i \wedge T_n)$ is the $(2n-1)$-skeleton of $\prod_i^w (X_i \wedge T_n) := \mathrm{colim}_j \prod_i^j (X_i \wedge T_n)$ and $\pi_k \prod_i^w (X_i \wedge T_n) = \bigoplus_i \pi_k(X_i)$ (the w indicates that this colimit is of course not the usual product - only finitely many coordinates are not the basepoint).

$\square$

Definition 1.91. A prespectrum T is a ***Ω-prespectum*** if the adjoints $\overline{g_n} : T_n \to \Omega T_{n+1}$ of $g_n : \Sigma T_n \to T_{n+1}$ are weak equivalences.

Theorem 1.92. *If T is a Ω-prespectum then*

$$H^k(X) = \begin{cases} [X, T_k]\ , & k \geq 0 \\ [X, \Omega^{-k} T_0]\ , & k < 0 \end{cases}$$

yields an axiomatic cohomology theory.

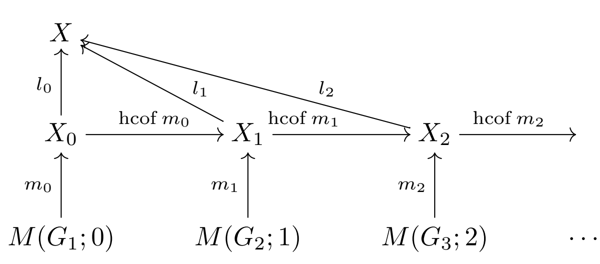
Proof. The suspension axiom follows from our assumption that T is a Ω-prespectrum. Namely, $\Sigma : [\Sigma X, T_k] \cong [X, \Omega T_k] \xrightarrow{(g_n)_*} [X, T_{k-1}]$. The other axioms are straightforward to verify. $\square$

Example 1.93. Existence of cellular homology cohomology now follows by setting $T_n = K(n;\mathbb{Z})$ in the above: $K(n;\mathbb{Z})$ are called ***Eilenberg-Maclane spaces*** and determined by the conditions $\pi_n K(n;\mathbb{Z}) = \mathbb{Z}$ and $\pi_{m\neq n} K(n;\mathbb{Z}) = 0$. They can be build in the same spirit as the proof of Theorem 1.65 (CW approximation) and they can be proved to have unique homotopy type (recalling we are in the context of CW complexes, see e.g. [Hat02] Prop 4.30). Clearly, with this choice for T_n the dimension axiom is satisfied, and thus the theorem gives the unique cellular homology (resp. cohomology) theory.

Construction 1.94 (Postnikov systems). From the definition in the previous example we have

$$[S^n, K(m;\mathbb{Z})] = \delta_{mn}\mathbb{Z}$$

which exhibits ordinary (cellular) cohomology as ***Eckmann-Hilton dual*** of ordinary homotopy of spheres, that is $K(m;\mathbb{Z})$ ("cohomology spheres") behave for $\pi_n = [S^n, -]$ in the same way as S^n ("homotopy spheres") do contravariantly for $H^m = [-, K(m;\mathbb{Z})]$. For the following recall (from 1.2.1) that the cofiber sequences for cohomology are the dual analogues of the fiber sequences for homotopy. We restrict our attention to 1-connected spaces [7] so that we can use Lemma 1.95. First, we recall what we did in Theorem 1.65 (CW approximation) in a more conceptual way. Consider



where $M(G;n)$ denote Moore-spaces *for cohomology* generalizing homotopy spheres to "homotopy spheres with coefficients", i.e. such that $H^k(M(G;n)) = \delta_{kn}G$

[7] This is for eliminating non-abelian behaviour of $\pi_1 X$, i.e. π becomes a functor into **AbGp**. Probably the argument still works for the class of *simple spaces*.

(they don't exist for all G - in contrast to $K(G;n)$). In the more familiar form of Theorem 1.65, we have $G_n = \bigoplus_{I_n} \mathbb{Z}$ and $M(\bigoplus_{I_n} \mathbb{Z}; n-1) = \bigvee_{I_n} S^{n-1}$ and m_n describes the attaching maps - these fully characterise $\mathrm{colim}_n X_n$.

In the above approximation we require that $(l_n)_*$ is an n-equivalence for cohomology and X_n are ***cohomology*** ***n-types***, i.e. $H^{k>n}X_n = 0$ (also see Lemma 1.95). This determines that $M(G;n)$ is a ***Moore space*** by inspection of the cofiber sequences of m_n (but G only up to an extension problem). Dualizing this (i.e. reversing arrows, replacing homotopy cofibers by fibers, $(l_n)_*$ are now *weak* n-equivalences and $\pi_{k>n}X_n = 0$ are ***n-types***) yields the notion of ***Postnikov systems*** and $K(n+1; \pi_n(X))$ then take the role of $M(G;n)$ by inspection of the fiber sequences. A Postnikov system exists for a CW complex X iff $\pi_1(X)$ acts trivially on higher homotopy groups (see [Hat02] §4.3). The characterising maps (dual to m_n) are often denoted by k_n and called ***k-invariants***. We will not use them in this essay, but they play a role e.g. in the arguments of [SP14].

Lemma 1.95. *Let X, Y be 1-connected spaces. Recalling Definition 1.56, the notion of weak equivalences is "self dual" in the following way*

(i) Let $f: X \to Y$ be a $(n+1)$-equivalence of homotopy. Then f^* is a isomorphism on H^k for $k < n$ and is injective for $k = n$, i.e. f is a n-equivalence of cohomology.

(ii) Let $f: X \to Y$ be a $(n+1)$-equivalence of cohomology such that X, Y have finitely generated homology groups. Then f is a n-equivalence for homotopy.

Proof (Sketch). (i) By cofibrant replacement assume f is an inclusion, and CW approximate (Y, X) by a complex such that Y/X is has one 0-cell and no other k-cells for $k \leq n$, i.e. X is the n-skeleton of Y. The statement then follows from the cellular cochain complex or Corollary 1.78 together with the universal coefficient theorem (see [Hat02] §2).

(ii) By cofibrant replacement assume f is an inclusion and CW approximate. Then $H^{k\leq n+1}(Y, X) = 0$ by assumption (passing to unreduced homology). The universal coefficient theorem allows a decomposition of H^k which shows that $H_{k\leq n}(X, Y) = 0$ since it is finitely generated. By assumption $\pi_1(X, Y)$ is abelian (namely zero). By a relative version of Theorem 1.80 (Hurewicz) we know the the first non-zero $\pi_k(Y, X)$ coincides with $H_k(Y, X)$ and thus the statement follows.

□

1.2.4. *Classifying spaces.*

Definition 1.96. A ***principal G-bundle*** $p: Y \to B$ for a group G is a fiber bundle with right free action of G on the fibers such that $p: Y \to Y/G = B$ is the projection by quotienting orbits. It is a ***universal*** such bundle if Y is simply connected. In this case we refer to the base space B as a ***classifying space*** of G, and it is usually denoted as BG. A ***morphism*** of such bundles Y and Y' is a map $f: Y \to Y'$ such that $f \circ p = p' \circ f$. It is a ***G-morphism*** if it is G-equivariant on each fiber.

We have seen the construction of the 1-object delooped category for a group G in Remark 1.1 which we now again denote by G (in order to not confuse it with the *space* BG that we will construct now). We can associate to each category a topological space by the following "bar construction":

$$B : \mathbf{Cat} \xrightarrow{N} \mathbf{Set}^{\Delta^{\mathrm{op}}} = \mathbf{sSet} \xrightarrow{|-|} \mathbf{Top} \xrightarrow{\gamma} \mathrm{Ho}(\mathbf{Top})$$

Here the geometric realisation $|-|$ was defined as a Yoneda extension in 1.1.3. An explicit expression for the geometric realisation can be derived using our explicit formulas for left Kan extensions. In particular for $X \in \mathbf{Set}^{\Delta^{\mathrm{op}}}$ its realisation $|-|$ is a gluing construction of the form

$$|X| = \left(\coprod_{n \geq 0} \Delta^n \times X(n) \right) / \Delta$$

with obvious notation for the equivalence relation induced by morphisms in Δ. This straightforwardly generalizes to the topologically enriched case $X \in \mathbf{Top}^{\Delta^{\mathrm{op}}}$ (also note that we have a change of base functor $D : \mathbf{Set} \hookrightarrow \mathbf{Top}$ using the discrete topology):

$$B : \textbf{Top-Cat} \xrightarrow{N} \mathbf{Top}^{\Delta^{\mathrm{op}}} \xrightarrow{|-|} \mathbf{Top} \xrightarrow{\gamma} \mathrm{Ho}(\mathbf{Top})$$

It can be further used to define B on *internal* topological categories, i.e. those which are carrying a topology on their set of objects as well, and we will make use of this once below. More about internal categories and groupoids can be found in most standard references on category theory.

As will be further explained below, we have a nice confluency of notation: The bar construction B on the delooped category of a (topological) group G then gives model BG for a classifying space. That means there is a universal principal G-bundle $p : EG \to BG$ as constructed below, or e.g. in [Hat02] §1.B. Note that the above discussion also makes B (and $|-|$) a product and pullback preserving functor into $\mathrm{Ho}(\mathbf{Top})$. In the case of ordinary presheaves this follows from N being right adjoint, the formula for left Kan extensions and the statement being true on simplices.

The construction of $EG \to BG$ below will elucidate what information the bar construction on a group G represents in $\mathrm{Ho}(\mathbf{Top})$, and in particular it will motivate the following theorem quite naturally. We first state the theorem: Note that we have a functor $\mathcal{P}G : \mathrm{Ho}(\mathbf{Top})^{\mathrm{op}} \to \mathbf{Set}$ mapping B to G-isomorphism classes of principal G-bundles and $\mathcal{P}G(f : B \to B')$ maps $Y \in \mathcal{P}G(B')$ to $f^*Y \in \mathcal{P}G(B)$ (the bundle obtained by the pullback along f). Since pullback of a principal bundle along homotopic maps yields equivalent bundles this is well defined (equivalence meaning a G-isomorphism which restricts to the identity on the base space, see e.g. [May99]). The principal G-bundle functor is then representable as follows

Theorem 1.97. *Given a universal principal G-bundle $p : Y \to B$ we have an representation*

$$[X, B] \cong \mathcal{P}G(X)$$

induced by mapping $f \in [X, B]$ to f^*Y.

In particular we deduce that $\mathcal{P}G$ is representable by BG. Another direct consequence is that the base spaces of universal principal G-bundles have unique homotopy type. So BG is a model for all of these spaces. Keeping this in mind we can give the following alternative approach to universal bundles

Construction 1.98 (generalized universal bundles). The construction of the mapping cocylinder E_f from section 1.2.1 can be performed in every closed monoidal (homotopical) category $\mathcal{C}$ with a suitable *intervall object* I (an object allowing at least morphisms $i_0, i_1 : * \to I$, and preferable also encoding homotopies in the sense of section 2 - e.g. in **Top** we can choose $I = [0,1]$). Namely, E_f is given as the pullback

$$\begin{array}{ccc} E_f & \longrightarrow & A \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle f} \\ [I,B] & \xrightarrow{\mathrm{ev}_0} & B \end{array}$$

If $b : * \to B$ is pointed the *generalized universal B-bundle* $p : E \to B$ is given by

$$\begin{array}{ccc} E & \longrightarrow & * \\ \big\downarrow & \lrcorner & \big\downarrow \\ [I,B] & \xrightarrow{\mathrm{ev}_0} & B \\ {\scriptstyle \simeq}\big\downarrow{\scriptstyle \mathrm{ev}_1} & \nearrow{\scriptstyle 1} & \\ B & & \end{array} \qquad p : E \to B$$

A *generalized principal B-bundle* with *classifying map* g can then be obtained via the pullback

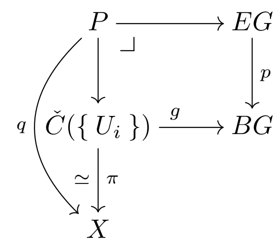
$$\begin{array}{ccc} P & \longrightarrow & E \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle p} \\ X & \xrightarrow{g} & B \end{array}$$

We will be able to reformulate these diagrams once we discuss homotopy limits in section 2.4.1. They also admit a further generalisation to *generalized groupoid principal bundles* which we won't need here (essentially generalizing pointedness $* \to B$ to "multiple pointedness" $B_0 \to B$). To illustrate the above definitions we consider one of the cases that interest us most

Example 1.99. Let $\mathcal{C} = \mathbf{Cat}$, and $B = BG$ the groupoid corresponding to some discrete group G. We choose I to be the free interval groupoid having a single invertible arrow. Then $G \xrightarrow{i} EG \xrightarrow{p} BG$ is exact in the sense of pointed categories $* \to BG$, and $EG = G//G$ is called the *action groupoid* of G acting by left multiplication on itself: $\operatorname{obj} EG$ are just elements of G, morphisms $f : g \to h$ are such that $fg = h$.

Now, to finish our motivation of the preceding theorem and to make the connection with the ordinary definition of principal bundles we first define the *Čech groupoid* $\check{C}(\{U_i\})$ associated to some cover $\{U_i\}$ of a topological space X: This is the internal topological groupoid having the object space $\coprod_i U_i$ and the morphism space $\coprod_{i,j} U_i \cap U_j$ with obvious domain and codomain maps. A *classifying map* $g : \check{C}(\{U_i\}) \to BG$ then picks out a transition element $g_{i,j}(x) \in G$ for each $x \in U_i \cap U_j$ in a continuous way. This is precisely the data that specifies an ordinary principal G-bundle. Pulling back along g (in internal topological groupoids, X being the trivial internal groupoid with $\operatorname{obj} X = \operatorname{mor} X = X$ below) we obtain the bundle

P classified by g



This construction already very much looks the statement of Theorem 1.97 (replacing B with our explicit model BG). The final step is just to take the geometric realisation of topological groupoids. As noted above this will preserve the pullback, preserve exactness of $G \xrightarrow{i} EG \xrightarrow{p} BG$ and clearly realize X and its Čech groupoid as homotopy equivalent spaces (recall that for the theorem we also passed to homotopy classes of maps). A formal treatment of the sketchy argument presented here was first given by Segal.

Remark 1.100. The map π in the above diagram was labelled with an equivalence symbol $\simeq$. This is to be understood as an internal categorical equivalence: it is essentially surjective and fully faithful in an internal sense. However, it does not have an inverse $X \to \check{C}(\{ U_i \})$. This situation is not unusual: The part consisting of the lower 3 objects in the above diagram is an example of an *anafunctor* $X \to BG$, which (in its homotopical interpretation) we will generalize to arbitrary zigzags of morphisms with all arrows that point "backwards" being weak equivalences (see Construction 3.4 about Hammock Localisations).

The usefulness of Theorem 1.97 is further increased by the following observation:

Construction 1.101 (associated principal bundles). Given a space F with effective left G-action (i.e. $(\forall x, gx = x) \Rightarrow g = 1$) a F-bundle $p : E \to B$ with structure group G is a fiber bundle with fiber F such that transition between fibers over x of different charts ϕ_α, ϕ_β is given by an action of a group element $g_{\alpha,\beta}(x) \in G$ and these "clutching functions" into G are continuous.

Every such bundle has a unique (up to G-isomorphism) ***associated principal** G**-bundle*** Y, and every principal G-bundle Y gives rise to an F bundle with structure group G. We give a rough sketch of how this correspondence works: starting from a F-bundle with structure group G, say $p : E \to B$, we can obtain Y over B as the G-bundle of admissible fiber inclusions $f : F \to E$ (e.g. the n-frames in a fiber of a tangent bundle for $F = \mathbb{R}^n$) - admissible means that this inclusion coincides with one determined by some chart of $p : E \to B$ up to an action by $g \in G$. It should be evident how to pick a base $b \in B$ for such a admissible inclusion to make this into a bundle. Conversely, starting from Y we can form the F-bundle as the quotient bundle $Y \times_G F$ of the trivial $(G \times F)$-bundle $Y \times F$ by the diagonal left action $g(y, f) = (yg^{-1}, gf)$.

This correspondence implies that given such F we have:

$$[X, BG] \cong \mathcal{P}G(X) \cong \mathcal{E}_F G(X)$$

where $\mathcal{E}_F G(X)$ denotes isomorphism classes of the F-bundles with structure group G over X.

One of the most basic applications follows from the fibration

$$O(n) \to V_n(\mathbb{R}^\infty) \to G_n(\mathbb{R}^\infty)$$

where denotes $G_n(\mathbb{R}^m)$ is the Grassmanian variety (n-planes in $\mathbb{R}^m$) and $V_n(\mathbb{R}^m)$ the Stiefel variety (n-frames in $\mathbb{R}^m$). $V_n(\mathbb{R}^\infty)$ being contractible implies that $G_n(\mathbb{R}^\infty)$ is a model for $BO(n)$. By the above, we deduce $[-, G_n(\mathbb{R}^\infty)]$ classifies ***n-plane bundles*** $\mathcal{E}_{\mathbb{R}^n}O(n)$, where $F = \mathbb{R}^n$ comes with an obvious effective left $O(n)$-action. Note that usually clutching functions of n-plane bundles are only required to live in $\mathrm{GL}(n)$. But with certain niceness assumptions (e.g. bundles allowing countable chart covers or base spaces being paracompact) every such "$\mathrm{GL}(n)$ n-plane bundle" admits a global Euclidean metric. Choosing local orthonormal basis then shows that $\mathrm{GL}(n)$ clutching functions can be equivalently regarded as $O(n)$ clutching functions. The converse is trivial.

In the above case of $F = \mathbb{R}^n$ and $G = O(n)$ we usually denote $\mathcal{E}_{\mathbb{R}^n}O(n)$ by $\mathcal{E}_n$. Denote by ϵ the trivial bundle (over any base space X). We have a natural transformation $- \oplus \epsilon : \mathcal{E}_n \to \mathcal{E}_{n+1}$. Since this is a transformation between representable functors we obtain a classifying map

$$- \oplus \epsilon : BO(n) \to BO(n+1) \tag{1.102}$$

and one can show that this map lives in the homotopy class of $B(i : O(n) \hookrightarrow O(n+1))$ where i denotes $O \mapsto O \oplus 1 \in O(n+1)$. Similarly one can classify other common operations on vectors bundles.

Remark 1.103 (classifying vs. representing). The words classifying object and representing object are essentially synonymous. The former e.g. also appears when representing functors (or "theories") from the 2-category of Grothendieck toposes, geometric morphisms and their transformation by a *classifying topos*. We will give a brief explanation of sheafs and Grothendieck toposes in the next section. In algebraic geometry one also uses the related term *moduli space* (**though** in contrast to classifying spaces it can live in a bigger category: It might be a higher "generalized space"/stack in the sense which we will explore now).

1.3. Motivation of higher topos theory.

We first give a brief definition of sheafs and Grothendieck topoi.

Construction 1.104. We think about a category $\mathcal{C}$ in the context of spaces. Given an object $X \in \mathcal{C}$ a map $f : X \to Y$ exhibits that the object X has an image in Y, we say Y contains a (X-like) ***generalised object*** f. A *(generalised) space built from* $\mathcal{C}$ is an F built from Y-like objects, for different $Y \in \mathcal{C}$: Precisely, to every Y we associate a set FY of Y-like objects in F and if Y contains an X-like object f this functorially yields a map $Ff : FY \to FX$ of Y-like to X-like objects in F. In other words, a ***generalized space*** is a presheaf. For instance, **sSet** are the generalized spaces built from Δ, but Δ were also building blocks for **Cat** and **Top** (in a precise sense, stated in 1.1.3) $-$ so **sSet** generalizes both them. This unifying perspective made it possible to define e.g. quasicategories on **sSet** easily in Definition 1.31. **Set** on the other hand is the category of generalized spaces built from a point $*$.

Dually, a ***generalised quantity*** on $\mathcal{C}$ is a Q quantifying objects in $\mathcal{C}$, that is it associates to every object Y a quantifying set QY such that if Y contains an X-like object f this gives a function Qf of QX to a subset of QY $-$ so Q is a copresheaf.

Spaces and quantities satisfy a duality called Isbell duality which is given by an adjunction with suggestively named functors

$$\mathcal{O} := \mathrm{Lan}_Y y = \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}(-, y-) \dashv \mathbf{Set}^{\mathcal{C}}(-, Y-) := \mathrm{Spec} : \mathrm{Quantities}^{\mathrm{op}} \to \mathrm{Spaces}$$

Since representables preserve limits this again immediately follows from Example 1.24 about Yoneda extensions. For instance, let $\mathcal{C} = \mathrm{CartSp} \subset \mathrm{Diff}$ be the full subcategory of cartesian spaces $\mathbb{R}^n$ in the category of manifolds, and say $A = \mathrm{Diff}(X, -)$ and $X = \mathrm{Diff}(-, X)$. Then $\mathcal{O}(X)(\mathbb{R}) = \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}(X, y\mathbb{R})$ gives the "C^∞-ring of functions" on X. On the other hand, given the smooth algebra A, $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}(y\mathbb{R}, \mathrm{Spec}(A)) = \mathrm{Spec}(A)(\mathbb{R}) = \mathbf{Set}^{\mathcal{C}}(A, Y\mathbb{R})$ describes the "spectrum" of A. Isbell duality as stated above or used in the previous example seems quite unexciting. It gets much more interesting if Quantities and Spaces come somewhat disguised and/or are understood in the enriched setting and/or are combined with further reflective localisations (see Remark 1.38, the notion of sheafs will be discussed in the next paragraph). The following list of prominent examples strikingly demonstrates this

Examples 1.105.
(i) *(Stone Duality).* $\mathcal{O} \dashv \mathrm{Spec} : \mathrm{Frm}^{\mathrm{op}} \to \mathrm{Top}$. $\mathcal{O}$, Spec are corepresented by the *schizophrenic object* **2** (i.e. it is part of both categories; these objects also go under the name of dualising objects) called the Sierpinski space. Units and counits are given by evaluation. Restricting to spatial lattices and sober spaces (which constitute the *Isbell dual* objects in this example) gives an equivalence of categories usually called *Stone duality*.
(ii) *(Gelfand-Naimark Duality).* $\mathcal{O} \dashv \mathrm{Spec} : C^*\text{-}\mathrm{Alg}^{\mathrm{op}} \to \mathrm{KTop}$. $\mathcal{O}$, Spec are again corepresented by a schizophrenic object, namely $\mathbb{C}$.
(iii) *(Functor of points duality).* $\mathcal{O} \dashv \mathrm{Spec} : \mathrm{CRng}^{\mathrm{op}} \to \mathrm{Sch}$. $\mathcal{O}$, Spec are corepresented by the schizophrenic object $\mathrm{Spec}(\mathbb{Z}[x])$ according to Martin Brandenburg on mathoverflow. We come back to this in Example 1.107.

To understand the last example in the list we now need to refine our notion of generalized space.

If an object $X \in \mathcal{C}$ admits a family of X_i-like objects $f_i : X_i \to X$ in X, $i \in \mathcal{I}$, we get a notion of ***families of*** $\{ f_i \}$***-compatible objects in presheafs*** as follows: For $i, j \in \mathcal{I}$ consider W-like objects $g : W \to X_i$ and $h : W \to X_j$ in X_i and X_j respectively, such that they correspond to the same W-like object $f_i g = f_j h : W \to X$ in X. A family of X_i-like objects $x_i \in FX_i$ in the generalized space $F \in \mathrm{PSh}(\mathcal{C})$ modelled on $\mathcal{C}$ is said to be $\{ f_i \}$-compatible if for all such W, g, h, g and h correspond to the same W-like object $Fg(x_i) = Fh(x_j) \in FW$ in F. We can think of such $\{ x_i \}$ as being "glued together" in F in the same way as f_i are in X.

A ***sheaf*** G for the family $\{ f_i \}$ is a presheaf such that the map from X-like objects $x \in GX$ to the set of $\{ f_i \}$-compatible families $\{ x_i \in GX_i \}$, mapping x to $\{ Ff_i(x) \}$, is a bijection. If $\mathcal{C}$ has pullbacks the set of $\{ f_i \}$-compatible families of G is just the equalizer of

$$\prod_{i \in \mathcal{I}} GX_i \rightrightarrows \prod_{i,j \in \mathcal{I}} G(X_i \times_X X_j)$$

and so in this case GX with it's canonical cone needs to be an equalizer of this diagram. This is a more familiar form of the *descent condition* for sheafs. A collection of "covering" families $\{ f_i \}$ for objects in $\mathcal{C}$ is called a ***coverage***. A category $\mathcal{C}$ with a coverage is called a ***site***. A presheaf which is a sheaf for all families in such a coverage is simply called a sheaf. The full subcategory of $\mathrm{PSh}(\mathcal{C})$ of sheafs is denoted by $\mathrm{Sh}(\mathcal{C})$. A category of the form $\mathrm{Sh}(\mathcal{C})$ for some site $\mathcal{C}$ is called a ***Grothendieck topos***. Note that this topos is a *subtopos* of $\mathrm{PSh}(\mathcal{C})$ meaning that the inclusion $i : \mathrm{Sh}(\mathcal{C}) \hookrightarrow \mathrm{PSh}(\mathcal{C})$ is a so-called *geometric morphism*, i.e. it has a left adjoint "sheafification" functor L. But we characterised such reflective localisations in Remark 1.38 and Lemma 1.40: $\mathrm{Sh}(\mathcal{C})$ is the localisation of $\mathrm{PSh}(\mathcal{C})$ at the local equivalences $L^{-1}(Isom)$. Sheafs are precisely local objects with respect to these local equivalences − this gives us a second equivalent descent condition, and a second way to think about sheafs, namely as reflective localisations of presheafs.

In the presence of a coverage (basically, always) we will be saying "generalised spaces" as referring to sheafs most of the time.

Example 1.106. Every topological space X admits a canonical site $\mathrm{op}(X)$: The underlying category is the poset of open subsets $U \subset X$ of X and the coverage consists of actual covering families $\{ U_i \subset U \}, \bigcup_i U_i = U$. Saying "sheafs on X" usually refers to the site we just defined.

Example 1.107 (functor of points). As an example we describe how to build a "geometry from algebra and topology". Given a notion of algebra (e.g. the category **CRing**) we can form affine spaces by considering their opposite category (e.g. $\mathbf{Aff} = \mathbf{CRing}^{\mathrm{op}}$). Given a notion of topology (i.e. a site) on the generalized spaces modelled on affine spaces (e.g. $\mathrm{PSh}(\mathbf{Aff})$) we obtain the topos of sheafs which we refer to as geometry. In case of the Zariski topology for $\mathbf{Aff} = \mathbf{CRing}^{\mathrm{op}}$ we recover (up to requiring a cover of affine subfunctors) the notion of schemes and with it ordinary algebraic geometry.

What follows is a rough sketch of how generalized spaces, generalized cohomology and higher structure work together.

We have seen in section 1.2.3 that the invariant $H^1(X; G)$ for some discrete (abelian) group has a representation $[X, K(G, 1)]$. Note that for any topological G from the LES of the fibrations $G \to EG \to BG$ and $\Omega BG \to BG^I \to BG$ we deduce by contractibility of EG and BG^I that $G \simeq \Omega BG$. So in particular for G a discrete group BG has the homotopy type of $K(G, 1)$. But this, by Theorem 1.97, gives us a third description of $H^1(X; G)$ as isomorphisms classes of principal G-bundles on X. Now, following the discrete Grothendieck construction from Construction 1.9 we can regard the latter as a **Set**-valued sheaf on X equipped with a G-action. It should be clear that the "sheaf" perspective on $H^1(X; G)$ is of much greater generality (e.g. doesn't depend on niceness of topological spaces) than the notion of $H^1(X; G)$ that we started with.

