Table of contents

Generalized differential cohomology

0 Goal

understand that cohesive ∞-toposes admit an internal notion of generalized diff. cohom.

$\rightsquigarrow$ apply this to $Sh(Mfld, Spt)$

$\rightsquigarrow$ recover ordinary diff. cohom.

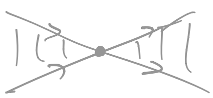
1 Generalities

1.1 Some terminology

- $x \in \mathcal{C}$ $\nearrow$ $\mathcal{C}(-,x)$: right ... / $\searrow$ $\mathcal{C}(x,-)$: left ...

  s|| groupoid

  adjoints / Kan extension / ...



(sometimes both cases are interesting, or the other one has a universal member [see adjunctions]) [see adjunctions]

- a lot of 1-category theory generalizes to $(\infty,1)$-category theory

  $\rightsquigarrow$ replace "iso" by "equivalence"

  $\rightsquigarrow$ replace "unique" by "up to contractible choice"

  $\rightsquigarrow$ replace "Set" by "Spc"

- we need some words

  - $(\infty\text{-})$ topos :
$$Sh(C) \simeq T \underset{\perp}{\overset{\text{lex}}{\leftrightarrows}} PSh^{\mathcal{V}}(C)$$

  (in correspondence with topologies on C) eg: trivial coverage $\rightsquigarrow T = PSh$

  keep base topos $\mathcal{V}$ implicit

  T inherits internal language from base topos

  - universally turn PSh into Sh "by quotient"
  - written $\mathcal{V}$ for generality
  - special instance is stable

  - presentable $(\infty\text{-})$ category :
$$P \underset{\perp}{\overset{\text{filt. rex}}{\leftrightarrows}} PSh^{\mathcal{V}}(C)$$

  small

  - good for adjointness

  accessible $\equiv$ generated by a set of objects

  presentable $\equiv$ accessible and has colimits

  e.g. $Spc = PSh(*)$

  - filtered colim :
    - generate all obj. in present. cat.

Skip, or shift to last

- commute with limits

- Stable (∞-) category :

$$\begin{tikzcd} S \arrow[r, hook] \arrow[r, phantom, shift left=2, "\perp"] & \mathrm{PSh}^{Spt}(C) \arrow[l, bend right, "\mathrm{lex}"'] \end{tikzcd}$$

note :

$\rightsquigarrow$ (co) lim being computed "pointwise" means that S inherits a lot of struct from Spt

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\ 0 \arrow[r] & X \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

$\rightsquigarrow$ Since Spt is Spt-enriched all stable ∞-categories are Spt enriched !

RECOLLECTION ON SPECTRA :

Ω = shifts htype

Σ = freely generates "higher" monoidal htype

$$\begin{tikzcd} \mathrm{Spc} \arrow[loop right, "\Sigma\Omega \dashv \Sigma"] \arrow[d, shift right, "\Sigma^\infty"'] \arrow[d, phantom, "\dashv"] \\ \mathrm{Spt} \arrow[u, shift right, "\Omega"'] \arrow[loop right, "\Sigma\Omega \dashv \Sigma"] \end{tikzcd}$$

commutes !

Check this on sets X:

$\pi_1 \Sigma X$ = free gp on X

{ "free symmetric mon struct." !

connective spectra

= E_∞ spaces

= E_∞ ∞-groupoids

= symm. mon. ∞-groupoids

keep stable homotopy hyp in mind:

e.g. abelian gps $\simeq$ stable 0-types

$\equiv$ Eilenberg-MacLane Spectra

generally

$\Pi_\infty X_{2n} \mapsfrom X$ stable n-type

(if $X_i \simeq \Omega X_i$ then must stabilize here)

(1.1.) More on sheafs

for concreteness : consider sheaves on topological spaces.

