(very basic) Enriched Category Theory

Part III essay

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1 Introduction

Motivation. Enrichment of categories generalizes the notion of ordinary categories by abstracting the role played by the category of sets in the definition of an ordinary category. The general question is to what $\mathcal{V}$ we can abstract, or in other words, in what $\mathcal{V}$ we can enrich. The classical and best known answer for ordinary categories is a monoidal category $\mathcal{V}$. We replace Hom sets by 'Hom objects' of $\mathcal{V}$, and since a monoidal $\mathcal{V}$ has a 'tensor product' we will easily define a 'two-variable' composition map as an arrow in $\mathcal{V}$.

Contents of this Essay. The central object of study of this essay will be the 2-category $\mathcal{V}$-**CAT** of $\mathcal{V}$-enriched categories, where $\mathcal{V}$ is a symmetric monoidal closed (and later complete) category. We give the basic defintions of $\mathcal{V}$-categories, $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations in section 2 after a short introduction to monoidal categories. In section 3 we then transfer parts of our theory of **CAT** to $\mathcal{V}$-**CAT**, the 2-category of $\mathcal{V}$-categories, such as showing $\mathcal{V}$-**CAT** to have symmetric monoidal structure, describing $\mathcal{V}$ itself as a $\mathcal{V}$-category and defining Hom $\mathcal{V}$-functors. For the latter, through the notion of closedness we will first internalize Hom sets to Hom objects in $\mathcal{V}$ and then lift canonical ordinary functors to $\mathcal{V}$-functors. In section 4 we first define (and explain) the concept of extraordinary $\mathcal{V}$-naturality and apply it to canonical families of maps previously defined. Equipped with this tool we can prove a first (weak) Yoneda Lemma with ease, and this version will be enough to recover important parts of the theory of adjunctions from **CAT** in $\mathcal{V}$-**CAT**. To strengthen this Yoneda lemma we will proceed to internalize ordinary $\mathcal{V}$-functor categories to $\mathcal{V}$-categories in $\mathcal{V}$-**CAT** by explicit construction using ends in section 5. We will not only gain (partial) closedness of $\mathcal{V}$-**CAT** in further analogy to **CAT** but will then also be able to state our stronger Yoneda lemma and various consequences. Finally, an outlook is appended to this essay which hints towards a generalized theory of enrichment.

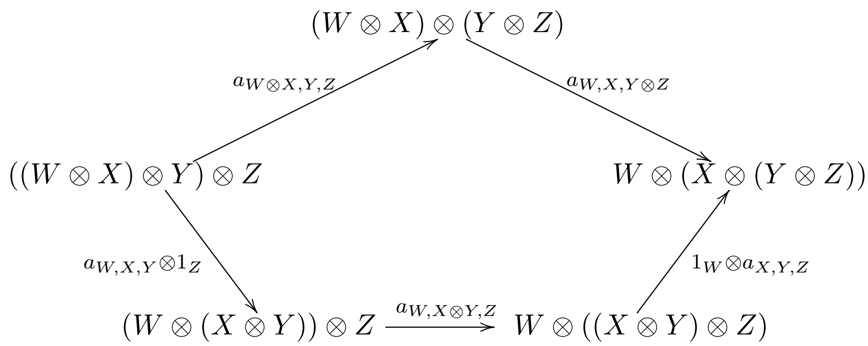
Notation. We will basically follow the notation of [2] but sometimes make it more complicated, in order to possibly achieve more clarity about the objects at interest. Also in many easy but larger diagrams we will not write out all the objects if there is only one reasonable way to fill them in. We sometimes denote partial functors by T_{A-} where the hyphen indicates the free slot. A justification for this notation will be given in section 4. The view on the difference between $\mathcal{V}$-**Cat** and $\mathcal{V}$-**CAT** taken in this essay is explained in section 5.4 but does not have considerable impact on the rest of this work.

Development of the Subject. First papers developing the idea of enrichment in monoidal categories were written in the 1960's by various authors including Eilenberg, Maclane and Kelly. The book [2] by Kelly written in it's original version 1982 was the first 'connected account' on the topic, and summarized more than 15 years of developement.

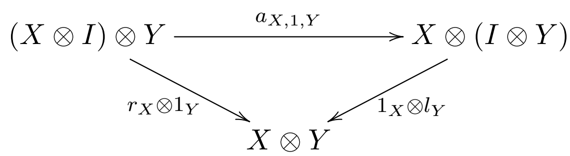
Acknowledgments. I would like to thank Julia for supervising me and suggesting the great book by Tom Leinster [5], although in the end I could not include much of it in this essay. The content and structure of content strongly follows the first two Chapters of [2] (but less so the style of presentation, in particular we do present some of the tedious verifications). Examples and proofs from [3], [4], [7] were included when it was suitable, and the nLab has proven to be a valuable reference as well. The outlook is based on [5] and [6].

2 Monoidal Categories

Definition 2.1. A monoidal category consists of a six-tuple $(\mathcal{V}_0, \otimes, a, l, r, I)$, where $\mathcal{V}_0$ is an ordinary category containing the object I, $\otimes$ is a bifunctor $\mathcal{V}_0 \times \mathcal{V}_0 \longrightarrow \mathcal{V}_0$ called the tensor map, and a, l, r are natural isomorphisms as follows: $a\colon ((- \otimes -) \otimes -) \longrightarrow (- \otimes (- \otimes -))$, $l\colon (- \otimes I) \longrightarrow 1_{\mathcal{V}}$ and $r\colon (I \otimes -) \longrightarrow 1_{\mathcal{V}}$ satisfying the following coherence axioms:



(V1)



(V2)

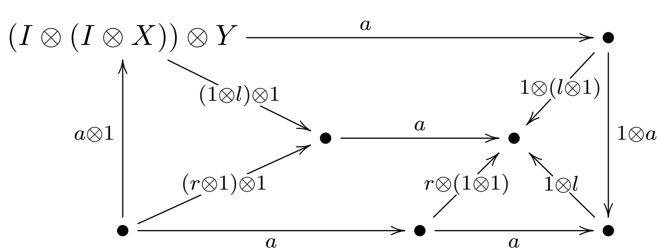
This axioms are of course not chosen randomly, but such that all possible diagrams formed of $a, l, r, 1$ via tensoring commute. This is formalized in a coherence theorem for monoidal categories. As an example we quickly derive commutativity of two small diagrams in the next claim. These two examples also show that it will be tedious to handle a, l, r without a general theorem of coherence.

Claim 2.2 (consequences of coherence axioms). *The coherence axioms imply that*

a) $l \otimes 1 = la : (I \otimes X) \otimes Y \longrightarrow X \otimes Y$ *and* $(1 \otimes r)a = r : (X \otimes Y) \otimes I \longrightarrow X \otimes Y$

b) $l = r : I \otimes I \longrightarrow I$

Proof.



(1)

The outer square commutes by (V1). All inner diagrams except the right most triangle commute by (V2) and naturality. We thus deduce commutativity of the right most triangle; i.e. $1 \otimes (1 \otimes l) = 1 \otimes la$. After composing it with l and using naturality of l, it is equivalent to the first statement of Part *a)*. The second statement follows similarly when setting the two objects on the right equal to I.

Part *b)* follows after noting $l_I = 1 \otimes l_I : I \otimes (I \otimes I) \longrightarrow I$ as a consequence of naturality of l. Thus we have $rlar^{-1} = r(1 \otimes l)ar^{-1} : I \otimes I \longrightarrow I$. This yields the statement after substituting $(1 \otimes l)a = r \otimes 1$ by (V2) on the right hand side, part *a)* on the left hand side, and commuting r to the right via it's naturality. $\square$

We give a rough *sketch* of a proof of coherence from the above claim: Part *a)* does already provide us with an important tool, namely the possibility of commuting a with instances of l, r. And after having commuted all instances of l, r to the right, we see by Part *b)* that we don't have to worry about differences in instances of l, r in places where they can occur interchangebly (i.e. on $I \otimes I$). Thus (argued sketchily) it remains to prove coherence for instances of a and 1 alone. Using this reduction argument a full proof, based on the construction of a free weak monoidal category on one generator, can be found in [3] (however, without the explicit proof of the above claim and actually *assuming* Part *b)*).

Handling 'almost identities' seems difficult. Do we need a, l, r at all? If a, l, r are identities then we recover the notion of a strict monoidal category. Now it is indeed possible to show that every (weak) monoidal category is equivalent to *some* strict monoidal category and this actually does provide us with an alternative proof of coherence (see [5] for a sketch). But still, the notion of a monoidal category is *non-trivial* in that we cannot just regard $\mathcal{V}_0$ as a strict monoidal category by quotienting out isomorphic objects. In Example 2.3 f) below we construct a non-strict monoidal category where all isomorphic objects are equal.

Finally, we can ask (the technical question) why we don't require full coherence in the definition of monoidal category from the beginning. In a more general setting, that is in higher categorical structures, this indeed seems to be the way to go (cf. [5]). But technically, it is easier to check two diagrams than all possible diagrams.

After these theoretic considerations, which show that monoidal categories as defined above are actually interesting, let us give examples of monoidal categories:

Examples 2.3. (**examples of monoidal categories**)

a) An example of a strict monoidal category is the category of endofunctors $[\mathcal{C}, \mathcal{C}]$ of an ordinary category $\mathcal{C}$. Tensoring is composition, I is the identity functor and a, l, r clearly become identities in this case.

b) A category with all finite products and a terminal object I is monoidal: Fix a product $A \times B$ with projections maps π_1, π_2 for each pair of objects A, B and make it the value of our tensor map. $f \times g$ is then uniquely induced via $\pi_1(f \times g) = f\pi_1$ and $\pi_2(f \times g) = g\pi_2$. Universality of the product makes $\times$ a functor and induces the maps a, l, r. The coherence axioms (V1) and (V2) also follow from universality (i.e. uniqueness of factorizing morphisms). We call such categories *cartesian*. This example includes for instance **Set**, **Top** and **Cat**.

c) Categories with an 'actual' tensor product, e.g. **Ab** and **R-Mod**: The case is much like the previous example, because we need to use the universal property (by which $\otimes$ is defined) to derive the monoidal structure.

d) Posets $\mathbf{2} = \{0, 1\}$ and $\overline{\mathbb{R}}_+$: In the category **2** we write $1 = true$, $0 = false$, $\leq \; = \; \vdash$. It becomes monoidal when we set $\otimes = \wedge$, $I = 1$. In particular a, l, r are identities. For the poset $\overline{\mathbb{R}}_+$ (reversely ordered positive reals with ∞) we set $I = 0$, $\otimes = +$ and see again a, l, r to be identities. (We will reuse these examples, but their potential power was developed by Lawvere in [7].)

e) For a topological space X, we get a monoidal category (X, x) with loops as objects and homotopy classes of loop homotopies as morphisms. $\otimes$ is concatenation of loops.

f) Take a skeleton $\mathbf{Set}_0 \subset \mathbf{Set}$ (i.e. the full subcategory of a collection of representatives of isomorphism classes in $\mathbf{Set}$). This has products and is monoidal. All isomorphic objects are equal. We show it is **non-strict**: Take *the* countably infinite set $N \in \mathbf{Set}$. Then N is stable under products $N \times N = N$. Assume all a are identities so that $(f \times g) \times h = f \times (g \times h) : N \times N \longrightarrow N$ for all $f, g, h : N \longrightarrow N$. But then applying $\pi_{i=1,2}$ to this equality yields $f\pi_1 = (f \times g)\pi_1$, $h\pi_2 = (g \times h)\pi_2$. Projections are epi in $\mathbf{Set}$. So we deduce $f = f \times g = g$ for all f, g which is not true.

We define the **underlying set functor** V as the representable functor $\mathcal{V}(I, -)$. Applying this to our above examples we see that it indeed gives us *a notion* of elements of objects of $\mathcal{V}$ (in particular for the examples of the usual cartesian categories and categories of models of algebraic theories with tensor products). But it is definitely *not* always faithful (e.g. for $\mathcal{V} = \mathbf{Cat}$). We will write $f \in X$ for $f : I \longrightarrow X$ but keep in mind that X is an 'object of something' and not in general a set.

2.1 $\mathcal{V}$-enriched categories

The third example from the above revives the spirit of our initial motivation: In monoidal categories we can express 'multivariable operations' (e.g. a binary composition) as actual morphisms in $\mathcal{V}$. We now give the central definition of

Definition 2.4 (**$\mathcal{V}$-categories**). A $\mathcal{V}$-category $\mathcal{A}$ consists of a set of objects ob$\mathcal{A}$, Hom objects $\mathcal{A}(A, B) \in \mathcal{V}$ for each pair of objects A, B, a multiplication law

$$M = M^{\mathcal{A}}_{ABC} : \mathcal{A}(B, C) \otimes \mathcal{A}(A, B) \longrightarrow \mathcal{A}(A, C)$$

and an identity element $j = j^{\mathcal{A}}_A : I \longrightarrow \mathcal{A}(A, A)$ satisfying the following axioms:

$$\begin{array}{ccccc}
(\mathcal{A}(C,D) \otimes \mathcal{A}(B,C)) \otimes \mathcal{A}(A,B) & & \xrightarrow{\quad a \quad} & & \mathcal{A}(C,D) \otimes (\mathcal{A}(B,C) \otimes \mathcal{A}(A,B)) \\
\downarrow{\scriptstyle M\otimes 1} & & & & \downarrow{\scriptstyle 1\otimes M} \\
\mathcal{A}(B,D) \otimes \mathcal{A}(A,B) & \xrightarrow[M]{} & \mathcal{A}(A,D) & \xleftarrow[M]{} & \mathcal{A}(C,D) \otimes \mathcal{A}(A,C)
\end{array} \qquad \text{(M1)}$$

$$\begin{array}{ccccc}
\mathcal{A}(B,B) \otimes \mathcal{A}(A,B) & \xrightarrow{M} & \mathcal{A}(A,B) & \xleftarrow{M} & \mathcal{A}(A,B) \otimes \mathcal{A}(A,A) \\
\uparrow{\scriptstyle j_B\otimes 1} & \nearrow{\scriptstyle l} & & \nwarrow{\scriptstyle r} & \uparrow{\scriptstyle 1\otimes j_A} \\
I \otimes \mathcal{A}(A,B) & & & & \mathcal{A}(A,B) \otimes I
\end{array} \qquad \text{(M2)}$$

With our notion of elements, be can now proceed to 'plug in' elements into our composition map M. We can define

Definition 2.5 (**pre- and postcomposition**). Let $f \in \mathcal{B}(B, C), g \in \mathcal{B}(D, A)$. Call

$$\mathcal{B}(A, f) : \mathcal{B}(A, B) \xrightarrow{l^{-1}} I \otimes \mathcal{B}(A, B) \xrightarrow{f\otimes 1} \mathcal{B}(B, C) \otimes \mathcal{B}(A, B) \xrightarrow{M} \mathcal{B}(A, C)$$

$$\mathcal{B}(g, B) : \mathcal{B}(A, B) \xrightarrow{r^{-1}} \mathcal{B}(A, B) \otimes I \xrightarrow{1\otimes g} \mathcal{B}(A, B) \otimes \mathcal{B}(D, A) \xrightarrow{M} \mathcal{B}(D, B)$$

post- and precompostion. Write $\mathcal{B}(1, f)$ and $\mathcal{B}(g, 1)$ if no reference to a specific A and B in a statement is needed or if they are clear from context.