Passing to higher n, by the above argument about the homotopy type of classifying spaces we deduce that $B^n G \simeq K(G, n)$ is a representing object for $H^n(X; G)$. Thus for nice spaces, we again have a correspondence of $H^n(X; G)$ to isomorphism classes of principal $B^{n-1}G$-bundles. But the fibers $B^{n-1}G$ have non-trivial higher homotopy groups (precisely they are $(n-1)$-types) and a description as **Set**-valued presheafs is therefore insufficient[8]. Instead, the analogous Grothendieck construction now yields

[8]Consider e.g. $B^2\mathbb{Z} = BU(1)$. By the above we have two perspectives on $[-, BU(1)]$: It represents $H^2(-; \mathbb{Z})$ and classifies $U(1)$ bundles, i.e. circle bundles - thus the fiber has non-trivial fundamental group $\mathbb{Z}$. In particular there are a bunch of inequivalent bundles $[S^2, BU(1)] = \mathbb{Z}$ over the 2-sphere, e.g. $1 \in \mathbb{Z}$ is the Hopf bundle. But from the perspective of Set-valued sheafs they all look trivial as S^2 is simply connected.

so-called higher *gerbes* banded by $B^{n-1}G$: these are in particular "higher sheafs" on X valued in $(n-1)$-**Grpd** (the $(n,1)$-category of homotopy $(n-1)$-types) such that in the limit of restricting to a point in X we recover the fiber $B^{n-1}G \in (n-1)$-**Grpd**. In general, such $(n,1)$-*sheafs* a.k.a. n-*stacks* are $(\infty,1)$***-presheafs*** into spaces (which using notation from section 3 we can write as $\mathrm{PSh}_{(\infty,1)} := \mathrm{Map}(N_\Delta(\mathrm{op}(X)), S)$) satisfying an appropriate higher descent condition[9] and having images in n-types.

These higher sheafs are precisely the ***higher generalized spaces*** that we previously referred to. We can ask two questions:

- How are $(\infty,1)$- and $(n,1)$-sheafs defined precisely? What sort of category do they form?
- What kind of information do generalized spaces represent?

Answering the first question is the main content of the book [Lur09a] by Lurie, where all details to the concepts mentioned here can be found (in particular notions like hypercovers and the descent condition for $(\infty,1)$-sheafs). Just as sheafs formed a Grothendieck topos, $(\infty,1)$-sheafs form what is called an $(\infty,1)$-topos. As noted above this encompasses the notion of $(n,1)$-topos by letting sheafs be valued in the subcategory of n-types. In this essay, we will only cover the basics of $(\infty,1)$-categories in section 3 without further discussion of sheafs.

For the second question we already claimed that generalized spaces have great use in algebraic geometry as moduli spaces or stacks. Another example is the following: Above we found that special n-stacks (namely the analogues of principal $B^{n-1}G$-bundles, n-gerbes banded by the constant $B^{n-1}G$-sheaf) are what is classified by $H^n(X;G)$. This has a further generalisation to the notion of sheaf cohomology $H^n(X;\mathcal{G})$ with coefficients in a sheaf $\mathcal{G}$ of abelian groups, which is classifying n-gerbes banded by a sheaf $\mathcal{G}$. Finding classifying spaces of this functor is only possible if we take into account the available higher structure of generalized spaces. That is, in our case for an appropriate spectrum of generalized spaces B^nA one can provide a representation of classical sheaf cohomology by considering path components $\pi_0\mathrm{Sh}_{(\infty,1)}(X, B^nA)$ of the mapping space functor in $\mathrm{Sh}_{(\infty,1)}$. A third simple example where such a representation is only possible when taking into account higher structure are classifying spaces of circle bundles with connection.

Finally, one should note that in general giving such representations firstly gives a useful different perspective on what structure we are classifying (and yields evidence that our structures are in a way "part of the theory" or conversely that our theory/category is large enough to capture that structure) and secondly provides a moral reason for similarities in the behaviour of possibly different looking invariants/cohomology theories. However, this will not be further discussed here. We will from now on concern ourselves with finding frameworks do first of all speak about $(\infty,1)$-categories - in the next section we will do so from a 1-categorical perspective.

2. Model categories

In this section, starting from a category with a notion of weak equivalence we will formalize the additional structure which enables one to work with fibrations, cofibrations as we found it for **Top** and **CW** in section 1.2 and thereby for instance

[9] As before, higher sheafs can be characterised as local objects in $\mathrm{PSh}(\mathcal{C})$ for some class $\mathcal{W}$ of local equivalences, usually determined by *hypercovers*.

gives us notions of paths objects, cylinder object and homotopy classes. Our 1-categorical formalisation will be restricted to the unenriched case until section 2.5. In the unenriched setting, the formalisation of homotopy classes and is quite technical, see for instance Theorem 2.8. The enriched approach is more natural in that a $\mathcal{V}$-enriched category $\mathcal{M}$ for a homotopical $\mathcal{V}$ automatically inherits a notion of homotopy between elements of its hom objects. This approach is presented in detail in [Rie14]. We will mainly content ourselves with the classical approach mostly following [DS95] (in particular the latter contains more detail for most statements made here, at least until section 2.5). Technical proofs containing not much topological motivation will only be sketched to keep the presentation concise.

2.1. **Definitions.** A ***category with weak equivalences*** is a category $\mathcal{C}$ with a class of morphisms $\mathcal{W} \in \operatorname{mor} \mathcal{C}$ satisfying the property

(2of3) If any 2 of f, g and gf are in $\mathcal{W}$ then so is the third.

As remarked before this is a very minimal setting to do homotopy theory. The definitions of paths and cylinder objects below can be adapted to this setting as indicated already in section 1.2. As an example to keep in mind we saw for instance $\mathbf{Top}_*$ and $\mathbf{CW}_*$ with weak equivalences defined via homotopy groups in section 1.2.2.

However, there is the following drawback of the above axiom: suppose we have f, g such that that fg and gf are "homotopic" to the identity. Any reasonable homotopy theory would deduce that such homotopy equivalences are weak equivalences themselves (as they have two-sided "homotopy inverses"). E.g. taking $\mathcal{W}$ to be isomorphisms this is always the case. But our (2of3) property is too weak to capture this behaviour! Instead the idea of homotopy inverses (together with the usual (2of3) compositional structure) is described in the following axiom of a ***homotopical category***:

(2of6) If $gf, hg \in \mathcal{W}$ then $f, g, h, hgf \in \mathcal{W}$

We will be interested in homotopical categories that allow an additional compatible structure which will help to build the homotopy category (localizing weak equivalences) by so-called (co)fibrant replacement functors: A ***model category*** is a complete and cocomplete category $\mathcal{M}$ with weak equivalences satisfying (2of3) and two other classes $\mathcal{C}, \mathcal{F}$ of morphism called cofibrations and fibrations which are closed under composition. These satisfy:

(retr) If f is a retract of g, i.e. $f \underset{(r,r')}{\overset{(i,i')}{\rightleftarrows}} g$ such that $(r,r')(i,i') = 1_f$ in $\mathrm{Arr}\mathcal{M}$ as expressed in the the diagram

$$\begin{array}{ccccc} A & \xrightarrow{i} & B & \xrightarrow{r} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ C & \xrightarrow{i'} & D & \xrightarrow{r'} & C \end{array}$$

then whenever g is a weak equivalence or cofibration or fibration so is f.

(lift) There is a solution to the lifting problem

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ {\scriptstyle f}\downarrow & \overset{l}{\nearrow} & \downarrow{\scriptstyle g} \\ C & \xrightarrow{k} & D \end{array}$$

whenever f is a cofibration and g is a fibration in $\mathcal{W}$ (called trivial fibration) or f is cofibration in $\mathcal{W}$ (called trivial cofibration) and g is a fibration.

(repl) For every morphism $h : A \to B$ there is a cofibrant replacement $h : A \xrightarrow{f} C \xrightarrow{g} B$ for f a trivial cofibration and g a fibration, and a fibrant replacement $h : A \xrightarrow{f} D \xrightarrow{g} B$ for f a cofibration and g a trivial cofibration.

Further we say a object $X \in \mathcal{M}$ is ***fibrant*** if $X \to *$ is a fibration, and $A \in \mathcal{M}$ is called ***cofibrant*** if $\emptyset \to A$ is a cofibration (here $*$ and $\emptyset$ denote the terminal and initial object respectively) and ***bifibrant*** if it is both fibrant and cofibrant.

Note that if $\mathcal{M}$ is a model category than $\mathcal{M}^{\mathrm{op}}$ **is a model category** with the roles of cofibrations and fibrations interchanged. Thus, proofs and statements below have a dual analogue. Also note, based on this axiomatisation there is some redundance in our definition. Namely, if we determine the class of cofibrations then it uniquely determines (trivial) fibrations by (repl) and (retr) and vice versa: This follows from the next Lemma. In more generality given a family of morphisms $\mathcal{F}$ in $\mathcal{M}$ a morphism $g : B \to D$ is said to satisfy the ***right lifting property*** (RLP) for $\mathcal{F}$ (respectively the ***left lifting property*** (LLP) for $\mathcal{F}$) if the following diagram has a lifting solution l for all $f \in \mathcal{F}, h, k \in \operatorname{mor} \mathcal{M}$

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ {\scriptstyle f}\downarrow & \overset{l}{\nearrow} & \downarrow{\scriptstyle g} \\ C & \xrightarrow{k} & D \end{array} \quad \text{resp.} \quad \begin{array}{ccc} B & \xrightarrow{h} & A \\ {\scriptstyle g}\downarrow & \overset{l}{\nearrow} & \downarrow{\scriptstyle f} \\ D & \xrightarrow{k} & C \end{array}$$

Precisely, we then have

Lemma 2.1. *(i) f is a cofibration $\iff$ f has the LLP for trivial fibrations*
(ii) f is a trivial cofibration $\iff$ f has the LLP for fibrations
(iii) the duals of (i) and (ii)

Proof. "$\Rightarrow$" is clear in both cases. Assume f has the LLP for trivial fibrations, and fibrantly replace $f : A \to B$ by $f : A \xrightarrow{i} C \xrightarrow{r} B$. Then let l be a lifting solution for the lifting problem $(i, 1) : f \to r$. Then $f \xrightarrow{(1,l)} i \xrightarrow{(1,r)} f$ exhibits f as retract of i and thus f is a cofibration. For (ii) the same argument holds when using the cofibrant replacement instead. □

Lemma 2.2. *(i) (trivial) cofibrations are preserved by pushouts*
(ii) (trivial) fibrations are preserved by pullbacks

Proof. This is an immediate consequence of the characterisation in the previous Lemma 2.1 and the universal property of a pushout (or pullback). □

The notion of cofibration and fibrations allow us to give a refined description of different types of paths and cylinder objects as follows

Definition 2.3. A ***path object*** X^I for X is an object of $\mathcal{M}$ equipped with maps $c \in \mathcal{W}$ ("constant paths") and p_i ("evaluation at $i = 0, 1$") as follows

$$\begin{array}{ccc} X & \xrightarrow[\sim]{c} & X^I \\ \Vert & \swarrow{\scriptstyle p_i} & \downarrow{\scriptstyle p=\binom{p_1}{p_2}} \\ X & \xleftarrow[\pi_i]{} & X \times X \end{array}$$

Dually, a ***cylinder object*** $A \wedge I$ for $A \in \mathcal{M}$ is equipped with maps $r \in \mathcal{W}$ ("retraction") and i_j ("inclusion at $j = 0, 1$") as follows

$$\begin{array}{ccc} A & \xleftarrow[\sim]{r} & A \wedge I \\ \Vert & \nearrow{\scriptstyle i_j} & \uparrow{\scriptstyle i=(i_1,i_2)} \\ A & \xrightarrow[\iota_j]{} & A \sqcup A \end{array}$$

We say a cylinder object is ***good*** if i is a cofibration, it is called ***very good*** if in addition r is a fibration. Dual definitions hold for paths objects (when c is a cofibration and p is a fibration). A ***left homotopy*** $H : f \simeq_l g : A \to X$ is given by a map $H : A \wedge I \to X$ for some cylinder object such that $Hi_0 = f$ and $Hi_1 = g$. H is called ***good*** or ***very good*** if $A \wedge X$ is good or very good respectively. Dually, a ***right homotopy*** $H : f \simeq_r g : A \to X$ is a map $H : A \to X^I$ such that $p_0 H = f$ and $p_1 H = g$, and H is good or very good depending on whether X^I is.

Lemma 2.4. *(i) If $f \simeq_l g : A \to X$ then there is a good $H : f \simeq_l g$*
(ii) If A is cofibrant, $A \wedge I$ good then i_j is a trivial cofibration
(iii) If X is fibrant, $f \simeq_l g : A \to X$ then there is a very good $H : f \simeq_l g$
(iv) The duals of (i), (ii), (iii)

Proof. For (i) we just cofibrantly replace i. For (ii) note that ι_j are cofibrations by Lemma 2.2 and $A \sqcup A$ being a pushout $A \sqcup_\emptyset A$. The statement follows from (2of3) and $\mathcal{C}$ being closed under composition applied to the two factorisations of i_j given in its definition. For (iii) we first apply (i), then cofibrantly replace $r = r'w$ and finally find a new very good homotopy H from the (good) old one H' by X being fibrant:

$$\begin{array}{ccc} (A \wedge I)' & \xrightarrow{H'} & X \\ {\scriptstyle w}\downarrow & \nearrow{\scriptstyle H} & \downarrow \\ A \wedge I & \longrightarrow & * \end{array}$$

□

Considering the the transitive closure of $\simeq_l$ and $\simeq_r$ we can define the left and right ***homotopy classes of maps*** in $\mathcal{M}(A, X)$ denoted by $\pi_l(A, X)$ and $\pi_r(A, X)$. But with the notion of (co)fibrant object we can also give a condition when $\simeq_l$ and $\simeq_r$ are transitively closed by their own definition:

Lemma 2.5. *(i) If A is cofibrant then $\simeq_l$ is an equivalence relation*
(ii) Dually, if X is fibrant then $\simeq_r$ is an equivalence relation

Proof. Symmetry ($A \wedge I \equiv A, H = f$) and reflexitiy (by swapping the indices) are clear. Transitivity can be proved straightforwardly following our topological intuition: First glue the cylinder object of $H : f \simeq_l g$, $H' : g \simeq_l h$ together by a pushout construction

$$\begin{array}{ccc} A & \xrightarrow{i_0'} & (A \wedge I) \\ {\scriptstyle i_1}\downarrow & & \downarrow \\ A \wedge I & \longrightarrow & (A \wedge I)'' \end{array}$$

noting that i_0', i_1 are trivial cofibrations by Lemma 2.4 and thus Lemma 2.2 applies making $A \to (A \wedge I)''$ a trivial cofibration. H'' and r'' can be constructed from the universality of the pushout and their properties can be easily verified. □

From our considerations of the category **CW** we see that it is quite essential for $\pi_l(-,-)$ and $\pi_r(-,-)$ to be functorial. We have

Lemma 2.6. *Let $p : X \to Y$ and $h : B \to A$. Then*

(i) Postcomposition $p_* : \pi_l(A, X) \to \pi_l(A, Y)$ is well defined. If further A is cofibrant and p is a trivial fibration then p_* is a bijection.

(ii) If X is fibrant, then precomposition is $h^* : \pi_l(A, X) \to \pi_l(B, X)$ is well-defined, and thus by (i) so is the composition map $\pi_l(B, A) \times \pi_l(A, X) \to \pi_l(B, X)$.

(iii) Dually, precomposition $h^* : \pi_r(A, X) \to \pi_r(B, X)$ is well defined. If further X is fibrant and h is a trivial cofibration then h^* is a bijection. On the other hand, if A is cofibrant then $p_* : \pi_r(A, X) \to \pi_r(B, X)$ is well-defined, and thus so is the composition map $\pi_r(A, X) \times \pi_r(X, Y) \to \pi_r(A, Y)$.

Proof. For (i), we note that $H : f \simeq_l g$ implies $pH : pf \simeq_l pg$ showing well-definedness. Assuming A is cofibrant and p a trivial fibration the lifting diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & X \\ \downarrow & \overset{f}{\nearrow} & \downarrow{\scriptstyle p} \\ A & \xrightarrow{g} & Y \end{array}$$

shows surjectivity of p_*. And the diagram

$$\begin{array}{ccc} A \sqcup A & \xrightarrow{(f,f')} & X \\ {\scriptstyle i}\downarrow & \overset{H'}{\nearrow} & \downarrow{\scriptstyle p} \\ A \wedge I & \xrightarrow{H} & Y \end{array}$$

shows injectivity of p_*.

For (ii) suppose we have $H : f \simeq_l f' : A \to X$. Since X is assumed fibrant, by Lemma 2.4 we can assume H to be very good (i.e. r below is a trivial fibration). Also by fibrant replacement of $B \sqcup B \xrightarrow{(1,1)} B$ we choose a very good cylinder object $B \wedge I$ (i.e. i' below is a cofibration). Assuming $f \simeq_l f'$ we want to find $H' : fh \simeq_l f'h$ over this fixed cylinder object. This can be obtained following topological intuition and the (lift) axiom in the following diagram:

$$\begin{array}{ccccc} B \sqcup B & \xrightarrow{h \sqcup h} & A \sqcup A & \xrightarrow{i} & A \wedge I \\ {\scriptstyle i'}\downarrow & & & & \downarrow{\scriptstyle r} \\ B \wedge I & \xrightarrow{r'} & B & \xrightarrow{h} & A \end{array}$$

to find the canonical map $(h \wedge 1) : B \wedge I \to A \wedge I$. Then set $H' = H(h \wedge 1)$. □

In **CW** we didn't have to make a distinction between left and right homotopies because path objects were related to cylinder objects by an adjunction. The following describes the interplay between $\simeq_l$ and $\simeq_r$ in general

Lemma 2.7. *Let $f, g : A \to X$*

(i) If A is cofibrant then $f \simeq_l g \Rightarrow f \simeq_r g$

(ii) Dually, if X is fibrant then $f \simeq_r g \Rightarrow f \simeq_l g$

(iii) For A cofibrant, X fibrant we set $\pi(A, X) := \pi_l(A, X) = \pi_r(A, X)$ and all homotopies can be taken with respect to fixed good path or cylinder objects.

Proof. For (i) let $H : f \simeq_l g$. By cofibrant replacement of $(X \xrightarrow{\Delta} X \times X) = (X \xrightarrow{c} X^I \xrightarrow{p} X \times X)$ find a *fixed* good path object X^I. A being cofibrant implies that $A \xrightarrow{i_0} A \wedge I$ is a trivial cofibration as usual by Lemma 2.4. Then consider the lift

$$\begin{array}{ccc} A & \xrightarrow{cf} & X^I \\ {\scriptstyle i_0}\downarrow & \overset{K}{\nearrow} & \downarrow{\scriptstyle p} \\ A \wedge I & \xrightarrow[\binom{fr}{H}]{} & X \times X \end{array}$$

and observe that (e.g. again by analogy with the topological case) the homotopy we are looking for should be Ki_1 which can be easily verified.

The statement (iii) follows directly from the construction of the fixed path and cylinder objects in the proof of (i) and its dual. □

Finally, we have

Theorem 2.8. *For A, X bifibrant we have $f : A \to X$ is a weak equivalence if and only if it has a homotopy inverse $g : X \to A$.*

Proof (Sketch). Assuming f is a weak equivalence consider its cofibrant replacement $f : A \xrightarrow{i} B \xrightarrow{p} X$. Both i and p are trivial by (2of3) and B is bifibrant. Thus it suffices to exhibit a homotopy inverse for one of them and then argue by duality. Then the statement follows from $i^* : \pi(B, A) \to \pi(A, A)$ and $i_* : \pi(B, B) \to \pi(A, B)$ being bijections in analogy to the proof of Theorem 1.62.

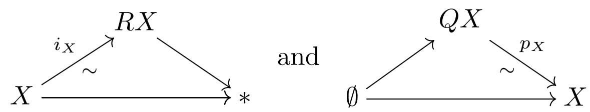
Conversely, assume a homotopy inverse g with $H : gf \simeq 1$, $H' : fg \simeq 1$. Again using the cofibrant replacement if f this time it is enough to show that p is a weak equivalence by (2of3). After finding a canonical s with $ps = 1$ by lifting $H' : 1 \simeq fg = pig$ along the fibration p to $H' : s \simeq ig$, one can show $sp \simeq 1$ using the homotopy inverse of i (which exists by the previous "⇒" direction since i is a weak equivalence). But we have the following:

Claim 2.9. *If $h : C \to D$ is a weak equivalence then $h \simeq_l k$ implies that k is a weak equivalence (i.e. $\mathcal{W}$ is closed under $\simeq_l$ and dually under $\simeq_r$).*

Proof. Indeed i_0, i_1 are weak equivalences by (2of3) and their definition. Employing (2of3) for $h = Hi_0$, $k = Hi_1$ the statement follows.

So sp is a weak a equivalence, and so is p being a retract of sp via $p \xrightarrow{1,s} sp \xrightarrow{1,p} p$ using $ps = 1$. □

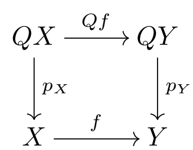
2.2. **The homotopy category.** Let $\mathcal{M}_c$, $\mathcal{M}_f$ and $\mathcal{M}_{cf}$ denote the full categories of subcategories of cofibrant, fibrant and bifibrant objects respectively. Passing the right resp. left homotopy classes we obtain categories $\pi_r\mathcal{M}_c$ and $\pi_l\mathcal{M}_f$ resp. by Lemma 2.7 and similarly a category $\pi\mathcal{M}_{cf}$. We also define fibrant and cofibrant replacement maps on objects R and Q by fixing cofibrant resp. fibrant replacements as follows:



where the cofibration i_X and fibration p_X are weak equivalences. For fibrant X we fix $RX = X$, and for cofibrant X we fix $QX = X$. R and Q can be made into functors as follows:

Lemma 2.10. *(i) Q as above gives rise to a functor $Q : \mathcal{M} \to \pi_r\mathcal{M}_c$ which in turn induces a functor $Q : \pi_l\mathcal{M}_f \to \pi\mathcal{M}_{cf}$.*
(ii) Dually, R yields functors $R : \mathcal{M} \to \pi_l\mathcal{M}_f$ and $R : \pi_r\mathcal{M}_c \to \pi\mathcal{M}_{cf}$.

Proof. By Lemma 2.6 $(p_Y)_*$ is a bijection on left homotopy classes and thus Qf is induced in the following diagram unique up to left homotopy



By QX being cofibrant and Lemma 2.7 (implying $\simeq_l \Rightarrow \simeq_r$) Qf is unique up to right homotopy as well. This implies $Q : \mathcal{M} \to \pi_r\mathcal{M}_c$ sending morphisms f to $[Qf]$ is functorial. Secondly, if Y is fibrant p_X^* is well defined as in Lemma 2.6 and thus Qf only depends on the left homotopy class of f showing well-definedness of the induced functor $Q : \pi_l\mathcal{M}_f \to \pi\mathcal{M}_{cf}$. □

Definition 2.11. The ***homotopy category*** $\mathrm{Ho}(\mathcal{M})$ of a model category $\mathcal{M}$ is the category with objects $\mathrm{obj}\,\mathcal{M}$ and morphisms

$$[X, Y] \equiv \mathrm{Ho}(\mathcal{M})(X, Y) := \pi(RQX, RQY)$$

with composition induced from composition in $\mathcal{M}$. There is a canonical embedding $\gamma : \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$ mapping $X \mapsto X$ and $f \mapsto RQf$.

Remark 2.12. (i) Firstly, from the definition of $f \mapsto Qf$ and $f \mapsto Rf$ we see that they preserve weak equivalences. Thus by Theorem 2.8 we have that γ maps *weak equivalences to isomophisms.*
(ii) Secondly, if X, Y are bifibrant we have $\pi(X, Y) \xrightarrow[\sim]{\gamma} \pi(RQX, RQY) = [X, Y]$ by our choice of R, Q on objects and on morphisms in Lemma 2.10.

The first item in the previous remark has an important consequence. Since $\gamma(i_Z)$, $\gamma(p_Z)$ (i_Z, p_Z being maps from the (co)fibrant replacement) are isomorphism they induce bijections by pre- and postcomposition between Hom spaces in $\mathrm{Ho}(\mathcal{M})$ as follows

$$[X, Y] \cong [RQX, RQY] = \mathrm{Ho}(\mathcal{M})(RQX, RQY)$$

But $\gamma : \pi(RQX, RQY) \cong [RQX, RQY]$ by the second part of the remark. We immediately deduce

Lemma 2.13. *Every $f \in [X, Y]$ can be decomposed for some $f' \in \pi(RQX, RQY)$ as follows*

$$f = \gamma(p_Y)\gamma(i_{QY})^{-1}\gamma(f')\gamma(i_{QX})\gamma(p_X)^{-1}$$

In particular, natural transformations defined on the image of γ, i.e. $t_\gamma : F\gamma \to G\gamma$ for $G : \mathrm{Ho}(\mathcal{M}) \to \mathcal{D}$ give rise to "full" natural transformations $t : F \to G$. □

Further, a functor F mapping weak equivalences to isomorphisms satisfies the following implication (and its dual)

$$f \simeq_l g \Rightarrow F(f) = F(g) \tag{2.14}$$

since $f = Hi_0$, $g = Hi_1$ and $F(i_0) = F(r)^{-1} = F(i_1)$. Equipped with these observations we can easily prove an extension to Remark 2.12

Lemma 2.15. *For A cofibrant, X fibrant we have $\gamma : \pi(A, X) \cong \mathrm{Ho}(\mathcal{M})(A, X)$.*

Proof. We have a zig-zag of isomorphism (using Lemma 2.6, (2.14) and Remark 2.12 in that order)

$$\pi(A, X) \xleftarrow{p_X^*(i_A)_*} \pi(RA, QX) \xrightarrow{\quad\gamma\quad} [RA, QX] \xrightarrow{\gamma(p_X)^*\gamma(i_A)_*} [A, X]$$

which composes to give γ on morphisms by its functoriality. □

Finally, we can show that all our work so far leading up to the definition of $\mathrm{Ho}(\mathcal{M})$ was independent of our chosen notion of cofibrations and fibrations

Theorem 2.16. *$\gamma : \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$ is the localisation of $\mathcal{M}$ at $\mathcal{W}$.*

Proof. Recall that $(\mathcal{C}, \gamma : \mathcal{M} \to \mathcal{C})$ being a localisation of $\mathcal{M}$ at $\mathcal{W}$ means that every $G : \mathcal{M} \to \mathcal{D}$ that maps $f \in \mathcal{W}$ to isomorphisms in $\mathcal{D}$ factors uniquely (up to natural isomorphism) over γ. Now let $\mathcal{C} = \mathrm{Ho}(\mathcal{M})$ and γ as previously constructed. Given a $G : \mathcal{M} \to \mathcal{D}$ mapping $\mathcal{W}$ to isomorphisms suppose we have G' such that $G = G'\gamma$. Clearly in this case we must have $GX = G'X$, and by writing out $f \in \mathrm{Ho}(C)(X, Y)$ as in Lemma 2.13 in terms of images of γ we find that

$$G'(f) = G\gamma(p_Y)G\gamma(i_{QY})^{-1}G\gamma(f')G\gamma(i_{QX})G\gamma(p_X)^{-1}$$

showing that G' is unique if it exists. On the other hand, we can use the above equation as definition for G' to prove its existence. □

2.3. **The small object argument.** The difficult part in establishing a model structure on a category with weak equivalences is to guarantee fibrant and cofibrant replacements (i.e. choosing the classes of (co)fibrations large enough) while still have solutions to all lifting problems (i.e. choosing the classes not *too large*). Therefore in many cases, it is easiest to generate these classes starting from a set of morphisms which are "small" members of one class. More precisely, a weak factorisation system (L, R) on $\mathcal{M}$ (i.e. classes L, R such that $L = \mathrm{LLP}(R)$, $R = \mathrm{RLP}(L)$ and $\forall f \in \mathrm{mor}\,\mathcal{M}, \exists l \in L, r \in R$ such that $f = rl$) is said to be ***cofibrantly generated*** by I if $R = \mathrm{RLP}(I)$ and $L = \mathrm{LLP}(R)$. Note that

Claim 2.17. *The class $\mathcal{F} = \mathrm{LLP}(I)$ is stable under pushouts, large pushouts, retracts, composition and transfinite composition (i.e. passing to colimits in a diagram of shape α for some ordinal α regarded as a poset).*

Proof. All claims follow straightforwardly from appropriate concatenation of diagrams and invoking universal properties (with the first case already discussed in Lemma 2.2). □

The ***small object argument*** which will be given in the next construction can now be summarised as follows

Theorem 2.18 (Small object argument)**.** *For a set $I \subset \operatorname{mor}\mathcal{M}$ with either $\mathcal{M}$ locally presentable or all domains in I compact (or otherwise suitably "small" as in Construction 2.21), we can factor each $f \in \mathcal{M}$ as $f : A \xrightarrow{i_\infty} C \xrightarrow{p_\infty} B$ where $p_\infty \in \mathrm{RLP}(I)$ and $i_\infty \in \mathrm{cell}(I)$. The latter is the class of transfinite compositions of pushouts of I which will be explicitely constructed in Construction 2.21.*

Note that by definition $I \subset \mathrm{LLP}(\mathrm{RLP}(I))$ and therefore by Claim 2.17 $\mathrm{cell}(I) \subset \mathrm{LLP}(\mathrm{RLP}(I))$. Conversely (following exactly the proof of Lemma 2.1) we can factor any $f \in \mathrm{LLP}(\mathrm{RLP}(I))$ as in Theorem 2.18 $f \in \mathcal{M}$ as $f : A \xrightarrow{i} C \xrightarrow{p} B$ and apply its lifting property to exhibit is as a retract of $i \in \mathrm{cell}(I)$. Thus, for the small object construction below we have

$$\mathrm{LLP}(\mathrm{RLP}(I)) = \mathrm{retracts}(\mathrm{cell}(I)) =: \mathrm{cof}(I) \tag{2.19}$$

Now in a category with weak equivalences $\mathcal{W}$ which are closed under retracts, assume we have two (again "small" enough for the small object Construction 2.21) sets of morphisms I, J. We define $\mathrm{cof}(I)$ as above and $\mathrm{cof}(J)$ to be similarly the class of cofibrations associated to J. Assume I and J are such that $\mathrm{RLP}(J) = \mathrm{RLP}(I) \cap \mathcal{W}$ can be verified. By Theorem 2.18, (2.19) and Claim 2.17 it only remains to verify that $\mathrm{cof}(I) \subset \mathcal{W}$ to obtain a model structure: Indeed since $\mathrm{RLP}(J) \subset \mathrm{RLP}(I)$ we deduce $\mathrm{cof}(I) \subset \mathrm{cof}(J)$ and thus $\mathrm{cof}(I) \subset \mathrm{cof}(J) \cap \mathcal{W}$. It remains to show $\mathrm{cof}(J) \cap \mathcal{W} \subset \mathrm{cof}(I)$ as we can then set $\mathcal{C} = \mathrm{cof}(J)$ and $\mathcal{F} = \mathrm{RLP}(I)$. So take $f \in \mathrm{cof}(J) \cap \mathcal{W}$, and factor it as $f = pi$ for $i \in \mathrm{cof}(I) \subset \mathcal{W}$ and $p \in \mathrm{RLP}(I)$. By (2of3) $p \in \mathcal{W}$, thus $p \in \mathrm{RLP}(I) \cap \mathcal{W} = \mathrm{RLP}(J)$. The solution to the lifting problem $(i, 1) : f \to p$ then exhibits f as retract of i and thus $f \in \mathrm{cof}(I)$ as required.