consider $f : Y \overset{inj}{\hookrightarrow} X \rightsquigarrow$

$$\begin{tikzcd} \mathrm{Sh}(Y) \arrow[r, hook, "f_*"'] \arrow[r, phantom, shift left=2, "\perp"] & \mathrm{Sh}(X) \arrow[l, bend right, "f^*"'] \end{tikzcd}$$

- recall : pushforward

"direct image"

$$\left(\mathrm{Op}(X) \xrightarrow{F} \mathrm{Set} \right) \mapsto \underbrace{\left(\mathrm{Op}(X) \xrightarrow{f^{-1}} \mathrm{Op}(Y) \xrightarrow{F} \mathrm{Set} \right)}_{f_* F}$$

- recall : pull back

"inverse image"

$$\begin{tikzcd} & \mathrm{Op}(Y) \arrow[dr, dashed, "\mathrm{Lan}_{f^{-1}} F"] & \\ \mathrm{Op}(X) \arrow[ur, "f^{-1}"] \arrow[rr, "F"'] & & \mathrm{Set} \end{tikzcd}$$

$\mathrm{Lan}_{f^{-1}} F =: f^* F$

- prepared you for the appearance of a Kan ext., but maybe lets be more concrete
- fibers are the stalks of the sheafs

Tot

Concrete interpretation : $Sh(X) \xleftarrow[\Gamma]{\sim} Et/X$

$$\rightsquigarrow \mathrm{Tot}\, f^*F = \begin{tikzcd} f^*E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "\mathrm{Tot}"] \\ X \arrow[r] & Y \end{tikzcd}$$

for general sheaf toposes totalization yields

$$F \mapsto \begin{array}{c} (Sh(\mathcal{C})/F) \\ \downarrow \pi \\ Sh(\mathcal{C}) \end{array}$$

if $f: X \to *$:

(think of bundles over $\mathcal{C}$, base topos determines fibers)

think of sheaf toposes as "generalized spaces modeled on $\mathcal{C}$" and as the "space of those spaces" i.e. a space itself.

($\rightsquigarrow$ each space X is determined by the spaces that can be modeled on it)

- recall global sections:

$$\Gamma_* : Sh(X) \longrightarrow Sh(*) \simeq Set \quad (Spc, Spt, \dots)$$

$$F \longmapsto F(X) = \mathrm{Hom}_{Sh(X)}(\underbrace{y(X)}_{\text{terminal in } Sh(X)}, F)$$

$\rightsquigarrow$ generalize:

$$\Gamma_* = \mathrm{Hom}(*,-) : Sh(\mathcal{C}) \longrightarrow Sh(*) = \text{base topos}$$

$$F \longmapsto \mathrm{Hom}(*, F) = F(*)$$

$* \equiv y(*)$

- recall constant sheaf

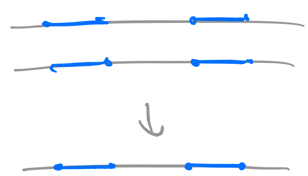
$$\Gamma^* : Set \longrightarrow Sh(X)$$

$$\begin{array}{c} E \\ s \downarrow \\ * \end{array} \longmapsto \begin{tikzcd} f^*E \arrow[r] \arrow[d, "\Gamma^*s"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\ X \arrow[r, "f"'] & * \end{tikzcd}$$

$$S \longrightarrow (U \longmapsto S^{\pi_0 U})$$

"sheafs turn colimits into limits"

$$(- \sqcup -) \longmapsto (- \times -)$$



cotensor: $\mathrm{Hom}_{\mathcal{V}}(v, w^R) \simeq \mathrm{Hom}_{\infty}(R, \mathrm{Hom}_{\mathcal{V}}(v, w))$

"easy to map into"

$\rightsquigarrow$ cotensor is not a "choice"

$\rightsquigarrow$ generalize:

$$\begin{tikzcd} \mathcal{T} = \mathrm{Sh}^{\mathcal{V}}(\mathcal{C}) \arrow[r, "\mathcal{T}_*"'] & \mathcal{V} \arrow[l, bend right, "\mathcal{T}^*"'] \arrow[l, phantom, bend right=20, "\perp"] \end{tikzcd}$$

"terminal geometric morphism"

Spc / Mfld / site of lhpy types

presentable ∞-cat cotensored over Spc

obj. w/ h. struct.