(M1) and (M2) are nothing but generalizations of the associativity and identity axioms of ordinary categories. Then, by naturality and coherence of a, l, r, pre- and postcomposition obey the same laws as they do in ordinary category theory. We use this observation throughout the essay and formalize it as follows

Claim 2.6 (**pre- and postcomposition**). *Let $f \in \mathcal{B}(A,B), g \in \mathcal{B}(B,C), h \in \mathcal{B}(C,D)$. Then*

a) $\mathcal{B}(1,g)f = \mathcal{B}(f,1)g =: g.f$ *and* $\mathcal{B}(1,j_B)f = \mathcal{B}(j_A,1)f = f$

b) $\mathcal{B}(1,g)\mathcal{B}(1,f) = \mathcal{B}(1,g.f)$ *(similarly for $\mathcal{B}(g.f,1)$) and $\mathcal{B}(1,f)$, $\mathcal{B}(h,1)$ commute*

c) $M(\mathcal{B}(f,1)\otimes 1) = M(1\otimes \mathcal{B}(1,f))$

d) $M(\mathcal{B}(1,f)\otimes 1) = \mathcal{B}(1,f)M$ *(similarly for $\mathcal{B}(f,1)$)*

Proof. Let us just record to corresponding statements in an ordinary category which will make them easy to remember for further use:

$$\begin{array}{l} a)\ g\circ f \ \text{ is well def. and } \ 1\circ f = f\circ 1 = f \\ b)\ g\circ(f\circ -) = (g\circ f)\circ - \ \text{ and } \ f\circ(-\circ h) = (f\circ -)\circ h \\ c)\ (-\circ f)\circ - = -\circ(f\circ -) \quad d)\ (f\circ -)\circ - = f\circ(-\circ -) \end{array}$$

In the ordinary case these are trivial applications of the axioms of a category. The proofs in the enriched setting just follow along these lines (now based on (M1) and (M2)) by coherence and naturality. To demonstrate this let us proof the first statement:

$$\mathcal{B}(1,g)f = M(g\otimes 1)l^{-1}f = M(g\otimes f)l^{-1} = M(1\otimes f)r^{-1}grl^{-1} = \mathcal{B}(f,1)g$$

□

We can generalize axioms for ordinary functors on the same ground

Definition 2.7 (**$\mathcal{V}$-functors**). A $\mathcal{V}$-functor $T : \mathcal{A} \longrightarrow \mathcal{B}$ is defined by a map of objects $T : \mathrm{ob}\mathcal{A} \longrightarrow \mathrm{ob}\mathcal{B}$ and morphisms on Hom objects $T = T_{AB} : \mathcal{A}(A,B) \longrightarrow \mathcal{B}(TA,TB)$ in $\mathcal{V}$, such that

$$\begin{array}{ccc} \mathcal{A}(B,C)\otimes\mathcal{A}(A,B) & \xrightarrow{M^{\mathcal{A}}} & \mathcal{A}(A,C) \\ {\scriptstyle T\otimes T}\downarrow & & \downarrow{\scriptstyle T} \\ \mathcal{B}(TB,TC)\otimes\mathcal{B}(TA,TB) & \xrightarrow[M^{\mathcal{B}}]{} & \mathcal{B}(TA,TC) \end{array} \tag{VF1}$$

and

$$\begin{array}{ccc} I & \xrightarrow{j^{\mathcal{A}}} & \mathcal{A}(A,A) \\ & {\scriptstyle j^{\mathcal{B}}}\searrow & \downarrow{\scriptstyle T} \\ & & \mathcal{B}(TA,TA) \end{array} \tag{VF2}$$

We call T fully faithful (f.f.) if all T_{AB} are isomorphisms in $\mathcal{V}$.

Let us apply the definition to our previous example

Examples 2.8. (**examples of $\mathcal{V}$-functors**)

a) In the case $\mathcal{V} = \mathbf{Set}$ we recover the notion of ordinary functors (Hom objects are sets of arrows and T_{AB} are functions between them compatible with composition).

b) By enriching in $\mathcal{V} = \mathbf{Cat}$ we obtain (or define) 2-Categories. We call **Cat**-functors correspondingly 2-functors, and **Cat**-natural transformations 2-natural (see Def. 2.9). Composition becomes a functor $\mathcal{A}(B,C) \times \mathcal{A}(A,B) \longrightarrow \mathcal{A}(A,C)$. If T is a 2-functor then T_{AB} is an ordinary functor between 'Hom categories' compatible with the composition functors. A one object 2-Category $\mathcal{A}$ is a strict monoidal category when identifying $\mathcal{A}(A,A) = \mathcal{V}_0$, $M = \otimes$ (and the corresponding weak notion is obtained through bicategories: cf. Example 6.4).

c) A category enriched in **Ab** is a preadditive category (see [1]).

d) For a category d enriched in the poset $\overline{\mathbb{R}}_+$ denote the Hom objects by $d(x,y)$. The existence of a multiplication law then reads $d(y,z) + d(x,y) \geq d(x,z)$ and the existence of a unit $0 \geq d(x,x)$. Thus we recover the notion of a generalized metric space (**without** requiring: $d(x,y) = d(y,x)$, $x \neq y \Rightarrow d(x,y) \geq 0$ and $d(x,y) < \infty$). $\overline{\mathbb{R}}_+$-Functors are contracting maps: $d(x,y) \geq d'(Tx,Ty)$ for $T : d \longrightarrow d'$

e) A category P enriched in **2** gives us: $P(y,z) \wedge P(x,y) \vdash P(x,z)$ and $1 \vdash P(x,x)$ via composition law and identity element. Thus, we can write $P(x,y)$ as the statement $x \leq y$ (being $true$ or $false$) and obtain that P is a preorder. A functor $T : P \longrightarrow P'$ is an order preserving map: $x \leq y \vdash Tx \leq Ty$

Definition 2.9 (**$\mathcal{V}$-natural transformations**)**.** A $\mathcal{V}$-natural transformation $\alpha : T \longrightarrow S : \mathcal{A} \longrightarrow \mathcal{B}$ is a family of elements $\alpha_{A \in \mathcal{A}} \in \mathcal{B}(TA, SA)$ such that:

$$\begin{array}{ccc} \mathcal{A}(A,B) & \xrightarrow{T} & \mathcal{B}(TA,TB) \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle \mathcal{B}(1,\alpha_B)} \\ \mathcal{B}(SA,SB) & \xrightarrow[\mathcal{B}(\alpha_A,1)]{} & \mathcal{B}(TA,SB) \end{array} \tag{VN1}$$

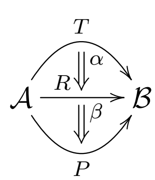
With these definitions at hand we can now consider the category of $\mathcal{V}$-categories.

3 The 2-Category $\mathcal{V}$-CAT

We first have to consider how the objects defined above compose.

1. $\mathcal{V}$-functors $T : \mathcal{A} \longrightarrow \mathcal{B}$, $S : \mathcal{B} \longrightarrow \mathcal{C}$ can clearly be composed as $(ST)_{AB} = S_{TATB}T_{AB}$ (since $STM^{\mathcal{A}} = SM^{\mathcal{B}}(T \otimes T) = M^{\mathcal{C}}(ST \otimes ST)$)

2. We need two types of composition for $\mathcal{V}$-natural transformations

   (a) *vertical composition*: Given  we can define $\beta \cdot \alpha : T \longrightarrow P$ as:

$$(\beta \cdot \alpha)_A = \beta_A . \alpha_A$$

since by claim 2.6 we have

$$\mathcal{B}(1, \beta_B.\alpha_B)T = \mathcal{B}(1, \beta_B)\mathcal{B}(1, \alpha_B)T = \mathcal{B}(1, \beta_B)\mathcal{B}(\alpha_A, 1)R$$
$$= ... = \mathcal{B}(\beta_A.\alpha_A, 1)P$$

(b) *horizontal composition*: Given $\mathcal{A} \underset{P}{\overset{T}{\Downarrow \alpha}} \mathcal{B} \underset{S}{\overset{Q}{\Downarrow \beta}} \mathcal{C}$ we can pre- and postcompose with functors as follows: $(Q\alpha)_A := Q_{TAPA}\alpha_A$ gives a $\mathcal{V}$-natural transformation $QT \longrightarrow QP$ (since by (VF1) we get $Q\mathcal{C}(1, \alpha) = \mathcal{C}(1, Q\alpha)Q$). Trivially, $(\beta_P)_A := \beta_{PA}$ gives a $\mathcal{V}$-natural transformation $QP \longrightarrow SP$ as well. Thus we define

$$\beta * \alpha = \beta_P \cdot Q\alpha$$

(Associative) vertical composition implies the existence of an ordinary category of $\mathcal{V}$-functors, which we denote by $[\mathcal{A}, \mathcal{B}]_0$. Composition of $\mathcal{V}$-functors and horizontal composition then show that the collection of $\mathcal{V}$-categories $\mathcal{V}$-**CAT** can be regarded as 2-category in the sense of Definition 2.4. For we can give the following defintion of composition and identity:

Definition 3.1 ($\mathcal{V}$-**CAT**)**.** Let $\mathcal{V}$-**CAT** denote the 2-category, whose objects are $\mathcal{V}$-categories $\mathcal{A}, \mathcal{B}$ and whose Hom objects are $[\mathcal{A}, \mathcal{B}]_0$ with composition being the functor

$$M : [\mathcal{B}, \mathcal{C}]_0 \times [\mathcal{A}, \mathcal{B}]_0 \longrightarrow [\mathcal{A}, \mathcal{C}]_0 \ , \ (Q, T) \ \mapsto \ QT \ , \ (\beta, \alpha) \ \mapsto \ \beta * \alpha$$

and identity $j_{\mathcal{A}} \equiv 1_{\mathcal{A}} \in [\mathcal{A}, \mathcal{A}]_0$.

(M1) follows from associativity of the above compositional calculus. Pre- and postcomposing with elements as in Def. 2.5 gives compostion with $\mathcal{V}$-functors: e.g. $\mathcal{V}$-$\mathbf{CAT}(1, Q)T = QT$, $\mathcal{V}$-$\mathbf{CAT}(1, Q)\alpha = 1_Q * \alpha = Q\alpha$. (M2) follows from definition of $j_{\mathcal{A}}$.

Let $\mathcal{I} \in \mathcal{V}$-**CAT** be defined by ob$\mathcal{I} = \{1\}$, $\mathcal{I}(1, 1) = I$ and with canonical M, j. For $A \in \mathcal{A}$ there is a unique $\mathcal{V}$-functor $\mathcal{I} \longrightarrow \mathcal{A}$ mapping 1 to A, which we call J^A.

From the theory of 2-Categories we can define a '**underlying category functor**' as the 2-representable functor $(-)_0 := \mathcal{V}$-$\mathbf{CAT}(\mathcal{I}, -) : \mathcal{V}$-$\mathbf{CAT} \longrightarrow \mathbf{CAT}$. We will learn about $\mathcal{V}$-representables soon. Until then, we have to give an explicit construction of $(-)_0$: Let $\alpha : T \longrightarrow S : \mathcal{A} \longrightarrow \mathcal{B}$

1. $\mathcal{A}_0 = (\mathcal{A})_0 = \mathcal{V}$-$\mathbf{CAT}(\mathcal{I}, \mathcal{A})$ has objects $J^A \equiv A$ and morphisms $(f : J^A \longrightarrow J^B) \equiv f \in \mathcal{A}_0(A, B) = V\mathcal{A}(A, B)$ with composition law $g \circ f = g.f$ and identity j_A.

2. $T_0 = (T)_0$ maps J^A to TJ_A and f to Tf. Thus $(T_0)_{AB} : f \mapsto Tf$ equals VT_{AB}. Therefore, if V is faithful the assignment $T \mapsto T_0$ is injective.

3. $\alpha_0 = (\alpha)_0 : (\alpha_0)_A = \alpha_A \in \mathcal{B}_0(T_0A, S_0A) = V\mathcal{B}(TA, SA)$ and naturality follows by applying V to (VN1).

These definitions clearly make $(-)_{0,\mathcal{AB}} : [\mathcal{A},\mathcal{B}]_0 \longrightarrow [\mathcal{A}_0,\mathcal{B}_0]$ a functor. (VF1) and (VF2) then follow from our definition of M noting that a functor is fully defined by it's action on objects and morphisms.

Remark. We will sometimes adopt the notation $f : A \longrightarrow B$ for an element f of $\mathcal{A}(A,B)$, i.e. $f : I \longrightarrow \mathcal{A}(A,B)$. There are of course no morphisms in the $\mathcal{V}$-category $\mathcal{A}$, so there is also nothing we could $f : A \longrightarrow B$ confuse with.

3.1 Monoidal structure of $\mathcal{V}$-CAT

In analogy to **CAT** we want to be able to speak of multivariable functors. The monoidal structure of **CAT** allows us do internalize multivariable functors to ordinary functors via cartesian products of ordinary categories. Thus we are seeking a tensor map in $\mathcal{V}$-**CAT**. The following would be a natural definition:

$$\begin{aligned} &\mathrm{ob}(\mathcal{A}\otimes\mathcal{B}) = \mathrm{ob}\mathcal{A}\times\mathrm{ob}\mathcal{B} \\ &(\mathcal{A}\otimes\mathcal{B})((A,B),(A',B')) = \mathcal{A}(A,A')\otimes\mathcal{B}(B,B') \end{aligned} \tag{VT1}$$

and an identity element $j^{\mathcal{A}\otimes\mathcal{B}} = (j^{\mathcal{A}}\otimes j^{\mathcal{B}})l^{-1}$. But then in order to apply our 'two variable' composition $M^{\mathcal{A}}\otimes M^{\mathcal{B}}$ we need to be able to rearrange our tensor product of Hom objects, to define $M^{\mathcal{A}\otimes\mathcal{B}}$ as follows:

$$\begin{array}{ccc} (\mathcal{A}(A',A'')\otimes\mathcal{B}(B',B''))\otimes(\mathcal{A}(A,A')\otimes\mathcal{B}(B,B')) & \xrightarrow{M^{\mathcal{A}\otimes\mathcal{B}}} & \mathcal{A}(A,A'')\otimes\mathcal{B}(B,B'') \\ \downarrow m & \nearrow_{M^{\mathcal{A}}\otimes M^{\mathcal{B}}} & \\ (\mathcal{A}(A',A'')\otimes\mathcal{A}(A,A'))\otimes(\mathcal{B}(B',B'')\otimes\mathcal{B}(B,B')) & & \end{array} \tag{VT2}$$

The construction of m can be achieved via

Definition 3.2 (symmetry). A monoidal category is symmetric if it is equipped with an additional natural isomorphism $c_{XY} : X\otimes Y \longrightarrow Y\otimes X$ (called the symmetry) and satisfying some appropriate axioms of commutativity.

Of course, the axioms are chosen such that a, l, c, r satisfy a corresponding coherence theorem; and we have already said enough on this topic (it should be clear that m can be defined in any reasonable way).

We want to complete the construction of the **tensor map in $\mathcal{V}$-CAT**. So far, we have defined in (VT1) and (VT2) a tensor map on 0-cells (objects, i.e. $\mathcal{V}$-categories). This enables us to speak already about $\mathcal{V}$-**CAT** $\times$ $\mathcal{V}$-**CAT** for instance (obtained by 'tensoring' 2-categories). We further define on 1-cells and 2-cells

$$T\otimes P : \mathcal{A}\otimes\mathcal{B}\longrightarrow\mathcal{C}\otimes\mathcal{D} \quad \text{by} \quad (T\otimes P)_{ABA'B'} := (T_{AA'}\otimes P_{BB'})$$

and

$$\alpha\otimes\beta : T\otimes P\longrightarrow Q\otimes S \quad \text{by} \quad (\alpha\otimes\beta)_A := (\alpha_A\otimes\beta_A)l^{-1}$$

By this definition (and in the second step using the definition of $M^{\mathcal{C}\otimes\mathcal{D}}$) we see that

$$\begin{aligned}((\alpha'\otimes\beta')_A.(\alpha\otimes\beta)_A) &:= (\alpha'_A\otimes\beta'_A)l^{-1}.(\alpha_A\otimes\beta_A)l^{-1}\\ &= ((\alpha'_A.\alpha_A)\otimes(\beta'_A.\beta_A))l^{-1}\end{aligned}$$

This is just stating that $(-\otimes-)_{(\mathcal{A}\mathcal{B})(\mathcal{A}'\mathcal{B}')}\colon [\mathcal{A},\mathcal{A}']_0\times[\mathcal{B},\mathcal{B}']_0\longrightarrow[\mathcal{A}\otimes\mathcal{B},\mathcal{A}'\otimes\mathcal{B}']_0$ is an (ordinary) functor and thus a morphism in **Cat**. It remains to verify the $\mathcal{V}$-functor axioms for our tensor map. We can deduce from the above definitions

$$T'T\otimes S'S = (T'\otimes S')(T\otimes S) \quad\text{and}\quad (\alpha' * \alpha)\otimes(\beta' * \beta) = (\alpha'\otimes\beta') * (\alpha\otimes\beta)$$

And this expresses compatibilty (VF1) of $(-\otimes-)_{(\mathcal{A}\mathcal{B})(\mathcal{A}'\mathcal{B}')}$ with the composition functors M (multiplication in $\mathcal{V}$-**CAT** $\times$ $\mathcal{V}$-**CAT** and in $\mathcal{V}$-**CAT** from Def. 3.1 and (VT2)). It clearly preserves identities as in (VF2). We have thus defined a 2-functor:

$$\otimes : \mathcal{V}\text{-}\mathbf{CAT}\times\mathcal{V}\text{-}\mathbf{CAT}\longrightarrow\mathcal{V}\text{-}\mathbf{CAT} \tag{2}$$

The **symmetric monoidal structure** can be inferred as follows: The maps a, l, r, c of $\mathcal{V}$ induce 2-natural isomorphisms which will again be denoted by a, l, r, c between appropriate instances of the above tensor 2-map and with identity object $\mathcal{I}$.