This argument provides is the first half of the following theorem. The second half has an analogous (and dual) proof.

Theorem 2.20 (Recognition theorem)**.** *Given $\mathcal{M}, \mathcal{W}$ and I, J as above they establish a model structure on $\mathcal{M}$ if one of the following holds;*

(i) $\mathrm{RLP}(J) = \mathrm{RLP}(I) \cap \mathcal{W}$ *and* $\mathrm{cof}(I) \subset \mathcal{W}$
(ii) $\mathrm{cof}(I) = \mathrm{cof}(J) \cap \mathcal{W}$ *and* $\mathrm{RLP}(J) \subset \mathrm{RLP}(I) \cap \mathcal{W}$

Proof. More details than given above can be found in [Hir03] §11.3. □

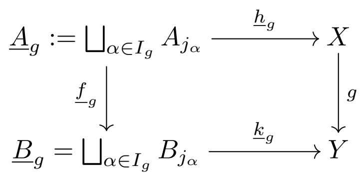
Construction 2.21 (Small object construction)*.* Let $I = \{ f_j : A_j \to B_j \}_{j \in \mathcal{J}}$ such that $\mathrm{Hom}(A_i, -)$ preserves sequential colimits (i.e. of shape $\mathbb{N}$) which is the case for instance when A_i are compact. For $g : X \to Y$ set

$$I_g = \{ \alpha \mid (h_\alpha, k_\alpha) : f_{j_\alpha} \to g \text{ for } h_\alpha, k_\alpha \in \operatorname{mor}\mathcal{M}, j_\alpha \in \mathcal{J} \}$$

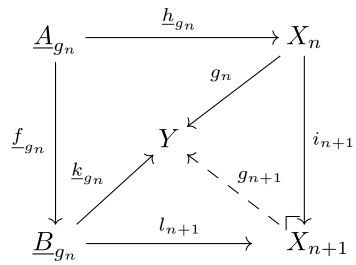
be the set indexing lifting problems for I and g, and define

$$\underline{h}_g := (..., h_\alpha, ...)_{\alpha \in I_g} \;,\; \underline{k}_g := (..., k_\alpha, ...)_{\alpha \in I_g} \;,\; \underline{f}_g := (... \sqcup f_{j_\alpha} \sqcup ...)_{\alpha \in I_g}$$

such that



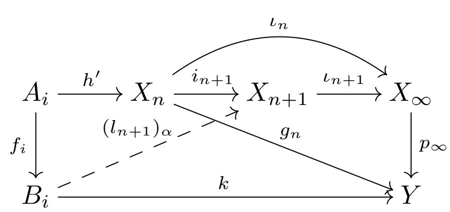
Now set $X_0 = X$, $g_0 = g$ and define X^{n+1}, g_{n+1} and l_{n+1} inductively as the pushout



We should regard this as a gluing many B_i's to the space X_n which will be elucidated further when considering the example of **CW**. Also note that by using "individual gluing operations" X_{n+1} can also be written as large pushout along maps obtained by small pushouts, exhibiting i_{n+1} as a composition of pushout maps of $f_i \in I$ which will lead to the definition of cell(I) in Theorem 2.18. Passing to $X_\infty = \operatorname{colim}(X_0 \xrightarrow{i_1} X_1 \xrightarrow{i_2} X_2 \to ...)$ induces a map $p_\infty : X_\infty \to Y$ factoring $g_n = p_\infty \iota_n$ where $\iota_n : X_n \to X_\infty$ are the cocone maps. Note $i_\infty := \iota_0$ is the transfinite composition of the i_n. We obtained a factorisation

$$g = g_0 : X \xrightarrow{i_\infty} X_\infty \xrightarrow{p_\infty} Y$$

We have $i_\infty \in \operatorname{cell}(I)$ by construction and we need to show that $p_\infty \in \operatorname{RLP}(I)$. So consider a lifting problem $(h, k) : f_j \to p_\infty$ for $f_j : A_j \to B_j$. Since A_j is compact with respect to sequential colimits we can factor $h : A_j \xrightarrow{h'} X_n \xrightarrow{\iota_n} X_\infty$ for some n and thus have a lift $l = \iota_{n+1}(l_{n+1})_\alpha$ as follows



since $h' = (\underline{h}_{g_n})_\alpha, k = (\underline{k}_{g_n})_\alpha$ for some $\alpha \in I_{g_n}$ in our construction of X_{n+1}. This is the statement of Theorem 2.18.

2.3.1. *Examples.* We use the Recognition Theorem 2.20 to discuss two basic examples. Both show that the notions of *disks and spheres* can play a deep role in the definition of a model structure.

(i) First let $\mathcal{M} = \operatorname{Ch}_{\geq 0}(\mathbf{R})$ the category of non-negatively graded chain complexes of $\mathbf{R}$-modules with weak equivalences being quasisomorphisms (i.e. chain maps inducing isomorphisms on homology). We define a ***disk*** D^n ***in*** $\operatorname{Ch}_{\geq 0}(\mathbf{R})$ to be a chain complex C with $C_n = C_{n-1} = R$, $C_m = 0$ otherwise and $d_{n-1} = 1$.

We define a ***sphere*** S^n ***in*** $\mathrm{Ch}_{\geq 0}(\mathbf{R})$ to be a chain complex C with $C_n = R$ and $C_m = 0$ otherwise. We let $I = \{ 0 \hookrightarrow D^n \}$ and $J = \{ S^{n-1} \hookrightarrow D^n \}$. Clearly, both I and J are small in the sense of Construction 2.21. Noting that D^n has trivial homology it is straightforward to see that i_∞ in Theorem 2.18 (used for the family I) is a weak equivalence. More generally we see that $\mathrm{cell}(I) \subset \mathcal{W}$ and thus $\mathrm{cof}(I) \subset \mathcal{W}$ since $\mathcal{W}$ is closed under retracts. Verifying the first condition of the recognition Theorem 2.20 (i) it is then straightforward. Thus we deduce the model structure axioms with $\mathrm{cof}(J)$ being the class of cofibrations, and $\mathrm{RLP}(I)$ the class of fibrations.

Using the explicit forms of S^n, D^n we have a more explicit characterisation of theses classes as follows: Fibrations $f : C \to C'$ are chain maps such that f_k is epi for $k \geq 0$. Cofibrations $f : C \to C'$ are chain maps such that f_k is mono with projective cokernel (i.e. they are inclusions in to direct sums). Note that this makes all objects fibrant and it follows that a cofibrant replacement $QC \xrightarrow[\sim]{p_C} C$ of a chain complex C with a single non-zero module in degree 0 is just a projective resolution of this module. Having a notion of spheres, disks, cofibrations and fibrations we can start to recover cohomology theory from section 1.2.3. E.g. let us first establish the analogues of Eilenberg-Maclane spaces. Set $K(n, B)_n = B$ and 0 everywhere else. We claim

$$[S^m, K(n, B)] = \pi(S^m, K(n, B)) = \delta_{nm} B$$

In the first step we used that S^n is already cofibrant and $K(n, B)$ already fibrant. For the second step we use Lemma 2.7 (iii) to fix the canonical path object $K(n, B)^I$ to be the following chain

$$\ldots \to 0 \to K(n, B)^I_n := B \oplus B \xrightarrow{(r,s) \mapsto r-s} B =: K(n, B)^I_{n-1} \to 0 \to \ldots$$

with $c : K(n, B)_n \to K(n, B)^I_n, r \mapsto (r, r)$ and $p_i : K(n, B)^I_n \to K(n, B)$ mapping $(r_1, r_2) \mapsto r_i$. With this one can verify that a homotopy $H : X \to K(n, B)^I$ of maps $f, g : X \to K(n, B)$ gives the usual notion of chain homotopies h from X to $K(n, B)$:

$$h_{n-1}\partial - \partial h_n = p_1 H_n - p_2 H_n = f - g$$

Now $\mathcal{M}(S^m, K(n, B)) = \delta_{nm} B$ and passing to chain homotopy classes we have $[S^m, K(n, B)] = \delta_{nm} B$ as claimed. Similarly for $K(m, A)$ in place of S^m using a projective resolution $QK(m, A)$ one can then find

$$[K(m, A), K(n, B)] = \pi(QK(m, A), K(n, B)) = \mathrm{Ext}^{n-m}(A, B)$$

by definition of Ext. Thus the Ext-functors are nothing but ordinary cohomology $H^k = \mathrm{Ho}(\mathcal{M})(-, K(k, B))$ with coefficients in B evaluated on Eilenberg-Maclane spaces, i.e. $\mathrm{Ext}^{n-m}(A, B) = \mathrm{Ho}(\mathcal{M})(K(m, A), K(n, B))$. This example demonstrates how the concepts of homotopical algebras subsume homological algebra as "homotopy theory on $\mathrm{Ch}_{\geq 0}(\mathbf{R})$".

(ii) The next example is $\mathcal{M} = \mathbf{CW}$ with weak equivalences being weak equivalences from Definition 1.56. We have already seen the notions of Hurewicz cofibrations and fibrations. Instead, similar to the previous example we set $I = \{ D^n \times \{ 0 \} \hookrightarrow D^n \times [0, 1] \}$ and $J = \{ S^{n-1} \hookrightarrow D^n \}$. The corresponding collections of (co)fibrations are called Serre (co)fibrations. Clearly, both I and J are small at least for the sequential colimits appearing in Construction 2.21: Our i_∞ now takes the form of an actual (transfinite) gluing construction of either cylinders (in case of I) or cells (in the case of J). It is easy to see that for the family I i_∞ becomes a weak equivalence (by contracting the cylinders). Again we deduce $\mathrm{cell}(I) \subset \mathcal{W}$ and by verifying the first condition of the recognition Theorem 2.20 (i) one proves the model structure axioms with $\mathrm{cof}(J)$ being the class of cofibrations, and $\mathrm{RLP}(I)$ the class of fibrations. As shown in Theorem 2.16 this gives the same homotopy category as our previous notion of Hurewicz fibrations.

(iii) **sSet** has a canonical model structure called Quillen or Kan model structure obtained by *transferring* the model structure along the geometric realisation/singular nerve adjunction. Precisely, we set $f \in \mathcal{W}$ if $|f|$ (its geometric realisation) is a weak homotopy equivalence, and $g : X \to Y$ is a cofibration if it is pointwise mono. Fibrations for this choice of model structure are called ***Kan fibrations*** - they satisfy the lifting property with respect to horn inclusions.

(iv) Finally, one can also put a model structure on the category of spectra **Sp** with weak homotopy equivalences as defined in section 1.2.3. Recall that homotopy (co)fibers were special instances of homotopy (co)limits and thus can be translated to any category with weak equivalences admitting these limits. In particular observe that for pointed spaces:

$$\begin{array}{ccc} X & \xrightarrow{c} & * \\ \downarrow & & \downarrow{\scriptstyle \mathrm{hcof}(c)} \\ * & \longrightarrow & \Sigma X \end{array} \quad \text{and} \quad \begin{array}{ccc} \Omega X & \xrightarrow{\mathrm{hfib}(c)} & * \\ \downarrow & & \downarrow{\scriptstyle c} \\ * & \longrightarrow & X \end{array}$$

and using these diagrams as a definition we obtain a notion of ***loop objects*** and ***suspension objects*** in every pointed category with weak equivalences (once we discussed homotopy (co)limits in the next sections). If they exist for all X these objects gives rise to functors in an adjunction $\Sigma \dashv \Omega : \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{M})$. If this adjunction is an equivalence we call $\mathcal{M}$ a ***stable model category*** (we also remark that this makes $\mathrm{Ho}(\mathcal{M})$ a triangulated category but will not make further use of this notion). Since we have an inverse suspension Σ^{-1} in **Sp**, the category of spectra is a (prototypical) example of a stable model category. We will abstract the process of stabilisation based on our construction of **Sp** in section 3.

2.4. **Derived functors.** Recall Definition 2.11 of the homotopy category and the localisation $\gamma_{\mathcal{M}} : \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$, and the definition of Kan extensions from section 1.1.3. Let $\mathcal{M}, \mathcal{N}$ be model categories, $\mathcal{A}$ a category and $F : \mathcal{M} \to \mathcal{A}, G : \mathcal{M} \to \mathcal{N}$ functors.

Definition 2.22. We define the ***left derived functor*** $(\mathrm{L}F, t)$ of F to be the right Kan extension $(\mathrm{Ran}_{\gamma_{\mathcal{M}}} F, t^F)$ where $t^F : \mathrm{Ran}_{\gamma_{\mathcal{M}}} \gamma_{\mathcal{M}} F \to F$ is the universal natural transformation. The ***total left derived functor*** denoted by $\mathbf{L}G$ of G is defined as the left derived functor of the composition $\gamma_{\mathcal{N}} G$. Dually, we define the right derived functor of F as $\mathrm{R}F = \mathrm{Lan}_\gamma F$ and the total right derived functor of G as $\mathbf{R}G = \mathrm{Lan}_{\gamma_{\mathcal{M}}}(\gamma_{\mathcal{N}} G)$

We have seen criteria for the existence of Kan extensions, but will give a more general criteria to construct LF in the setting of extensions along γ. First, we will state a lemma specializing (2.14)

Lemma 2.23. *If $F|_{\mathcal{M}_c}$ (F restricted to the cofibrant subcategory) maps trivial cofibrations to isomorphisms then for maps f, g in $\mathcal{M}_c$ we have $f \simeq_r g : A \to B \Rightarrow Ff = Fg$.*

Proof. Since A is cofibrant we can choose a very good path object B^I for the homotopy, which is cofibrant since B is. Then $p_i : B^I \to B$ are both mapped to the isomorphism $F(c : X \xrightarrow{\sim} X^I)^{-1}$ and the statement follows dually to (2.14). □

Now the essential tool to construct derived functors on model categories is the following

Lemma 2.24. *If $F|_{\mathcal{M}_c}$ maps weak equivalences to isomorphisms then LF exists and $t^F_{X \in \mathcal{M}_c} = 1_X$ for fibrant X and cofibrant A.*

Proof. By the previous Lemma 2.23 F induces a functor $F' : \pi_r \mathcal{M}_c \to \mathcal{N}$, and thus a composite $F'Q : \mathcal{M} \to \mathcal{N}$. Recall our functor Q maps f to $[Qf]_r$ where

$$\begin{array}{ccc} QX & \xrightarrow{Qf} & QY \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle p_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

and thus $F'Q$ maps weak equivalences to isomorphisms. By Ho($\mathcal{M}$) being a localisation at $\mathcal{W}$ (Theorem 2.16) we obtain a factorisation L$F\gamma = F'Q$ of $F'Q$ over γ. By applying F to the above diagram we also obtain a natural transformation $t^F : \mathrm{L}F\gamma \to F$ with $t^F_X = F(p_X)$. We claim that (LF, t^F) is the left derived functor of F. Indeed given $s : G\gamma \to F$ suppose we have $s' : G \to \mathrm{L}F$ such that $s = ts'$. Then s' and s satisfy the following equality

$$\begin{array}{ccccc} G\gamma(QX) & \xrightarrow{s'_{QX}} & \mathrm{L}F(QX) & \xrightarrow{t_{QX}=F(1)=1} & F(QX) \\ {\scriptstyle G\gamma(p_X)}\downarrow\cong & & \downarrow{\scriptstyle \mathrm{L}F(\gamma(p_X))=\mathrm{L}F([1])=1} & & \downarrow{\scriptstyle F(p_X)} \\ G\gamma(X) & \xrightarrow{s'_X} & \mathrm{L}F(X) & \xrightarrow{t_X=F(p_X)} & F(X) \end{array} \quad = \quad \begin{array}{ccc} G\gamma(QX) & \xrightarrow{s_{QX}} & F(QX) \\ \cong\downarrow{\scriptstyle G\gamma(p_X)} & & \downarrow{\scriptstyle F(p_X)} \\ G\gamma(X) & \xrightarrow{s_X} & F(X) \end{array}$$

and thus $s_{QX} = s'_{QX}$ and $s'_X = s'_{QX}(G\gamma(p_X))^{-1}$. So s' is unique but on the other hand can be defined this way showing its existence. □

The following is the central theorem of this section. It gives a condition when the total derived functors of an adjunction between model categories gives rise to an adjunction between their respective homotopy categories - such pairs of functors are called ***left*** and ***right Quillen functors***. It also gives a condition for when the derived adjunction is actually an equivalence, in which case this is then called a ***Quillen equivalence***. So suppose we have an adjunction $F \dashv G : \mathcal{N} \to \mathcal{M}$ given by a natural isomorphism

$$\pi : \mathcal{M}(A, GB) \cong \mathcal{N}(FA, B)$$

we denote $\pi(f) = \overline{f}, \pi^{-1}(g) = \overline{g}$ in the following discussion. Note that the adjunction also translates lifting problems between categories as follows:

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ GB & \xrightarrow{Gg} & GB' \end{array} \quad \overset{\overline{(-)}}{\longleftrightarrow} \quad \begin{array}{ccc} FA & \xrightarrow{Ff} & FA' \\ {\scriptstyle \overline{h}}\downarrow & & \downarrow{\scriptstyle \overline{k}} \\ B & \xrightarrow{g} & B' \end{array}$$

which by Lemma 2.1 (characterizing (co)fibrations by lifting properties) implies that the following 3 conditions are equivalent

(i) F preserves cofibrations, G preserves fibrations
(ii) F preserves cofibrations and trivial cofibrations
(iii) G preserves fibrations and trivial fibrations

Theorem 2.25. *If any of the above conditions holds we obtain an adjunction* $\mathbf{L}F \dashv \mathbf{R}G : \mathrm{Ho}(\mathcal{N}) \to \mathrm{Ho}(\mathcal{M})$ *of total derived functors. If furthermore* $\overline{(-)}$ *preserves weak equivalences then this adjunction is a Quillen equivalence.*

Proof. Firstly, by equivalence of the above conditions F preserves weak equivalences as long as they are also cofibrations. Then we claim

Claim 2.26 (Ken Brown's Lemma)**.** $F|_{\mathcal{M}_c}$ *preserves weak equivalences*

Indeed, given a weak equivalence $f : A \to B$ between cofibrant objects, the cocone $(f, 1_B)$ over the pushout diagram $A \sqcup_\emptyset B = A \sqcup B$ induces a map $A \sqcup B \xrightarrow{(f,1)} B$ which we fibrantly factor as $A \sqcup B \xrightarrow{q} C \xrightarrow{p} B$. Composing with p and using (2of3) we see $qi_A : A \to C$, $qi_B : B \to C$ are weak equivalences. They are also cofibrations by A, B being cofibrant and thus the pushout i_A, i_B are cofibrations. It follows $F(f) = F(p)F(qi_A) = F(qi_B)^{-1}F(qi_A)$ is a weak equivalence which proves the claim.

As a consequence, by Lemma 2.24 we can now construct $\mathbf{L}F = \mathrm{L}\gamma_{\mathcal{N}}F$. Dually, we find that $\mathbf{R}G$ exists.

Secondly, note that being a left adjoint F preserves colimits and by assumptions of the theorem therefore it also preserves cofibrant objects and good cylinder objects. Dually, G preserves fibrant objects and good paths objects. Given a cofibrant A and fibrant B the adjunction then induces an isomorphism of homotopy classes

$$\pi(A, GB) \xrightarrow[\sim]{\overline{(-)}} \pi(FA, B)$$

which is well-defined and bijective since homotopies $H : A \wedge I \to GB$ carry over to give homotopies $\overline{H} : F(A \wedge I) = F(A) \wedge I \to B$ and dually for G. Using Lemma 2.15 and the construction of derived functors in Lemma 2.24 we have a zig-zag of isomorphisms:

$$[A, \mathbf{R}GX]_{\mathcal{M}} \xrightarrow{\gamma(p_A)^*} [QA, \mathbf{R}GX]_{\mathcal{M}} \xleftarrow{\gamma} \pi(QA, GRX) \xrightarrow{\overline{(-)}} \pi(FQA, RX) \quad \ldots$$

$$\xrightarrow{\gamma} [\mathbf{L}FA, RX]_{\mathcal{N}} \xleftarrow{(\gamma i_X^{-1})_*} [\mathbf{L}FA, X]_{\mathcal{N}}$$

By definition of p_A and i_X this is clearly natural in A, X as objects of $\mathcal{M}^{\mathrm{op}} \times \mathcal{N}$. Then by Remark 2.12 (ii) the first statement of the theorem follows.

For the second statement it is enough to show that units and counits of the adjunction are isomorphisms. Since every object in $\mathrm{Ho}(\mathcal{M})$ and $\mathrm{Ho}(\mathcal{N})$ is isomorphic to a cofibrant one, it is enough to do so for cofibrant objects. So let $A \in \mathcal{M}$ be cofibrant (in particular $\mathbf{L}FA = F'QA = F'A = FA$ from Lemma 2.24). Then tracing $1_{\mathbf{L}FA} \in [\mathbf{L}FA, \mathbf{L}FA]_{\mathcal{N}}$ backwards through the isomorphisms we obtain the unit

$$\eta_A = \gamma(p_A)^* \gamma(\overline{i_{FA}})$$

which as required is an isomorphism by assumptions on the adjunction in the theorem and γ turning weak equivalences into isomorphisms. The dual argument works for the counit. □

2.4.1. *Global homotopy (co)limits.*

Construction 2.27. We will only give a short sketch of the idea of "global" homotopy limits and colimits and how they relate to our previous notions of homotopy fibers and cofibers. Note that based on the observation of Theorem 2.16 the definitions of derived functors go through for any category $\mathcal{M}$ with weak equivalences. For a (small) diagram category $\mathcal{D}$ we can then make $\mathcal{M}^{\mathcal{D}}$ naturally a category with weak equivalences by requiring them to be objectwise weak equivalences (cf. Remark 1.34 and the notion of derivators therein). Diagrams F, J obtained from each other by replacing objects along weak equivalences will then become weakly equivalent, and a cone over the diagram F will thus correspond to a cone over J commuting *up to weak equivalence*.

Secondly, note that if $\mathcal{M}$ has $\mathcal{D}$-shaped colimits it admits an adjunction

$$\operatorname{colim} \dashv \Delta : \mathcal{M} \to \mathcal{M}^{\mathcal{D}}$$

where Δ is the constant diagram functor. Then the ***homotopy colimit*** of a diagram $J \in \mathcal{M}^{\mathcal{D}}$ is defined via the total left derived functor of colim

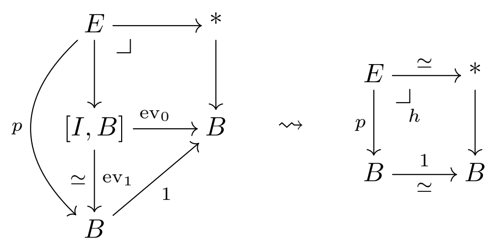
$$\operatorname{hocolim} J = \mathbf{L} \operatorname{colim}(J)$$

Note that this is a "global" definition of a homotopy colimit, meaning that we are defining homotopy colimits of all diagrams $\mathcal{M}^{\mathcal{D}}$ (or of a "weakly saturated" full subcategory containing $\operatorname{im}\Delta$) at once. Dually, the global notion of ***homotopy limit*** is just the derived limit functor. If $\mathcal{M}$ has a model structure then under certain conditions we can give $\mathcal{M}^{\mathcal{D}}$ model structures (weak equivalences are still objectwise weak equivalences) enabling us to calculate the derived (co)limit functors. We will only name these structures here:

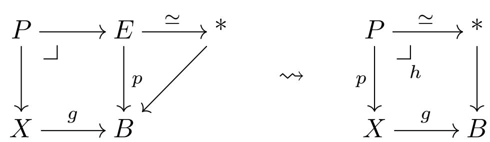
- If $\mathcal{M}$ is cofibrantly generated, then we can put the *projective model structure* on $\mathcal{M}^{\mathcal{D}}$ in which fibrations are objectwise fibrations. It is clear that Δ then preserves fibrations and trivial fibrations, so the adjunction $\operatorname{colim} \dashv \Delta$ satisfies the conditions of Lemma 2.24. Cofibrations are determined by the LLP (in the the case of easier diagrams like pushout diagrams these have an explicit description, see e.g. [DS95]).
- If $\mathcal{M}$ is combinatorial we can put the *injective model structure* on $\mathcal{M}^{\mathcal{D}}$ in which cofibrations are objectwise cofibrations. It is clear that Δ then preserves cofibrations and trivial cofibrations, so the adjunction $\Delta \dashv \lim$ satisfies the conditions of Lemma 2.24. Fibrations are determined by the RLP (in the the case of easier diagrams like pullback diagrams these have again a more explicit description).
- If $\mathcal{D}$ is a *Reedy category*, then we can put a Reedy model structure on $\mathcal{M}^{\mathcal{D}}$

In each case we now know how to construct $\mathbf{L}\operatorname{colim}(F)$ or $\mathbf{L}\lim(F)$ seeing the conditions of Lemma 2.24 are satisfied: Namely, we need to pass to a cofibrant replacement of the diagram F first and then apply colim. This is conceptually what we did when constructing homotopy cokernels and kernels in section 1.2.1 by first passing to (co)fibrant replacements of the maps at issue and then calculating their (co)kernel.