- set is a 0-category $\rightarrow$ no higher structure
  $\mathcal{C} \to \mathcal{D} = k\text{-Trunc}(\mathcal{C}) \to \mathcal{D}$
  k-trunc

$\rightsquigarrow \mathcal{T}^*: \quad \mathcal{V} \longrightarrow \mathrm{Sh}^{\mathcal{V}}(\mathcal{C})$

$\qquad S \longmapsto \left(M \longmapsto S^{\Pi_\infty(M)} \right)$

Chapter 4 has the details

- if higher structure then cotensor changes to this

2. Generalized differential cohomology

2.1 Cohesion

Cohesive ∞-topos

$$\begin{tikzcd} \mathrm{Sh}^{\mathcal{V}}(\mathcal{C}) \arrow[r, "\mathcal{T}_*"] \arrow[r, bend left=60, "\Pi_!"] & \mathcal{V} \arrow[l, bend right, "\mathcal{T}^*"'] \arrow[l, bend left=60, "\Pi^!"] \end{tikzcd}$$

$\Pi_!$ --- "geometric realization" (must preserve finite products)

also called "discrete functor": think about the embeddings in the Set case.

$\Pi^!$ --- "codiscrete functor"

. why? very similar to $\mathcal{T}^*$ but "dual" in flavour

- let's state a relevant example

example: $\mathcal{C} = \mathrm{Mfld}^{(\mathrm{sm})} \equiv \mathcal{M}$, $\mathcal{V} = \mathrm{Spc}$ (or Spt)

- codiscrete functor:

$$\Pi^! S : M \longmapsto S^{\mathrm{pt}(M)} = \prod_{\mathrm{pt}(M)} S$$

w/ counit:

$$\rightsquigarrow \mathcal{T}_* \Pi^! S = S \simeq S$$

$$\Rightarrow \Pi^! \text{ is fully faithful}$$

--- general property for

cohesive ∞-toposes

$\xRightarrow{\text{abstract non-sense}}$ Γ^* is fully faithful

- geometric realization:

  - what is a natural transformation
  - need a limit … an end!

$$\mathrm{Sh}^{\mathcal{V}}(\mathcal{M})(F, \Gamma^* S) \simeq \int_{\mathcal{M}^{op}} \mathcal{V}(FM, S^{\Pi_\infty M}) \quad \text{cotensored}$$

$$\simeq \int_{\mathcal{M}^{op}} \mathrm{Sp}(\Pi_\infty M, \mathcal{V}(FM, S)) \quad \text{tensor}$$

$$\simeq \int_{\mathcal{M}^{op}} \mathcal{V}(\Pi_\infty M \cdot FM, S)$$

$$\simeq \mathcal{V}\left(\int^{\mathcal{M}^{op}} \Pi_\infty M \cdot FM, S\right)$$

$$=: \mathcal{V}(\Gamma_! F, S)$$

compare this to:

$$\int^{\Delta} \Delta^n \cdot X(n) = |X|$$

$\Pi_\infty \Delta^n = *$

aside: define coends and ends via

$$\int^{c} - \vdash (-)^{\mathcal{C}(-,-)}$$

$$\int_{c} - \dashv \mathcal{C}(-,-) \cdot (-)$$

$$\xrightarrow{\Delta^\bullet \hookrightarrow \mathcal{M}} \Gamma_! F = \mathrm{colim}_{\Delta} F\Delta^\bullet$$

manifolds glue together from simplices and F is a sheaf

$\boxed{\text{Content of Chapter 5}}$

2.2. $\mathbb{R}$-invariance

Q: What is the image of $\Gamma^* : \mathcal{V} \hookrightarrow \mathrm{Sh}(\mathcal{M})$?