For instance, the $\mathcal{V}$-functor $a_{\mathcal{A}\mathcal{B}\mathcal{C}} : (\mathcal{A}\otimes\mathcal{B})\otimes\mathcal{C}\longrightarrow\mathcal{A}\otimes(\mathcal{B}\otimes\mathcal{C})$ is given by $((A,B),C)\mapsto(A,(B,C))$ on objects and by $(a_{\mathcal{A}\mathcal{B}\mathcal{C}})_{((A,B),C))((A',B'),C')} = a_{\mathcal{A}(A,A')\mathcal{B}(B,B')\mathcal{C}(C,C')}$ on Hom objects. Similar definitions hold for l, r, c. We deduce that a, l, r, c satisfy the coherence relation because their maps on Hom objects do so in $\mathcal{V}$. We also see that they are 2-natural: Plugging e.g. a into (VN1) gives

$$\begin{array}{ccc} [\mathcal{A},\mathcal{A}']_0\times[\mathcal{B},\mathcal{B}']_0\times[\mathcal{C},\mathcal{C}']_0 & \xrightarrow{((-\otimes-)\otimes-)} & [(\mathcal{A}\otimes\mathcal{B})\otimes\mathcal{C},(\mathcal{A}'\otimes\mathcal{B}')\otimes\mathcal{C}']_0 \\ {\scriptstyle(-\otimes(-\otimes-))}\downarrow & & \downarrow{\scriptstyle\mathcal{V}\text{-}\mathbf{CAT}(1,\alpha_{\mathcal{A}'\mathcal{B}'\mathcal{C}'})} \\ [\mathcal{A}\otimes(\mathcal{B}\otimes\mathcal{C}),\mathcal{A}'\otimes(\mathcal{B}'\otimes\mathcal{C}')]_0 & \xrightarrow{\mathcal{V}\text{-}\mathbf{CAT}(a_{\mathcal{A}\mathcal{B}\mathcal{C}},1)} & [(\mathcal{A}\otimes\mathcal{B})\otimes\mathcal{C},\mathcal{A}'\otimes(\mathcal{B}'\otimes\mathcal{C}')]_0 \end{array} \tag{3}$$

We then have to verify an equality of ordinary functors. Recall that e.g. $\mathcal{V}$-$\mathbf{CAT}(1, a_{\mathcal{A}\mathcal{B}\mathcal{C}})$ is a functor acting by postcomposing the $\mathcal{V}$-functor $a_{\mathcal{A}\mathcal{B}\mathcal{C}}$ to $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations. Then the above equality follows directly from naturality of a and coherence. Similarly for l, r, c.

We have finally arrived at the following:

$\mathcal{V}$-**CAT** *is a symmetric monoidal 2-category*

Of course the words 'functor' and 'natural' in the definitions of monoidality and symmetry now need to be replaced by 2-functor and 2-natural. This statement allows to transfer the following constructions from the case $\mathcal{V}$ = **Set**:

Definition 3.3 (partial functors). Let $T : \mathcal{A}\otimes\mathcal{B}\longrightarrow\mathcal{C}$ be a bifunctor. We define it's partial functors as follows:

$$T_{A-} = T(A,-) : \mathcal{B}\xrightarrow{l^{-1}}\mathcal{I}\otimes\mathcal{B}\xrightarrow{J^A\otimes 1_{\mathcal{B}}}\mathcal{A}\otimes\mathcal{B}\xrightarrow{T}\mathcal{C}$$

$$T_{-B} = T(-,B) : \mathcal{A}\xrightarrow{r^{-1}}\mathcal{A}\otimes\mathcal{I}\xrightarrow{1_{\mathcal{A}}\otimes J^B}\mathcal{A}\otimes\mathcal{B}\xrightarrow{T}\mathcal{C}$$

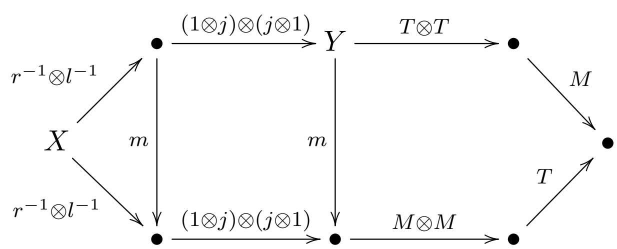
As for $\mathcal{V}$ =**Set** we now have

Proposition 3.4 (**Compatibility condition of partial functors**). *Let $T_{A-} : \mathcal{B} \longrightarrow \mathcal{C}$ and $T_{-B} : \mathcal{A} \longrightarrow \mathcal{C}$ be families of $\mathcal{V}$-functors satisfying $T_{-A}B = T_{B-}A =: T(A, B)$ on objects. Then there is a unique 'full' $\mathcal{V}$-functor $T : \mathcal{A} \otimes \mathcal{B} \longrightarrow \mathcal{C}$ with partial functors T_{A-} and T_{-B} if and only if we have*

$$\begin{array}{ccc}
\mathcal{A}(A,A') \otimes \mathcal{B}(B,B') & \xrightarrow[T_{-B'} \otimes T_{A-}]{} & \mathcal{C}(T(A,B'),T(A',B')) \otimes \mathcal{C}(T(A,B),T(A,B')) \\
\Big\downarrow c & & \Big\downarrow M \\
 & & \mathcal{C}(T(A,B),T(A',B')) \\
 & & \Big\uparrow M \\
\mathcal{B}(B,B') \otimes \mathcal{A}(A,A') & \xrightarrow{T_{A'-} \otimes T_{-B}} & \mathcal{C}(T(A',B),T(A',B')) \otimes \mathcal{C}(T(A,B),T(A',B))
\end{array} \tag{4}$$

Note that the corresponding condition in **Set** would read $T(g,1)T(1,f) = T(1,f)T(g,1)$

Proof. (cf. [4]) Suppose T exists. The upper leg of diagram (4) is by Definition 3.3 of partial functors equal to the upper leg of the following diagram:



where $X = \mathcal{A}(A, A') \otimes \mathcal{B}(B, B')$ and $Y = (\mathcal{A}(A, A') \otimes \mathcal{B}(B', B')) \otimes (\mathcal{A}(A, A) \otimes \mathcal{B}(B, B'))$. The inner diagrams (from left to right) commute by coherence, naturality of m, and functoriality of T. But by claim 2.6 the lower leg yields $T_{(AB)(A'B')}$. Thus if T exists it equals the upper leg of (4). A similar diagram (obtained by folding in c to the above) shows T is equal to the lower leg of diagram (4). Thus diagram (4) commutes and T is equal to its diagonal.

Conversely, assume T_{A-}, T_{-B} are functors compatible as in (4). We set $T_{(AB)(A'B')}$ equal to the diagonal of (4) (clearly there is just one way to define T on objects). We have to verify the $\mathcal{V}$-functor axioms for T. Since T is defined via composition of it's partial functors, the proofs follow from axioms of composition, functoriality of partial functors and (4). As an example (VF2) follows by:

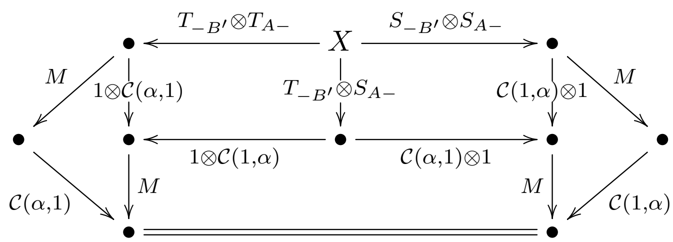
$$Tj^{\mathcal{A}\otimes\mathcal{B}} := M(T_{-B'} \otimes T_{A-})(j \otimes j)l^{-1} = M(j \otimes j)l^{-1} = M(j \otimes 1)l^{-1}j = j$$

□

Proposition 3.5 (**Naturality is verified variable by variable**). *Let $\alpha_{AB} \in \mathcal{C}(T(AB), S(AB))$ and $T, S : \mathcal{A} \otimes \mathcal{B} \longrightarrow \mathcal{C}$. Then α being a $\mathcal{V}$-natural transformation $T \longrightarrow S$ is equivalent to $\alpha_{A\cdot} : T_{A-} \longrightarrow S_{A-}, \alpha_{\cdot B} : T_{-B} \longrightarrow S_{-B}$ being families of $\mathcal{V}$-natural transformations.*

Proof. (cf. [4]) Given α $\mathcal{V}$-natural, the corresponding $\alpha_{A\cdot}, \alpha_{\cdot B}$ are obtained by precomposition with functors as Def. 3.3, and thus natural.

Conversely, assume that $\alpha_{A\cdot}, \alpha_{\cdot B}$ are natural $\forall A, B$. Let $X = \mathcal{A}(A, A') \otimes \mathcal{B}(B, B')$ as before. Then



The outer legs of this diagram give us the statement we seek. The upper squares commute by $\mathcal{V}$-naturality of $\alpha_{A\cdot}, \alpha_{\cdot B}$. All other parts commute by Claim 2.6.

□

We quickly consider **underlying bifunctors** of $\mathcal{V}$-bifunctors. Let $T : \mathcal{A} \otimes \mathcal{B} \longrightarrow \mathcal{C}$ as before. Then the underlying bifunctor defined through the underlying partial functors is not the same as T_0, since $(\mathcal{A} \otimes \mathcal{B})_0 \neq \mathcal{A}_0 \times \mathcal{B}_0$ in general. We can however reconcile those two pictures if we precompose T_0 with the canonical embedding $u : \mathcal{A}_0 \times \mathcal{B}_0 \longrightarrow (\mathcal{A} \otimes \mathcal{B})_0$ given by $(A, B) \mapsto (A, B)$ and $(f, g) \mapsto (f \otimes g)l^{-1}$. The partial functors of $T_0 u$ are now the underlying functors of the partial $\mathcal{V}$-functors of T. (This can be seen tracing through our definition of partial functors, and defining $I \cong \mathcal{I}_0$ and thus $\mathcal{B}_0 \cong \mathcal{I}_0 \times \mathcal{B}_0$ in the only reasonable way).

Another consequence of symmetry is the following **duality**. We define an involutive operation $(-)^{\mathrm{op}}$ on $\mathcal{V}$-**CAT**: $\mathcal{A}^{\mathrm{op}}$ lives on the same set of objects as $\mathcal{A}$ and has Hom objects $\mathcal{A}^{\mathrm{op}}(A, B) = \mathcal{A}(B, A)$. Then multiplication can clearly be given as $M^{\mathcal{A}^{\mathrm{op}}} = M^{\mathcal{A}} c$ and the identity element stays the same. We deduce from the above definitions that: $(\mathcal{A} \otimes \mathcal{B})^{\mathrm{op}} = \mathcal{A}^{\mathrm{op}} \otimes \mathcal{B}^{\mathrm{op}}$ and $(\mathcal{A}^{\mathrm{op}})_0 = (\mathcal{A}_0)^{\mathrm{op}} =: \mathcal{A}_0^{\mathrm{op}}$. Every $\mathcal{V}$-functor $T : \mathcal{A} \longrightarrow \mathcal{B}$ has a canonical opposite functor $T^{\mathrm{op}} : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{B}^{\mathrm{op}}$, every $\mathcal{V}$-natural transformation $\alpha : T \longrightarrow S$ has a canonical $\alpha^{\mathrm{op}} : S^{\mathrm{op}} \longrightarrow T^{\mathrm{op}}$.

Examples 3.6. An example of a non-symmetric monoidal category is the category of bimodules over a noncommutative ring.

3.2 Closedness and internal Homs

Drop symmetry for now. We note that **Cat** is itself a **Cat**-category, and more trivially **Set** is itself a **Set**-category. That means, both admit to lift their Hom-Sets to Hom-Objects. We will call these internal Homs. In the **Set** case the underlying idea is the idea of currying; a two-variable function can be seen as an indexed family of functions. Thus to every to 2-variable function there corresponds a unique function from the 'index set' to the set of functions in the second variable (the internal Hom). We generalize the idea as follows:

Definition 3.7 (closedness). A monoidal category is called closed if the functor $-\otimes Y : \mathcal{V}_0 \longrightarrow \mathcal{V}_0$ has right adjoint $[Y, -] : \mathcal{V}_0 \longrightarrow \mathcal{V}_0$ for all $Y \in \mathcal{V}_0$.

We denote this adjunction (natural in X and Z) by

$$\pi = \pi^Y_{XZ} : \mathcal{V}_0(X \otimes Y, Z) \cong \mathcal{V}_0(X, [Y.Z]) \tag{5}$$

and the unit and counit correspondingly:

$$d = d^Y_X : X \longrightarrow [Y, X \otimes Y] \quad , \quad e = e^Y_Z : [Y, Z] \otimes Y \longrightarrow Z \tag{6}$$

Drawing from our intuition from **Set** (where e evaluates variables from Y on functions from $[Y, Z]$) we call the counit *evaluation*. And with the same intuition in mind we will refer to the adjunction as *currying* in one direction, and *uncurrying* in the other.

To see that such an adjunction lifts underlying Sets to internal Homs consider setting $X = I$ in eq. (5) to obtain a map ι (Y above an arrow denotes the usual Yoneda embedding):

$$\iota^X : \mathcal{V}_0(X, Z) \xrightarrow{Y(l)} \mathcal{V}_0(I \otimes X, Z) \xrightarrow{\pi^X} \mathcal{V}_0(I, [X, Z]) =: V[X, Z] \Rightarrow \iota^X : \mathcal{V}_0(X, Z) \cong V[X, Z] \tag{7}$$

We thus call $[X, Z]$ an **internal Hom**. Even more is possible: We can lift ι and π to maps between internal Homs. For this we need the following technical lemma. The result of this discussion is given in claim 3.9.

Lemma 3.8 (technical lemma). *Let $F \dashv G : \mathcal{D} \longrightarrow \mathcal{C}$ and $F' \dashv G' : \mathcal{D} \longrightarrow \mathcal{C}$ be adjunctions with their corresponding natural isomorphisms and counits denoted by π, ϵ and π', ϵ'. Let $s : F \longrightarrow F'$ be a natural isomorphism. Then there exist a unique natural isomorphism $t : G \longrightarrow G'$ such that $\pi' \circ Y(s^{-1}) = Y^{\mathrm{op}}(t) \circ \pi : \mathcal{D}(FA, B) \longrightarrow \mathcal{C}(A, G'B)$.*

Proof. Let $f : FA \longrightarrow B$. From $\pi' \circ Y(s^{-1}) = Y^{\mathrm{op}}(t) \circ \pi$ we deduce:

$$(\pi')^{-1}(t\pi(f)) = (\pi')^{-1}(\pi'(fs^{-1})) = fs^{-1} \tag{8}$$

On setting $f = \epsilon_B$ we find $(\pi')^{-1}(t) = \epsilon_B s^{-1}$. Thus if a natural isomorphism t as above exists it is unique. Similarly we deduce (exchange the roles of primed and non primed) that $\pi^{-1}(t^{-1}) = \epsilon' s$ must hold. Conversely, we can take this as definitions for t,t^{-1} and need to verify naturality and that the definitions give mutual inverses. This is a straight forward check using bijectiveness and naturality of our adjunctions.

□

We can now apply this lemma to derive the liftings of π and ι. For our first application we use our natural isomorphism $r_Z : Z \otimes I \longrightarrow Z$. Take $r = s$, $i^{-1} := t$, and adjunctions $\pi : (- \otimes I) \dashv [I, -]$, $\pi' : 1_{\mathcal{V}_0} \dashv 1_{\mathcal{V}_0}$. Since $r : (- \otimes I) \cong 1_{\mathcal{V}_0}$ have a unique i:

$$i : 1_{\mathcal{V}_0} \cong [I, -] \quad \text{or} \quad i = i_Z : Z \cong [I, Z] \tag{9}$$

such that $\pi^{-1}(i_Z) = r_Z$ or $i_Z = \pi(r_Z)$.