Example 2.28. We can reformulate are definitions of generalized principal bundles as follows. We first note that for $\mathcal{D} = A \xrightarrow{f} B \xleftarrow{g} C$ (i.e. pullback diagrams) a diagram F is fibrant if it's legs are fibrant. A theorem in the appendix of [Lur09a] then states that for computing the homotopy pullback as an ordinary pullback it is actually sufficient to have FA, FB fibrant objects and Fg fibrant. Then we have



and therefore



Finally, note that the last condition of Theorem 2.25 is in general not satisfied, i.e. hocolim $\not\dashv$ $\mathbf{R}\Delta$. In particular hocolim is not just the colimit functor on $\operatorname{Ho}(\mathcal{M})$. Instead we will see in the next sections (2.5 and 4.1) that (in $\operatorname{Ho}(\mathcal{M})$) hocolim F represents up to weak equivalence cocones under F commuting up to weak equivalence (as explained in the beginning of this section)[10].

2.5. **A more general story.** We have noted in the previous discussion that all of our central constructions were actually independent of the choices of classes of cofibrations and fibrations.

We have seen that to construct (left) derived functors functors we tested the behaviour of functors F when restricted to the cofibrant objects $\mathcal{M}_c$: left derived functors could then be easily constructed if $F|_{\mathcal{M}_c}$ mapped weak equivalences to isomorphism. Thus, we can regard a choice of cofibrations and thereby cofibrant replacement Q as a deformation or "correction" for the specific functor F, i.e. instead of thinking of $\mathcal{M}_c$ as being fixed we regard it as a class that we would like

[10]The definition of a right Kan extension doesn't seem to lend itself as a good explanation for why the universal property of hocolim is just the "weakened" universal property of colim - it only recovers a "localized Yoneda lemma". Instead most standard texts conceptually proceed as follows: Show that colim has a right Kan extension/left derived functor by exhibiting it is left deformable (see next section), then use the mapping spaces of objects in this deformation to understand the universal property of hocolim. It seems like finding the deformation (the "constructive" part in defining the Kan extension) is more fundamental in the setting of derived functors than the universal property they satisfy.

to choose in order to *homotopically correct* a specific functor. This leads to the following definition

Definition 2.29. A ***(left) deformation*** on a homotopical category $\mathcal{C}$ is a functor $Q : \mathcal{C} \to \mathcal{C}$ and a natural weak equivalence $q : Q \xrightarrow{\sim} 1$. A functor $F : \mathcal{C} \to \mathcal{D}$ of homotopical categories is a ***homotopical functor*** if it maps weak equivalences to weak equivalences. Then for general F we say Q is a ***left deformation for*** F if F is homotopical on the full image of Q (i.e. the full subcategory spanned by the object in the image of Q).

It now follows that all the constructions from the last section carry over to this more general setting. That is, given a left deformation Q for F in

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\delta} \\ \mathrm{Ho}(\mathcal{C}) & \xrightarrow[\mathbf{L}F]{} & \mathrm{Ho}(\mathcal{D}) \end{array}$$

then by the same arguments as before $\mathbf{L}F : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$ corresponds via γ (i.e. the universal property of the localisation) to the homotopical functor $\delta\mathbb{L}F := \delta FQ : \mathcal{C} \to \mathrm{Ho}(\mathcal{D})$. This is just Lemma 2.24. By abuse of terminology we call $\mathbb{L}F = FQ$ the ***left derived functor*** of F. The very same argument of the lemma together with the observation that any functor preserves isomorphisms shows that $\delta\mathbb{L}F$ constructed in this way is actually an ***absolute right Kan extension*** - that is, it is preserved by any functor (not only representables as it is the case for pointwise Kan extensions). This in turn yields a much simpler proof of the first part of Theorem 2.25 by elementary formal considerations of (absolute) Kan extensions of an adjoint pair to obtain counit and unit between $\mathbf{L}F$ and $\mathbf{R}G$.

We see that the real challenge in calculating derived functors lies in finding the right deformations. An “unreasonably effective”[11] way to construct certain deformations is the (co)bar construction which we will now use in a short sketch of how one can compute homotopy colimits in this general setting of homotopical categories. I.e. we want to find a deformation on diagram category $\mathcal{M}^{\mathcal{D}}$ for colim. With terminology to emphasize the analogy to the classical theory (cf. the Dold-Kan correspondence Example 1.27) this will be achieved by considering the “category of chains” $(\mathcal{M}^{\mathcal{D}})^{\Delta^{\mathrm{op}}}$ (which by Example 1.27 is naturally homotopical) and constructing a weakly equivalent cofibrant replacement $B(\mathcal{D}, \mathcal{D}, F)$ to the chain concentrated in 0-degree associated to $F \in \mathcal{M}^{\mathcal{D}}$. As suggested by the notation this replacement $B(\mathcal{D}, \mathcal{D}, F)$ will be built from objects $\mathcal{D}$ which are “free modules” in $\mathcal{M}^{\mathcal{D}}$ in a sense explained below.

First some notation: We introduce a ***functor tensor*** on $\mathcal{M}^{\mathcal{D}}$ [12] which will also make our notation much more elegant in the following. Given a $\mathcal{V}$-tensored category $\mathcal{M}$, i.e. $\mathcal{M}$ comes equipped with a bifunctor $- \otimes - : \mathcal{V} \times \mathcal{M} \to \mathcal{M}$, and a diagram

[11] Which is a perspective taken by many authors though cf. our Remark 2.38.

[12] They generalize many tensor constructions: see e.g. `http://ncatlab.org/nlab/show/Yoneda+reduction`

$F : \mathcal{D} \to \mathcal{M}$ with weight $G : \mathcal{D}^{\mathrm{op}} \to \mathcal{V}$ we denote their functor tensor by

$$G \otimes_{\mathcal{D}} F = \int^{d \in \mathcal{D}} Gd \otimes Fd$$

The reader should be reminded that we are working in the enriched context in this section: For our purposes usually $\mathcal{V} = \mathbf{sSet}$ and $\mathcal{M}$ is bicomplete, **sSet** enriched and bitensored (i.e. equipped with a tensor and cotensor action of **sSet**). If we work with the enriched version $\underline{\mathcal{M}}$ of a category we will denote this by an underline, otherwise $\mathcal{M}$ denotes the underlying ordinary category. Further, we denote the Yoneda embedding $\mathcal{D} \to \mathbf{Set}$ (as well as sometimes its opposite $\mathcal{D}^{\mathrm{op}} \to \mathbf{sSet}$) by $\mathcal{D}$ itself.

Example 2.30 (functor tensors). By comparing the definition of functor tensors to our theory of (unenriched) weighted limits, which we were able to express in a coend formula, we get for $\mathcal{V} = \mathbf{Set}$, $\otimes$ the copower and $G = W$ a weight

$$\operatorname{colim}^W F = W \otimes_{\mathcal{D}} F$$

Recall that pointwise Kan extensions from section 1.1.3 could be expressed as weighted limits as well. Thus we obtain:

$$\operatorname{Lan}_K Fb = \mathcal{B}(K-, b) \otimes_{\mathcal{A}} F$$

In particular, if $\mathcal{B}$ is the terminal category we obtain

$$\operatorname{colim} F = \operatorname{Lan}_* Fb = * \otimes_{\mathcal{A}} F \tag{2.31}$$

Along the same lines we can reformulate our coYoneda lemma from Example 1.21. This can be rephrased by the above definition as

$$Fd = \mathcal{D}(-, d) \otimes_{\mathcal{D}} F \tag{2.32}$$

or, by letting d vary

$$F = \mathcal{D} \otimes_{\mathcal{D}} F$$

In this sense $\mathcal{D}$ are "free modules". Indeed, consider e.g. in the case of $\mathcal{D} = \mathbf{R}$ a one object **Ab**-enriched category i.e. a ring. Here $F : \mathbf{R} \to \mathbf{Ab}$, $G : \mathbf{R}^{\mathrm{op}} \to \mathbf{Ab}$ are left resp. right **R**-modules and $\otimes$ is the monoidal product of **Ab**. $- \otimes_{\mathbf{R}} -$ then becomes the tensor product over **R** in a nice confluency of notation. Then the above statement just says that $\mathbf{R} = \operatorname{Hom}(\mathbf{R}, \mathbf{R}) \in \mathbf{Ab}$ is an actual free **R**-module.

We are now in the position to give the (co)bar construction. The ***simplicial bar object*** is the simplicial object defined by

$$B_n(G, \mathcal{D}, F) = \coprod_{\vec{d} \in N(\mathcal{D})} Gd_n \otimes Fd_0$$

where $\vec{d} = d_0 \xrightarrow{f_1} d_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} d_n$ is a string $[n] \to \mathcal{D}$. The face and degeneracy maps of the above simplicial object are "obvious" maps between components indexed by corresponding simplices in $N(\mathcal{D})$ under these faces/degeneracies: For instance (the first d_n is denoting the nth face map)

$$d_n : (d_0 \xrightarrow{f_1} d_1 \xrightarrow{f} f_1 \dots \xrightarrow{f_n} d_n) \to (d_0 \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} d_{n-1})$$

is just given by (on these components of $B_n(G, \mathcal{D}, F)$ and $B_{n-1}(G, \mathcal{D}, F)$)

$$Gf_n \otimes 1 : Gd_n \otimes Fd_0 \to Gd_{n-1} \otimes Fd_0$$

Note that this makes $B_\bullet(G,\mathcal{D},F)$ a simplicial object in $\mathcal{M}$. The ***geometric realisation*** of a simplicial object X in $\mathcal{M}$, i.e. $X:\Delta^{\mathrm{op}}\to\mathcal{M}$, is defined as

$$|X| = \Delta \otimes_{\Delta^{\mathrm{op}}} X$$

And this allows us to define the ***simplicial cobar construction*** by

$$B(G,\mathcal{D},F) = |B_\bullet(G,\mathcal{D},F)| \in \mathcal{M}$$

We want to firstly exploit the functoriality in F of this resolution, and secondly replace G by free modules: Denoting by $\mathcal{D}$ the opposite Yoneda embedding as declared above, we finally obtain a functor

$$B(\mathcal{D},\mathcal{D},-):\mathcal{D}\times\mathcal{M}^{\mathcal{D}}\to\mathcal{M} \quad \text{or} \quad \mathcal{M}^{\mathcal{D}}\to\mathcal{M}^{\mathcal{D}}$$

which looks like a good candidate for a deformation on the diagram category $\mathcal{M}^{\mathcal{D}}$. Also note that all this actually reduces (for certain $G,\mathcal{D},F$) to the usual bar complex via the Dold-Kan correspondence.

Remark 2.33 (the dual situation). Our definitions and constructions above of tensoredness, functor tensors and the bar construction can be regarded as constructions for $\mathcal{M}^{\mathrm{op}}$. This yields their opposite notions: cotensoredness, ***functor cotensors*** and the ***cobar construction***.

2.5.1. *Local homotopy (co)limits.* The previous construction yields a deformation for the colim functor on a *simplicial model category* $\mathcal{M}$. We will not need to define the latter precisely, since the main theorem below depending on their properties will not be proved here - it will be enough to think of a simplicial model category as a bitensored simplicially enriched category with model structure on its underlying category that is suitably compatible with the enrichment (e.g. homotopy equivalences are weak equivalences, cf. Lemma 3.8.6 in [Rie14]). More precisely we have

Theorem 2.34. *Given a simplicial model category $\mathcal{M}$ (which in particular implies it is bicomplete,* ***sSet****-enriched and bitensored and has a cofibrant replacement functor Q), then $B(\mathcal{D},\mathcal{D},Q-)$ is a left deformation for* colim $:\mathcal{M}^{\mathcal{D}}\to\mathcal{M}$*. In particular we have*

$$\operatorname{hocolim} F := \mathbb{L}\operatorname{colim} F = \operatorname{colim} B(\mathcal{D},\mathcal{D},QF) = B(*,\mathcal{D},QF)$$

and for a pointwise cofibrant F this becomes

$$\operatorname{hocolim} F := B(*,\mathcal{D},F)$$

Note that in the equality $\operatorname{colim} B(\mathcal{D},\mathcal{D},QF) = B(*,\mathcal{D},QF)$ we used (2.31) and commuted colimits as described in step 2 below. To reasonably simplify the expression involving the bar construction we need to take the following two steps:

Step 1 (Exercise 4.1.8 in [Rie14]) We need to compute the so called geometric realisation $|X|$ of a bisimplicial set $X:\Delta^{\mathrm{op}}\to\mathbf{sSet}$.
Solution. Given a bisimplicial set X, we have $X_{m,n}\in\mathbf{Set}$ and we just need to put together the definiton of geometric realisations and the coYoneda lemma from (2.32) as follows:

$$\begin{aligned}|X|_n &= \Delta(n,-)\otimes_{\Delta^{\mathrm{op}}} X_{-,n}\\ &= \Delta^{\mathrm{op}}(-,n)\otimes_{\Delta^{\mathrm{op}}} X_{-,n}\\ &\cong X_{n,n}\end{aligned}$$

We conclude that the geometric realisation of a bisimplicial set is its *diagonal*, i.e. $\Delta^{\mathrm{op}} \xrightarrow{\Delta} \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \xrightarrow{X} \mathbf{Set}$. □

STEP 2 (Exercise 4.2.6 in [Rie14]) Let $\mathcal{V} = \mathcal{M} = \mathbf{Set}$. Show that $B_\bullet(\mathcal{D}(-,d), \mathcal{D}, *) \cong N(\mathcal{D}/d)$ where N denotes the nerve of a category.
SOLUTION. In our situation the defintion reads

$$B_n(\mathcal{D}(-,d), \mathcal{D}, *) = \coprod_{\vec{d}:[n]\to\mathcal{D}} \mathcal{D}(d_n, d) \times * = \coprod_{N(D/d)_n} *$$

since a string of n morphisms augmented by an element of $\mathcal{D}(d_n, d)$ is just an element of $N(D/d)_n$. Face and degeneracy maps are easily checked to coincide. □

Dually we have,

$$B_n(*, \mathcal{D}, \mathcal{D}(d, -)) = \coprod_{\vec{d}:[n]\to\mathcal{D}} * \times \mathcal{D}(d, d_1)$$

gives $N(d/\mathcal{D})_n$. Now setting back $\mathcal{V} = \mathbf{sSet}$ by using $\mathbf{Set} \subset \mathbf{sSet}$ (as constant functors) we can regard the opposite Yoneda embedding as mapping into discrete simplicial sets $\mathcal{D} : \mathcal{D}^{\mathrm{op}} \to \mathbf{sSet}^{\mathcal{D}}$. Then $N(d/\mathcal{D})_n$ becomes a (discrete) simplicial set $N(d/\mathcal{D})_{n,\bullet}$, and $N(d/\mathcal{D})_{\bullet,\bullet}$ a (horizontally discrete) bisimplicial set. This gives

Theorem 2.35. *We have a natural isomorphism*

$$B(*, \mathcal{D}, F) \cong N(-/\mathcal{D}) \otimes_{\mathcal{D}} F$$

Proof. By first using what was just said about step 2 and then by applying our result from step 1 we find

$$B(*, \mathcal{D}, \mathcal{D}) := |B_\bullet(*, \mathcal{D}, \mathcal{D})| = |N(-/\mathcal{D})_{\bullet,\bullet}| = N(-/\mathcal{D})_\bullet$$

is a functor into simplicial sets. Then we have

$$N(-/\mathcal{D}) \otimes_{\mathcal{D}} F = B(*, \mathcal{D}, \mathcal{D}) \otimes_{\mathcal{D}} F = B(*, \mathcal{D}, \mathcal{D} \otimes_{\mathcal{D}} F) = B(*, \mathcal{D}, F)$$

where in the second step we used Fubini and commuted coproducts and coends (colimits of diagram colimits can be commuted, if the reader does not believe this there is an explicit calculation in Riehl's book on p. 74). In the final step we just applied coYoneda (2.32) once more. □

If F is pointwise cofibrant we saw that there was no need to apply the deformation $Q : \mathcal{M} \to \mathcal{M}$, thus by Theorem 2.34 for such F that "don't need correction" we have

$$\mathrm{hocolim}_{\mathcal{D}} F = N(-/\mathcal{D}) \otimes_{\mathcal{D}} F$$

This in particular leads to one popular example

$$\mathrm{hocolim}_{\Delta^{\mathrm{op}}} X \cong N(\Delta, -)^{\mathrm{op}} \otimes_{\Delta^{\mathrm{op}}} X$$

It now turns out (somewhat expectedly) that our discussion discussion of functor tensors and weighted limits canonically carries over to the enriched case (yielding *enriched functor tensors* and *enriched weighted limits*). But then in our case $N(-/\Delta)$ can be regarded as a weight in $\mathbf{sSet}$ for a diagram in our $\mathbf{sSet}$-enriched $\mathcal{M}$. Explicitely, with the enriched definitions we have the following equality

$$N(-/\mathcal{D}) \otimes_{\mathcal{D}} F = N(-/\mathcal{D}) \otimes_{\underline{\mathcal{D}}} F = \mathrm{colim}^{N(-/\mathcal{D})} F$$

Here, in the first step we are first freely enriching $\mathcal{D}$ in **sSet** (which is possible since **sSet** has coproducts and amounts effectively to a change of base along $\mathbf{Set} \hookrightarrow \mathbf{sSet}$) to then rewrite our expression as an enriched functor tensor. Then in the second step we apply the enriched analogue of reformulating ends withs weighted limits. We however know the the universal property of **sSet** enriched weighted limits, and this leads us to a central statement of

$$\underline{\mathcal{M}}(\mathrm{hocolim}_{\mathcal{D}} F, m) \cong \mathbf{sSet}^{\mathcal{D}^{\mathrm{op}}}(N(-/\mathcal{D}), \underline{\mathcal{M}}(F-, m)) \tag{2.36}$$

And (writing $c = \mathrm{hocolim}_{\mathcal{D}} F$) with some reflection this should be regarded as a reasonable object for specifying "homotopy coherent" cocones, i.e.

$$\underline{\mathcal{M}}(c, m) \cong \mathrm{hoCocones}(F, m)$$

This is in the usual universal-property-form of conical colimits which we will also recover in the setting of quasicategories quite easily as we will see in section 3 (and which is described in more detail e.g. in [Lur09a]). Indeed, the above notion of homotopy colimit will coincide with the notion of colimit in the quasicategory $N_\Delta(\mathcal{M})$ underlying $\mathcal{M}$. We will come back to the latter statement in section 4.1. Technical details of the above argument (regarding the passage to enriched categories) are discussed in in chapter 7 of [Rie14].

2.5.2. *Further remarks.*

Remark 2.37 (Classical homological algebra). With our new terminology of deformations and new notation for derived functors we can make our treatment of classical homological algebra a bit nicer. It is well known that a R-module M posseses a projective resolution $P_\bullet$. This means the associated chain $M_\bullet$ concentrated in degree zero admits a quasi-isomorphism from $P_\bullet$. With more care such a projective replacement can be constructed for any $A_\bullet \in \mathbf{Ch}_{\geq 0}(R)$ yield a deformation $Q : \mathbf{Ch}_{\geq 0}(R) \to \mathbf{Ch}_{\geq 0}(R)$. Given a additive functor $F : \mathbf{Mod}_R \to \mathbf{Mod}_S$ it induces a functor $\hat{F} : \mathbf{Ch}_{\geq 0}(R) \to \mathbf{Ch}_{\geq 0}(S)$ which preserves chain homotopies. Since quasi-isomorphisms between chains of projectives are homotopy equivalences Q is a deformation for $\hat{F}$. So we can construct $\mathbb{L}\hat{F} = FQ$ and in particular examine the resulting chain by means of homology. The classical ith right derived functor of F is then given by

$$\mathbf{Mod}_R \xrightarrow{\deg_0} \mathbf{Ch}_{\geq 0}(R) \xrightarrow{\mathbb{L}\hat{F}} \mathbf{Ch}_{\geq 0}(S) \xrightarrow{H_i} \mathbf{Mod}_S$$

Note that $Q \circ \deg_0$ is exact. On the other hand if F is right exact, the 0th derived functor above coincides with F. Starting from a short exact sequence $0 \to M_1 \to M \to M_2 \to 0$ this lets us recover the usual form of associated long exact sequences ending in $... \to FM \to FM_2 \to 0$. There is a dual discussion involving injective resolutions and left exact functors.

Remark 2.38 (Comment on the bar construction). The general simplicial bar construction can be regarded as the composite of a (weighted) nerve $\mathcal{L} \to \mathcal{M}^{\Delta^{\mathrm{op}}}$ and realisation functor $\mathcal{M}^{\Delta^{\mathrm{op}}} \to \mathcal{N}$. In this sense it is a two step process: We first disassemble objects of $\mathcal{L}$ into simplices and then reassemble it (in a possibly different category). The dissambly on its own is called simplicial bar construction and can in most (or rather all important) cases be phrased in terms of a simplicial object associated to a monad. Depending on the realiziation of simplices in that category reassembly of such a simplicial object will yield a nicely behaved object. More on this can be found in [Shu09]. In this sense, classifying spaces ($\mathcal{L} = \mathbf{Top\text{-}Cat}, \mathcal{M} = \mathcal{N} = \mathbf{Top}$) are an example of a bar construction, which is the reason for our terminology in 1.2.4 (and also served to distinguish classifying spaces from their explicit model provided by the bar construction).

3. $(\infty,1)$-CATEGORIES

In this section we will dicuss how to generalize many constructions we saw before (e.g. homotopy (co)limits, monoidal structure, duals and stabilisation) in the context of quasicategories. Before we go into details of how they model these constructions on $(\infty, 1)$-categories we will compare them on the level of "homotopy theories" as established in the previous chapter to another intuitive approach to model $(\infty, 1)$-categories: The category of **sSet**-enriched categories $\mathbf{Cat}_\Delta$ with Bergner model structure.

3.1. **Joyal and Bergner model structures.** We already gave a definition of quasicategories in the introduction to these notes in Construction 1.28. Our construction was based on characterizing the category-type simplicial sets $\cong N(\mathcal{C})$, and the space-type simplicial sets $\cong S_{\text{top}}(X)$ (yielding the notion of Kan complexes) and then weaken the notion of composition in the former by this characterisation of the latter. Precisely we recall,

Definition 3.1. A quasicategory is a simplicial set such that all inner horns have a (possibly non-unique) filler.

One would expect that another approach to model $(\infty, 1)$-categories would proceed through enriching categories in Kan complexes. Indeed, we can consider the category of **sSet**-enriched categories for this purpose $\mathbf{Cat}_\Delta := \mathbf{sSet\text{-}Cat}$ with the following model structure: Weak equivalences and fibrations are just the usual *Dwyer-Kan equivalences* and *isofibrations*. More explicitely,

(i) $F : \mathcal{C} \to \mathcal{D} \in \mathbf{Cat}_\Delta$ is a weak equivalence if the corresponding underlying **Set**-functor[13] F_0 is essentially surjective and $F_{AB} : \mathcal{C}(A, B) \to \mathcal{D}(FA, FB)$ is a weak equivalence in the Quillen model structure of **sSet**.
(ii) $F : \mathcal{C} \to \mathcal{D} \in \mathbf{Cat}_\Delta$ fibration if the corresponding underlying **Set**-functor F_0 is an isofibration[14] and $F_{AB} : \mathcal{C}(A, B) \to \mathcal{D}(FA, FB)$ is a fibration in the Quillen model structure of **sSet**.

The full model structure obtained from these definitions is called the ***Bergner model structure***. It follows from the above that its fibrant objects are exactly those categories whose Hom objects are Kan complexes.

Remark 3.2 (Bergner fibrant replacement). Given the definition of fibrations, there are different ways of choosing explicitely a fibrant replacement functor R on $\mathbf{Cat}_\Delta$. For this we need a product preserving Kan fibrant replacement of the hom spaces. On obvious possible choice is $S\,|{-}|$ with S the simplicial nerve. Another popular choice is Ex^∞, where $\mathrm{Ex} : \mathbf{sSet} \to \mathbf{sSet}$ is the right adjoint to the subdivision

[13] recall that a monoidal functor $U : \mathcal{V} \to \mathcal{V}'$ induces a *change of base functor* $U : \mathcal{V}\text{-}\mathbf{Cat} \to \mathcal{V}'\text{-}\mathbf{Cat}$. In particular here we use the 0-th projection functor $\pi_0 : \mathbf{sSet} \to \mathbf{Set}$ which also is called the underlying set functor

[14] Recall an isofibration $F : \mathcal{C} \to \mathcal{D}$ is the categorical analogue of a fibration: Every isomorphism (or "paths") $Fc \xrightarrow{\sim} d$ has a lift in $\mathcal{C}$, i.e. is the image of some $c \xrightarrow{\sim} c'$ under F for some c.

functor sd[15]. This comes equipped with a natural map $X \to \mathrm{Ex}(X)$ and thus we can pass to the colimit Ex^∞.

To relate this approach to our previous approach of quasicategories we realize Δ in $\mathbf{Cat}_\Delta$ by a functor S and use the associated nerve adjunction $N_\Delta \dashv C[-]$ to compare (or in our case to transfer) the Bergner model structure to a model structure on **sSet**. An appropriate right Quillen nerve N_Δ should therefore map Kan-enriched categories to quasicategories (i.e. preserve fibrant objects). A realisation achieving this goal is $S(\Delta^n) \in \mathbf{Cat}_\Delta$ called *simplicial thickening* of Δ^n. We will reconsider it's construction in section 4.1 discussing homotopy coherence. For now, we will define it follows:

- $S[n]$ has the same objects as $[n]$
- $S[n](i,j)$ is the nerve of a poset $P_{i,j}$ having as objects of all paths $I \subset (i+1, ...j-1)$ ordered by inclusion

Denote the corresponding nerve by N_Δ and its adjoint via Example 1.23 by $C[-]$. Note that by the properties of Yoneda extension $C[-]$ is an actual extension of S and can be calculated from simplices.

We can now transfer our model structure: Fibrant objects in **sSet** are images of fibrant objects under N_Δ (indeed these are precisely quasicategories). And we define weak equivalences $f : X \to Y$ of $\mathbf{sSet}_{Joyal}$ to correspond to weak equivalences $C[f]$ in $\mathbf{Cat}_\Delta$. Again, this can be made into a full model structure which is called the *Joyal model structure*. With this cheated definition of the Joyal model structure as a "transferred structure" along $N_\Delta \dashv C[-]$ the following should be less of a surprise

Proposition 3.3. *With the Joyal and Bergner model structures in place, the adjunction* $C[-] \dashv N_\Delta : \mathbf{Cat}_\Delta \to \mathbf{sSet}$ *gives rise to a Quillen equivalence.*

We remark that for a cofibrant A and fibrant X in a *simplicial* model category $\mathcal{M} \in \mathbf{Cat}_\Delta$ the axioms (namely $\mathcal{M}(A,X) \to *$ needs to be a Kan fibration) actually guarantee that $\mathcal{M}_{cf}$, the subcategory of bifibrant objects, is fibrant in $\mathbf{Cat}_\Delta$ i.e. Kan complex enriched and thus $N_\Delta(\mathcal{C}_{cf})$ is a quasicategory.

Construction 3.4 (Simplicial localisation). Note that not every quasicategory can be obtained in this way - but every quasicategory $\mathcal{V}$ can be obtained up to equivalence from a homotopical category $\mathcal{C}$ by an appropriate localisation at its weak equivalences. We call $\mathcal{V}$ the ***underlying quasicategory*** of $\mathcal{C}$. This procedure in general is also called *simplicial localisation*, and specifically we will describe the ***Hammock localisation*** $L^H\mathcal{C}$ of $\mathcal{C}$: $L^H\mathcal{C}$ is a **sSet**-enriched category that has the same objects as $\mathcal{C}$, and its hom spaces $L^H\mathcal{C}(A,B)$ are simplicial sets obtained as the nerve of the following category

- $L^H\mathcal{C}(A,B)$ has as objects finite "paths" of morphisms between A, B

$$A \xrightarrow{f_0} C_1 \xleftarrow{\sim} ... \xleftarrow{\sim} C_n \xrightarrow{f_n} B$$

  i.e. such that we are allowed to walk backwards along weak equivalences, up to the equivalence relation of setting two paths to be equal if there are obtained from each other via composition, inserting identities or canceling weak equivalences with their inverses.