defn: $F \in \mathrm{Sh}(\mathcal{M})$ is $\mathbb{R}$-invariant if it preserves homotopy equivalences.

rmk: $\iff$ it preserves the equ. $M \times \mathbb{R} \to M$

$\rightsquigarrow$ "concordance invariance"

⇝ obtain $Sh_{\mathbb{R}}(M) \hookrightarrow Sh(M)$

⇝ check $\left(T^*X : M \mapsto X^{\Pi_\infty M}\right) \in Sh(M)$

<u>obs.</u>:

$$\begin{tikzcd} Sh_{\mathbb{R}}(M) \arrow[r, "T_*"'] \arrow[r, phantom, shift left=2, "\perp^{\simeq}\ T"] & \mathcal{V} \arrow[l, bend right, "T^*"'] \end{tikzcd}$$

(co)complete! presentable

is an adjoint equivalence!

indeed: T^* f.f. ✓

T_* conservative on $Sh_{\mathbb{R}}(M)$ ✓ (+ nonsense)

$$\left(E \simeq E' \overset{\text{sheafs}}{\Leftrightarrow} E|_{Euc} \simeq E'|_{Euc} \overset{\mathbb{R}\text{-inv}}{\Leftrightarrow} E|_* \simeq E'|_*\right)$$

<u>defn</u>: obtain <u>adjoints</u>

ℝ-invariance stable under computing limits and colimits

$$\begin{tikzcd} Sh_{pu}(M) \arrow[r, hook] & Sh(M) \arrow[r, bend left, "L_{hi}"] \arrow[r, bend right, "R_{hi}"'] & Sh_{\mathbb{R}}(M) \arrow[l, hook'] \arrow[l, phantom, shift left=3, "\perp"] \arrow[l, phantom, shift right=3, "\perp"] \end{tikzcd} \simeq \mathcal{V}$$

s.t. $\forall F$, $T_*(F) = 0$

"$F \in \ker T_*$"

$$\rightsquigarrow T_! = T_* L_{hi}$$

Say in words:

<u>rough idea</u>: assemble sheafs from their "ℝ-invariant" and their "pure" part

<u>think</u>: generalized cohomology ⫴ homotopical ⫴ differential

(e.g. diff cohom.)

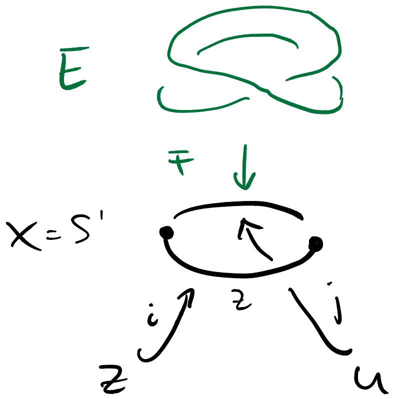
2.3 Recollements (apparently goes back to Grothendieck)