Our second application uses our isomorphism $a_{XYZ} : (X \otimes Y) \otimes Z \longrightarrow X \otimes (Y \otimes Z)$. Take $a = s$, $p^{-1} := t$, and adjunctions $\pi : (- \otimes X) \otimes Y \dashv [X, [Y, -]]$, $\pi' : - \otimes (X \otimes Y) \dashv [X \otimes Y, -]$. As before, we derive natural isomorphisms:

$$p = p^{XY}_Z : [X \otimes Y, Z] \cong [X, [Y, Z]] \tag{10}$$

such that $p_Z = \pi(e^{X\otimes Y}a_{XYZ})$ and $p_Z^{-1} = \pi'(e^X(e^Y \otimes 1)a_{XYZ}^{-1}))$.

We adopt the **notation** $\overline{(-)}$ **for** π^Y from now on, in particular dropping reference to Y. For later use we sum up the above discussion in the following simple equations:

$$p^{-1} = \overline{e(e \otimes 1)a} \quad , \quad i = \overline{r} \tag{11}$$

Now p, i are *liftings* of π and ι in the following sense.

Claim 3.9. *The $\mathcal{V}$ morphisms p and i given in eq.* (10) *and* (9)*, and defined by eq.* (11)*, fulfill $Vi = \iota^I : VZ \longrightarrow V[I, Z]$ and $Vp = \iota\pi\iota^{-1} : V[X \otimes Y, Z] \longrightarrow V[X, [Y, Z]]$.*

Proof. We will proof the first statement. Take $f : I \longrightarrow Z$. Then:
$Vi(f) = \bar{r}f$. On the other hand $\iota^I(f) = \overline{fl} = \overline{fr} = \overline{r(f \otimes 1)} = \bar{r}f$. ☐

Examples 3.10. a) **Set** is of course closed. The adjunction reduces to the rule of currying (or lambda conversion) as described before.

b) **Top** is unfortunately not closed. Instead one can work with a 'convenient category of topological spaces' (see the same-named article in nLab for more information).

c) In the poset **2** the internal Hom is implication $[y, z] = (y \Rightarrow z)$: The unit and counit are the (always existing) entailments $x \vdash (y \Rightarrow x \wedge y)$ and $(y \Rightarrow z) \wedge y \vdash z$ ('modus ponens'). It follows that we have an adjunction $x \wedge y \vdash z$ *iff* $x \vdash (y \Rightarrow z)$. This is just seen to be the 'deduction theorem' (cf. [7]).

d) In the poset $\overline{\mathbb{R}}_+$ we have $x + y \geq z$ *iff* $x \geq z - y$ and thus set $[y, z] = \max\{z - y, 0\}$ (cf. [7]). Indeed unit and counit just express the inequalities $x \geq \max\{x + y - y, 0\}$ and $\max\{z - y, 0\} + y \geq z$.

e) If we have a symmetry c we have the adjunction $Y \otimes - \dashv [Y, -]$ via $\pi' = \pi Y(c)$. More generally a closed $\mathcal{V}$ with a right adjoint for $Y \otimes -$ is called biclosed (thus we deduced symmetric $\Rightarrow$ biclosed).

We can now return to our initial goal to establish $\mathcal{V} \in \mathcal{V}$-**CAT**. In order to avoid confusion we will call the $\mathcal{V}$-category lifting of $\mathcal{V}$ that we are about to construct by the name $\hat{\mathcal{V}}$. From our discussion the following definition should be natural

Definition 3.11 (**construction of $\hat{\mathcal{V}}$ as $\mathcal{V}$ category**)**.** Let $\hat{\mathcal{V}}$ have objects ob$\mathcal{V}$, and Hom objects $\hat{\mathcal{V}}(Y, Z) = [Y, Z]$. Multiplication is given via currying as successive evaluation, explicitly: $M : [Y, Z] \otimes [X, Y] \longrightarrow [X, Z]$ corresponds under the adjunction π^X to:

$$([Y, Z] \otimes [X, Y]) \otimes X \xrightarrow{a} [Y, Z] \otimes ([X, Y] \otimes X) \xrightarrow{1 \otimes e} [Y, Z] \otimes Y \xrightarrow{e} Z \tag{12}$$

The unit j_X corresponds under currying to $l : I \otimes X \longrightarrow X$. Then $\hat{\mathcal{V}} \in \mathcal{V}$-**CAT**.

For this definition to hold we need to verify (M1) and (M2). As an example and introduction to the type of proofs that will follow we proof (M1).

Proof. The statement we need is $M(1 \otimes M)a = M(M \otimes 1)$. The definition of M reads $\overline{M} = e(M \otimes 1) := e(1 \otimes e)a$. If we uncurry our statement we get:

$$\begin{aligned} & e(1 \otimes e)a((1 \otimes M) \otimes 1)(a \otimes 1) = e(1 \otimes e)a((M \otimes 1) \otimes 1) \\ \Leftrightarrow \; & e(1 \otimes e(M \otimes 1))a(a \otimes 1) = e(M \otimes 1)((1 \otimes 1) \otimes e)a \\ \Leftrightarrow \; & e(1 \otimes e)(1 \otimes (1 \otimes e)) = e(1 \otimes e)(1 \otimes (1 \otimes e)) \end{aligned}$$

using (tacitly as always) coherence in the last step. (Note that the indices can be easily filled in by drawing the diagram corresponding to the above equations.) □

Claim 3.12 (**$\hat{\mathcal{V}}_0$ and $\mathcal{V}_0$ are equivalent**)**.** *The underlying category $\hat{\mathcal{V}}_0$ and $\mathcal{V}_0$ are equivalent.*

Proof. We define the **hat functor** $\hat{(\cdot)} : \mathcal{V}_0 \longrightarrow \hat{\mathcal{V}}_0$ as follows

$$X \mapsto X \quad \text{and} \quad (f : Y \longrightarrow Z) \mapsto (\hat{f} = \iota(f) \in [Y, Z]) \tag{13}$$

Recall $\iota(f) = \overline{fl}$. Clearly $\hat{1_X} = \bar{l} = j_X$. For functoriality it remains to check $\iota(f).\iota(g) = \iota(fg)$. Uncurried this statement reads:

$$e(1 \otimes e)a((\overline{fl} \otimes \overline{gl})l^{-1} \otimes 1) = fgl \tag{14}$$

which is easily verified using $e(\bar{h} \otimes 1) = h$. Now the hat map is clearly full and faithful (ι is iso) and bijective on objects which completes the proof.

□

We will denote the **inverse hat functor** as $\check{(\cdot)} : \hat{\mathcal{V}}_0 \longrightarrow \mathcal{V}_0$ mapping $g \in [X, Y] \mapsto \bar{g}l^{-1}$. Of course all this being very bijective, we can just identify the two categories. And the reader is invited to *ignore* the small hats here and there. But when we want to reach an explicit expression in one or the other representation we need to refer back to these isomorphisms.

3.3 Hom and tensor as $\mathcal{V}$-functors

Having our $\mathcal{V}$-category $\hat{\mathcal{V}}$ at hand, we can now go on to lift representables to $\mathcal{V}$-functors with codomain $\hat{\mathcal{V}}$. Roughly speaking, representables should be curried versions of composition. More formally,

Definition 3.13 (**lifting representables**)**.** Define $\mathcal{A}(A, -) : \mathcal{A} \longrightarrow \hat{\mathcal{V}}$ by $\mathcal{A}(A, -)(B) = \mathcal{A}(A, B)$ (written as $\mathcal{A}(A, B)$) on objects and $\mathcal{A}(A, -)_{BC} : \mathcal{A}(B, C) \longrightarrow [\mathcal{A}(A, B), \mathcal{A}(A, C)]$ corresponds under adjunction to

$$M : \mathcal{A}(B, C) \otimes \mathcal{A}(A, B) \longrightarrow \mathcal{A}(A, C) \tag{15}$$

Again we need to verify that this is a functor, which follows from our definition and axioms of composition in both $\mathcal{A}$ and $\hat{\mathcal{V}}$. We verify (VF1):

Proof. We will sometimes *abbreviate* $\mathcal{A}$ for $\mathcal{A}(A, -)_{BC}$in proofs for legibility. The definition of $\mathcal{A}(A, -)_{BC}$ then reads $e(\mathcal{A} \otimes 1) = M^{\mathcal{A}}$. (VF1) states $M^{\hat{\mathcal{V}}}(\mathcal{A} \otimes \mathcal{A}) = \mathcal{A}M^{\mathcal{A}}$. Uncurry the RHS to get $M^{\mathcal{A}}(1 \otimes M^{\mathcal{A}})a$ and the LHS to get $M^{\mathcal{A}}(M^{\mathcal{A}} \otimes 1)$. Thus the statement holds by (M1) in $\mathcal{A}$. □

Keeping the analogy to the **Set** case, we want our representables to be bifunctors: Consider eq. (15) of Def. 3.13. We would like to fix the right, instead of the left argument of M. This was shown to be possible in the presence of symmetry in Example 3.10 e). Thus assume $\mathcal{V}$ to be symmetric and define

$$\mathcal{A}(-,B) := \mathcal{A}^{\mathrm{op}}(B,-) : \mathcal{A}^{\mathrm{op}} \longrightarrow \hat{\mathcal{V}} \tag{16}$$

These functors are compatible by definition of opposite composition and associativity axioms and therefore we have

Claim 3.14 (existence of Hom). *There is a functor $\mathcal{A} = \mathrm{Hom}_{\mathcal{A}} : \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \longrightarrow \hat{\mathcal{V}}$ with partial functors $\mathcal{A}(A,-)$ and $\mathcal{A}(-,B)$*

Proof. By Proposition 3.4 we need to verify that $M(\mathcal{A}_{-B'} \otimes \mathcal{A}_{A-}) = M(\mathcal{A}_{A'-} \otimes \mathcal{A}_{-B})c$ (with the notation for partial functors introduced in Def. 3.3). Uncurry to get

$$e(1 \otimes e)a((\mathcal{A}_{-B'} \otimes \mathcal{A}_{A-}) \otimes 1) = e(1 \otimes e)a((\mathcal{A}_{A'-} \otimes \mathcal{A}_{-B}) \otimes 1)(c \otimes 1)$$

write $\mathcal{A}_{-B}$ as $\mathcal{A}^{\mathrm{op}}_{B-}$ and plug in the definition $e(\mathcal{B} \otimes 1) = M^{\mathcal{B}}$ using $M^{\mathcal{A}^{\mathrm{op}}} = M^{\mathcal{A}}c$. The above statement then holds by (M1). □

Recall our short discussion of underlying bifunctors and the inclusion map u at the end of the last subsection. It assures that the partial functors of the underlying ordinary bifunctor will be the underlying functors of the partial $\mathcal{V}$-bifunctor.

Definition 3.15 (the hom functor). Define $\mathrm{hom}_{\mathcal{A}}$ to be the ordinary bifunctor $(\mathrm{Hom}_{\mathcal{A}})_0 u$ post-composed with our hat isomorphism. Explicitely $\mathrm{hom} = \mathrm{hom}_{\mathcal{A}} : \mathcal{A}_0^{\mathrm{op}} \times \mathcal{A}_0 \longrightarrow \mathcal{V}_0$ is given by:

$$\mathrm{hom}_{\mathcal{A}} : \mathcal{A}_0^{\mathrm{op}} \times \mathcal{A}_0 \xrightarrow{u} (\mathcal{A}^{\mathrm{op}} \otimes \mathcal{A})_0 \xrightarrow{(\mathrm{Hom}_{\mathcal{A}})_0} \hat{\mathcal{V}}_0 \xrightarrow{\check{(\cdot)}} \mathcal{V}_0 \tag{17}$$

In accordance with our discussion of underlying functors, we denote $\mathrm{hom}_{\mathcal{A}}(g,f) = \mathcal{A}(g,f)$ and keep in mind that the RHS of this statement should be identified with a map to $\mathcal{V}_0$. Then we derive from eq. (17) and def. of $\check{(\cdot)}$

$$\mathrm{hom}(A,g) = \overline{\mathcal{A}g}l^{-1} = M(g \otimes 1)l^{-1} \equiv \mathcal{A}(A,g) \tag{18}$$

and

$$\mathrm{hom}(g,A) = \mathrm{hom}_{\mathcal{A}^{\mathrm{op}}}(A,g) = Mc(g \otimes 1)l^{-1} = M(1 \otimes g)r^{-1} \equiv \mathcal{A}(g,A) \tag{19}$$

Our notation turns out to be perfectly *consistent* with section 1.

Recall that composition in $\mathcal{A}_0$ was just given by the above expressions (18) and (19). We thus obtain the following ordinary bifunctor equality (proved by comparing partial functors):

$$\mathrm{Hom}_{\mathcal{A}_0} = V\,\mathrm{hom}_{\mathcal{A}} \equiv V(\mathrm{Hom}_{\mathcal{A}})_0 \tag{20}$$

(surpressing u and hats on the right.) Thus our initial Definition 3.13 describes indeed a lifting of the ordinary Hom functor of the underlying category!

We now consider the special case $\mathcal{A} = \hat{\mathcal{V}}$. Then we can *interchangebly* write $\hat{\mathcal{V}}(\hat{g},\hat{f})$ and $[f,g]$. This is the content of the following claim:

Claim 3.16 (the internal Hom functor $[-,-]$). *There exists $[-,-] : \mathcal{V}_0^{\text{op}} \times \mathcal{V}_0 \longrightarrow \mathcal{V}_0$ with partial functor $[Y,-]$ being the right adjoint of $- \otimes Y$ as before. $[-,-]$ is given by* hom *when we identify $\hat{\mathcal{V}}_0 \cong \mathcal{V}_0$ in it's domain. Explicitly,*

$$[-,-] = \text{hom}((\hat{-}), (\hat{-}))$$

Proof. We need to show that the partial functor $\text{hom}(Y,-)$ agrees with $[Y,-]$. Clearly on objects we have equality $\text{hom}_{\hat{\mathcal{V}}}(Y,Z) = [Y,Z]$. Thus, we have to show equality on morphisms: uncurry the statement $\text{hom}_{\hat{\mathcal{V}}}(Y,\hat{f}) = [Y,f]$ to get (using $e(\hat{f} \otimes 1) = fl$ and eq. (18)):

$$\text{LHS} = e(1 \otimes e)a((\hat{f} \otimes 1)l^{-1} \otimes 1) = fe$$

which equals the RHS by definition of our adjunction. □

Note that the internal Hom bifunctor could also be defined along the lines of the 'technical lemma' (but without isomorphisms). This is reflected in the first part of the following

Lemma 3.17 (properties of $\text{hom}_{\hat{\mathcal{V}}}$). *Let $f : X \longrightarrow Y$ and $\alpha : I \longrightarrow \mathcal{A}(A,B)$. We have*

a) $\hat{\mathcal{V}}(\hat{f},1) = [f,1] = \overline{e(1 \otimes f)}$ and $\hat{\mathcal{V}}(1,\hat{f}) = [1,f] = \overline{fe}$

b) $[\alpha,1]\mathcal{A}(A,-) = i\mathcal{A}(\alpha,1)$ and $[1,\alpha]\mathcal{A}(A,-) = i\mathcal{A}(1,\alpha)$ (dually, $[\alpha,1]\mathcal{A}(-,A) = \ldots$)

Proof. a) Recall $\hat{\mathcal{V}}(\hat{f},1) = M(1 \otimes \hat{f})r^{-1}$. Uncurry to get

$$e(1 \otimes e)a((1 \otimes \hat{f})r^{-1} \otimes 1)) = e(1 \otimes f)$$

as required. The second statement follows directly from definition of the adjunction.

b) From part *a)* we derive that the uncurried LHS reads $e(1 \otimes \alpha)(\mathcal{A} \otimes 1) = M^{\mathcal{A}}(1 \otimes \alpha)$. On the RHS employ our definition $\bar{i} = r$ to obtain the same expression. The second statement follows similarly.