[15]This can be defined as the nerve of the poset of all non-degenerate subsimplices of some simplicial set X.

- Morphisms are "homotopies of paths with fixed endpoints" (respecting the equivalence relation just defined), i.e.

$$\begin{array}{ccccccccc} A & \xrightarrow{f_0} & C_1 & \xleftarrow{\sim} & \dots & \xleftarrow{\sim} & C_n & \xrightarrow{f_n} & B \\ \| & & \downarrow{\sim} & & \vdots & & \downarrow{\sim} & & \| \\ A & \xrightarrow{f_0} & C_1 & \xleftarrow{\sim} & \dots & \xleftarrow{\sim} & C_n & \xrightarrow{f_n} & B \end{array}$$

Passing to the fibrant replacement to obtain a Kan-enriched $RL^H\mathcal{C}$ as explained in Remark 3.2, and then applying N_Δ we finally get the quasicategory $\mathcal{V}_\mathcal{C} = N_\Delta(RL^H\mathcal{C})$ canonically associated to $(\mathcal{C}, \mathcal{W})$. It is important to note, that in the case of a simplicial model category $\mathcal{C}$ this construction coincides up to Joyal weak equivalence with $N_\Delta(\mathcal{C}_{cf})$. It is also important to note (see section 4.1) that the notion of homotopy (co)limits in $\mathcal{C}$ coincide with the notion of (co)limits in $\mathcal{V}_\mathcal{C}$ that we will define in the next section.

Having to different perspectives on $(\infty,1)$-categories and a way to translate between those two makes many construction straight-forward now:

Example 3.5. Let **sSet** with usual Quillen model structure be regarded as sSet-category (cf. the end of Construction 1.26 where Map is defined) yielding in particular a simplicial model category. We have $\mathbf{sSet}_{cf} = \mathbf{sSet}_f = Kan$ the subcategory of Kan complexes. Thus, we define $\mathcal{S} = N_\Delta(Kan)$ to be the ***quasicategory of spaces***.

Since objects of $(\mathbf{Cat}_\Delta)_f$ are enriched in Kan the correct notion of presheaves for quasicategories should then be the following

Definition 3.6. Given an quasicategory $\mathcal{C}$ we define its ***presheaf category*** $\mathrm{PSh}(\mathcal{C})$ to be the mapping space $\mathrm{Map}(\mathcal{C}, S)$. More generally, we define the ***functor category*** $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) = \mathrm{Map}(\mathcal{C}, \mathcal{D})$ for quasicategories $\mathcal{C}, \mathcal{D}$ and call 0-simplices of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ functors and refer to 1-simplices as natural transformations.

Detailed verifications of this definition being reasonable (in particular yielding quasicategories) and that they satisfy an analogue of the Yoneda and coYoneda Lemma described in Example 1.21 can be found in [Lur09a], the latter being discussed in §5.1.

Another application of the Bergner model structure is the following: The **sSet**-enriched subcategory $\widehat{\mathbf{Cat}}_\infty$ of **sSet** with mapping spaces $\widehat{\mathbf{Cat}}_\infty(X, Y)$ being the maximal Kan complexes contained in the corresponding mapping space $\mathrm{Map}(X, Y)$ of **sSet** gives a natural model of the $(\infty,1)$-category of quasicategories in $\mathbf{Cat}_\Delta$. We can then define

Definition 3.7. The associated quasicategory $\mathbf{Cat}_\infty := N_\Delta(\widehat{\mathbf{Cat}}_\infty)$ is called the ***quasicategory of $(\infty,1)$-categories***.

Finally, taking an analogue of the Yoneda and coYoneda lemma for granted (which needs in particular a discussion of colimits which we will provide in the next section) we can revisit a previous statement about the characterisation of the homotopy theory of spaces in Remark 1.34. Analogously to the statement of Example 1.23 we have:

$$\mathcal{S} = \mathrm{Map}(*, \mathcal{S}) = \mathrm{PSh}(*) \Rightarrow \mathrm{Fun}^L(\mathcal{S}, \mathcal{D}) \simeq \mathrm{Fun}(*, \mathcal{D}) = \mathcal{D}$$

where Fun^L denotes colimits preserving functors, which we will discuss now. In words, the $(\infty, 1)$-category of spaces is the $(\infty, 1)$-category modelled on a point, i.e. the free $(\infty, 1)$-colimit completion of the terminal category.

3.2. **Constructions on quasicategories.** We first give some intuition about the nature of quasicategories: Given a quasicategory X we call its 0-simplices ***objects*** and its 1-simplices ***morphisms***. In the spirit of Remark 1.32 we will now regard higher simplices as the higher coherence rules for composition of morphisms. It will turn out that our inner horn filling condition garantuees exactly that higher rules are invertible and unique up to a "contractible" choice as we expect from enriching in topological spaces: In a 2-simplex $\alpha : f \circ g \to h$ we call h a ***candidate composition*** of $f \circ g$. We call a 2-simplex $\sigma : f \circ g \to h$ a ***homotopy*** $h : h \simeq f$ if $g = 1$, i.e it is degenerate. By the inner horn filling property of an appropriate 3-simplex it is easy to see that $\sigma : h \simeq f \Rightarrow \exists \sigma' : f \simeq h$. The notion of composition and n-homotopies of $(n-1)$-homotopies can be defined in exactely the same way. Further, all candidate compositions are related by homotopies (and these themselves by higher homotopies and so forth) which again follows by the inner horn filling property. More formally we summarise this as

Lemma 3.8. *The restriction* $\mathrm{Map}(\Delta^2, X) \to \mathrm{Map}(\Lambda^2_1, X)$ *is a trivial Kan fibration.*

We call a morphism f a weak equivalence if it is invertible up to homotopy in the above terminology. Then to each quasicategory $\mathcal{C}$ we can associate its ***homotopy category*** $\mathrm{Ho}(\mathcal{C})$ having the same objects as $\mathcal{C}$ and its morphism are homotopy classes of morphisms in $\mathcal{C}$. The check that this is well-defined is once more a natural consequence of the inner horn filling condition. Also, weak equivalence now correspond to isomorphisms in $\mathrm{Ho}(\mathcal{C})$ as we are used to from the case of model categories. We can now give the following

Definition 3.9. A quasicategory $\mathcal{C}$ is an ∞***-groupoid*** if $\mathrm{Ho}(\mathcal{C})$ is a groupoid.

Since weak equivalence allow filling of outer horns by "cancelling" them with their homotopy inverse we obtain

Lemma 3.10. *$\mathcal{C}$ is an ∞-groupoid if and only if it is a Kan complex.*

In the Construction 1.14 of cographs and their special case of joins we have seen a conceptual approach to their construction. For the case of simplicial sets we give a more explicit characterisation. The ***join*** $X \star Y$ of two simplicial sets X, Y is obtained by first defining $\Delta^n \star \Delta^m$ to be isomorphic to Δ^{n+m+1} and then extending these isomorphisms in a colimit preserving way when viewed as functors $- \star X$ or $X \star -$ from **sSet** to $X/$**sSet**. More geometrically one could explain the join of two simplicial sets X, Y as follows: To obtain $X \star Y$ we are connecting all k-simplicies of X with l-simplices of Y by $(k + l + 1)$-simplices. In particular the reader should verify that for $X = \Delta^n$ and $Y = \Delta^m$ we obtain Δ^{m+n+1} in this way. Our construction is analogous to our previous definition of joins, more precisely in the case of categorical nerves they coincide:

$$N(\mathcal{C} \star \mathcal{C}') \cong N(\mathcal{C}) \star N(\mathcal{C}')$$

The construction of joins of categories gave us the universal characterisation of categories over and under a diagram in (1.15), e.g. the categories of (co)cones of diagrams, which are otherwise hard to define in a general context. More abstractly, we saw in Construction 1.14 the notion of proarrow equipment from which joins could be constructed, and hom functors were naturally in place as identities - of course, hom functors will be ultimately necessary for discussing universal properties within a quasicategory. We are going the reverse path here (!): We will see a notion of mapping space based on our explicit (but not unmotivated) definition of joins above leading to a characterisation of over-categories. Indeed, this universal property can be naturally carried over from (1.15) to the setting of quasicategories

Definition 3.11. Given a diagram $F : \mathcal{D} \to \mathcal{C}$ the ***quasicategory of cones*** $\mathcal{C}_{/F}$ ***over*** F is defined by the universal property

$$\mathbf{sSet}(X, \mathcal{C}_{/F}) \cong \mathbf{sSet}_F(X \star \mathcal{D}, \mathcal{C})$$

where $\mathbf{sSet}_F$ denotes the full subcategory of functors which restrict to F on $\mathcal{D} \hookrightarrow X \star \mathcal{D}$. Similarly we can define a ***category of cocones*** $C_{F/}$. The terminology here comes from our observation in (1.16).

The existence of such a set can be easily established by probing it with simplicies Δ^n. But of course it then still needs to be checked that we actually obtain a quasicategory which is done in [Lur09a] and will be omitted here (like many other verifications in this section). We can now state as promised

Definition 3.12. Given a quasicategory $\mathcal{C}$, and objects $c_1, c_2 \in \mathcal{C}$ we define their ***mapping space*** to be the simplicial set in the pullback

$$\begin{array}{ccc} \mathrm{Map}(c_1, c_2) & \longrightarrow & C_{/c_2} \\ \downarrow & \lrcorner & \downarrow \\ c_1 & \longrightarrow & C \end{array}$$

where the map $C_{/F} \to C$ is the canonical "forgetful functor" obtained from considering $1 \in \mathbf{sSet}(\mathcal{C}_{/F}, \mathcal{C}_{/F})$ in the above universal property.

Further we set $\mathcal{C}^{\triangleleft} = * \star \mathcal{C}$ and $\mathcal{C}^{\triangleright} = \mathcal{C} \star *$. Thus for instance $(\Lambda^2_2)^{\triangleleft}$ is just a "pullback cone". We are finally in the position to define the quasicategory analogues of limits and colimits

Definition 3.13. $c \in \mathcal{C}$ is ***terminal*** if the forgetful map $\mathcal{C}_{/c} \to \mathcal{C}$ is a trivial Kan fibration, i.e. $\mathrm{Map}(c', c)$ are contractible for all $c' \in \mathcal{C}$. Similarly an ***initial*** object c satisfies that $\mathcal{C}_{c/} \to \mathcal{C}$ is a trivial Kan fibration. A ***limit*** of a diagram $J : \mathcal{D} \to \mathcal{C}$ is a terminal object in the category $C_{/F}$ and a ***colimit*** an initial object in $\mathcal{C}_{F/}$. A quasicategory is called ***(co)complete*** if it admits suitably small (co)limits.

Note that the full subcategory of terminal or initial objects is a contractible Kan complex analogous to such objects being determined up to isomorphism in ordinary categories.

Now that we have a notion of joins, cones, limits and colimits, we can repeat our considerations about accessibility from section 1.1.4 and obtain completely analogous notions of $(\infty,1)$-filteredness, $(\infty,1)$-ind-categories, accessibility and presentability. There is an interplay between 1-categorical presentability of model categories and presentability of quasicategories. For this we first define

Definition 3.14. A quasicategory is ***presentable*** if it is accessible and cocomplete. A model category is ***combinatorial*** if it is cofibrantly generated (recall section 2.3) and presentable.

With these definitions one can then show (e.g. in [Lur09a] Ch. 5)

Theorem 3.15. *A quasicategory $\mathcal{C}$ is presentable if and only if it is equivalent to the localisation $N_\Delta(\mathcal{M}_{cf})$ of some combinatorial simplicial model category $\mathcal{M}$.*

3.3. **Monoidal structure.** We want to find the analogue of a monoidal structure for quasicategories. As for notions of limits and colimits we need to reformulate our **Set**-based notion of monoidal structure in a "theory invariant" way, i.e. solely based on concepts available also for quasicategories. Recall that leading up to the Construction 1.28 of quasicategories we argued that passing from Δ_2 to Δ for modelling generalized spaces gave us the generality to describe **Cat** and **Top** on a common base. But in fact, this generality also gave an appealingly easy condition for categories (namely unique inner horn filling) within **sSet**. We will try the same for monoidal categories:

(i) We know ordinary monoidal categories are monoid objects in **Cat**, which can be defined as $M : \Delta_2^{\text{op}} \to \mathbf{Cat}$ subject to the condition that $M(d_{i=0,2} : [1] \to [2])$ are projections (i.e. they induce an isomorphism $M([2]) \cong M([1]) \times M([1])$ such that we can set $M(d_1 : [1] \to [2]) = \otimes$ to be the multiplication) and additional conditions for weak associativity and the unit $M(s_0 : [1] \to [0])$. We want to generalize this to $M : \Delta^{\text{op}} \to \mathbf{Cat}$ with an appropriate (and appealing) condition for M to be a monoidal category.

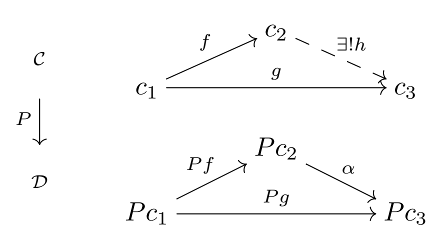
(ii) By use of the Grothendieck construction (1.10)

$$\mathrm{Fib}(\mathcal{A}) = \mathrm{opFib}(\mathcal{A}^{\text{op}}) \simeq \mathrm{Fun}(\mathcal{A}^{\text{op}}, \mathbf{Cat}) \tag{3.16}$$

this has an equivalent formulation in terms of a fibration over Δ^{op}.

It is the second idea that yields a good intuition about how to encode monoidal structure, so we start with (ii) and then go back to (i). For this purpose we will keep our promise and now make the Grothendieck construction for opfibrations explicit in the non-discrete case (the discrete case was discussed in Construction 1.9).

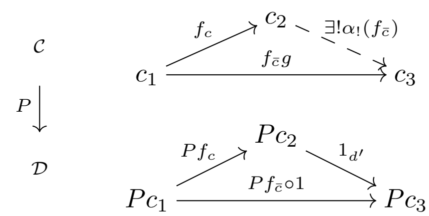
Definition 3.17. Given $P : \mathcal{C} \to \mathcal{D}$ in **Cat** we say an $f : c_1 \to c_2, f \in \operatorname{mor} \mathcal{C}$ is ***coCartesian*** if it satisfies the following unique 0-th horn filling condition for all α, g in the diagram below



Thus equivalently, the fibers of the canonical map from $\mathcal{C}_{f/}$ (consisting of "triangles" with first edge f) to $\mathcal{C}_{c_1/} \times_{\mathcal{D}_{/p(c_1)}} \mathcal{D}_{Pf}$ are the singleton category $*$.

Definition 3.18. An ***opfibration*** $P : \mathcal{C} \to \mathcal{D}$ is such that for all $c \in \mathcal{C}$ all arrows $\alpha : Pc \to d$ in $\mathcal{D}$ have coCartesian lifts.

This is exactly the condition to make fibers depend covariantly on $\mathcal{D}$. Given a (non-empty) fiber $\mathcal{C}_d = P^{-1}(1_d)$ which is always a category in the ordinary case, and $\alpha : d \to d'$ then for each $c \in \mathcal{C}_d$ we chose a coCartesian lift $f_c : c \to f(c) \in \mathcal{C}_{d'}$ and define $\alpha_! : \mathcal{C}_d \to \mathcal{C}_{d'}$ on objects by $\alpha(c) = f(c)$ and on morphisms $g : c \to \bar{c}$ in $\mathcal{C}_d$ by the universal property of coCartesian lifts



Such a choice of coCartesian lifts is called a ***cleavage***. By employing this universal property once more we see that our choice of coCartesian lifts is actually unique up to isomorphism. Since composition of coCartesian lifts are coCartesian lifts we deduce a *unique* natural isomorphism of functors

$$(\beta \circ \alpha)_! \cong \beta_! \circ \alpha_! \tag{3.19}$$

This finally yields one direction of (3.16). Namely, given an opfibration $P : \mathcal{C} \to \mathcal{D}$ as in the definition above we obtain a weak 2-functor $(-)_! : \mathcal{D} \to \mathbf{Cat}$ mapping objects d to $\mathcal{C}_d = P^{-1}(d)$ and morphisms α to the functors $\alpha_!$ constructed above (and we can choose $1_! = 1$). The natural isomorphisms commute with the (strict) associators of $\mathbf{Cat}$ by their uniqueness. Thus we indeed have the claimed pseudofunctor $(-)_! \in \mathrm{Fun}(\mathcal{D}, \mathbf{Cat})$. Conversely, starting from a $F \in \mathrm{Fun}(\mathcal{D}, \mathbf{Cat})$ the Grothendieck construction that leads to a opfibrations $F : \mathcal{C} \to \mathcal{D}$ is very similar to the discrete case. Instead of discussing this further we will give an example to show the intricacies of the right hand side of (3.19) being the 2-category of pseudofunctors, natural transformations and *modifications* (see e.g. [SP14] for some basics about bicategories).

Example 3.20. Let $\mathcal{D} = \mathbb{Z}_2$ i.e. a 1-object category with $\operatorname{mor} \mathcal{D} = \{ 1, g \}$ and $g \circ g = 1$. Consider 2-functors F, G of $\mathcal{D}$ into $\mathbf{Cat}$ with image $\mathbb{Z}$ considered as an additive group. Then $FgFg \cong F(gg) = F1 \cong 1_{\mathbb{Z}}$ implies that $Fg : \mathbb{Z} \to \mathbb{Z}$ equals $1_{\mathbb{Z}}$. Thus such functors essentially only consist of data specifying a natural isomorphism $\phi_1 : 1_{\mathbb{Z}} \xrightarrow{\sim} (1_{\mathbb{Z}} = F1)$ witnessing functoriality on the unit and a $\phi_{gg} : FgFg \xrightarrow{\sim} F1$ (all remaining 3 witnesses for composition are then determined by the axioms of a pseudofunctor). We note $\phi_1, \phi_{gg} \in \mathbb{Z}$ since group homomorphisms $1_{\mathbb{Z}} \to 1_{\mathbb{Z}}$ are given by some group element. Now a 2-natural transformation consists of 1-cells $\sigma_d : \mathbb{Z} \to \mathbb{Z}$ (functors in $\mathbf{Cat}$) making the natural commutation square commutative up to a 2-cell σ_f (natural transformations). $(F, \phi), (G, \psi)$ are equivalent when related by σ with σ_d being equivalences, thus in our case $\sigma_d = 1_{\mathbb{Z}}$. By the axioms for a 2-natural transformation the naturality witnesses $\sigma_f \in \mathbb{Z}$ need to be compatible with the functoriality witnesses which gives two equations

$$\sigma_1 + \phi_1 = \psi_1$$
$$\sigma_1 + \phi_{gg} = \psi_{gg} + 2\sigma_g$$

Substracting shows $\phi_1 - \phi_{gg} \equiv \psi_1 - \psi_{gg} \mod 2$ when worked out in diagrams. We that see that there are two equivalence classes of 2-functors mapping $\mathbb{Z}_2$ into

Cat with image $\mathbb{Z}$ depending on whether $\phi_1 - \phi_{gg} = 1 \mod 2$ or $0 \mod 2$. This corresponds to two non-equivalent fibrations with fiber $\mathbb{Z}$, namely:

$$P_F \equiv (\pi : \mathbb{Z} \to \mathbb{Z}/(2\mathbb{Z})) : \mathbb{Z} \to \mathbb{Z}_2$$
$$P_G \equiv p_1 : \mathbb{Z}_2 \times \mathbb{Z} \to \mathbb{Z}_2$$

(note that above when we constructed $(-)_!$ we implicitly chose $\phi_1 : 1_{\mathbb{Z}} \to ((1)_! = 1_{\mathbb{Z}})$ to be zero. This gives the slightly nicer condition $\phi_{gg} \equiv \psi_{gg} \mod 2$)

3.3.1. *Monoidal quasicategories.* Every ordinary monoidal category $(\mathcal{V}, \otimes)$ gives rise to a fibration over $P : \mathcal{V}^{\otimes} \to \Delta^{\text{op}}$ as follows: Fibers have objects $\vec{M}_n = (M_1, ...M_n) \in \mathcal{V}^n =: \mathcal{V}^{\otimes}_{[n]}$ and morphism are

$$f = (\alpha : [k] \to [n], f_{1 \leq i \leq k} : M_{\alpha(i-1)+1} \otimes \cdots \otimes M_{\alpha(i)} \to L_i) : \vec{M}_n \to \vec{L}_k$$

So in words, in the tuple $(\vec{f}, \alpha)$ α indicates how to partition $(M_1, ..., M_n)$ for a map into $(L_1, ...L_k)$. Clearly this admits coCartesian lifts (just choose f_i to be isomorphisms). It also satisfies that the cartesian lifts $(\iota^n_i)_!$ of $\iota^n_i : [1] \to [n], 0 \mapsto i-1, 1 \mapsto i$ are projections, i.e. they induce an isomorphism

$$\mathcal{V}^{\otimes}_{[n]} \cong \left(\mathcal{V}^{\otimes}_{[1]}\right)^n$$

Conversely, we want to verify the following

Definition 3.21. Given a fibration $P : \mathcal{V}^{\otimes} \to \Delta^{\text{op}}$ such that $(\iota^n_i)_!$ are projections inducing

$$\mathcal{V}^{\otimes}_{[n]} \cong \left(\mathcal{V}^{\otimes}_{[1]}\right)^n$$

then $\mathcal{V}$ is a ***monoidal category***.

Indeed, it is a straightforward argument setting $\otimes \colon \mathcal{V}_{[1]} \times \mathcal{V}_{[1]} \cong \mathcal{V}^{\otimes}_{[2]} \xrightarrow{(d_1)_!} \mathcal{V}^{\otimes}_{[1]}$ to verify associatal and unital laws (cf. the argument for the diagram (4.12)). The important bit is to notice is that the projection condition in the above definition implies $\mathcal{V}^{\otimes}_{[0]}$ is a one object category giving a natural choice for the unit and that actually all coCartesian lifts of ***convex maps*** become projections: Here, by convex map we mean and injective $\alpha : [k] \to [n], k < n$ in Δ such that the image of α spans an intervall $[i, i+k]$. We note that by the preceeding definition we succeded in finding an appealing reformulation of monoidal structures on a category.

To transfer this to the world of quasicategories we first need to reconsider the notion of fibration. Given a map between simplicial sets $p : X \to Y$ we call it an ***inner fibration*** if its fibers are quasicategories and maps between them have composites, i.e. more concisely there are solutions to the lifting problem (where i is an inner index)

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & \mathcal{D} \end{array}$$

So we should require our fibrations for quasicategories to be at least inner fibrations. After this observation, we can transfer the notion of coCartesian arrow and opfibrations in a straighforward manner

Definition 3.22. Given a functor of quasicategories $P : \mathcal{C} \to \mathcal{D}$ a morphism $f \in \operatorname{mor}\mathcal{C}, f : c_1 \to c_2$ is called ***coCartesian*** if the canonical map

$$\mathcal{C}_{f/} \to \mathcal{C}_{c_1/} \times_{\mathcal{D}_{Pc_1/}} \mathcal{D}_{Pf/}$$

is a trivial Kan fibration (this should be compared to the reformulation at the end of our Definition 3.17 of ordinary coCartesian lifts). P is an ***opfibration*** of quasicategories if it is an inner fibration and for every $c_1 \in \operatorname{obj}\mathcal{C}$, every morphism $\alpha : Pc_1 \to d$ has a coCartesian lift $f : c_1 \to c_2$.

Finally we can define

Definition 3.23. Given an opfibration $P : \mathcal{V}^{\otimes} \to N(\Delta^{\mathrm{op}})$ such that $(\iota_i^n)_!$ are inducing

$$\mathcal{V}^{\otimes}_{[n]} \xrightarrow{\sim} \left(\mathcal{V}^{\otimes}_{[1]}\right)^n \tag{3.24}$$

we say that $\mathcal{V} = \mathcal{V}^{\otimes}_{[1]}$ is a ***quasicategory with monoidal structure***.

Monoidal functors should preserve either the isomorphisms (3.24) or all of the monoidal structure (i.e. the covariant dependence of the fibers on $N(\Delta^{\mathrm{op}})$ in the fibration). We obtain the following two notions

Definition 3.25. Given two monoidal quasicategories $\mathcal{V}^{\otimes} \to N(\Delta^{\mathrm{op}})$, $\mathcal{W}^{\otimes} \to N(\Delta^{\mathrm{op}})$ a functor $\mathcal{F} : \mathcal{V} \to \mathcal{W}$ such that

$$\begin{array}{ccc} \mathcal{V}^{\otimes} & \xrightarrow{\mathcal{F}} & \mathcal{W}^{\otimes} \\ & \searrow \quad \swarrow & \\ & N(\Delta^{\mathrm{op}}) & \end{array}$$

is called ***lax monoidal*** if it preserves coCartesian arrows over convex morphisms, and it is called ***strong monoidal*** if it preserves all coCartesian arrows.

Note that all terms in the following inclusion (with the obvious terminology)

$$\mathrm{Fun}^{\mathrm{mon}}(\mathcal{V}, \mathcal{W}) \subset \mathrm{Fun}^{\mathrm{lax}}(\mathcal{V}, \mathcal{W}) \subset \mathrm{Fun}_{/N(\Delta^{\mathrm{op}})}(\mathcal{V}^{\otimes}, \mathcal{W}^{\otimes}) \subset \mathrm{Fun}(\mathcal{V}, \mathcal{W})$$

are quasicategories as expected.

Now let us motivate the transition from step (ii) to step (i) as promised in the beginning of this section in the setting of quasicategories, i.e. let us exhibit monoidal categories in our "simplicial encoding" of monoid objects in $\mathbf{Cat}_\infty$ (called algebra objects below). Indeed, it turns out there is a quasicategorical analogue of the Grothendieck construction (3.16) as established by Lurie − though this seems to be quite non-trivial. In our case of monoidal structures, this should state roughly that

$$\mathrm{opFib}(N(\Delta^{\mathrm{op}})) \simeq \mathrm{Fun}(N(\Delta^{\mathrm{op}}), \mathbf{Cat}_\infty) \tag{3.26}$$

Recall, we did put an extra condition on objects of the left hand side of this equation in Definition 3.23 - namely coCartesian arrows over convex morphism induce an equivalence in order for an opfibration to encode monoidal structure. Note that that $\mathbf{Cat}_\infty$ should have *cartesian monoidal* structure $\mathbf{Cat}_\infty^{\times} \to N(\Delta^{\mathrm{op}})$ (so projections and their induced Segal maps can actually *live in* $\mathbf{Cat}_\infty$) and therefore conditions on the maps into $\mathbf{Cat}_\infty$ on the right hand side should have equivalent conditions as maps into $\mathbf{Cat}_\infty^{\times}$ as used below. Also note that $N(\Delta^{\mathrm{op}}) \to N(\Delta^{\mathrm{op}})$ encodes the "terminal monoidal quasicategory". The corresponding conditions we are looking for (and which can again be easily motivated from the 1-categorical case) are then given in the following definition:

Definition 3.27. An ***algebra object*** $A : N(\Delta^{\mathrm{op}}) \to \mathcal{V}^{\otimes}$ in a monoidal quasicategory $\mathcal{V}$ is a section of $\mathcal{V}^{\otimes} \to N(\Delta^{\mathrm{op}})$ mapping convex arrows to coCartesian arrows in $\mathcal{V}^{\otimes}$. In other words, an algebra object is a lax monoidal functor from $N(\Delta^{\mathrm{op}})$ to $\mathcal{V}^{\otimes}$.