(A) microcosm / macrocosm reconstruction

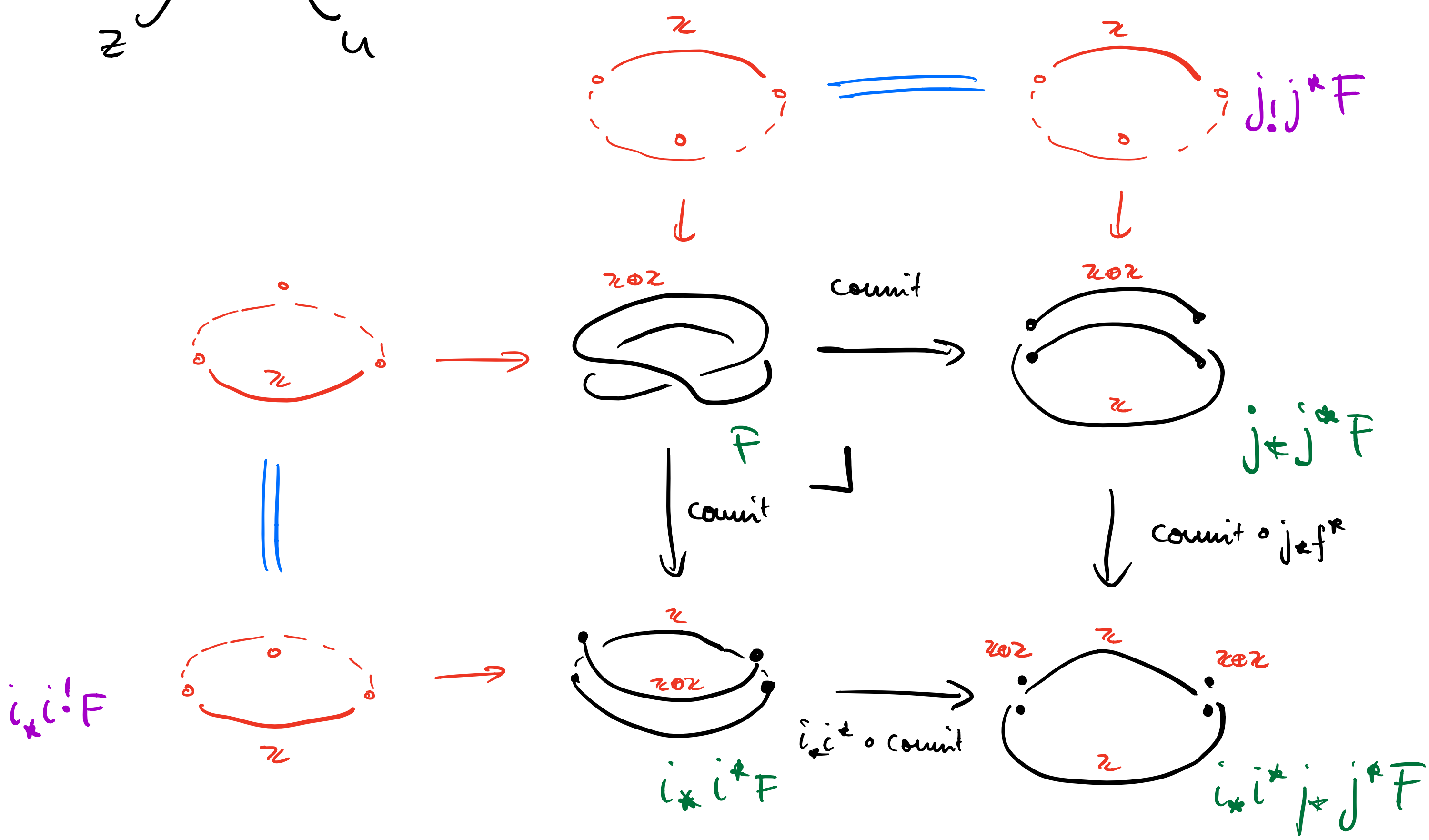
(B) ...

(B) the extended fracture square

Motivation: $Sh^{Set}(X) \longrightarrow Sh^{Ab}(X)$



$$\rightsquigarrow \quad \begin{tikzcd} Sh(U) \arrow[r, bend left=60, "j_!"] \arrow[r, "j_*"'] & Sh(X) \arrow[l, bend right, "j^*"'] \arrow[r, bend left, "i^*"] \arrow[r, bend right=60, "i^!"'] & Sh(Z) \arrow[l, "i_*"] \end{tikzcd}$$



defn: <u>recollement</u> is diag. of ∞-cat

$$\begin{tikzcd} U \arrow[r, "j_*"'] & X \arrow[l, bend right, "j^*"'] \arrow[r, bend left, "i^*"] & Z \arrow[l, "i_*"] \end{tikzcd}$$

in geometric morphism left adjoints are always exact. (in this case follows since limits are computed pointwise and limits commute w/ limits)

s.t.
- j_*, i_* are ff.
- j^*, i^* are left exact adjoints
- $j^* i_*$ is terminal
- i^*, j^* jointly conservative