□

So far we constructed an 'internalized' $\mathcal{V}$-functor Hom for all $\mathcal{V}$-categories $\mathcal{A}$ and showed that it reduced to the ordinary Hom by applying the underlying category and underlying set functor accordingly. We should now also *internalize tensoring* in the case $\mathcal{A} = \mathcal{V}$. Again, intuition is drawn from the **Set** case for the following definition:

Definition 3.18 (lifting the tensor map). Define $\text{Ten} : \hat{\mathcal{V}} \otimes \hat{\mathcal{V}} \longrightarrow \hat{\mathcal{V}}$ on objects by $\text{Ten}(X,Y) = X \otimes Y$. Let $\text{Ten}_{(XY)(X'Y')} : [X,X'] \otimes [Y,Y'] \longrightarrow [X \otimes Y, X' \otimes Y']$ be the map corresponding to 'separate evaluation' as follows:

$$([X,X'] \otimes [Y,Y']) \otimes (X \otimes Y) \xrightarrow{m} ([X,X'] \otimes X) \otimes ([Y,Y'] \otimes Y) \xrightarrow{e \otimes e} X' \otimes Y'$$

Recast this definition in the form $e(\text{Ten} \otimes 1) = (e \otimes e)m$. As for $\mathcal{A}(A,-)_{BC}$ this allows us to (uncurry and) proof $\mathcal{V}$-functor axioms (VF1) and (VF2). We can then proceed as in the case of Hom when constructing hom: Consider $\text{Ten}_0\, u$ and postcompose with the hat isomorphism. Call this map ten. The absurdly chosen name becomes obsolete because: Just as $\text{hom}_{\hat{\mathcal{V}}}(-,-)$ was shown equal to $[-,-]$, it follows that $\text{ten}(-,-)$ is equal to $- \otimes -$, when we identify $\hat{\mathcal{V}}_0$ and $\mathcal{V}_0$ in their domains. This gives the first part of the following

Lemma 3.19 (properties of Ten). *Our tensoring $\mathcal{V}$-functor* Ten *has the properties*

a) $-\otimes-$ *is given through* $\mathcal{V}_0 \times \mathcal{V}_0 \cong \hat{\mathcal{V}}_0 \times \hat{\mathcal{V}}_0 \xrightarrow{u} (\hat{\mathcal{V}} \otimes \hat{\mathcal{V}})_0 \xrightarrow{\mathrm{Ten}_0} \hat{\mathcal{V}}_0 \cong \mathcal{V}_0$. *In particular* $g \otimes f = \mathrm{Ten}(u(\hat{g}, \hat{f})) \equiv \mathrm{Ten}(g, f)$ *(surpressing u and hats).*

b) Ten *allows us to represent* $\mathrm{Hom}_{\mathcal{A}\otimes\mathcal{B}}$ *as follows*

$$\begin{array}{ccc} (\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B}^{\mathrm{op}}) \otimes (\mathcal{A} \otimes \mathcal{B}) & \xrightarrow{\mathrm{Hom}_{\mathcal{A}\otimes\mathcal{B}}} & \hat{\mathcal{V}} \\ \Big\downarrow m & & \Big\uparrow \mathrm{Ten} \\ (\mathcal{A}^{\mathrm{op}} \otimes \mathcal{A}) \otimes (\mathcal{B}^{\mathrm{op}} \otimes \mathcal{B}) & \xrightarrow[\mathrm{Hom}_{\mathcal{A}}\otimes\mathrm{Hom}_{\mathcal{B}}]{} & \hat{\mathcal{V}} \otimes \hat{\mathcal{V}} \end{array} \tag{21}$$

Proof. a) was explained above.

b) We only need to verify equality on partial tensor by Prop. 3.4. We want

$$\begin{array}{ccc} (\mathcal{A}(C, C') \otimes \mathcal{B}(D, D')) & \xrightarrow{(\mathcal{A}\otimes\mathcal{B})((AB),-)} & \hat{\mathcal{V}} \\ & \searrow^{\mathcal{A}(A,-)\otimes\mathcal{B}(B,-)} & \Big\uparrow \mathrm{Ten} \\ & & \hat{\mathcal{V}} \otimes \hat{\mathcal{V}} \end{array} \tag{22}$$

Uncurrying the lower leg gives $(e \otimes e)m((\mathcal{A} \otimes \mathcal{B}) \otimes (1 \otimes 1)) = (M^{\mathcal{A}} \otimes M^{\mathcal{B}})m$ as required. Dually, the second partial tensor equality follows.

□

An immediate consequence of *both* parts of this lemma is

$$(\mathcal{A} \otimes \mathcal{B})(u(f, g), u(f', g')) = \mathrm{Ten}(\mathcal{A}(f, f'), \mathcal{B}(g, g')) = \mathcal{A}(f, f') \otimes \mathcal{B}(g, g') \tag{23}$$

This equation is of course derived from the functorial equality from part *b)*, and precomposed with elements under use of part *a)*.

Remark (another coherence theorem). The reader will have inevitably noticed the similarity of the above proofs. Just as translating trivial applications of ordinary category axioms to $\mathcal{V}$-enriched categories for a monoidal $\mathcal{V}$, the proofs followed along the lines of verifications in **Set** via our notion of currying. And the feeling that "the statements just had to be right" can be put into another coherence theorem encompassing many of the verifications above.

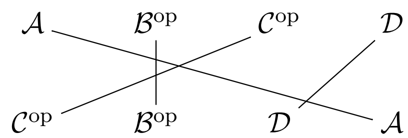
4 The weak Yoneda Lemma

4.1 Extraordinary $\mathcal{V}$-naturality

Let us recall our definition of $\mathcal{V}$-naturality. Given $\mathcal{V}$-functors $T, S : \mathcal{A} \longrightarrow \mathcal{B}$ and a family of maps $\alpha_A : TA \longrightarrow SA$ (i.e. elements of $\mathcal{B}(TA, SA)$), this constitutes a $\mathcal{V}$-natural transformation if the following commutes

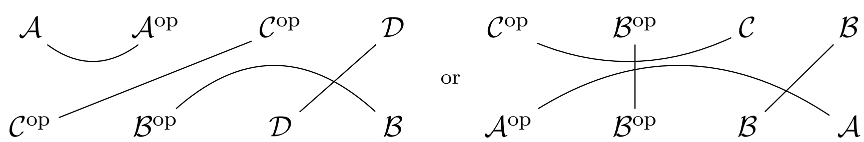
$$\begin{array}{ccc} \mathcal{A}(A,B) & \xrightarrow{T} & \mathcal{B}(TA,TB) \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle \mathcal{B}(1,\alpha_B)} \\ \mathcal{B}(SA,SB) & \xrightarrow[\mathcal{B}(\alpha_A,1)]{} & \mathcal{B}(TA,SB) \end{array} \tag{VN1}$$

A short motivation for what will follow: In the above we are 'moving between images' of functors depending covariantly on one variable $\mathcal{A}$. By Prop. 3.5 the multivariable case can be treated 'variable by variable' and we denote it in the following string diagram:



(24)

where the upper row denotes the arguments of T, and the lower row the arguments of S. We obtain more general transformations if we allow bending of these comparing lines: This requires co- and contravariant dependence on an argument, and 'bending' the rules for α accordingly as we will soon define. For instance we obtain:



(25)

This is *essentially what this section will be about*, and it provides a nice (and correct) intuition for composition of these generalized natural transformations: namely, vertically concatenation of the string diagrams (if possible, as it is in eq. (25) **if** $\mathcal{C} = \mathcal{D}$). We will prove this at the end of this section.

Remark. 1. We are not braiding above (there is no crossing 'above' or 'below').

2. There is a fourth way to 'reasonably arrange four arrows' in a square. This is for instance used in Riemannian geometry to express $T_pM \cong T_p^*M$ via g (see [3], p. 220). But it does not generalize to the diagrammatic calculus above. All cases can be subsumed in the notion of dinatural transformations (which Max Kelly did not like, cf. nLab).

We now define **extraordinary $\mathcal{V}$-naturality** sketched above. Given a bifunctor $S : \mathcal{A}^{\text{op}} \otimes \mathcal{A} \longrightarrow \mathcal{B}$ and a familiy $\alpha_A : K \longrightarrow S(A,A)$ we call it $\mathcal{V}$-natural if

$$\begin{array}{ccc} \mathcal{A}(A,B) = \mathcal{A}^{\text{op}}(B,A) & \xrightarrow{S_{A-}} & \mathcal{B}(S(A,A),S(A,B)) \\ {\scriptstyle S_{-B}}\downarrow & & \downarrow{\scriptstyle \mathcal{B}(\alpha_A,1)} \\ \mathcal{B}(S(B,B),S(A,B)) & \xrightarrow[\mathcal{B}(\alpha_B,1)]{} & \mathcal{B}(K,S(A,B)) \end{array} \tag{VN3}$$

commutes. Dually (i.e. bending lines in the domain of T) we call $\alpha_A : T(A,A) \longrightarrow K$, for

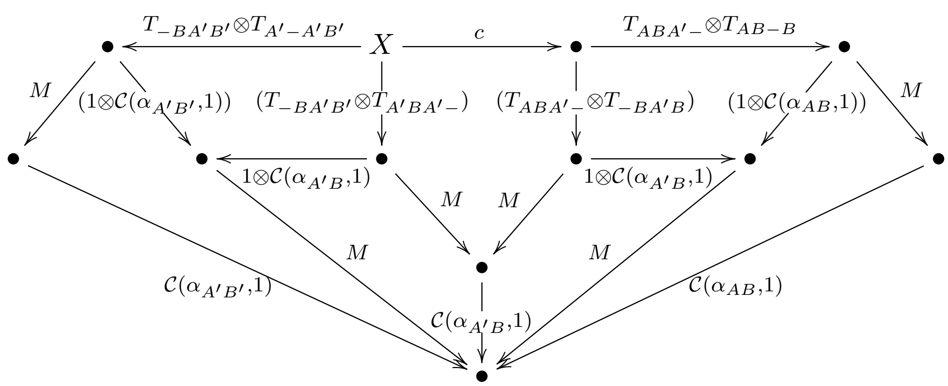
$T : \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \longrightarrow \mathcal{B}$, $\mathcal{V}$-natural if we have

$$\begin{array}{ccc} \mathcal{A}(B,A) & \xrightarrow{T_{A-}} & \mathcal{B}(T(A,B),T(A,A)) \\ {\scriptstyle T_{-B}}\downarrow & & \downarrow{\scriptstyle \mathcal{B}(1,\alpha_A)} \\ \mathcal{B}(T(A,B),T(B,B)) & \xrightarrow[\mathcal{B}(1,\alpha_B)]{} & \mathcal{B}(T(A,B),K) \end{array} \qquad \text{(VN2)}$$

Let us first establish that we can again verify naturality variable by variable. In other words drawing the above string diagrams makes sense.

Proposition 4.1 (extraordinary $\mathcal{V}$-naturalilty is verified var-by-var). *Given* $T : (\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B}^{\mathrm{op}}) \otimes (\mathcal{A} \otimes \mathcal{B}) \longrightarrow$ *Then* $\alpha_{AB} : K \longrightarrow T((A,B),(A,B))$ *is* $\mathcal{V}$*-natural in* (AB) *iff it is* $\mathcal{V}$*-natural in each variable.*

Proof. We proof the non-obvious direction. Let $X = (\mathcal{A} \otimes \mathcal{B})((AA')(BB')) = \mathcal{A}(A,A') \otimes \mathcal{B}(B,B')$. Consider (with our notation for partial functors $T_{-BA'B'} = T((-,B),(A',B')))$ the following:



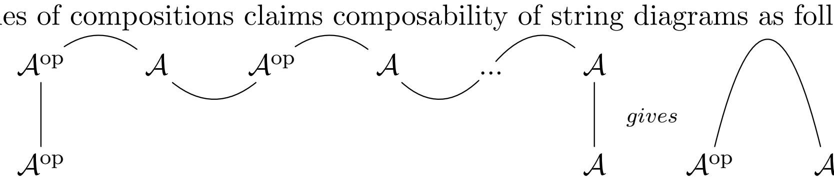
(26)

The outer diagram gives the statement we want. All lower shapes commute by claim 2.6. The left and right trapezoids commute by $\mathcal{V}$-naturalilty of partial transformations. Thus the above commutes if the pentagon commutes, which is seen to be a *compatibilty condition* on partial functors of T. $\square$

As in the case of ordinary $\mathcal{V}$-natural transformation we can **pre- and postcompose with** (adequate) **functors** P, Q to yield new $\mathcal{V}$-natural transformation $Q\alpha$ and α_P (compare construction of $\mathcal{V}$-**CAT** in section 3). It amounts to post- or precomposing with a single string in our diagram. Finally we can, as in the case of (VN1) $\mathcal{V}$-naturality, **tensor** our natural maps, to obtain a map between the corresponding tensored functors. This follows directly from equation (23) and by tensoring the naturality diagrams.

We now consider **vertical composition** $\beta \cdot \alpha$, i.e. composition of the string diagrams below. By our Prop 4.1 we can always fix variables, and thus can assume w.l.o.g. α has at most one argument in it's domain. The verification of the statement 'string diagrams can be composed' then splits up in the following three subcases of composition series:

1. The first series of compositions claims composability of string diagrams as follows:

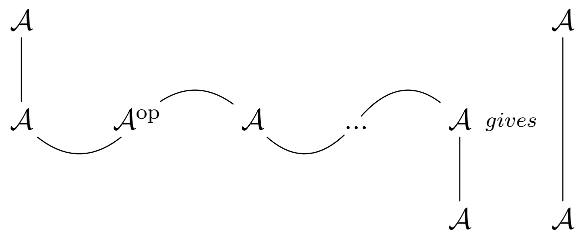


(27)

example: $\alpha_{AB} : K \longrightarrow T(A,A,B,B)$, $\beta_{ABC} : T(A,B,B,C) \longrightarrow S(A,C)$ for T, S with codomain $\mathcal{B}$ and domain as above. The composite is $\beta_{AAA}.\alpha_{AA} : K \longrightarrow S(A,A)$.

2. The second series is dual to the first (read the diagram upside down).

3. The third series claims composability of:



(28)

example: $\alpha_{AB} : T(A) \longrightarrow S(A,B,B)$, $\beta_{AB} : S(A,A,B) \longrightarrow R(B)$ giving the composite $\beta_{AA}.\alpha_{AA} : T(A) \longrightarrow R(A)$

Proof. Recall that $\mathcal{B}(\beta.\alpha, 1) = \mathcal{B}(\alpha,1)\mathcal{B}(\beta,1)$ and $\mathcal{B}(\alpha,1)$ commutes with $\mathcal{B}(1,\beta)$ by claim 2.6. The proofs are then straightforward; e.g. for the first term of the first series we alternatingly apply $\mathcal{V}$-naturality and commute our terms to get from $\mathcal{B}(\alpha_B,1)\mathcal{B}(\beta_{BB},1)S_{-B}$ to $\mathcal{B}(\alpha_A,1)\mathcal{B}(\beta_{AA},1)S_{A-}$. $\square$

4.2 Naturality of canonical maps

We will now give a range of useful and canonical examples for the concepts developed in the last section.

Lemma 4.2 ($\mathcal{V}$-naturality of general maps). *Let $R, S : \mathcal{A} \longrightarrow \mathcal{B}$, $T : \mathcal{A} \otimes \mathcal{B} \longrightarrow \mathcal{C}$ and $\alpha : R \longrightarrow S$. Conditions in the definition of these objects can be reformulated as follows*

a) $\alpha_A : I \longrightarrow \mathcal{B}(RA, SA)$ satisfies (VN2) iff $\alpha : R \longrightarrow S$ satisfies (VN1)

b) $S_{AB} : \mathcal{A}(A,B) \longrightarrow \mathcal{B}(SA, SB)$ satisfies (VN1) in B iff S satisfies (VF1). The same holds for $\mathcal{V}$-naturality in A.

c) $T_{-B,AA'} = T(-,B)_{AA'} : \mathcal{A}(A,A') \longrightarrow \mathcal{C}(T(A,B), T(A',B))$ satisfies (VN2) in B iff T fulfills the compatibility condition of Prop. 3.4.