We will denote the functor category $\mathrm{Fun}^{\mathrm{lax}}(N(\Delta^{\mathrm{op}}), \mathcal{V})$ by $\mathrm{Alg}(\mathcal{V})$ in light of the previous definition. In particular, we can now state that the quasicategory of monoidal quasicategories which we denote $\mathbf{Cat}_{\infty}^{\mathrm{mon}}$ can be equivalently described as the quasicategory $\mathrm{Alg}(\mathbf{Cat}_{\infty})$. We close this section with two remarks

Remark 3.28.
- Using our considerations of opfibrations and monoidal structure in the 1-categorical case, it can be seen that the homotopy category of a monoidal category $\mathcal{V}^{\otimes}$ canonically inherits ordinary monoidal structure.
- In Theorem 3.15 we stated that a quasicategory is presentable if and only if it is equivalent to the coherent nerve of a combinatorial model category. This can be extended to the case when both carry monoidal structure as follows: A bilinear monoidal quasicategory is presentable if and only if it is equivalent as a monoidal category to the coherent nerve of a combinatorial monoidal model category (where monoidal and model structure are suitably compatible): Here *bilinear* means that the monoidal product preserves colimits in each variable.

3.3.2. *Symmetric monoidal quasicategories.* In this section we will refine our encoding of data of a monoidal structure to the case of a symmetric monoidal structure. We have actually already done most of the conceptual work in the previous section: The idea for finding such an encoding (by passing to opfibrations via the Grothendieck construction) is completely analogous to the case of monoidal structure.

In the latter case, we then used Δ^{op} to encode partial partitioning of n-tuples of objects. In the symmetric case, we not only need to partially partition n-tuples of objects (to encode tensor and units) but also need to have coCartesian morphisms for permutations to encode uses of the symmetry morphism. A natural canditate for this task is the category of finite pointed sets $\mathcal{F}in_*$: We can encode partial partitions *and* permutations faithfully by the preimages of $n \in \langle n \rangle_* = \{ 1, 2, .., n, * \}$. The role of projections projecting out a single object $\langle n \rangle_* \to \langle 1 \rangle_*$ should then be taken by the following morphisms

$$\alpha_i^n(j \in \langle n \rangle_*) = \begin{cases} 1 & , \quad \text{if } j = i \\ * & , \quad \text{otherwise} \end{cases}$$

More generally, projections should be coCartesian arrows lying over ***collapsing maps*** in $\mathcal{F}in_*$: a map $\beta : \langle n \rangle_* \to \langle m \rangle_*$ is called collapsing if it's preimages of $j \neq *$ are singleton sets (thus each partition in the partial partitioning contains exactly one element, exactly as it was the case for convex maps in the previous section for convex morphisms). With this in mind we give the following definition

Definition 3.29. Given a opfibration $\mathcal{V}^{\otimes} \to N(\mathcal{F}in_*)$ such that $(\alpha_i^n)_!$ are inducing

$$\mathcal{V}_{\langle n \rangle_*} \xrightarrow{\sim} \left(\mathcal{V}^{\otimes}_{\langle 1 \rangle_*}\right)^n$$

then $\mathcal{V}$ is called a quasicategory with ***symmetric monoidal structure***.

Again, the verifications that certain $m_!$, $s_!$ and $u_!$ defined below behave as the monoidal product, symmetry and unit can be easily done by hand and are consequences of this definition and of the universality of coCartesian morphisms (or more precisely of the resulting pseudofunctoriality of $(-)_!$). As before one should observe that this makes all $\beta_!$ for a collapsing map β are projections (also cf. (4.12)). Explicitely, m, t and u can be given as follows

$$\begin{aligned} m &: \langle 2\rangle_* \to \langle 1\rangle_* \ , \ m(j) = 1 \\ s &: \langle 2\rangle_* \to \langle 2\rangle_* \ , \ t(j) = j+1 \mod 2 \\ u &: \langle 0\rangle_* \to \langle 1\rangle_* \end{aligned}$$

Similarly, we can carry over our discussion of functor categories and algebra objects to the symmetric case.

Definition 3.30. Given symmetric monoidal categories $\mathcal{V}^\otimes \to N(\mathcal{F}in_*), \mathcal{W}^\otimes \to N(\mathcal{F}in_*)$ a functor $F : \mathcal{V}^\otimes \to \mathcal{W}^\otimes$ such that

$$\begin{array}{ccc} \mathcal{V}^\otimes & \xrightarrow{\mathcal{F}} & \mathcal{W}^\otimes \\ & \searrow \quad \swarrow & \\ & N(\mathcal{F}in_*) & \end{array}$$

is called a ***lax monoidal*** functor $F : \mathcal{V} \to \mathcal{W}$ if it preserves coCartesian arrows over collapsing morphisms, and it is called a ***strong monoidal*** functor $F : \mathcal{V} \to cW$ if it preserves all coCartesian arrows. A ***commutative algebra object*** A in $\mathcal{V}$ is a lax monoidal functor $A : N(\mathcal{F}in_*) \to \mathcal{V}$.

Again we obtain functor categories and categories of commutative algebra objects which we will denote as we did for monoidal quasicategories

$$\mathrm{Fun}^{\mathrm{mon}}(\mathcal{V},\mathcal{W}) \subset \mathrm{Fun}^{\mathrm{lax}}(\mathcal{V},\mathcal{W}) \subset \mathrm{Fun}_{/N(\mathcal{F}in_*)}(\mathcal{V}^\otimes,\mathcal{W}^\otimes) \subset \mathrm{Fun}(\mathcal{V},\mathcal{W})$$
$$\mathrm{CAlg}(\mathcal{V}) = \mathrm{Fun}^{\mathrm{lax}}(N(\mathcal{F}in_*),\mathcal{V})$$

Remark 3.31. Analogous to a previous remark the homotopy category of a symmetric monoidal quasicategory carries canonically an ordinary symmetric monoidal structure. But in the symmetric case a comparison result, similar to the one in the previous remark, of symmetric monoidal structure on presentable quasicategories and combinatorial simplicial model categories has so far not been worked out.

3.3.3. *Dualizable objects.* In this short section we will carry our definition of (left and right) duals from equation (1.4) over to the setting of monoidal quasicategories. Our definition will have to be adapted for the case of (∞,n)-categories to the notion of "fully dualizable" objects in section 4 as used by Lurie in the proof of the cobordism hypothesis. First recall that in the case of ordinary monoidal categories (regarded as 1-object bicategories) we say that 2 objects (X, Y) are a ***dual pair*** (i.e. left resp. right dual to another) if we have a unit and counit such that the following composes to give identities

$$X \xrightarrow{\sim} X \otimes I \xrightarrow{1\otimes\eta} X \otimes Y \otimes X \xrightarrow{\epsilon\otimes 1} 1 \otimes X \xrightarrow{\sim} X \tag{3.32}$$
$$Y \xrightarrow{\sim} I \otimes Y \xrightarrow{\eta\otimes 1} Y \otimes X \otimes Y \xrightarrow{1\otimes\epsilon} Y \otimes 1 \xrightarrow{\sim} Y$$

X is called left dual to Y and conversely, Y is called right dual to X. As in the general case of adjoints in bicategories they mutually determine each other up to isomorphism. We note that in a symmetric monoidal category the pair (X, Y) is a pair of duals if and only if the pair (Y, X) has a pair of duals. In this case we say X is ***dualizable*** if it has either right or left dual.

If we want to translate this definition into one which holds up to coherent homotopy this just means to transfer the above to the homotopy category (*We remark*: The above definition involves equalities of composites of arrows, and equality up to homotopy then becomes an equality in the homotopy category. So this approach is reasonable. In contrast, e.g. constructing global homotopy (co)limits (e.g. as a left kan extension of functors from the underlying (model) category) could as we saw not be constructed in this naive way. It involves information about arrows *before* passing to 0th homotopy on hom spaces!). We thus define

Definition 3.33. An object X in a symmetric monoidal quasicategory $\mathcal{V}$, $p : \mathcal{V}^{\otimes} \to N(\mathcal{F}in_*)$, is called dualizable if it is dualizable in the ordinary symmetric monoidal category $\mathrm{Ho}(\mathcal{V})$.

We remark that every symmetric monoidal quasicategory $\mathcal{V}$ admits a full subcategory (closed under the monoidal product $\otimes$) $\mathcal{V}^{fd} \subset \mathcal{V}$ of (fully) dualizable objects. We will make further use of these notions in section 4.

3.4. **Spectra and stabilisation.** We will now turn our attention to the second purpose on this very short and basic overview of infinity categories: Namely, we want to finish our discussion of the category of spectra that started in section 1.2.3, now moving to the setting of quasicategories. We will abstract the process of stablisation that we demonstrated for topological spaces. For this section we assume to have established an appropriate notion of adjunctions and Kan extensions in the theory of quasicategories.

The $\mathcal{C}$ be a pointed homotopical category. Recall the defintion of loop and suspension objects in a pointed model category $\mathcal{M}$ via homotopy (co)kernels in 2.3.1:

$$\begin{array}{ccc} X & \xrightarrow{c} & * \\ \downarrow & & \downarrow{\scriptstyle \mathrm{hcof}(c)} \\ * & \longrightarrow & \Sigma X \end{array} \quad \text{and} \quad \begin{array}{ccc} \Omega X & \xrightarrow{\mathrm{hfib}(c)} & * \\ \downarrow & & \downarrow{\scriptstyle c} \\ * & \longrightarrow & X \end{array} \tag{3.34}$$

This can be regarded as (co)limits in the underlying quasicategories of $\mathcal{M}$ by Construction 3.4, or we can just adapt the notion of loop and suspension objects entirely for pointed quasicategories, i.e. those having an object $*$ which is both initial and terminal.

From now on we will assume $\mathcal{V}$ to be a *bicomplete* and *pointed* quasicategory, so that in particular the above (co)kernels exists. Then, let $\mathcal{V}^{\Sigma}$ and $\mathcal{V}^{\Omega}$ denote the categories of pushouts resp. pullbacks that are of the form of the left resp. right diagram in (3.34). In the classical theory we have seen that ΣX (and ΩX), being homotopy colimits, are uniquely determined up to weak equivalence. This of course should still hold in the case of $\mathcal{V}^{\Sigma}$ and $\mathcal{V}^{\Omega}$ which was formalized by Lurie (using (∞,1)-Kan extensions) in the following

Lemma 3.35. *The maps of quasicategories* $\mathrm{ev}_{(0,0)} : \mathcal{V}^{\Sigma} \to \mathcal{V}$ *and* $\mathrm{ev}_{(1,1)} : \mathcal{V}^{\Omega} \to \mathcal{V}$ *are trivial Kan fibrations.*

We can choose essentially unique sections $s_\Sigma : \mathcal{V} \to \mathcal{V}^\Sigma$, $s_\Omega : \mathcal{V} \to \mathcal{V}^\Omega$ by pulling back e.g. $1_\mathcal{C} : * \to \mathrm{Map}(\mathcal{C}, \mathcal{C})$ along the induced trivial Kan fibration $\mathrm{Map}(\mathcal{C}, \mathcal{C}^\Sigma) \to \mathrm{Map}(\mathcal{C}, \mathcal{C})$. Thus, up to this contractible choice we define the functors

$$\Sigma : \mathcal{V} \xrightarrow{s_\Sigma} \mathcal{V}^\Sigma \xrightarrow{\mathrm{ev}_{(1,1)}} \mathcal{V}$$
$$\Omega : \mathcal{V} \xrightarrow{s_\Omega} \mathcal{V}^\Omega \xrightarrow{\mathrm{ev}_{(0,0)}} \mathcal{V}$$

One can show that in analogy to the classical case these give an adjunction $\Sigma \dashv \Omega$. In particular we can translate our previous definition of stable model categories as follows:

Definition 3.36. $\mathcal{V}$ is called ***stable*** if cokernels in $\mathcal{V}$ are equivalently kernels and vice versa.

This in particular implies that by their definition $\Sigma \dashv \Omega$ become a pair of adjoint equivalences, which was the corresponding characterisation we gave for stable model categories in the classical case (indeed it turns out to be equivalent to the above definition).

With a notion of stability at hand we can now start looking for a general process of stabilisation of bicomplete quasicategories $\mathcal{C}$. Inspired by Construction 1.84 the definitions we give should be easily motivated for the reader. We note however that (cf. Remark 1.88) we now use a slightly different terminology and will say spectrum for the type of construction that we previously called Ω-prespectrum.

Definition 3.37. A ***prespectrum*** in a pointed bicomplete quasicategory $\mathcal{C}$ is a functor

$$T : N(\mathbb{Z} \times \mathbb{Z}) \to \mathcal{C}$$

which satisfies $T(i,j) = *$ whenever $i \neq j$. Here, $\mathbb{Z}$ is considered as a partially ordered set (and not a group). The image of T is thus of the form

$$\begin{array}{ccccccc}
 & & & & * & \longrightarrow & T(i+1,i+1) & \quad \dots \\
 & & & & \uparrow & & \uparrow & \\
 & & * & \longrightarrow & T(i,i) & \longrightarrow & * & \\
 & & \uparrow & & \uparrow & & & \\
\dots & & T(i-1,i-1) & \longrightarrow & * & & &
\end{array}$$

We write T_i for $T(i,i)$. We denote the category of spectra by $\mathrm{PSp}(\mathcal{C}) \subset \mathrm{Fun}(N(\mathbb{Z} \times \mathbb{Z}, \mathcal{C})$. A ***spectrum below n*** E in a pointed bicomplete quasicategory $\mathcal{C}$ is a prespectrum such that the induced maps $E_i \to \Omega E_{i+1}, i \leq n$ are weak equivalences. We denote the corresponding category by $\mathrm{Sp_n} \subset \mathrm{PSp}$. A ***spectrum*** is a spectrum below n for every n, and their category is denoted by Sp.

If $\mathcal{C}$ is not pointed we can still make sense of the above by using instead $\mathcal{C}_* = \mathcal{C}_{*/}$ for a terminal object $*$. We then define its ***stabilisation*** $\mathrm{Stab}(\mathcal{C})$ to be

$$\mathrm{Stab}(\mathcal{C}) = \mathrm{Sp}(\mathcal{C}_*)$$

In the particular case of $\mathcal{C} = \mathrm{S} = N_\Delta(Kan)$ our quasicategory of spaces, we define the ***quasicategory of spectra*** Sp to be Sp(S). We define $\Omega^\infty : \mathrm{Sp}(\mathcal{C}_*) \to \mathcal{C}_*$ to be the evaluation functor at $(0,0)$. If $\mathcal{C}$ is presentable one can show using the quasicategorical analogue of Proposition 1.35 that it has a left adjoint functor Σ^∞ which we will call ***suspension spectrum*** functor. In the case of spaces this actually coincides with the suspension spectrum that we previously defined in the 1-categorical setting. Now, if we restrict our attention to a stable, presentable $\mathcal{D}$, and $\mathrm{Stab}(\mathcal{C})$ for presentable $\mathcal{C}_*$ Lurie showed that $\mathrm{Stab}(\mathcal{C})$ satisfies the following universal property

$$Pr^L_\infty(\mathrm{Sp}(\mathcal{C}_*), \mathcal{D}) \xrightarrow[\sim]{(\Sigma^\infty)*} Pr^L_\infty(\mathcal{C}_*, \mathcal{D})$$

Here $Pr_\infty \subset \mathbf{Cat}_\infty$ denotes the full subcategory of presentable quasicategories, and Pr^L_∞ its restriction to colimit preserving functors. In the case $\mathcal{C} = S$ an application of the Yoneda lemma yields

$$Pr^L_\infty(\mathrm{Sp}, \mathcal{D}) \xrightarrow[\sim]{(\Sigma^\infty)^*} Pr^L_\infty(\mathrm{S}_*, \mathcal{D}) \simeq \mathcal{D}$$

and so we deduce that Sp is the ***free presentable stable quasicategory*** generated by one object, namely the sphere spectrum (the image of a point under Σ^∞).

So far, all of the above is easily motivated from recalling how we arrived at the description of generalized cohomology theories on $\mathbf{Top}_*$ using Ω-prespectra (called spectra in the above). Actually, we have the following equivalent characterisation of $\mathrm{Sp}(\mathcal{C}_*)$ as a sequential limit in $\mathbf{Cat}_\infty$

$$...\mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \to ... \tag{3.38}$$

This recovers the idea of $\mathrm{Sp}(\mathcal{C}_*)$ as the quasicategory of infinite loop objects in $\mathcal{C}_*$.

We now want give an radically rough sketch how a smash product for Sp, i.e. a (symmetric) monoidal structure, can be constructed since we promised such an construction in earlier sections. The idea is that Pr^L_∞ carries a tensor product structure with monoidal unit S, which is inherited by the subcategory of stable quasicategories $SPr^L_\infty \hookrightarrow Pr^L_\infty$ with monoidal unit being Sp. In particular Sp becomes an initial algebra object in SPr^L. We then refer back to our remarks about the quasicategorical Grothendieck construction following equation (3.26), which establish that such an (commutative) algebra object is a quasicategory with (symmetric) monoidal structure.

The evidently essential bit in this argument is to establish the tensor product structure on Pr^L_∞. We will do so by the analogy of $S \in Pr^L_\infty$ and $\mathbf{Set} \in Pr^L$. We are seeking a monoidal structure such that **Set**, becomes a unit $-$ the hom functors of $Pr^R \subset Pr$ (with limit preserving morphsim) provide a natural choice of "bilinear" functors for which **Set** behaves like a unit. Indeed we have

$$Pr^R(\mathbf{Set}, \mathcal{D}) \simeq \mathcal{D}$$
$$Pr^R(\mathcal{D}^{\mathrm{op}}, \mathbf{Set}) \simeq \mathcal{D}$$

where the first equivalence follows from the Yoneda lemma as usual (it is actually an isomorphism), and the second equivalence from Example 1.25. Fixing the variance in the first equivalence we are led to set

$$\mathcal{C} \otimes \mathcal{D} := Pr^R(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$$

With some further use of Proposition 1.35 one can then indeed go on to prove that this leads to

$$Pr^L(Pr^R(\mathcal{C}^{\text{op}}, \mathcal{D}), \mathcal{A}) \simeq Pr^L(\mathcal{C}, Pr^L(\mathcal{D}, \mathcal{A}))$$

showing that Pr^L actually becomes monoidal closed with this tensor structure (also note Pr^L is cartesian closed).

All the statements involved in this above derivation (precisely our use of Yoneda, presheafs, presentability and adjoints) have correspoding statements for quasicategories as noted in previous remarks with S in place of **Set**. We thus can guess statement (i) in the following theorem

Theorem 3.39. *(i) $Pr^L_\infty \subset \mathbf{Cat}_\infty$ has structure of a symmetric monoidal quasicategory with tensor product $\mathcal{C} \otimes \mathcal{D} = Pr^L_\infty(\mathcal{C}^{\text{op}}, \mathcal{D})$ and monoidal unit S.*
(ii) $SPr^L_\infty \subset Pr^L_\infty$ has structure of a symmetric monoidal quasicategory with tensor product $\mathcal{C} \otimes \mathcal{D} = SPr^L_\infty(\mathcal{C}^{\text{op}}, \mathcal{D})$ and monoidal unit Sp.

Proof. We still need to proof (ii): $\text{Sp} \otimes \mathcal{D} \simeq \mathcal{D}$ follows from Sp being the free presentable stable quasicategory. On the other hand $\mathcal{D} \otimes \text{Sp} \simeq \mathcal{D}$ follows from representables preserving limits and part (i):

$$\begin{aligned}\mathcal{D} \otimes \text{Sp} &\simeq SPr^L_\infty(\mathcal{D}^{\text{op}}, \text{Sp}) = Pr^L_\infty(\mathcal{D}^{\text{op}}, \text{Sp}) \\ &\simeq Pr^L_\infty(\mathcal{D}^{\text{op}}, \lim(... \xrightarrow{\Omega} S_* \xrightarrow{\Omega} ....)) \\ &\simeq \lim(... \xrightarrow{\Omega_*} Pr^L_\infty(\mathcal{D}^{\text{op}}, S_*) \xrightarrow{\Omega_*} ....)\end{aligned}$$

So noting $SPr^L_\infty(\mathcal{D}^{\text{op}}, S_*) \simeq SPr^L_\infty(\mathcal{D}^{\text{op}}, S) \simeq \mathcal{D}$, the discussion for (3.38) and $\text{Stab}(\mathcal{D}) \simeq \mathcal{D}$ (stabilisation is indempotent) the statement follows. □

As remarked above as a direct consequence of this theorem Sp becomes an initial algebra object in SPr^L_∞ which by the Grothendieck construction can be regarded as Sp having monoidal structure $\text{Sp}^\otimes$. We remark the following two facts about this structure

(i) The monoidal structure on Sp is essentially uniquely determined by requiring the sphere spectrum to be the monoidal unit, and the monoidal product to be bilinear (i.e. colimit preserving in each variable or in other words an arrow in SPr^L_∞).
(ii) With a more careful argument it can be shown that Sp actually obtains a symmetric monoidal structure from this construction.

This enables us the give the final definition of this section, namely special (highly structured) spectra in Sp

Definition 3.40. An $\mathcal{A}_\infty$***-ring*** is an object of $\mathcal{A}_\infty := \text{Alg}(\text{Sp})$. An $\mathcal{E}_\infty$***-ring*** is an object of $\mathcal{E}_\infty := \text{CAlg}(\text{Sp})$.

The theory of modules over such algebra objects was worked out in detail by Lurie in his book on "Higher Algebra".

Remark 3.41. In a similar way to copying over the definition of loop and suspension functors we can also copy the definition of homotopy fibers and cofibers (recall these were functors on Arr(**Top**)) to obtain functors on $\mathrm{Map}(\Delta^1, \mathcal{C})$. This yields an adjoint pair

$$\mathrm{cof} \dashv \mathrm{fib} : \mathrm{Map}(\Delta^1, \mathcal{C}) \to \mathrm{Map}(\Delta^1, \mathcal{C})$$

and if $\mathcal{C}$ is stable it forms an equivalence. As expected this shows we can translate much of the homotopy theory in $\mathbf{Top}_*$ done in section 1.2 to general $(\infty,1)$-categories or "homotopy theories".

4. Model independence

In this section we will start by relating the different notions of homotopy limits presented in sections 1.2, 2 and 3. In the second part, we will then turn our attention to higher (one-sorted) non-invertible morphisms. This is an area of active research. We have already seen that first of all there might be naturally different sorts of morphisms (e.g. already in the context bicategories we found to have lax, oplax and strong monoidal functors/pseudofunctors; double categories are another example of categories with more than one sort of morphisms). Secondly, though there are some generally accepted criteria for a theory of one-sorted (∞,n)-categories like the homotopy and stabilisation hypothesis, it was hard to compare different approaches until the axiomatisation by Barwick and Schommer-Pries in [BSP11]. We will sketch the axioms involved in the final part of this section, without giving details on the proofs.

4.1. **Homotopy coherence.** We have seen how to formalize our treatment of algebraic topology in 1.2 via model structures in section 2, or more generally categories with weak equivalences. We then saw how to canonically make functors I on the underlying categories ***homotopy invariant*** in a universal way, that means up to weak equivalence *blind* to replacements by weak equivalences and uniquely factoring every other such homotopy invariant functor "extending" I:

Precisely, given a functor $I : \mathcal{C} \to \mathcal{C}'$ carrying information about some category $(\mathcal{C}, \mathcal{W})$ valued in $(\mathcal{C}', \mathcal{W}')$, this universal homotopy invariant functor was obtained by replacing I by its derived functor $\mathbb{L}I : \mathcal{C} \to \mathcal{C}'$ (as defined in section 2.5). In particular we applied this to the colimit and limit functors: Weak equivalences on diagram categories are objectwise weak equivalences - that is we want limits and colimits to be blind (up to weak equivalence) to weakly equivalent replacements of objects in diagrams.

Thus we defined "global" homotopy colimits as the derived functors of ordinary colimits. But in order to understand what kind of information about each diagram this derived functor actually carries we needed to find its "local" representation, i.e. the universal property satisfied by $\mathrm{hocolim}\, F$ of some $F \in \mathcal{M}^{\mathcal{D}}$. At least in the case of a simplicial model category $\mathcal{M}$ we found in (2.36) the following **sSet**-enriched representation:

$$\underline{\mathcal{M}}(\mathrm{hocolim}\, F, m) \cong \underline{\mathbf{sSet}}^{\mathcal{D}^{\mathrm{op}}}(N(-/\mathcal{D}), \underline{\mathcal{M}}(F-, m)) = \mathrm{hoCocones}_F(m)$$

where the diagram category $\mathcal{D}$ is freely enriched in **sSet**, i.e. by the inclusion $\mathbf{Set} \hookrightarrow \mathbf{sSet}$. Passing to homotopy categories of simplicially enriched categories, which equals a change of basis along $h_0 : \mathbf{sSet} \xrightarrow{|-|} \mathbf{Top} \xrightarrow{\pi_0} \mathbf{Set}$, we thus obtain the 1-categorical universal property we were looking for

$$\mathrm{Ho}(\mathcal{M})(\mathrm{hocolim}\, F, m) \cong h_0(\mathrm{hoCocones}_F(m)) \tag{4.1}$$

where we wrote out h_0 only on the right, while using standard notation for the homotopy category on the left.

We then translated categories with weak equivalences into quasicategories in section 3. The translation was given in Construction 3.4 by $\mathcal{C} \mapsto N_\Delta RL^H\mathcal{C}$ for a homotopical $\mathcal{C}$. Conversely, for a quasicategory $X \in \mathbf{sSet}$ we can set $X \mapsto \pi_0 C[X]$ (with $\pi_0 : \mathbf{sSet} \xrightarrow{\mathrm{ev}_{[0]}} \mathbf{Set}$ and a notion of homotopy and weak equivalence inherited from $\mathbf{sSet}$-enrichment). This gives indeed a mutual inverse translation up to weak equivalence by our discussion of Theorem 2.25 and Proposition 3.3 (assuming a canonical equivalence $C[X] \xrightarrow{\sim} L^H\pi_0 C[X]$).

In quasicategories we developed independently a natural notion of cones, cocones, limits and colimits. Since functors were naturally homotopy coherent (with our terminology from 3.2) a discrete diagram of shape $\mathcal{D} \in \mathbf{Cat}$ in a quasicategory $\mathcal{C}$ was just a map

$$F : N(\mathcal{D}) \to \mathcal{C}$$

i.e. $F \in \operatorname{obj}(\mathbf{sSet}(N(\mathcal{D}), \mathcal{C}))$. A homotopy coherent cocone over F was a similarly map $F^{\triangleright} \in \operatorname{obj} \mathcal{C}_{F/}$ or equivalently

$$F^{\triangleright} : N(\mathcal{D}) \star \Delta^0 = N(\mathcal{D} \star *) \to \mathcal{C}$$

(recall $N(\mathcal{D} \star \mathcal{D}') \cong N(\mathcal{D}) \star N(\mathcal{D}')$ by definition of the join). A colimit $c = \operatorname{colim} F$ in $\mathcal{C}$ was defined as the summit of an initial object in the quasicategory of such cocones $\mathcal{C}_{F/} = \mathbf{sSet}_F(N(\mathcal{D}) \star \Delta^0, \mathcal{C})$. Recall that we have a forgetful functor $U : \mathcal{C}_{F/} \to \mathcal{C}$. It's fiber over $m \in \mathcal{C}$ has as objects cocones $F^{\triangleright}$ with summit m. We denote the fiber over m by $\mathrm{Cocones}_F(m)$. The contractibility condition in Definition 3.13 for a colimit cocone implies[16] that

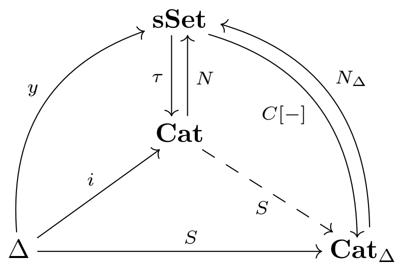
$$\mathrm{Ho}(\mathcal{C})(\operatorname{colim} F, m) \cong h_0(\mathrm{Cocones}_F(m)) \tag{4.2}$$

Both here and in (4.1) one should of course note that weak equivalences induce isomorphisms on the right hand side which then lifts to a functor on the homotopy categories. Comparing (4.1) and (4.2) this discussion is supposed to motivate the following: For understanding that the notions of hocolim and colim as defined above coincide, it suffices to understand that the notion ***homotopy coherent*** cones coincide. In the case of simplicial model categories, this can be seen quite easily in $\mathbf{Cat}_\Delta$ as it provides a unifying setting for both notions: We will first recover the classical notion of homotopy coherent diagram in the following construction. We will then illustrate its equivalence to the quasicategorical notion above by an example.