See example

above

thm.:

$$\begin{tikzcd}
\mathrm{id}_X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & j_*j^* \arrow[d] \\
i_*i^* \arrow[r] & i_*i^*j_*j^*
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
Sh(X) \arrow[r] \arrow[d] & Sh(U) \arrow[d, "i^*j_*"] \\
Sh(Z)^I \arrow[r, "ev_1"'] & Sh(Z)
\end{tikzcd}$$

"microcosm" reconstruction

"macrocosm" reconstruction

stable htpy 0-types = abelian gps
(truncation cause ker ≠ coker)

obs: if X is stable there exists further adjoints

$$\begin{tikzcd}
 & j_!j^* \arrow[r, equal] \arrow[d] & j_!j^* \arrow[d] \\
i_*i^! \arrow[r] \arrow[d, equal] & \mathrm{id}_X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & j_*j^* \arrow[d] \\
i_*i^! \arrow[r] & i_*i^* \arrow[r] & i_*i^*j_*j^*
\end{tikzcd}$$

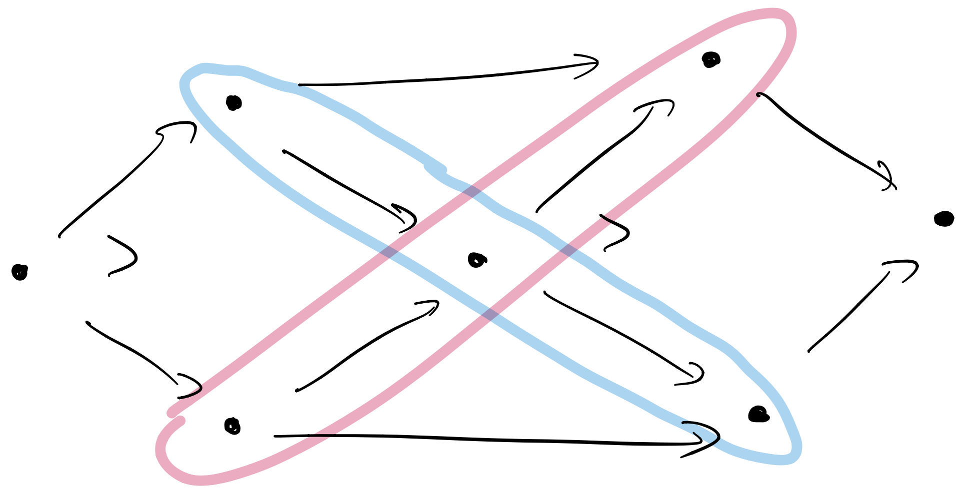
"the extended fracture square"

2.4 The hexagons, abstract and differential

For $E \in X$, the above rearranges as

$$\begin{tikzcd}
\Omega i_*i^*j_*j^*E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & i_*i^!E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\
{} & {} & {}
\end{tikzcd}$$

$$\begin{tikzcd}
j_!j^*E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & j_*j^*E \arrow[d] \\
0 \arrow[r] & i_*i^*E \arrow[r] & i_*i^*j_*j^*E
\end{tikzcd}$$

$\rightsquigarrow$ 

next: differential case $X = Sh^{Spt}(\mathcal{U})$ $\underline{\text{abstract}}$ hexagon ✓

<u>defn</u>: Def = diff. deform.
Cyc = diff. cycles
s.t.

$$\begin{tikzcd}[column sep=huge]
Sh_{pu}(\mathcal{U}) \arrow[r, "j_*"'] \arrow[r, bend left=60, dashed, "\mathrm{Def} = j_!"] & Sh(\mathcal{U}) \arrow[l, bend right=30, dashed, "\mathrm{Cyc} = j^*"'] \arrow[r, bend left, "L_{hi} = i^*"] \arrow[r, bend right, "R_{hi} = i_!"'] & Sh_{\mathbb{R}}(\mathcal{U}) \arrow[l, "i_*"]
\end{tikzcd}$$

is recollement.