Proof. a) Write out (VN2)

$$\begin{array}{ccccc}
\mathcal{A}(A,B) & \xrightarrow{S_{AB}} & \mathcal{B}(SA,SB) & \xrightarrow{\mathcal{B}(RA,-)} & [\mathcal{B}(RA,SA),\mathcal{B}(RA,SB)] \\
\downarrow{\scriptstyle R_{AB}} & & & & \\
\mathcal{B}(RA,RB) & & & & \downarrow{\scriptstyle [\alpha_A,1]} \\
\downarrow{\scriptstyle \mathcal{B}(-,SB)} & & & & \\
[\mathcal{B}(RB,SB),\mathcal{B}(RA,SB)] & & \xrightarrow[{[\alpha_B,1]}]{} & & [I,\mathcal{B}(TA,SB)]
\end{array} \tag{29}$$

and the statement follows immediately from lemma 3.17 b).

b) Write out (VN1) for the variable B (i.e. fix A in the statement):

$$\begin{array}{ccc} \mathcal{A}(C,D) & \xrightarrow{\mathcal{A}(A,-)} & [\mathcal{A}(A,C),\mathcal{A}(A,D)] \\ \downarrow{\scriptstyle \mathcal{B}(SA,-)S} & & \downarrow{\scriptstyle [1,S_{AD}]} \\ [\mathcal{B}(SA,SC),\mathcal{B}(SA,SD)] & \xrightarrow{[S_{AC},1]} & [\mathcal{A}(A,C),\mathcal{B}(SA,SD)] \end{array} \tag{30}$$

By Lemma 3.17 a) uncurrying the upper leg gives $S_{AD}e(\mathcal{A}\otimes 1) = S_{AD}M^{\mathcal{A}}$ and the lower leg gives $e(\mathcal{B}\otimes 1)(S_{CD}\otimes S_{AC}) = M^{\mathcal{B}}(S_{CD}\otimes S_{AC})$. Thus (VN1) in B is equivalent to (VF1). Considering S^{op} yields the statement in A

c) Fix A, A'. Set $Q(-,-) = \mathcal{C}(T(A,-),T(A',-))$. With our usual notation for partial functors, we want to verify

$$\begin{array}{ccc} \mathcal{B}(B,B') & \xrightarrow{Q_{B-}} & [Q(B,B),Q(B,B')] \\ {\scriptstyle Q_{-B'}}\downarrow & & \downarrow{\scriptstyle [T_{-B},1]} \\ [Q(B',B'),Q(B,B')] & \xrightarrow[{[T_{-B'},1]}]{} & [\mathcal{A}(A,A'),Q(B,B')] \end{array} \tag{31}$$

Employing Lemma 3.17 a) once more we can derive that this diagram corresponds under the adjunction to the equality of $M(T_{A'-}\otimes T_{-B}) = M(T_{-B'}\otimes T_{A-})c$ expressing the compatibility of partial functors.

□

We should dwell a bit on this lemma: We expressed *three seemingly different ideas* (naturality, functoriality, compatibility of partial functors) in terms of the same idea. Also, part c) gives a justification of the non-standard (but space saving) notation for partial functors in this essay.

Lemma 4.3 (**$\mathcal{V}$-naturality of canonical maps**)**.** *We have*

a) $\mathcal{A}(A,-)_{BC}$ *and* $\mathcal{A}(A,g)$ *are* $\mathcal{V}$*-natural in all variables*

b) $e^Y_Z : [Y,Z]\otimes Y \longrightarrow Z$ *(the LHS is the functor* $\text{Ten}(\hat{\mathcal{V}}(-,-),-)$ *evaluated at* (Y,Z,Y)*) is* $\mathcal{V}$*-natural in all variables. And so are* $a_{XYZ}, l_Z, r_Z, c_{XY}$ *(where the functors are formed of appropriate instances of* Ten*).*

c) $M_{ABC} : \mathcal{A}(B,C)\otimes\mathcal{A}(A,B)\longrightarrow\mathcal{A}(A,C)$ *is* $\mathcal{V}$*-natural in all variables (where the LHS is the functor* $\text{Ten}(\mathcal{A}(-,-),\mathcal{A}(-,-))$*)*

d) $j_A : I\longrightarrow\mathcal{A}(A,A)$ *is* $\mathcal{V}$*-natural. For* $\mathcal{A}=\mathcal{V}$*,* $f : A\longrightarrow B$ *this implies* $[1,f]j_A = [f,1]j_B$*.*

e) $p_{XYZ} : [X\otimes Y,Z]\longrightarrow[X,[Y,Z]]$ *(where the LHS is the functor* $\hat{\mathcal{V}}(\text{Ten}(-,-),-)$ *and the RHS is* $\hat{\mathcal{V}}(-,\hat{\mathcal{V}}(-,-))$*) is natural in all variables. And so are* d^Y_X, i_Z*.*

Proof. a) $\mathcal{A}(A,-)_{BC}$ is $\mathcal{V}$-natural in all variables by the previous lemma part b). So we have that $\mathcal{A}(A,-)_{BC}g : I\longrightarrow[\mathcal{A}(A,B),\mathcal{A}(A,C)]$, for $g : I\longrightarrow\mathcal{A}(B,C)$, is $\mathcal{V}$-natural in A, and thus by the previous Lemma part a) a natural transformation (which is identified with $\mathcal{A}(A,g)$).

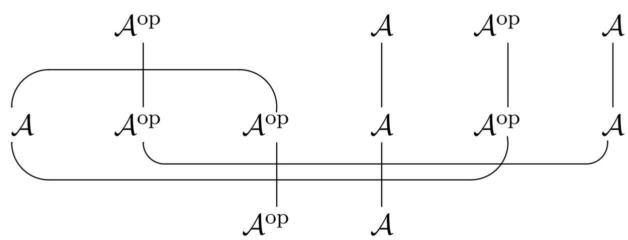
b) As an example of (VN3), we verify naturality of e in Y

$$\begin{array}{ccc} [Y,Y'] & \xrightarrow{\mathrm{Ten}([Y',Z],-)} & [[Y',Z]\otimes Y,[Y',Z]\otimes Y'] \\ \downarrow{\scriptstyle \hat{\mathcal{V}}^{\mathrm{op}}(Z,-)} & & \\ [[Y',Z],[Y,Z]] & & \downarrow{\scriptstyle [1,e_Z^{Y'}]} \\ \downarrow{\scriptstyle \mathrm{Ten}(-,Y)} & & \\ [[Y',Z]\otimes Y,[Y,Z]\otimes Y] & \xrightarrow{[1,e_Z^Y]} & [[Y',Z]\otimes Y,Z] \end{array} \tag{32}$$

Recall that we have $\mathrm{Ten}(-,Y)_{XX'} = \mathrm{Ten}(-,-)_{(XY)(X'Y)}(1\otimes j_Y)r^{-1}$ by definition of partial functors. Then we can use Lemma 3.17, as well as the definitions $e(\mathcal{V}^{\mathrm{op}}\otimes 1) = Mc$, $e(\mathrm{Ten}\otimes 1) = (e\otimes e)m$ and $e(j\otimes 1) = l$ to derive the statement by uncurrying the above diagram.

The cases of the maps a,l,r,c are treated in exactly the same spirit.

c) By definition $M_{ABC} = e(\mathcal{A}(A,-)_{BC}\otimes 1)$. To apply our compositional calculus we note: Firstly, by precomposing e_Z^Y in each label with the functor $\mathcal{A}(-,-)$ it becomes a $\mathcal{V}$-natural map of the form $e_{\mathcal{A}(C,D)}^{\mathcal{A}(A,B)} : S(A,B,C,D,A,B)\longrightarrow R(C,D)$. Secondly, precompose 1_A with $\mathcal{A}(-,-)$ and then tensor it with $\mathcal{A}(A,-)_{BC}$ (natural by a)) to obtain a natural map of the form $\mathcal{A}(A,-)_{BC}\otimes 1_{\mathcal{A}(D,E)} : T(B,C,D,E)\longrightarrow S(A,B,A,C,D,E)$. We can now draw a nice string diagram



(33)

to see that by the second and third composition series M composes to be a $\mathcal{V}$-natural map in all variables.

d) $j_A : I\longrightarrow\mathcal{A}(A,A)$ is a natural transformation transformation between the functors $1_{\mathcal{A}}$ and $1_{\mathcal{A}} : \mathcal{A}\longrightarrow\mathcal{A}$. Thus by the previous Lemma, it is (VN2) natural in A. Note that applying V to the (VN1) naturality of j means $\mathcal{A}(j_A,1)f = \mathcal{A}(1,j_B)f$ or equivalently by claim 2.6, $\mathcal{A}(1,f)j_A = \mathcal{A}(f,1)j_B$. Thus in the case $\mathcal{A}=\mathcal{V}$ we (re)derived for $f : A\longrightarrow B$

$$[1,f]j_A = [f,1]j_B \tag{34}$$

e) Following the spirit of part c) we just note the equalities : $p^{-1} = [1,e]\,\mathrm{Ten}(-,Y)$, $d = e(p\otimes 1)(j\otimes 1)l^{-1}$, $i = [1,r]d$ (proved, of course, by uncurrying and using our defining equations (11) of p^{-1} and i, and definitions of d and j in $\mathcal{V}$). Then $\mathcal{V}$-naturality follows from our compositional calculus as in part c).

□

Having shown d,e to be $\mathcal{V}$-natural enables us (together with our results about composition) to state the following general principle. Below A,B,C,D,E,F each denote a set of variables.

Lemma 4.4 (**a general principle for $\mathcal{V}$-naturality**)**.** *For T, S, R $\mathcal{V}$-functors with codomain $\hat{\mathcal{V}}$ we have that $\alpha : T(D,D,A,B) \otimes S(E,E,A,C) \longrightarrow R(F,F,B,C)$ is $\mathcal{V}$-natural if and only if $\overline{\alpha} : T(D,D,A,B) \longrightarrow [S(E,E,A,C), R(F,F,B,C)]$ is $\mathcal{V}$-natural (in all variables)*

Proof. $\alpha = e(\alpha \otimes 1)$. $\overline{\alpha} = [1,\alpha]d$. □

4.3 Forms of the weak Yoneda Lemma

Lemma 4.5 (**weak Yoneda lemma**)**.** *Let $\alpha : \mathcal{A}(K,A) \longrightarrow FA$ denote a natural transformation between $\mathcal{A}(K,-)$ and $F : \mathcal{A} \longrightarrow \hat{\mathcal{V}}$. Let $\eta : I \longrightarrow FK$ denote an element of FK. The following mapping gives a bijection between the set of natural transformations $\mathcal{A}(K,-) \longrightarrow F$ and the set of elements of FK:*

$$\alpha \mapsto \eta(\alpha) : I \xrightarrow{j_K} \mathcal{A}(K,K) \xrightarrow{\alpha_K} FK$$

$$\eta \mapsto \alpha(\eta)\ ,\ \alpha(\eta)_A : \mathcal{A}(K,A) \xrightarrow{F_{KA}} [FK, FA] \xrightarrow{[\eta,1]} [I, FA] \xrightarrow{i^{-1}} FA$$

Proof. 1. We want to show $\eta(\alpha(\eta)) = \eta$. Consider

$$\begin{array}{ccccc} [FK,FK] & \xrightarrow{[\eta,1]} & [I,FK] & \xrightarrow{i_{FK}^{-1}} & FK \\ {\scriptstyle j_{FK}}\uparrow & & \uparrow{\scriptstyle [1,\eta]} & & \uparrow{\scriptstyle \eta} \\ I & \xrightarrow{j_I} & [I,I] & \xrightarrow{i_I^{-1}} & I \end{array} \tag{35}$$

Using $j_I = i_I$ (by definition $\overline{j_I} = r = \bar{i}$), the lower leg gives η. The upper leg gives $\eta(\alpha(\eta))$. The diagram commutes by $\mathcal{V}$-naturatily of j (Lemma 4.3 d)) and naturality of i.

2. We want to show $\alpha(\eta(\alpha)) = \alpha$. Consider

$$\begin{array}{ccccccc} \mathcal{A}(K,A) & \xrightarrow{\mathcal{A}(K,-)} & [\mathcal{A}(K,K), \mathcal{A}(K,A)] & \xrightarrow{[j,1]} & [I, \mathcal{A}(K,A)] & \xrightarrow{i^{-1}} & \mathcal{A}(K,A) \\ {\scriptstyle F_{KA}}\downarrow & & {\scriptstyle [1,\alpha_A]}\downarrow & & {\scriptstyle [1,\alpha_A]}\downarrow & & \downarrow{\scriptstyle \alpha_A} \\ [FK,FA] & \xrightarrow{[\alpha_K,1]} & [\mathcal{A}(K,K), FA] & \xrightarrow{[j,1]} & [I,FA] & \xrightarrow{i^{-1}} & FA \end{array} \tag{36}$$

The diagram commutes by $\mathcal{V}$-naturality of α and naturality of i. The top leg of this diagram is $1_{\mathcal{A}(K,A)}$ since by Lemma 3.17 $[j,1]\mathcal{A}(K,-) = i$. Comparing the legs then yields the statement. □

Corollary 4.6 (**extra variable Yoneda**)**.** *Let $F : \mathcal{B}^{\mathrm{op}} \otimes \mathcal{A} \longrightarrow \mathcal{V}$, $K : \mathcal{B} \longrightarrow \mathcal{A}$ and $\alpha_{BA} : \mathcal{A}(KB, A) \longrightarrow F(B,A)$ be $\mathcal{V}$-natural in A. By Yoneda, let $\eta = \eta(\alpha_{B\,\cdot}) : I \longrightarrow F(B, KB)$. Then η_B is $\mathcal{V}$-natural in B if and only of α_{BA} is $\mathcal{V}$-natural in B*

Proof. Just write out the Yoneda bijection from the previous Lemma

$$\eta(\alpha_{B\,\cdot}) : I \xrightarrow{j_{KB}} \mathcal{A}(KB,KB) \xrightarrow{\alpha_{BKB}} F(B,KB)$$

$$\alpha(\eta_B)_A :\ \mathcal{A}(KB,A) \xrightarrow{F(B,-)_{(KB)A}} [F(B,KB), F(B,A)] \xrightarrow{[\eta_B,1]} [I, F(KB,A)] \xrightarrow{i^{-1}} F(B,A)$$

Our results for composition of $\mathcal{V}$-natural maps imply that if one of α_{BA}, η_B is $\mathcal{V}$-natural in B the other is as well. □

Corollary 4.7 (**Yoneda for representables**). *For $T : \mathcal{A} \longrightarrow \mathcal{B}$ take $F = \mathcal{B}(B, T-)$.*

a) A map $\eta : I \longrightarrow F(K) = \mathcal{B}(B, TK)$ corresponds under the Yoneda bijection to

$$\alpha(\eta)_A : \mathcal{A}(K, A) \xrightarrow{T_{KA}} \mathcal{B}(TK, TA) \xrightarrow{\mathcal{B}(\eta,1)} \mathcal{B}(B, TA) \tag{37}$$

b) Let $T = 1_{\mathcal{A}}$ (i.e $F = \mathcal{A}(B, -)$). Then $\alpha_A = \mathcal{A}(\eta, A)$ and $\alpha : \mathcal{A}(K, -) \longrightarrow F$ is an iso if and only if $\eta(\alpha)$ is an iso.

Proof. a) Apply Lemma 3.17 b) to the Yoneda bijection: $i^{-1}[\eta, 1]\mathcal{B}(B, -) = i^{-1}i\mathcal{B}(\eta, 1)$.

b) Suppose η is an iso. Then $\alpha = \mathcal{A}(\eta, 1)$ (by part a)) is an iso. Conversely suppose α is an iso. Then $\alpha\alpha^{-1} = \mathcal{A}(\eta(\alpha).\eta(\alpha^{-1}), 1) = \mathcal{A}(1, 1) = 1$ and $\alpha^{-1}\alpha = \mathcal{A}(\eta(\alpha^{-1}).\eta(\alpha), 1) = \mathcal{A}(1, 1)$. By bijectiveness of the Yoneda map we deduce $\eta(\alpha).\eta(\alpha^{-1}) = 1 (= j_B)$ and $\eta(\alpha^{-1}).\eta(\alpha) = 1 (= j_K)$. Thus $\eta(\alpha)$ is an isomorphism in $\mathcal{A}_0$.

□

Definition 4.8 (**representations**). $F : \mathcal{A} \longrightarrow \mathcal{V}$ is called representable if there is an $\mathcal{V}$-natural isomorphism $\alpha : \mathcal{A}(K, -) \longrightarrow F$. (K, α) is called a representation of F, $\eta(\alpha)$ is called its counit.

Note that by the previous Corollary any two representations $(K, \alpha), (K', \alpha')$ of F are isomorphic as follows: By part b) $(\alpha')^{-1}\alpha = \mathcal{A}(k, -)$ and this $k : K \longrightarrow K'$ is an iso.