Construction 4.3 (Vogt's homotopy coherent diagrams). We can combine the two Yoneda extensions involved in the preceding discussion and there adjoints N and

[16]A rough argument would be the following: Let λ be an initial object of $C_{F/}$ with summit $\operatorname{colim} F$. U induces a map $V : (\mathcal{C}_{F/})_{\lambda/} \to C_{\operatorname{colim} F/}$. Initiality of λ by definition implies $(\mathcal{C}_{F/})_{\lambda/} \to \mathcal{C}_{F/}$ is a trivial Kan fibration. $\mathrm{Cocones}_F(m) \hookrightarrow \mathcal{C}_{F/}$ is the fiber over m in $\mathcal{C}_{F/} \to \mathcal{C}$ and it has the thus the same 0-th homotopy as the fiber over m in $(\mathcal{C}_{F/})_{\lambda/} \to C_{F/} \to \mathcal{C}$. The latter is the same as the fiber over m in $(\mathcal{C}_{F/})_{\lambda/} \to C_{\operatorname{colim} F/} \to \mathcal{C}$ and is mapped into the fiber over m in the fibration $C_{\operatorname{colim} F/} \to \mathcal{C}$ defining $\mathrm{Map}(\operatorname{colim} F, m)$ in Definition 3.12. This can be verified to induce a bijection on path components by λ being initial. So composing these bijections after passage to 0-th homotopy gives us the map between fibers we were looking for.

N_Δ in one diagram as follows



Here we define the functor S as the composite $C[N-]$. Since $C[Ni-] \cong C[y-] \cong S$ this actually extends our original simplicial thickening functor $S : \Delta \to \mathbf{Cat}_\Delta$ as indicated by the dashed arrow. The other parts of the diagram commute up to weak equivalence. The definition of S allows us to reformulate the notion of homotopy coherent diagrams in the quasicategory $N_\Delta(\mathcal{M}_{cf})$ associated to a simplicial model category $\mathcal{M}$:

$$\mathbf{sSet}(N(\mathcal{D}), N_\Delta(\mathcal{M}_{cf})) \cong \mathbf{Cat}_\Delta(S(\mathcal{D}), \mathcal{M}_{cf}) \tag{4.4}$$

So up to a bifibrant replacement this naturally leads us to make the following definition:

Definition 4.5. A ***homotopy coherent diagram*** of shape $\mathcal{D}$ in a simplicially enriched category $\underline{\mathcal{M}}$ is a simplicial functor $F : S(\mathcal{D}) \to \underline{\mathcal{M}}$.

The original definition is due to Vogt in [Vog73], and used a description of S in terms of the simplicial object associated to the free category monad. This was later shown to be equivalent to the above definition e.g. by Riehl in [Rie11b]. The functor S has an intuitive description, since we know how to compute both N and $C[-]$ (this should be clear on the the image of y, then recall $X = \mathrm{colim}^X y = \mathrm{colim}_{\mathrm{el}X}\, y$ for $X \in \mathbf{sSet}$ by Example 1.21). Namely, $S(\mathcal{D})$ has the same objects as $\mathcal{D}$, and the simplicial set of morphisms $S(\mathcal{D})(a, b)$ is given as the nerve of the poset $P(a, b)$ defined as follows: Objects of $P(a, b)$ are strings of morphisms $f_1 f_2 ... f_n : a \to b$ such that $g_1 g_2 ... g_n \le f_1 f_2 ... f_n$ whenever the former can be obtained by composition from the latter. Composition is induced by string concatenation $P(b, c) \times P(a, b) \to P(a, c)$. Thus a homotopy coherent diagram $F : S(\mathcal{D}) \to \mathcal{M}$ consists of the following data

$$F : \mathrm{obj}(S(\mathcal{D})) \to \mathrm{obj}(\mathcal{M})$$
$$F_{i,j} : S(\mathcal{D})(i, j) \to \underline{\mathcal{M}}(Fi, Fj)$$

Morally, $F_{i,j}$ picks for each binary composition a 1-simplex in $\underline{\mathcal{M}}(Fi, Fj)$, *but also* for each higher coherence a higher simplex in $\underline{\mathcal{M}}(Fi, Fj)$ exactly as explained already in Remark 1.32.

Example 4.6 (two views on homotopy pushouts). We are now ready to compare the notions of homotopy coherence. Just for illustrative purpose we choose $\mathcal{D}$ to be a pushout diagram and $\mathcal{M} = \mathbf{Top}$: Recall that **Top** is a simplicial model category since we can change basis along the (right adjoint) singular nerve functor $S_\bullet : \mathbf{Top} \to \mathbf{sSet}$. Being right adjoint guarantees that $S_\bullet$ will transfer cotensoredness and tensoredness as is required for the definition of a simplicial model category (see e.g. [Rie14], §3 and §11).

(i) We have $\mathcal{D} = b \leftarrow a \rightarrow c$ and thus $\mathcal{D} \star *$ is the category

$$\begin{array}{ccccc} & & * & & \\ & {\scriptstyle l_b}\nearrow & \uparrow{\scriptstyle l_a} & \nwarrow{\scriptstyle l_c} & \\ b & \xleftarrow[f]{} & a & \xrightarrow[g]{} & c \end{array}$$

Choose a homotopy coherent diagram $F : S(\mathcal{D}) \to \mathbf{Top}$. A homotopy coherent cocone $F^{\triangleright}$ from the perspective of quasicategories translated to the **sSet**-enriched setting via (4.4) is then a **sSet**-functor $F^{\triangleright} : S(\mathcal{D} \star *) \to \mathbf{Top}$ such that $F^{\triangleright}|_{S(\mathcal{D})} = F$. In other words we need to specify the data

$$m := F^{\triangleright}(*) \in \mathbf{Top}$$
$$F^{\triangleright}_{i,*} : S(\mathcal{D} \star *)(i, *) \to \mathbf{Top}(F(i), m)$$

suitable compatible with composition for the diagram data of F. We compute

$$S(\mathcal{D} \star *)(a, *) = N(l_b f \leftarrow l_a \to l_c g) \cong \mathrm{sd}(\Delta^1)$$
$$S(\mathcal{D} \star *)(b, *) = N(l_b) \cong \Delta^0 \cong S(\mathcal{D} \star *)(c, *)$$

Compatibility of $F^{\triangleright}_{i,*}$ with composition then implies that e.g. in the case of precomposition with morphisms f (i.e. a 0-simplex of the hom spaces) we have:

$$\begin{array}{ccc} S(\mathcal{D} \star *)(b, *) & \xrightarrow{S(\mathcal{D}\star*)(f,1)} & S(\mathcal{D} \star *)(a, *) \\ {\scriptstyle F^{\triangleright}_{b,*}}\downarrow & & \downarrow{\scriptstyle F^{\triangleright}_{a,*}} \\ \mathbf{Top}(F(b), m) & \xrightarrow{-\circ Ff} & \mathbf{Top}(Fa, m) \end{array}$$

(ii) A homotopy coherent cocone from the perspective of simplicial model categories is an element $\hat{F}^{\triangleright} \in \mathbf{sSet}^{\mathcal{D}^{\mathrm{op}}}(N(-/\mathcal{D}), \underline{\mathcal{M}}(F-, m))$ and thus an **sSet**-enriched natural transformation. It consists of the following data

$$\hat{F}^{\triangleright}_i : N(i/\mathcal{D}) \to \mathbf{Top}(Fi, m)$$

We compute the images of $N(-/\mathcal{D})$

$$N(a/\mathcal{D}) = N(f \leftarrow 1 \to g) \cong \mathrm{sd}(\Delta^1)$$
$$N(b/\mathcal{D}) = N(1) = \Delta^0 = N(c/\mathcal{D})$$

In the above $N(-/\mathcal{D})$ and $\underline{\mathcal{M}}(F - .m)$ are **sSet**-functors, with its domain $\mathcal{D}$ freely **sSet**-enriched. $\hat{F}^{\triangleright}$ is in particular a natural transformation for their underlying functors, for instance the following commutes

$$\begin{array}{ccc} N(b/\mathcal{D}) & \xrightarrow[=-\circ f]{N(f/\mathcal{D})} & N(a/\mathcal{D}) \\ \downarrow{\scriptstyle \hat{F}^{\triangleright}_b} & & \downarrow{\scriptstyle \hat{F}^{\triangleright}_a} \\ \mathbf{Top}(Fb, m) & \xrightarrow{-\circ Ff} & \mathbf{Top}(Fa, m) \end{array}$$

This should be compared to the similar diagram from (i).

With a bit more work one can convince oneself that that $\hat{F}^{\triangleright}$ and $F^{\triangleright}$ specify the same data beyond the few verifications above (and for general $\mathcal{D}$) and that weak equivalence for cocones of the former sort can be translated to weak equivalences of the later and vice versa. In the light of (4.2) and (4.1) we thus obtain

Corollary 4.7. *Given a simplicial model category $\underline{\mathcal{M}}$ and a diagram $F : \mathcal{D} \to \mathcal{M}$ its homotopy colimit coincides up to weak equivalence with the quasicategorical limit of $F_{cf} : N(\mathcal{D}) \to N_\Delta(\underline{\mathcal{M}}_{cf})$ (obtained after objectwise bifibrant replacement of the original diagram yielding a homotopy coherent diagram $\hat{F}_{cf} : S(\mathcal{D}) \to \underline{\mathcal{M}}$).*

One can more generally define a notion of homotopy (co)limit on Kan complex enriched categories using *homotopy Kan extensions* (coinciding with our derived (co)limits in the case of simplicial model categories). Lurie then generalizes the result to hold for all such fibrant objects of $\mathbf{Cat}_\Delta$

Theorem 4.8 ([Lur09a] 4.2.4.1)**.** *The notions of homotopy (co)limit of a diagram $F \in \mathbf{Cat}_\Delta(\mathcal{D}, \mathcal{C})$ (with $\mathcal{C}$ Kan-enriched) coincides with the (co)limit of $N_\Delta(F)$ in the corresponding quasicategory $N_\Delta(\mathcal{C})$.*

Remark 4.9 (general model categories). So far we only considered the case of simplicial model categories as this allowed us to explicitely define homotopy coherent diagrams. But it would be nice to have a similar statement to Corollary 4.7 in the general case. This statement exists and can be guessed already from the fact every model category should be "almost" enriched in simplicial sets e.g. via the Hammock localisation L^H. The following very roughly sketched ideas are due to Dennis-Charles Cisinki in [Cis14]. Given a model category $\mathcal{M}$ we need two ingredients:

(i) Being "almost" enriched precisely means, that for each cofibrant $A \in \mathcal{M}$ we have a functor $\mathrm{Map}_\mathcal{M}(A, -) : \mathcal{M} \to \mathbf{sSet}$ ([Hov99]). Recall for X fibrant $\mathcal{M}(A, X)$ has a notion of homotopy. It turns out we then can then reconcile $\mathrm{Map}_\mathcal{M}(A, X) \simeq L^H\mathcal{M}(A, X)$ as expected. Also, $\mathrm{Map}_\mathcal{M}(A, -)$ is right Quillen and preserves homotopy limits $-$. It thus behaves like a "almost $\mathbf{sSet}$-enriched" hom functor.

(ii) We can then translate our discussion to the previous case by this Yoneda embedding and it's quasicategorical analogue. Indeed, the quasicategorical Yoneda embedding $y : \mathcal{C} \to \mathbf{sSet}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ preserves limits in the sense of quasicategories, and so does $\mathrm{ev}_A : \mathbf{sSet}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \to \mathcal{S}$. This implies $\mathrm{Map}_\mathcal{C}(A, -)$ preserves limits. But $\mathcal{S} = N_\Delta(\mathbf{sSet}_{cf})$, so Corollary 4.7 will apply.

We summarise the two ideas in the following diagram. Let $F : \mathcal{D} \to \mathcal{M}$ and suppose F takes values in $\mathcal{M}_{cf}$. Take the minimal homotopical structure $(\mathcal{D}, Isom)$ on $\mathcal{D}$. Then $\hat{F} = RL^HF : \underline{\mathcal{D}} \to RL^H\mathcal{M}$ is coherent on the nose ($\underline{\mathcal{D}}$ coincides with our previously used freely enriched $\underline{\mathcal{D}}$), and in particular lifts to $\hat{F} : S(\mathcal{D}) \to RL^H\mathcal{M}$. This corresponds to $F : N(\mathcal{D}) \to N_\Delta RL^H\mathcal{M} := \mathcal{C}$ by the adjunction $C[-] \dashv N_\Delta$ as was discussed previously in this section. Then

$$
\begin{array}{ccc}
X = \lim_{N(\mathcal{D})} F \text{ in } \mathcal{C} & \xrightarrow{\sim} & \hat{X} = \operatorname{holim} F \text{ in } \mathcal{M} \\
 & \Uparrow \text{Yoneda in } \mathrm{Ho}(\mathcal{M}) & \\
\Big\downarrow \mathrm{Map}_\mathcal{C}(A,-) & \forall A, \mathrm{Map}_\mathcal{C}(A, X) \simeq \mathrm{Map}_\mathcal{M}(A, \hat{X}) & \Big\downarrow \mathrm{Map}_\mathcal{M}(A,-) \\
 & \Uparrow & \\
\lim \text{ in } \mathcal{S} & \xleftrightarrow[\sim]{\qquad\qquad} & \operatorname{holim} \text{ in } \mathbf{sSet}
\end{array}
$$

where weak equivalence at the bottom follows from Corollary 4.7 for weakly equivalent diagrams (using $\mathrm{Map}(A, X) \simeq L^H\mathcal{M}(A, X)$ from the first ingredient (i)) and $\mathcal{S} = N_\Delta(\mathbf{sSet}_{cf})$. The first implication in the above diagram then holds for all cofibrant A. But in $\mathrm{Ho}(\mathcal{M})$ every object is isomorphic to some cofibrant A so the second implication holds.

4.2. **Unicity Theorem.** We would now like to introduce higher morphisms which are not necessarily invertible. The idea of homotopy coherence prevails: composition should still be coherent up to higher homotopy. This coherence needs to be encoded e.g. when we are enriching in quasicategories to obtain $(\infty,2)$-categories. We have already defined monoidal quasicategories $\mathcal{V}^\otimes$ and we remind ourself that $\otimes$ should then be regarded as composition of 1-morphisms in a 1-object $(\infty, 2)$-category. Let us thus take inspiration from section 3.3 where we defined monoidal structures:

- Strict monoidal categories were essentially regarded as algebra objects $\mathcal{V} : \Delta^{\mathrm{op}} \to \mathbf{Cat}$ such that $\mathcal{V}(\iota_i)$ induce an *isomorphism*

$$\mathcal{V}([n]) \cong \mathcal{V}([1]) \times ... \times \mathcal{V}([1]) \tag{4.10}$$

- Monoidal quasicategories were essentially regarded as algebra objects $\mathcal{V} : N(\Delta^{\mathrm{op}}) \to \mathbf{Cat}_\infty$ such that $\mathcal{V}(\iota_i)$ induce an *weak equivalence*

$$\mathcal{V}([n]) \xrightarrow{\sim} \mathcal{V}([1]) \times ... \times \mathcal{V}([1])$$

  (cf. remarks following (3.26) about cartesian monoidal structure)

Here $N(\Delta^{\mathrm{op}})$ is just the quasicategorical analogue of Δ^{op} (the strict quasicategory obtained by freely adding witnesses of higher coherence as constructed in Remark 1.32). But the essential difference is that our ambient $\mathbf{Cat}_\infty$ carries a notion of weak equivalence. To understand this we recall that $\mathbf{Cat}$ comes equipped with weak equivalenes (actually even with a model structure) as well - namely just the usual categorical equivalences (and isofibrations as fibrations). So let us change (4.10) to

$$\mathcal{V}([n]) \xrightarrow{\sim} \mathcal{V}([1]) \times ... \times \mathcal{V}([1]) \in \mathbf{Cat} \tag{4.11}$$

This gives us (as we will define in a moment) a 1-object *Tamsamani 2-category*. Horizontal composition can be *non-naturally* obtained from

$$\mathcal{V}([1]) \times \mathcal{V}([1]) \xleftarrow{\sim} \mathcal{V}([2]) \xrightarrow{\mathcal{V}(d_1)} \mathcal{V}([1])$$

by a choice of weak inverse $\mathcal{V}([1]) \times \mathcal{V}([1]) \xrightarrow{\sim} \mathcal{V}([2])$ and is thus only defined up to a natural isomorphism. Even without such a choice associativity is already expressed in the following commutative diagram[17]

$$
\begin{array}{ccccc}
\mathcal{V}([2]) & \xrightarrow{\mathcal{V}(d_1)} & \mathcal{V}([1]) & \xleftarrow{\mathcal{V}(d_1)} & \mathcal{V}([2]) \\
\big\downarrow{\sim} & \nwarrow & \uparrow & \nearrow & \big\downarrow{\sim} \\
\mathcal{V}([1]) \times \mathcal{V}([1]) & \longleftarrow & \mathcal{V}([3]) & \longrightarrow & \mathcal{V}([1]) \times \mathcal{V}([1]) \\
\big\uparrow{1\times\mathcal{V}(d_1)} & \swarrow{\sim} & \big\downarrow{\sim} & \searrow{\sim} & \big\uparrow{\mathcal{V}(d_1)\times 1} \\
\mathcal{V}([1]) \times \mathcal{V}([2]) & \xrightarrow{\sim} & \mathcal{V}([1])^{\times 3} & \xleftarrow{\sim} & \mathcal{V}([2]) \times \mathcal{V}([1])
\end{array}
\tag{4.12}
$$

Thus a choice of $\mathcal{V}([1]) \times \mathcal{V}([1]) \xrightarrow{\sim} \mathcal{V}([2])$ will make this diagram commute up to isomorphism giving us a notion of weakly associative composition, where (in the general case $\mathcal{V} : \Delta^{\text{op}} \to \mathcal{M} \neq \mathbf{Cat}$) "weak" is precisely the notion of weak equivalence in the ambient category.

Remark 4.13. The same ideas apply to $(\infty,1)$-categories: they should be weakly enriched in **Top** or $\mathbf{sSet}_{Quillen}$. By the above, this means our data should naturally look *"bisimplicial"*. The models we are about to present look exactly like this. But quasicategories in this sense are an "effective hack" encoding all mapping spaces in one simplicial set (as a drawback they don't come with a notion of composition).

4.2.1. *Models for (∞,n) categories.* We will now define weak enrichment formally. As a warm-up we will define Segal categories. Recall that for a poset T the notation Δ^T meant NT where N is the categorical nerve. Recall that the spine of Δ^n was the subsimplex

$$S_n := \Delta^{\{0,1\}} \cup_{\Delta^{\{1\}}} \Delta^{\{1,2\}} \cup_{\Delta^{\{2\}}} \ldots \cup_{\Delta^{\{n-1\}}} \Delta^{\{n-1,n\}} \subset \Delta^n$$

Also recall these inclusions corepresent the (**Set**) ***Segal maps*** for a simplicial set Z:

$$s_n : Z_n = \mathbf{sSet}(\Delta^n, Z) \to \mathbf{sSet}(S_n, Z) = Z_1 \times_{Z_0} Z_1 \times_{Z_0} \ldots \times_{Z_0} Z_1$$

For general simplicial objects they are equivalently induced by the n obvious non-degenerate convex inclusions of 1-simplices into an n-simplex which we denoted by ι_k earlier (in section 3.3).

The Segal maps have an alternative "objectwise" description. Define $Z_n(x_0, \ldots, x_n)$ by the pullback

$$
\begin{array}{ccc}
Z_n(x_0, \ldots, x_n) \lrcorner & \longrightarrow & Z_n \\
\big\downarrow & & \big\downarrow \\
(x_0, \ldots, x_n) & \longrightarrow & Z_0 \times \ldots \times Z_0
\end{array}
$$

which then induces the objectwise Segal maps (which we saw already in (1.33))

$$s_n : Z_n(x_0, \ldots, x_n) \to Z_1(x_0, x_1) \times \ldots \times Z_1(x_{n-1}, x_n)$$

[17] This diagram should have been drawn already in section 3.3 for verifying an essential part of Definition 3.23.

In the following we should think of $Z_n(x_0, ..., x_n)$ as the space of n composable arrows along the objects $x_0, ..., x_n$.

Definition 4.14 (Segal categories)**.** $Z \in \mathbf{Top}^{\Delta^{\mathrm{op}}}$ is a ***Segal category*** if Z_0 is a discrete topological space s_n are homotopy equivalences. The ***homotopy category*** hZ of a Segal category Z has $\mathrm{obj}\, hZ = Z_0$ and morphisms $hZ(a,b) = \pi_0 Z(a,b)$ composition is induced by

$$Z(a,b) \times Z(b,c) \xleftarrow{\sim} Z(a,b,c) \xrightarrow{Z(d_1)} Z(a,c)$$

upon passing to 0th homotopy.

Recall that a ***relative category*** is a even weaker notion than a category with weak equivalences: it is merely a category $\mathcal{C}$ with an distinguished collection of morphism $\mathcal{W}$ containing the identities and closed under composition. We keep referring to (and thinking of) $\mathcal{W}$ as weak equivalences. The *homotopy category* of a relative category $(\mathcal{C}, \mathcal{W})$ is defined analogously to be the localisation $h\mathcal{C} = \mathcal{C}[\mathcal{W}^{-1}]$. By the universal property of localisations it follows $h(\mathcal{C} \times \mathcal{C}') \simeq h\mathcal{C} \times h\mathcal{C}'$. Given a $\mathcal{C}$ we denote the ***homotopicial relative category*** by $\underset{\sim}{\mathcal{C}} = (\mathcal{C}, Isom)$. Finally, a ***relative functor*** is a functor of relative categories preserving weak equivalences.

The following defintions and constructions should be seen in analogy to the definitions of monoidal categories and the classical theory of enrichment.

Definition 4.15 (weak enrichment)**.** (i) A ***segalic relative category*** is a relative category $(\mathcal{C}, \mathcal{W})$ equipped with and relative "underlying set functor" $\pi_0 : (\mathcal{C}, \mathcal{W}) \to \underset{\sim}{\mathbf{Set}}$ such that $\mathcal{M}$ has finite products, π_0 preserves finite products, and $\mathcal{W}$ is closed under taking finite products and their coherences and symmetries (that is e.g. $A \times B \cong B \times A$ is in $\mathcal{W}$).

(ii) A ***(segalic) relative functor*** F is a relative functor of segalic categories which is "weakly cartesian", i.e. $F(x \times y) \xrightarrow{\sim} F(x) \times F(y)$ and $F(1) = 1$.

(iii) Given $S \in \mathbf{Set}$ the category of ***S-indexed simplices*** Δ_S is the comma category U/S where $U : \Delta \to \mathbf{Set}$ is the forgetful functor. In particular, objects of Δ_S are tuples $(s_0, ..., s_n)$ for some $n \in \mathbb{N}$, morphisms are just "ordered maps" between such tuples. We think of $(s_0, ..., s_n)$ as points along a spine.

Construction 4.16 (Segal construction). Given a segalic relative category $\mathcal{M}$, an $\mathcal{M}$ ***enriched precategory*** is a set of objects S together with a functor $X : \Delta_S^{\mathrm{op}} \to \mathcal{M}$ such that $X(s_0) = 1$. It is ***$\mathcal{M}$-enriched segalic category*** if all Segal maps are weak equivalences in $\mathcal{M}$, i.e.

$$s_n : X(s_0, s_1, ..., s_n) \xrightarrow{\sim} X(s_0, s_1) \times ... \times X(s_{n-1}, s_n)$$

A ***(segalic) functor*** $(f, F) : (S, X) \to (T, Y)$ of $\mathcal{M}$-enriched segalic categories (resp. precategories) consists of a map on objects $f : S \to T$, which in particular induces a map $\hat{f} : \Delta_S \to \Delta_T$, and a natural transformation $F : X \to \hat{f}^* Y : \Delta_S^{\mathrm{op}} \to \mathcal{M}$. Together these notions define the category of $\mathcal{M}$-enriched segalic categories denoted by $\mathbf{Seg}(\mathcal{M})$ (resp. $\mathbf{PreCat}(\mathcal{M})$).

Remark 4.17. We think of $X(s_0, s_1, ..., s_n)$ as the $\mathcal{M}$-object describing n-strings of composable arrows with spine points $(s_0, s_1, ..., s_n)$. $X(s_0) = 1$ then means there is just one empty string for each object.

Example 4.18. Let $\mathcal{M} = \underset{\sim}{\mathbf{Set}}$. In the discussion following Remark 1.32 we have seen that $\mathbf{PreCat}(\underset{\sim}{\mathbf{Set}}) \cong \mathbf{sSet}$: For this we set X_0 to be S and X_n to be the coproduct of all $X(x_0, ..., x_n)$ (in particular this agrees with S on X_0). We say more generally that $A \in \mathcal{M}$ is ***discrete*** if it can be written as a coproduct of copies of 1. Thus all objects of **Set** are discrete.

Example 4.19. If $\mathcal{M} = \underset{\sim}{\mathcal{C}}$ is segalic, then $\mathbf{Seg}(\mathcal{M}) = \mathcal{M}$-**Cat** - the usual category of $\mathcal{M}$-enriched categories (where $\mathcal{M}$ has cartesian monoidal structure). In particular, we have $\mathbf{Seg}(\underset{\sim}{\mathbf{Set}}) = \mathbf{Cat}$.

We need to make three essential observations

(i) Similarly to classical enrichment a segalic relative functor $F : \mathcal{M} \to \mathcal{M}'$ induces a *change of basis* functor between Segal categories

$$F_* : \mathbf{Seg}(\mathcal{M}) \to \mathbf{Seg}(\mathcal{M}')$$

Indeed, the map $X \to FX$ preserves the segal condition as F commutes with products up to weak equivalence. Now, by definition every segalic relative category $\mathcal{M}$ comes equipped with a segalic relative functor $\pi_0 : \mathcal{M} \to \underset{\sim}{\mathbf{Set}}$. The change of basis along π_0 defines the ***homotopy category functor***

$$h : \mathbf{Seg}(\mathcal{M}) \to \mathbf{Cat}$$

and the segalic underlying set functor of $\mathbf{Seg}(\mathcal{M})$

$$\pi_0^{\mathbf{Seg}(\mathcal{M})} : \mathbf{Seg}(\mathcal{M}) \to \underset{\sim}{\mathbf{Set}}$$
$$X \mapsto \pi_0(\mathrm{core}(hX)) = \mathrm{obj}(\mathrm{sk}(hX))$$

i.e. $\pi_0^{\mathbf{Seg}(\mathcal{M})}$ gives us the paths components of the maximal groupoid in hX. It can be easily verified to be relative and preserve products, after noting the next two items.

(ii) We have a notion of Dwyer-Kan equivalence. That is, given $F : X \to Y \in \mathbf{Set}(\mathcal{M})$ we say it is a weak equivalence in $\mathbf{Seg}(\mathcal{M})$ if hF is a equivalence and $F_{(s_0,s_1)} : X(s_0, s_1) \xrightarrow{\sim} Y(f(s_0), f(s_1))$ are weak equivalences in $\mathcal{W}$.