must have:

$$j_*j^* = \mathrm{cofib}(i_*i_! \longrightarrow \mathrm{id})$$

$$\rightsquigarrow \mathrm{Cyc} := \mathrm{cofib}(R_{hi} \longrightarrow \mathrm{id})$$

that's where we need it

constr: for each $E \in Sh^{Spt}(M)$ the abstract hexagon specializes to a "differential hexagon"

③ Recovering ordinary differential cohomology

Recollection:

- in Spc:
  - abstract cohomology: $\pi_0 \mathrm{Map}(-, X)$
  - ordinary cohomology: $\pi_0 \mathrm{Map}(-, K(\mathbb{Z}, k))$

(cohesive)

- in T ∞-topos,

$$\begin{tikzcd} T \arrow[r, bend right, "\Gamma_*"'] \arrow[r, bend left=60, "|-|"] & \mathrm{Spc} \arrow[l, bend right, "\Gamma^*"'] \arrow[r, dashed, "\Sigma^\infty"] & \mathrm{Spt} \end{tikzcd}$$

monoidal fct.

  - abstract — " —
  - ordinary cohomology: $\pi_0 \mathrm{Map}(-, \Gamma^* K(\mathbb{Z}, k))$

$$= \pi_0 \mathrm{Map}(|-|, K(\mathbb{Z}, k))$$

$$= \pi_{-k} \mathrm{Map}(\Sigma^\infty |-|, H\mathbb{Z})$$

Idea: assemble differential cohomology $\hat{H\mathbb{Z}}$ from

- the $\mathbb{R}$-invariant sheaf of ordinary cohomology $\Gamma^* H\mathbb{Z}$
- the "pure" sheaf $H\Omega^{\geq k}$, $k > 0$

why?

by Poincaré lemma $\Omega^\bullet(M) \simeq \mathbb{R}[0]$ $\Rightarrow H\Omega^\bullet = \Gamma^* H\mathbb{R}$

via pullback:

$$
\begin{tikzcd}
F \equiv & \hat{H\mathbb{Z}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cyc}(H\Omega^{\geq k}) \arrow[d, "\text{counit}"] & \equiv j_* j^* F \\
i_* i^* F \equiv & T^* H\mathbb{Z} \arrow[r] & T^* T_! \mathrm{Cyc}(H\Omega^{\geq k}) & \equiv i_* i^* j_* j^* F
\end{tikzcd}
$$

$H(\mathbb{Z} \subset \mathbb{R})$ $\quad \simeq T^* H\mathbb{R}$

they call $\hat{E}$ a differential refinement of $H\mathbb{Z}$

noting:

bc pure

$k \geq 1: \ \mathrm{Cyc}(H\Omega^{\geq k}) \simeq H\Omega^{\geq k}$

$k = 0: \ \mathrm{Cyc}(H\Omega^{\geq 0}) \simeq 0 \quad (\text{since } H\Omega^{\geq 0} \simeq H\mathbb{R})$

Defn:

$$\hat{H}^k(M) = \pi_{-k} \mathrm{Map}(y(M), \hat{H\mathbb{Z}})$$
$$= \pi_{-k} \hat{H\mathbb{Z}}(M)$$

$\rightsquigarrow$ plug $\hat{H\mathbb{Z}}$ into our differential cohomology hexagon (of sheafs!)

$\rightsquigarrow$ apply to M, compute e.g.

$$T^* H\mathbb{Z}(M) \underset{\substack{\text{recall } T^*}}{=} H\mathbb{Z}^{\Pi_\infty M} = \mathrm{Map}(\Sigma^\infty \Pi_\infty M, H\mathbb{Z})$$

$\rightsquigarrow$ apply π_{-k} to recover ordinary hexagon