If we 'index' a family of representables with an additional variable we get the following

Lemma 4.9 (**extra variable representations**). *Let $F : \mathcal{B}^{\mathrm{op}} \otimes \mathcal{A} \longrightarrow \mathcal{V}$ such that for all B $F(B, -)$ is representable by (KB, α_{BA}), where K is a map on objects. Then there is a unique way to make K a $\mathcal{V}$-functor $K : \mathcal{B} \longrightarrow \mathcal{A}$ (i.e. define K_{BC}) such that α_{BA} becomes $\mathcal{V}$-natural in B.*

Proof. By our 'extra variable Yoneda' Corollary 4.6 α_{BA} is $\mathcal{V}$-natural in B if and only if the corresponding family $\eta_B : I \longrightarrow F(B, KB)$ is. This determines K_{BC} is follows

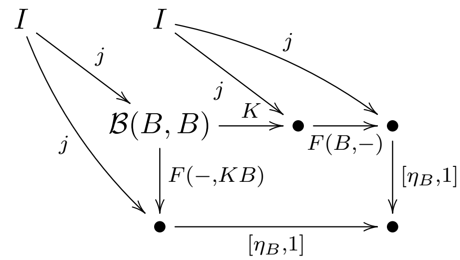
$$\begin{array}{ccccc}
\mathcal{B}(B,C) & \xrightarrow{\quad K_{BC} \quad} & \mathcal{A}(KB,KC) & & \\
& & {\scriptstyle \alpha_{B(KC)}} \swarrow \quad \searrow {\scriptstyle F(B,-)_{KBKC}} & & \\
\downarrow {\scriptstyle F(-,KC)_{CB}} & F(B,KC) & & [F(B,KB),F(B,KC)] & \\
& {\scriptstyle i_{F(B,KC)}} \searrow & & \swarrow {\scriptstyle [\eta_B,1]} & \\
[F(C,KC),F(B,KC)] & \xrightarrow{\quad [\eta_C,1] \quad} & [I,F(B,KC)] & &
\end{array} \tag{38}$$

The outer diagram just (VN2) for η_B. The right square is seen to follow from the Yoneda bijection. Since $F(B, -)$ is representable α is an isomorphism and we deduce

$$K_{BC} = \alpha^{-1}_{B(KC)} i^{-1}_{F(B,KC)} [\eta_C, 1] F(-, KC)_{CB} \tag{39}$$

Thus K_{BC} is seen to be $\mathcal{V}$-natural in B (not yet in C since in the equation above we don't know if K is a $\mathcal{V}$-functor).

But then, by Lemma 4.2 b) K_{BC} satisfies (VF1). It remains to show (VF2). Set $B = C$ in the above diagram. Then we have following commutative diagram



(40)

The outer legs of this diagram are equal. We deduce $[\eta_B, 1]F(B, -)Kj_B = [\eta_B, 1]F(B, -)j_{KB}$. Apply $\alpha^{-1}i^{-1}$ to get (VF2).

□

4.4 Applications of the weak Yoneda Lemma

We are now in the position to rediscover more parts of ordinary category theory. We have established that $\mathcal{V}$-**CAT** is a 2-category and in every 2-category (actually also in the weak case) we have the notion of adjunctions, defined below. But with the Yoneda Lemma at hand, we will see that this notion becomes richer and more powerful.

Definition 4.10 (adjunctions). An adjunction $T \dashv S$ consists of $\mathcal{V}$-functors $T : \mathcal{A} \longrightarrow \mathcal{B}$, $S : \mathcal{B} \longrightarrow \mathcal{A}$ and $\mathcal{V}$-natural transformations $\eta : 1 \longrightarrow TS$, $\epsilon : ST \longrightarrow 1$, called unit and counit, which satisfy the triangular identities $T\epsilon \cdot \eta_T = 1_T$ and $\epsilon_S \cdot S\eta = 1_S$.

We write out the triangular identities in variables:

$$T\epsilon_A.\eta_{TA} = j_{TA} \quad , \quad \epsilon_{SB}.S\eta_B = j_{SB} \tag{41}$$

Lemma 4.11. *Take T, S as above. Then specifying an adjunction $T \dashv S$ by natural transformations η, ϵ satisfying the triangular identities is equivalent to specifying a $\mathcal{V}$-natural isomorphism*

$$n_{AB} : \mathcal{A}(SB, A) \cong \mathcal{B}(B, TA) \tag{42}$$

natural in A and B.

Proof. Assume $T \dashv S$ is given by η, ϵ satisfying the triangular identities: Set $F(-,-) = \mathcal{B}(-, T-)$. Then $\eta_B : I \longrightarrow \mathcal{B}(B, TSB) = F(B, SB)$ corresponds by our 'extra-variable Yoneda' Corollary 4.6 to a family of maps $\alpha(\eta_B)_A$ $\mathcal{V}$-natural in A and B. By Corollary 4.7 these are of the form:

$$n_{AB} := \alpha(\eta_B)_A = \mathcal{B}(\eta_B, 1)T_{SBA} \quad , \quad n_{AB} : \mathcal{A}(SB, A) \longrightarrow \mathcal{B}(B, TA) \tag{43}$$

Dually (via $S^{\mathrm{op}} \dashv T^{\mathrm{op}} : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{B}^{\mathrm{op}}$) define a $\mathcal{V}$-natural map $\bar{n}$ as follows

$$\bar{n}_{AB} = \mathcal{A}(1, \epsilon_A)S_{BTA} \quad , \quad \bar{n}_{AB} : \mathcal{B}(B, TA) \longrightarrow \mathcal{A}(SB, A) \tag{44}$$

We need to show $\bar{n}_{AB}n_{AB} = 1_{\mathcal{A}(SB,A)}$. Equivalently 1 and $\bar{n}n$ need to correspond to the same map under the Yoneda bijection (for fixed B):

$$j_{SB} = 1_{\mathcal{A}(SB,SB)}j_{SB} \stackrel{!}{=} \bar{n}_{SBB}n_{SBB}j_{SB} = \mathcal{A}(1, \epsilon_{SB})S_{BTSB}\mathcal{B}(\eta_B, 1)T_{SBSB}j_{SB}$$

The RHS now is seen to be $\epsilon_{SB}.S_{BTSB}\eta_B$ if we use (VF2) for T, (VF1) for S and the rules of claim 2.6 (In other words, translate the ordinary steps to get from $\epsilon S(T(1)\eta)$ to $\epsilon S\eta$). Thus we have shown that $\bar{n}n = 1$ *is equivalent* to the second triangular identity. Dually, $n\bar{n} = 1$ is equivalent to the first triangular identity.

Conversely, we just define η_B, ϵ_A to correspond to n_{AB} and n_{AB}^{-1} by the Yoneda bijection. $\mathcal{V}$-naturality follows from Cor. 4.6, and the triangular identities from the argument above. □

If we combine this previous Lemma and Lemma 4.9 about 'parametrized representations' we obtain the following

Corollary 4.12. *$T : \mathcal{A} \longrightarrow \mathcal{B}$ has left adjoint if and only if $\mathcal{B}(B, T-)$ is representable for all B.*

Examples 4.13. For $\mathcal{V} = \mathbf{Set}$ we have a criterion for $\mathcal{B}(B, T-)$ being representable. Namely it is representable if and only if $(B \downarrow T)$ has initial object (SB, η) (cf. [1]). For general $\mathcal{V}$ there is no such criterion.

Definition 4.14 (**equivalences in $\mathcal{V}$-CAT**)**.** We call $\mathcal{A}$ and $\mathcal{B}$ equivalent, if there exist $\mathcal{V}$-functors $T : \mathcal{A} \longrightarrow \mathcal{B}$, $S : \mathcal{B} \longrightarrow \mathcal{A}$ and $\mathcal{V}$-natural ismorphisms $\eta : 1 \longrightarrow TS$, $\zeta : ST \longrightarrow 1$. We speak of an adjoint equivalence if $T \dashv S$.

Lemma 4.15. *Given an equivalence as above it can be made into an adjoint equivalence with unit η.*

Proof. Set $\epsilon = \zeta \cdot S\eta_T^{-1} \cdot \zeta_{ST}^{-1}$ to be the counit. As in the last proof translate the ordinary steps to the enriched case in order to verify the triangular identities. (cf. [1]) □

Lemma 4.16. *In an adjunction as above, T is fully faithful if and only if ϵ is iso.*

Proof. We have the equality

$$\begin{array}{ccc} \mathcal{A}(A, A') & \xrightarrow{T_{AA'}} & \mathcal{B}(TA, TA') \\ & {\scriptstyle \mathcal{A}(\epsilon_A, 1)} \searrow \quad \nearrow {\scriptstyle n_{TAA'}} & \\ & \mathcal{A}(STA, A') & \end{array} \tag{45}$$

since under the Yoneda bijection: $n_{TAA}\mathcal{A}(\epsilon_A, 1)j_A = n_{TAA}\epsilon_A = \mathcal{A}(\eta_{TA}, 1)T\epsilon_A = j_{TA} = T_{AA}j_A$. The second equality holds by def. of n, the third is the first triangular identity. Clearly ϵ_A being iso makes T fully faithful. Conversely, $n_{AA'}^{-1}T_{AA'}$ being iso makes ϵ_A iso by Corollary 4.7 b). □

Proposition 4.17 (**$\mathcal{V}$-category equivalence conditions**)**.** *$T : \mathcal{A} \longrightarrow \mathcal{B}$ is part of an equivalence if and only if it is fully faithful and essentially surjective*

Proof. If T is part of and equivalence it is essentially surjective. By the previous Lemmas 4.15 and 4.16 we may choose the equivalence adjoint, so that ϵ is iso, and T thus fully faithful.

Conversely, choose SB, and $\eta_B : B \longrightarrow TSB$ iso by T being essentially surjective. Then $n := \mathcal{B}(\eta_B, 1)T_{SBA}$ is iso (T is f.f.) and $\mathcal{V}$-natural in A. We can regard it as 'parametrized representation', and apply Lemma 4.9 to get functoriality for S and $\mathcal{V}$-naturality in B. This determines our adjunction by Lemma 4.11, and ϵ is iso by the triangular identities and T being full and faithful. □

Of course, adjunctions in $\mathcal{V}$-**CAT** *carry over* to ordinary adjunctions in the underlying categories, by applying $(-)_0$ to categories, functors and natural transformation. We just note:

$$n_0 \equiv \mathcal{B}_0(\eta, 1)T_0 f = V\mathcal{B}(\eta, 1)Tf = Vn \tag{46}$$

by our result on 'lifting the Hom' functors in equation (20).

In this section we have have recovered many basic result about adjunctions and equivalences from ordinary category theory. This was made possible by the presence of the Yoneda bijection that we established in the previous subsections. Finally, we note that as for ordinary categories, adjunctions in $\mathcal{V}$-**CAT** can be composed (by composition of n's) and form a category (actually even a 2-category through the definition of *mates*). We will end this section with 2 examples.

Examples 4.18. a) $p_{XYZ} : \hat{\mathcal{V}}(\mathrm{Ten}(X,Y), Z) \cong \hat{\mathcal{V}}(X, \hat{\mathcal{V}}(Y,Z))$ is $\mathcal{V}$-natural in all variables and thus gives rise to an adjunction with $S = \mathrm{Ten}(-,Y)$ and $T = \hat{\mathcal{V}}(Y,-)$. Thus by constructing p we have not only lifted π but also it's adjunction.

b) $q := p\hat{\mathcal{V}}(c,1)p^{-1} : \hat{\mathcal{V}}(X, \hat{\mathcal{V}}(Y,Z)) \cong \hat{\mathcal{V}}(Y, \hat{\mathcal{V}}(X,Z))$ is $\mathcal{V}$-natural and thus gives an adjunction with $S = (\hat{\mathcal{V}}(-,Z))^{\mathrm{op}} : \hat{\mathcal{V}} \longrightarrow \hat{\mathcal{V}}^{\mathrm{op}}$ and $T = \hat{\mathcal{V}}(-,Z) : \hat{\mathcal{V}}^{\mathrm{op}} \longrightarrow \hat{\mathcal{V}}$. The underlying adjunction (via eq. (46)) can be identified with

$$q_0 : \pi Y(c)_Z \pi^{-1} : \mathcal{V}_0(X, [Y,Z]) \cong \mathcal{V}_0(Y, [X,Z]) \tag{47}$$

This operation corresponds to 'changing the curried variable' (at least, if $\mathcal{V} = \mathbf{Set}$).

5 Functor categories

So far we internalized Hom sets of $\mathcal{V}$ to form $\hat{\mathcal{V}}$ and lifted the Hom functors of underlying categories, the tensor product of $\mathcal{V}_0$ and the closedness adjunction of $\mathcal{V}_0$. Now we would like to lift the Yoneda bijection. In particular we need to 'internalize' sets of $\mathcal{V}$-natural transformations to objects in $\mathcal{V}$. In other words, we want to internalize functor categories of $\mathcal{V}$-**CAT** to $\mathcal{V}$-categories. Instead of constructing this indirectly by an adjunction to the tensor map (as we did for $\mathcal{V}$ itself) we will give an explicit construction by the notion of ends. But in the section on closedness below we will reconcile these two approaches.

We assume $\mathcal{V}$ to be closed symmetric monoidal and complete.

5.1 Construction of functor $\mathcal{V}$-categories via ends

For the explicit construction of the internatlized functor categories, we seek a $K \in \mathcal{V}$ such that VK corresponds to the set of natural transformations $R \longrightarrow S$ (for R, S $\mathcal{V}$-functors).

Definition 5.1. Let $T : \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \longrightarrow \mathcal{V}$. A $\mathcal{V}$-natural family $\lambda_A : K \longrightarrow T(A,A)$ is called an end of T, if for all $\mathcal{V}$-natural families $\alpha_A : Y \longrightarrow T(A,A)$ there exists a unique factorizing $f : Y \longrightarrow K$ such that $\alpha_A = \lambda_A.f$. We call K or the tuple (K, λ) an end, λ it's counit, and denote K by $\int_{A \in \mathcal{A}} T(A,A)$. Write $\alpha = \alpha(f)$ and $f = f(\alpha)$ for the 1-1 correspondence by universality of λ.

We have an equivalent characterization of this definition as follows (recall eq. (47)):

Lemma 5.2 (ends as equalizers). (K, λ) *is an equalizer of*

$$K \xrightarrow{\lambda} \prod_{A \in \mathcal{A}} T(A, A) \underset{\sigma}{\overset{\rho}{\rightrightarrows}} \prod_{A, B \in \mathcal{A}} [\mathcal{A}(A, B), T(A, B)] \tag{E}$$

where $\pi_{AB}\rho := \rho_{AB}\pi_A$*,* $\rho_{AB} := q_0(T(A, -)_{AB})$ *and* $\pi_{AB}\sigma := \sigma_{AB}\pi_B$*,* $\sigma_{AB} = q_0(T(-, B)_{BA})$*, if and only if* $\lambda_A = \pi_A\lambda : K \longrightarrow T(A, A)$ *is an end of* T*.*

Proof. We claim the following diagrams are equivalent:

$$\begin{array}{ccc} \mathcal{A}(A,B) & \xrightarrow{T(A,-)_{AB}} & [T(A,A), T(A,B)] \\ {\scriptstyle T(-,B)_{BA}}\downarrow & & \downarrow{\scriptstyle [\lambda_A, 1]} \\ [T(B,B), T(A,B)] & \xrightarrow[{[\lambda_B, 1]}]{} & [K, T(A,B)] \end{array} \quad \text{(VN2)} \quad \Leftrightarrow \quad \begin{array}{ccc} K & \xrightarrow{\lambda_A} & T(A,A) \\ {\scriptstyle \lambda_B}\downarrow & & \downarrow{\scriptstyle \rho_{AB}} \\ T(B,B) & \xrightarrow[\sigma_{AB}]{} & [\mathcal{A}(A,B), T(A,B)] \end{array} \tag{48}$$

To see this apply q_0 to the left diagram (recalling $q_0 = \pi Y(c)\pi^{-1}$). By Lemma 3.17 we have $\pi^{-1}([\lambda, 1]T) = e(T \otimes \lambda)$. Thus

$$q_0([\lambda, 1]T) = \pi(e(T \otimes \lambda)c) = \pi(e(T \otimes 1)c(\lambda \otimes 1)) = \pi(e(T \otimes 1)c)\lambda = q_0(T)\lambda \tag{49}$$

Taking T to be $T(-, B)$ and $T(A, -)$ above (and λ accordingly) proofs our claim. It follows: A morphism $\alpha : Y \longrightarrow \prod_{A \in \mathcal{A}} T(A, A)$ satisfies $\sigma\alpha = \rho\alpha$ *if and only if* we have $\pi_{AB}\sigma\alpha = \pi_{AB}\rho\alpha$ *if and only if* $\pi_A\alpha$ satisfies (VN2) (by the equivalence above). Thus there is a one to one correspondence of cones (K, α) and $\mathcal{V}$-natural transformations α_A **via** $\alpha_A = \pi_A\alpha$. The universal property of a coequalizer of (E) and an end become equivalent. □

This Lemma provides us with a tool to discuss *existence of ends*. In particular, if $\mathcal{A}$ is small we immediately see ends to exist.