(iii) Noting $\Delta_S \times \Delta_T \cong \Delta_{S\times T}$ a product of precategories X, Y can be given by $X\pi_1 \times Y\pi_2$ where π_i project out the factors of $\Delta_{S\times T}$. This product on $\mathbf{PreCat}(\mathcal{M})$ descents to a product on $\mathbf{Seg}(\mathcal{M})$: Indeed for $\mathcal{M}$-enriched Segal categories X and Y we observe that

$$(s_n^X : X_n(s_0, .., s_n) \to X_1(s_0, s_1) \times ... \times X_1(s_{n-1}, s_n))$$
$$\times \ (s_m^Y : Y(t_0, ..., t_m) \to Y(t_0, t_1) \times ... \times Y(t_{n-1}, t_n))$$

is (up to product coherence relations) just a Segal map of $X \times Y$ and a weak equivalence by definition of $\mathcal{W}$.

These three observation allow us to state the central theorem of the above construction of weak enrichment

Theorem 4.20. *The Segal construction on a segalic relative category*

$$(\mathcal{M}, \mathcal{W}, \pi_0) \mapsto (\mathbf{Seg}(\mathcal{M}), \mathcal{W}_{\mathbf{Seg}(\mathcal{M})}, \pi_0^{\mathbf{Seg}(\mathcal{M})})$$

is an endofunctor of the category of segalic relativ categories.

Remark 4.21. The corresponding statement for classical enrichment says that for a symmetric monoidal (closed) $\mathcal{V}$ yields a symmetric monoidal (partially closed) $\mathcal{V}$-**Cat** and takes considerably more effort to prove - although we had to restrict ourselves to "cartesian" compositional structure in the above.

Our main examples are then the following

Example 4.22. Iterating the Segal construction n times for $\mathcal{M} = \underset{\sim}{\mathbf{Set}}$ gives a notion of (n,n)-categories $\mathbf{Cat}_n = \mathbf{Seg}(...\mathbf{Seg}(\mathcal{M}))$ (we discussed the case of bicategories in the introduction to this section). $\mathbf{Cat}_n$ are called ***Tamsamani*** n***-categories***.

Example 4.23. Iterating the Segal construction n times for $\mathcal{M} = \mathbf{sSet}$ gives a notion of (∞, n)-categories $\mathbf{Seg}_n = \mathbf{Seg}(...\mathbf{Seg}(\mathcal{M}))$. They are called ***Segal*** n***-categories***.

Observing that **sSet** has coproducts (and so does $\mathbf{Seg}_n$), in analogy to Example 4.18 there is the following equivalent and concise definition

Definition 4.24. A Segal n-category is a simplicial object $X : \Delta^{\mathrm{op}} \to \mathbf{Seg}_n$ in Segal $(n-1)$-categories such that X_0 is discrete and the Segal maps s_n are weak equivalences.

Taking Segal 0-categories to be simplicial sets this essentially recovers our warm-up definition of Segal categories in the case $n = 1$.

Remark 4.25. We have an nice and expected interplay of these notions of (∞, n) and (n, n)-categories

- $N : \mathbf{Cat}_{n-1} \hookrightarrow \mathbf{Cat}_n$ by change of bases starting with $\mathbf{Set} \hookrightarrow \mathbf{Cat}$
- $C : \mathbf{Cat}_n \hookrightarrow \mathbf{Seg}_n$ by change of bases starting with $\mathbf{Set} \hookrightarrow \mathbf{sSet}$
- $T : \mathbf{Seg}_n \to \mathbf{Cat}_n$ by change of base starting with $\mathbf{sSet} \to \mathbf{Set}, X \mapsto \pi_0 |X|$ (this is segalic since geometric realisation preserves products, and so does π_0).

Definition 4.26. Given a $X \in \mathbf{Seg}_n$ we define a 0-morphism to be an object $x \in S$ (or $x \in X_0$ depending on which definition we use). A k***-morphism*** of $X \in \mathbf{Seg}_n$ is a $(k-1)$-morphism of $X(a,b)$ for some 0-morphisms a, b.

4.2.2. *Cobordism hypothesis.* We first (as promised in section 3.3.3) will generalize the notion of dualizibility to (∞, n)-categories in a straight forward manner.

Definition 4.27. Given $X \in \mathbf{Seg}_n$ we define the ***homotopy 2-category*** $h_2 X$ as the category obtained from X by a change of basis as follows

$$(\pi_0^{\mathbf{Seg}_{n-2}})_* : \mathbf{Seg}_n = \mathbf{Seg}(\mathbf{Seg}(\mathbf{Seg}_{n-2})) \to \mathbf{Seg}(\mathbf{Seg}(\mathbf{Set})) = \mathbf{Cat}_2$$

We already spoke about horizontal composition of $X \in \mathbf{Cat}_2$ depending on a choice of categorical inverse to $X(a,b) \times X(b,c) \xleftarrow{\sim} X(a,b,c)$, and it can thus be regarded as well-defined up to isomorphism. Our definitions of left and right duals which we gave for bicategories then directly carry over to the Tamsamani 2-category $X \in \mathbf{Cat}_2$. And we can generalize this to the following definition

Definition 4.28. The 0th level homotopy 2-category of a Segal n-category X is $h_2^{(0)} X = h_2 X$. The kth level homotopy 2-category $h_2^{(k)} X$ is the $(k-1)$th level homotopy 2-category of the disjoint union of all the hom objects. Explicitely

$$h_2^{(k)} X = h_2^{(k-1)} \left(\bigsqcup_{a,b} X(a,b) \right)$$

A $(k+1)$-morphism f of X is then said to be ***left dual*** if it is left dual in the kth level 2-category of X (and similarly for ***right duals***). A $k+1$-morphisms is said to be dualizible if it has both left and right dual. Analogously, we can use kth level bicategories to define when a $(k+1)$-morphism is a ***weak equivalence*** of X. In analogy to Definition 3.9 we then say that X is a ∞***-groupoid*** if all morphisms are weak equivalences.

We can extend extend our discussion to 0-morphisms by introducing symmetric monoidal structures as usual (cf. (4.11)), i.e. via a functor

$$\mathcal{V} : \mathcal{F}in_* \to \mathbf{Seg}_n$$

such that the Segal maps are weak equivalences

$$s_n : \mathcal{V}(\langle n \rangle_*) \xrightarrow{\sim} \mathcal{V}(\langle 1 \rangle_*) \times ... \times \mathcal{V}(\langle 1 \rangle_*)$$

Comparing to our first definition of Segal n-categories, this effectively describes a 1-object $\mathbf{Seg}_{n+1}$ category with symmetric composition of 1-morphisms as one would expect. (Note there is a canonical functor $U : \Delta^{\text{op}} \to \mathcal{F}in_*$. Precomposing with this functor amounts to forgetting the symmetric structure.) With this at hand we further define

Definition 4.29. A symmetric monoidal $\mathcal{C} \in \mathbf{Seg}_n$ is l***-fully dualizable*** if all its k-morphisms for $0 \leq k \leq l$ are dualizable. If $l = n$ we say it is fully dualizable. Every $\mathcal{C}$ has a maximal fully dualizable subcategory $\mathcal{C}^{\text{f.d.}} \subset \mathcal{C}$ having as k-morphisms precisely the dualizable morphisms of $\mathcal{C}$. We note that every weak equivalence as defined in Definition 4.28 is dualizable - a short proof of this fact is provided in the Appendix 6.1. The maximal subcategory of weak equivalences is denoted by $\text{Core}(\mathcal{C})$ and called the ***core of*** $\mathcal{C}$.

Finally we note every functor $F : \mathcal{C} \to \mathcal{D}$ maps k-morphism f, $k \geq 1$ that are dualizable (resp. weak equivalences) to dualizable k-morphisms (resp. weak equivalences). This is because it induces a functor on the kth level homotopy 2-category, translating witness data of dualizibility (resp. weak equivalence) of f to the data for $F(f)$ accordingly. If F is monoidal it also preserves dualizability of 0-morphisms. Equipped with these notions we define

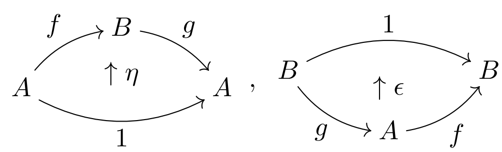
Definition 4.30. The category $\mathbf{Bord}_n^{\text{fr}}$ is defined as the free symmetric monoidal fully dualizable (∞, n)-category generated by a single object, that is for every symmetric monoidal category (∞, n)-category $\mathcal{C}$ we have

(4.31)

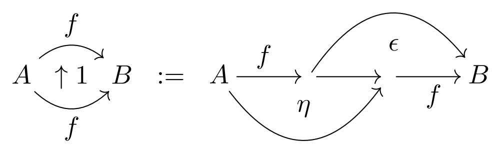
$$\text{Fun}^{\otimes}(\mathbf{Bord}_n^{\text{fr}}, \mathcal{C}) \simeq \text{Fun}^{\otimes}(\mathbf{Bord}_n^{\text{fr}}, \mathcal{C}^{\text{f.d.}}) \simeq \text{Fun}(*, \text{Core}(\mathcal{C}^{\text{f.d.}})) \simeq \text{Core}(\mathcal{C}^{\text{f.d.}})$$

To understand this definition (in particular the *second equivalence* in the above chain of equivalences) we will consider the symmetric monoidal 1-categorical case, which as usual we will regard as 1-object 2-categorical case, thus speaking about 1-cells instead of objects in the following discussion. The central observation is that duals are unique up to weak equivalence, or up to isomorphism in the corresponding homotopy 2-category.

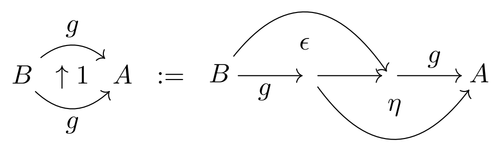
Fix a 1-cell $f : A \to B \in \mathcal{C}$. A dualizibility witness for f consists of a triple (g, η, ϵ) such that



satisfying the equations (3.32), which we now write out in 2-cells: The first triangle identity reads

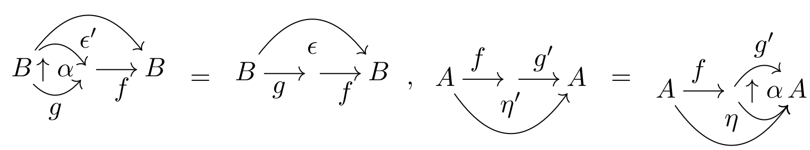


and the second states

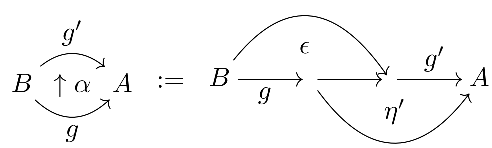


These witnesses are organized in a category of ***dualizibility data associated to*** f such that a morphism $\alpha : (g, \eta, \epsilon) \to (g', \eta', \epsilon')$ of two sets of dualizibility data is a natural transformation $\alpha : g \to g'$ such that

(4.32)

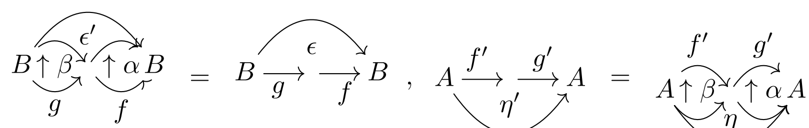


It is easy to see that $\alpha : (g, \eta, \epsilon) \to (g, \eta, \epsilon) \Rightarrow \alpha = 1$: Indeed we can just any of the triangle identities above to cancel out one of the relations in (4.32). On the other hand, for every given two sets of data we have a morphism $\alpha : (g, \eta, \epsilon) \to (g', \eta', \epsilon')$ between them given by



Similarly $\alpha^{-1} : (g', \eta', \epsilon') \to (g, \eta, \epsilon)$, which is necessarily the inverse of α as there is no non-trivial morphism from (g, η, ϵ) to itself. We conclude that the category of dualizability data associated to f is contractible.

We now consider the analogous ***category of all dualizibility data*** $\mathcal{D}_{AB}$ consisting of 4-tuples $(f : A \to B, g, \eta, \epsilon)$ exhibiting f and g as duals, with morphisms being tuples $(\alpha, \beta) : (f, g, \eta, \epsilon) \to (f', g', \eta', \epsilon')$ such that



This category comes naturally equipped with a forgetful 1-functor $U : \mathcal{D}_{AB} \to \mathcal{C}(A,B)$ mapping $(f,g,\eta,\epsilon) \mapsto f$. From the triangle equalities we once more derive that all such α, β are actually isomorphisms. It followes for instance that $(\alpha,\beta) = (\alpha,1) \circ (1,\beta)$ since the latter two provide valid morphisms by α, β being iso. It also follows that U factors as $U' : \mathcal{D}_{AB} \to \mathrm{Core}(\mathcal{C}(A,B))$. Since every object $A \in \mathcal{D}_{AB}$ is a witness of dualizibility for $U'A$, this functor further factors over the full subcategory of dualizible objects in $\mathcal{C}(A,B)$: $\hat{U} : \mathcal{D}_{AB} \to \mathrm{Core}(\mathcal{C}(A,B)^{\text{f.d.}}$. Clearly $\hat{U}$ is essentially surjective. It is also fully faithful: for every isomorphism $\alpha : f \to f'$, and for every pair of data $(f,g,\eta,\epsilon), (f',g',\eta',\epsilon')$ we can find a *unique* β (in the category of dualizibility data associated to f') such that $(1,\beta) \circ (\alpha,1) = (\alpha,\beta) : (f,g,\eta,\epsilon) \to (f',g',\eta',\epsilon')$ has image α under $\hat{U}$. Thus

Corollary 4.33. *We have an equivalence* $\hat{U} : \mathcal{D}_{AB} \simeq \mathrm{Core}(\mathcal{C}(A,B)^{\text{f.d.}})$ *of the category of dualizibility data (of morphisms between* A *and* B*) and the core of* $\mathcal{C}(A,B)^{\text{f.d.}}$.

Letting $\mathcal{C}$ be a 1-object bicategory, we regard $\mathcal{C}(A,A) = \mathcal{V}$ as a monoidal category. If the morphisms in $\mathcal{C}(A,A)$ commute up to isomorphism then $\mathcal{V}$ is symmetric, left and right dualizibility witnesses will coincide, but our discussion above remains still valid: The category of dualizibility data $\mathcal{D}_{\mathcal{V}} := \mathcal{D}_{AA}$ is equivalent to $\mathrm{Core}(\mathcal{V}^{\text{f.d.}}) = \mathrm{Core}(\mathcal{C}(A,A)^{\text{f.d.}})$.

On the other hand a functor F from the free fully dualizable symmetric monoidal category on $*$ into $\mathcal{V}$ is equivalent to picking a dataset in $\mathcal{D}_{\mathcal{V}}$: This will be the image of $*$ and freeness morally assures that the rest of the functor data is determined up to isomorphism as $\mathbf{Bord}_1^{\text{fr}}$ will be obtained from $*$ by free adjoining duals (no need for left and right by symmetry), units, counits, and tensors of these objects by the monoidal product. In fact, $\mathbf{Bord}_1^{\text{fr}}$, truncating higher morphisms for the present 1-categorical discussion, has a very simple presentation: It is the free symmetric monoidal category on

- two objects $* = +$ and $-$
- a counit morphism $\epsilon : (+ \otimes -) \to I$, and a unit morphism $\eta : I \to (- \otimes +)$
- with the only relations being the triangle identities

We can also see that a natural morphism of such functors F, F' yields a morphisms of datasets in $\mathcal{D}$. This motivates

$$\mathrm{Fun}^{\otimes}(\mathbf{Bord}_1^{\text{fr}}, \mathcal{V}) \simeq \mathrm{Core}(\mathcal{V}^{\text{f.d.}})$$

which is the 1-categorical analogue of (4.31).

Remark 4.34. (i) As our notation suggests $\mathbf{Bord}_n^{\text{fr}}$ is a category which has a description independent of the universal property (4.31). It is called the (∞, n)***-category of framed cobordisms***:

We first define the notion of framing and framed diffeomorphisms: We choose an ambient copy of $\mathbb{R}^n$ (and with it a canonical complete flag) classified by $n\epsilon \equiv \epsilon \oplus ... \oplus \epsilon : * \to BO(n)$, which was defined in section 1.2.4. We take $k\epsilon$ as the basepoint of $BO(k)$, $k \leq n$. Assume a compact m-manifold M and a m'-manifold M' with tangent bundles classified by $F : M \to BO(m)$ and $F' : M' \to BO(m')$ respectively. Using the map (1.102), we say M has a

n-framing if

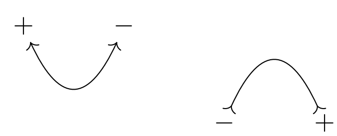
$$M \xrightarrow{F} BO(m) \xrightarrow{-\oplus(n-m)\epsilon} BO(n)$$

is null-homotopic, say via a homotopy $H : M \times I \to BO(n)$ with $H(m,1) = n\epsilon$. Assume M and M' have such trivialisations H and H' respectively. Let $m = m'$. Then an n-framed diffeomorphism $\phi : M \to M'$ is a diffeomorphism ϕ such that $h : H \simeq H'\phi$ with the conditions that $h(m,1,t) = n\epsilon$ and that $h(m,0,t)$ factors through $BO(m) \to BO(n)$ at all times t (this makes sure F and F' are actually homotopic via $h(m,0,-)$, i.e. their classified m-bundles are bundle isomorphic).

Example 4.35 ($\mathbb{R}$-framed point)*.* Take $M = M' = *$ and $n = 1$. Then a trivialisation is a null-homotopy of the map $\epsilon : M \to BO(0) = * \to BO(1) = \mathbb{R}P^\infty$, and thus a map based map $S^1 \to \mathbb{R}P^\infty$. Noting that $\pi_1 BO(1) = \mathbb{Z}_2$ we see that up to 1-framed diffeomorphism there are two ways to trivialize a point: We denoted them by $+$ and $-$ in the above.

We now roughly explain what $\mathbf{Bord}_n^{\mathrm{fr}}$ as a (∞,n)-category consists of: A k-morphism in $\mathbf{Bord}_n^{\mathrm{fr}}$ for $k \leq n$ is a n-framed compact k-manifold N with corners such that each boundary segment M_i has a co-orientation (i.e. a orientation of the normal 1-bundle) - we chose an inwards co-orientation to mean that M_i is *incoming*, while an outwards co-orientation means that M_i is outgoing. Note that the n-framing of N implies its boundary segments are n-framed as well. In particular M_i is a $(k-1)$-morphism by the definition we just gave. The monoidal tensor product on $\mathbf{Bord}_n^{\mathrm{fr}}$ is given by disjoint union, the monoidal unit is thus the empty 0-manifold. Domain and codomain of N are then just the disjoint union of incoming and respectively outgoing boundary segments. Vertical and horizontal composition are given by appropriate gluing operations.

Example 4.36*.* Coming back to our example of $\mathbf{Bord}_1^{\mathrm{fr}}$ we can now give a graphic representation of the generating morphisms ev and coev



and similarly for the relations they satisfy.

(ii) We would still need to verify that our description can be actually pressed into the definition of a Segal n-category that we previously gave. In his original paper [Lur09b] Lurie uses a different model for (∞,n) categories: n-fold ***Complete Segal spaces*** (CSS). n-fold CSS are encoding (∞,n)-categories in n-fold simplicial sets X, so they are similar to Segal n-categories. Indeed, the Segal condition on the Segal maps to be weak equivalences is still in place. But instead of requiring X_0 to be discrete we firstly require a CSS X to be a fibrant object in the so-called Reedy model structure. Then X_0 is required to be equivalent to the core of X, which is known as *completeness condition* (in particular X_0 as a $(n-1)$-fold CSS presents a ∞-groupoid). With these additional conditions, n-fold CSS can be regarded as certain internal category objects in $(n-1)$-fold CSS. This is in contrast to the idea of inductive weak enrichment underlying Segal n-categories, which required us to start from a discrete object set.

(iii) In the above we exhibited a simple presentation of $\mathbf{Bord}_n^{\mathrm{fr}}$. It is thought that higher cobordism categories admit such presentations as well, though finding them involves techniques of higher Morse theory. For the case of dimension two see [SP09].

4.2.3. *Axiomatisation.* In this section we will finally briefly describe, but by far not rigorously define, an axiomatisation of the homotopy theory of (∞, n)-categories and present some of it's very useful consequences. We have seen a couple of different models for the (homotopy) theory of $(\infty, 1)$ categories in this essay. Namely:

- Simplicially enriched categories $\mathbf{Cat}_\Delta$ with Bergner model structure
- Simplicial sets **sSet** with Joyal model structure
- Segal 1-categories $\mathbf{Seg}_1$

Other models include for instance categories with weak equivalences (or even weaker: relative categories) and complete Segal spaces. All these come equipped with a notion of weak equivalence and it should thus be possible to interpret all of these models as themselves living in one of these models. Actually, they all come with model structures so we could just regard all of them living in **Cat** and compare them via Quillen equivalences. This has indeed been done.

But firstly, all equivalences had to be established by hand and this means there is a *monodromy problem*: The Quillen equivalences don't necessarily commute. This would be solved if we would know the homotopy automorphism groups of each of these models.

Secondly, now passing to models for (∞, n)-categories, we see that it might be equally hard to first of all construct such an equivalence of theories. This on the other hand would be solved if we had a "universal property" of the theories that would guarantee us the *existence of equivalences* between any two theories satisfying this property.

In their paper [BSP11] solved both problems: Working within quasicategories, they give an axiomatisation of (presentable) theories of (∞, n)-categories such that firstly, purely from the axioms one can derive that the homotopy automorphism group of $\mathbf{Cat}_{(\infty,n)}$ is $(\mathbb{Z}/2)^n$ - this is corresponds precisely to levelwise inversion on n levels. Secondly, any (combinatorial) model categories representing such $\mathbf{Cat}_{(\infty,n)}$ theories are related by a zig zag of Quillen equivalences. The axiomatisation consists of the following four roughly sketched axioms

(A1) Working in quasicategories, consider our theory of $(\infty, 0)$ categories S defined in Example 3.5. We stated that $\mathcal{S}$ was the free homotopy colimit completion of a point: In other words the homotopy[18] Kan extension of the inclusion

[18] $\mathcal{S}$ being a quasicategory in our notation, we could have left out the qualifier "homotopy" for both colimits and Kan extensions, implicitely referring to their quasicategorical analogues as we previously did when talking about (co)limits. But in principle the above statements should hold in other models of $(\infty$,1)-categories though Barwick and Pries explicitely choose to work with quasicategories.

$i : * \hookrightarrow S$ along itself is equivalent to the identity

$$\begin{array}{ccc} & S & \\ {}^{i}\nearrow & & \searrow^{1} \\ * & \xrightarrow{\quad i \quad} & S \end{array}$$

Our first axiom is to require that such a set Υ_n of ***strong generators*** exists for $\mathbf{Cat}_{(\infty,n)}$ as well:

$$\begin{array}{ccc} & \mathbf{Cat}_{(\infty,n)} & \\ {}^{i}\nearrow & & \searrow^{1} \\ \Upsilon_n & \xrightarrow{\qquad i \qquad} & \mathbf{Cat}_{(\infty,n)} \end{array}$$

The next axiom will describe closer the nature of Υ_n.

(A2) We require $\mathbf{Cat}_{(\infty,n)}$ to contain a distinguished set of objects $\{ C_n \}$ called cells. We also define ***gaunt categories*** $\tau_{\leq 0}\mathbf{Cat}_{(\infty,n)}$ to be those categories which are homotopically discrete: X is gaunt if for all Y $\mathrm{Map}(X,Y)$ is homotopically discrete (in the models presented in this essay this just means that up to an equivalence of categories the only k-isomorphisms are identities). Then Υ_n is the smallest subcategory of the full subcategory of $\tau_{\leq 0}\mathbf{Cat}_{(\infty,n)}$ containing C_i and closed under retracts and pullbacks. The latter conditions ensures that we have basic glueing operations on cells, but we should make sure that cells do actually behave like cells under this operations: For this reason Barwick and Schommer-Pries require a finite list (growing $\sim n^2$) of colimit equations, including equations for having "completeness maps".

(A3) The third axiom morally states that we have higher categorified relations a.k.a higher correspondences. Recall from the end of Remark 1.13 that ordinary profunctor could be regarded as objects in the slice category $\mathbf{Cat}_{/C_1}$. The third axiom requires that this category and it's higher analogues $(\mathbf{Cat}_{(\infty,n)})_{/C_i}, i \leq n$, have internal homs. By our implicit assumption that $\mathbf{Cat}_{(\infty,n)}$ is presentable this is equivalent to saying that for all $i \leq n$ the functor $X \times_{C_i} - :$ $(\mathbf{Cat}_{(\infty,n)})_{/C_i} \to \mathbf{Cat}_{(\infty,n)}$ preserves colimits. (So taking again the case $n = 0$ this just means that S has internal homs.)

(A4) The last axiom states the $\mathbf{Cat}_{(\infty,n)}$ is *minimal* with respect to all previous axioms: Every other quasicategory $\mathcal{D}$ satisfying (A1) - (A3) is a reflective localisation (cf. Remark 1.38) of $\mathbf{Cat}_{(\infty,n)}$, i.e. $\mathbf{Cat}_{(\infty,n)} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{D}$.

Again taking the example of spaces as $\mathcal{S} = \mathbf{Cat}_{(\infty,0)}$ this is just saying that (homotopy) Yoneda extensions exist

$$\begin{array}{ccc} & S & \\ {}^{i}\nearrow & & \searrow^{|-|} \; \nwarrow^{N} \\ * & \xrightarrow{\quad y \quad} & \mathcal{D} \end{array}$$

and exhibits $\mathcal{D}$ as reflective localisation of $\mathcal{S}$ whenever $\mathcal{D}$ is strongly generated by $*$.

The reader should be aware that in the formulation of the axioms above a great amount of technical detail has been ommitted. Barwick and Schommer-Pries further show that (almost) all well-known models of (∞, n) categories satisfy the above for axioms, and one can thus deduce their equivalence in the sense explained in the beginning of this section.

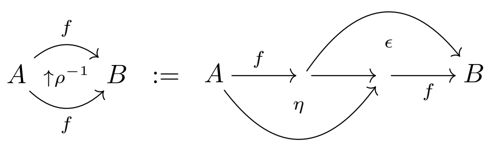
5. Conclusion

This is nothing but a hastily assembled bunch of ideas in higher category theory. But we hope that some of the ideas might be inspiring to explore further.

6. Appendix

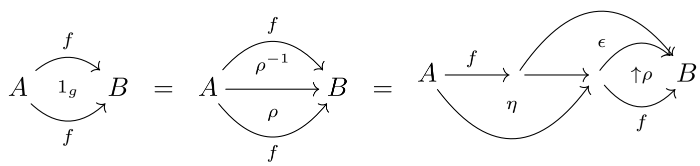
6.1. **Duals and equivalences.** *Statement.* Given 2-isomorphisms η, ϵ exhibiting $f : A \to B$ and $g : B \to A$ as equivalences there is a canonical way of exhibiting f, g as duals.

Proof. We define the *first obstruction to adjointness* ρ to be



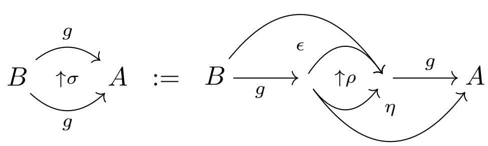
This is an isomorphism. In particular

(6.1)

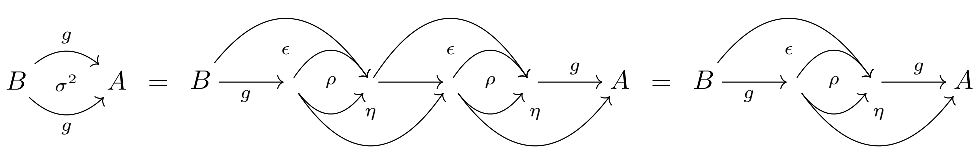


Define the *second obstruction* σ to be

(6.2)



This is clearly idempotent (using (6.1) in the second step)



But σ is iso, so being idempotent means it is the identity: $\sigma = \sigma(\sigma\sigma^{-1}) = \sigma^2\sigma^{-1} = \sigma\sigma^{-1} = 1$. Thus we can take (6.1) and (6.2) as the equations exhibiting f and g as duals. □

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