We can use the fact that our adjunction π preserves naturality (Lemma 4.4) to prove:

Lemma 5.3. *The right adjoint functor* $[X, -]$ *preserves ends.*

Proof. Let $(\int T(A, A), \lambda)$ be an end of T. By our adjunction we have a 1-1 correspondence of maps

$$\begin{array}{ccc} & Y & \\ {\scriptstyle f}\swarrow & & \searrow{\scriptstyle \alpha_A} \\ [X, \int T(A,A)] & \xrightarrow[{[1, \lambda_A]}]{} & [X, T(A,A)] \end{array} \quad \overset{\pi}{\longleftrightarrow} \quad \begin{array}{ccc} & Y \otimes X & \\ {\scriptstyle \overline{f}}\swarrow & & \searrow{\scriptstyle \overline{\alpha_A}} \\ \int T(A,A) & \xrightarrow[\lambda_A]{} & T(A,A) \end{array} \tag{50}$$

Now α_A is $\mathcal{V}$-natural if and only $\overline{\alpha_A}$ is. So given α as above there exists a unique f making the right diagram commute. Thus $[1, \lambda_A]$ is an end. □

We deduce an isomorphism

$$[X, \int_{A \in \mathcal{A}} T(A, A)] \cong \int_{A \in \mathcal{A}} [X, T(A, A)] \tag{51}$$

Note that the above proof resembles a proof of the fact that right adjoints preserve limits. But $\mathcal{V}$-natural families are not cones. To generalize the above Lemma we would probably require adjunctions that preserve $\mathcal{V}$-naturality, or some other compatibility in conjunction with Lemma 5.2.

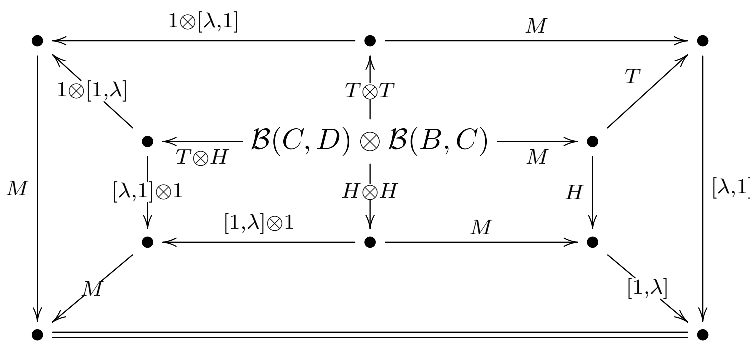
As in the case of ends, we can now consider the induced functors of an end with an additional indexing variable. So let $T : \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \otimes \mathcal{B} \longrightarrow \hat{\mathcal{V}}$

Lemma 5.4 (extra variable ends). *Let $\lambda_{AB} : H(B) := \int_A T(A, A, B) \longrightarrow T(A, A, B)$ be an end of T for all B. Then there is a unique way to make H a functor $H : \mathcal{B} \longrightarrow \hat{\mathcal{V}}$ such that λ_{AB} becomes $\mathcal{V}$-natural in B.*

Proof. Write out the $\mathcal{V}$-naturality condition for λ_{AB} in B:

$$\begin{array}{ccc} \mathcal{B}(B, C) & \xrightarrow{\;H_{BC}\;} & [HB, HC] \\ {\scriptstyle T(A,A,-)_{BC}} \downarrow & & \downarrow {\scriptstyle [1,\lambda_{AC}]} \\ [T(A, A, B), T(A, A, C)] & \xrightarrow{[\lambda_{AB},1]} & [HB, T(A, A, C)] \end{array} \qquad (52)$$

By the previous lemma $[1, \lambda_{\cdot C}]$ is an end. Since the lower leg is natural in A, there is a unique way to choose the H_{BC} to make the above diagram commute, for all B, C. We have to show this makes H functorial. This follows from universality of $[1, \lambda]$. For instance, (VF1) follows from



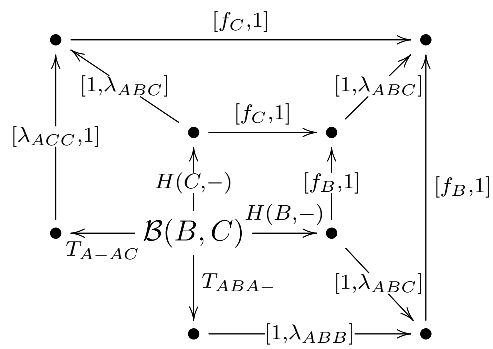
(53)

We want the right central square to commute. Everything else commutes by (52), (VF1) for T and claim 2.6. Thus, postcomposing the square with $[1, \lambda]$ and using uniqueness of factorizing maps gives the statement. ☐

Note that by the very definition of ends, there is a 1-1 correspondence between $\mathcal{V}$-natural families $\alpha_{AB} : Y \longrightarrow T(A, A, B)$ and induced maps $f_B : Y \longrightarrow H(B)$, vaguely reminding us of our (extra variable) Yoneda bijection. We can now discuss $\mathcal{V}$-naturalilty of f in B (if B stands for an appropriate set of variables). So the following Lemma can be seen in analogy to our 'extra variable Yoneda' Lemma

Lemma 5.5 (naturality in extra variables of ends). *Let $T : \mathcal{A}^{\mathrm{op}} \otimes \mathcal{B}^{\mathrm{op}} \otimes \mathcal{A} \otimes \mathcal{B} \longrightarrow \hat{\mathcal{V}}$ and $(H(B, C), \lambda_{\cdot BC})$ an end of $T(-, B, -, C)$ where $H : \mathcal{B}^{\mathrm{op}} \otimes \mathcal{B} \longrightarrow \hat{\mathcal{V}}$ as in the previous Lemma. Let $\alpha_{AB} : Y \longrightarrow T(A, B, A, B)$ be natural in A, and $f_B = f(\alpha_{\cdot B}) : X \longrightarrow H(B, B)$ the factorizing map. So $\alpha_{AB} = \lambda_{ABB} f_B$. Then, α is $\mathcal{V}$-natural in B if and only if f is $\mathcal{V}$-natural in B.*

Proof. Since $\alpha_{AB} = \lambda_{ABB} f_B$, f being $\mathcal{V}$-natural clearly implies α being $\mathcal{V}$-natural in B. Conversely assume the product $\lambda_{ABB} f_B = \alpha_{AB}$ is $\mathcal{V}$-natural in B. Consider the diagram

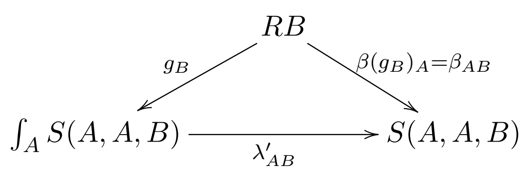


(54)

The inner square is the statement we want to show. The other parts commute trivially or by $\mathcal{V}$-naturality of λ. Now by our assumption, $\lambda_{ABB} f_B$ is $\mathcal{V}$-natural in B, and so the outer legs of this diagrams commute. Arguing by universality of $[1, \lambda]$ yields the statement.

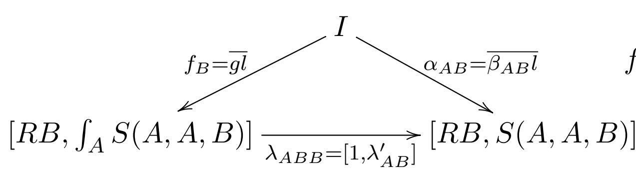
□

Let $R : \mathcal{B} \longrightarrow \hat{\mathcal{V}}$, $S : \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \otimes \mathcal{B} \longrightarrow \hat{\mathcal{V}}$. We claim that our previous Lemma implies



g_B $\mathcal{V}$-nat. $\Leftrightarrow$ β_{AB} $\mathcal{V}$-nat. in B (55)

Indeed, when identifying $RB \cong I \otimes RB$ via the $\mathcal{V}$-natural isomorphism l, under adjunction the above becomes equivalent to



f_B $\mathcal{V}$-nat. $\Leftrightarrow$ α_{AB} $\mathcal{V}$-nat. in B

which is just a special case of our previous Lemma.

Corollary 5.6 (**Fubini's Theorem**)**.** *With objects as in the previous Lemma.*

a) If $\int_A T(A, B, A, C)$ exists for all B, C then

$$\int_{(A,B)} T(A, B, A, B) \cong \int_B H(B, B) = \int_B \int_A T(A, B, A, B)$$

b) If in addition $\int_B T(A, B, D, B)$ exists for all A, D then

$$\int_A \int_B T(A, B, A, B) \cong \int_B \int_A T(A, B, A, B)$$

Proof. α_{AB} $\mathcal{V}$-natural in (A,B) corresponds to a $\mathcal{V}$-natural $f_B = f(\alpha_{\cdot B}) : Y \longrightarrow H(B,B)$ and factors uniquely as $\alpha_{AB} = \lambda_{AB} f_B$. Now f_B factors uniquely over $\int_B H(B,B)$. Thus the RHS of a) satisfies the universal property of an end of $T(--,--)$. Part b) follows immediately from a). $\square$

We have developed the tools we need and will construct functor categories now.

Definition 5.7. Let $T, S : \mathcal{A} \longrightarrow \mathcal{B}$. Define $[\mathcal{A},\mathcal{B}](T,S) := \int_{A\in\mathcal{A}} \mathcal{B}(TA,SA)$, if it exists, and call it's counit $E_A = E_{A,TS}$.

In accordance with our initial motivation we note that the elements $\alpha : I \longrightarrow [\mathcal{A},\mathcal{B}](T,S)$ of $[\mathcal{A},\mathcal{B}](T,S)$ are in 1-1 correspondence to $\mathcal{V}$-natural transformation $\alpha_A : I \longrightarrow \mathcal{B}(TA,SA)$ by the identification $\alpha_A = E_{A,TS}\alpha$. In analogy to the ordinary case, E_A can be seen as 'evaluation map' and we will refer to it as evaluation. We want $[\mathcal{A},\mathcal{B}]$ to be a $\mathcal{V}$-category, and E to be a functor. We thus require

$$\begin{array}{ccc} [\mathcal{A},\mathcal{B}](S.R)\otimes[\mathcal{A},\mathcal{B}](T,S) & \xrightarrow{M^{[\mathcal{A},\mathcal{B}]}} & [\mathcal{A},\mathcal{B}](T,R) \\ \downarrow{\scriptstyle E_A\otimes E_A} & & \downarrow{\scriptstyle E_A} \\ \mathcal{B}(SA,RA)\otimes\mathcal{B}(TA,SA) & \xrightarrow{M} & \mathcal{B}(TA,RA) \end{array} \quad \text{and} \quad \begin{array}{ccc} I & \xrightarrow{j^{[\mathcal{A},\mathcal{B}]}} & [\mathcal{A},\mathcal{B}](T,T) \\ & \searrow^{j} & \downarrow{\scriptstyle E_A} \\ & & \mathcal{B}(TA,TA) \end{array} \tag{56}$$

The lower legs of each diagram are $\mathcal{V}$-natural in A. By universality of E_A this defines $M^{[\mathcal{A},\mathcal{B}]}$ and $j^{[\mathcal{A},\mathcal{B}]}$ uniquely. And, once more by universality of E_A this implies the axioms (M1) and (M2) for $[\mathcal{A},\mathcal{B}]$ (we can 'evaluate' them down to the axioms in $\mathcal{B}$ and use naturality of a, l, r) .

We sum up our construction: $[\mathcal{A},\mathcal{B}]$ forms a $\mathcal{V}$-category of $\mathcal{V}$-functors with Hom objects being ends $[\mathcal{A},\mathcal{B}](T,S)$ if they exists for all $T, S : \mathcal{A} \longrightarrow \mathcal{B}$. Multiplication and identity are given through the defining equations (56) above, where E_A is the $\mathcal{V}$-functor $[\mathcal{A},\mathcal{B}] \longrightarrow \mathcal{B}$ mapping $T \mapsto TA$ and $E_{A,TS}$ is given as counit of $[\mathcal{A},\mathcal{B}](T,S)$.

If not all Hom objects of $[\mathcal{A},\mathcal{B}]$ exist, then we can still work with the full '$\mathcal{V}$-subcategory' of functors for which $[\mathcal{A},\mathcal{B}](T,S)$ does exist (here, subcategory is written in quotation marks, since there is no parent category). Or we can restrict our attention only to $\mathcal{V}$-functors parametrized by an extra variables, say $P(C,-)$ and $Q(D,-)$ for $P : \mathcal{C}\otimes\mathcal{A} \longrightarrow \mathcal{B}$, $Q : \mathcal{D}\otimes\mathcal{A} \longrightarrow \mathcal{B}$, such that $[\mathcal{A},\mathcal{B}](P(C,-),Q(D,-))$ exists for all C, D. In light of our extra variable results above call this end $H(C,D)$. We know H exists as a functor, but it is not given to us explicitely so far. This will change in the next section. First we record the statements of Lemma 5.5 and equation (55) for our current case:

$$\begin{array}{ccc} & Y & \\ {\scriptstyle f_B}\swarrow & & \searrow{\scriptstyle \alpha_{AB}} \\ H(B,B) & \xrightarrow[=E_{A,P_{B-}Q_{B-}}]{\lambda_{ABB}} & \mathcal{B}(P(B,A),Q(B,A)) \end{array} \quad \text{and} \quad \begin{array}{ccc} & R(B,C) & \\ {\scriptstyle g_{BC}}\swarrow & & \searrow{\scriptstyle \beta_{ABC}} \\ H(B,C) & \xrightarrow[=E_{A,P_{B-}Q_{C-}}]{\lambda_{ABC}} & \mathcal{B}(P(B,A),Q(C,A)) \end{array} \tag{57}$$

where f and α, g and β *mutually imply* $\mathcal{V}$-naturality in extra variables as before.

Finally, we remind ourselves that there was a reason for writing the index A in E_A downstairs, namely $\mathcal{V}$-naturality. In the light of Lemma 4.2 c) (naturality of variables in partial functors) we know that $E_{A,TS}$ will be part of a full evaluation functor $E(-,A) = E_A$ if we can specify a family of $\mathcal{V}$-functors $E(T,-)$ such that $E(T,-)(A) = TA = E_AT$ for all T and A. This is achieved by setting $E(T,-) = T : \mathcal{A} \longrightarrow \mathcal{B}$. Thus we obtained (of course, under the condition that $[\mathcal{A},\mathcal{B}]$ exists):

$$E : [\mathcal{A},\mathcal{B}] \otimes \mathcal{A} \longrightarrow \mathcal{B} \ , \ E(T,-) = T \ , \ E(-,A) = E_A \tag{58}$$

5.2 Partial closedness of $\mathcal{V}$-CAT

Evaluation maps are the counits of 'currying'-like adjunctions. So given this E we will now build an isomorphism in $\mathcal{V}$-**CAT**. Assume $[\mathcal{A},\mathcal{B}]$ exists. Consider the canonical map:

$$\phi_{\mathcal{ABC}} : \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{C},[\mathcal{A},\mathcal{B}]) \xrightarrow{(-\otimes\mathcal{A})_{\mathcal{C}[\mathcal{A},\mathcal{B}]}} \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{C}\otimes\mathcal{A},[\mathcal{A},\mathcal{B}]\otimes\mathcal{A}) \xrightarrow{\mathcal{V}\text{-}\mathbf{CAT}(1,E)} \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{C}\otimes\mathcal{A},\mathcal{B}) \tag{59}$$

By our result on 2-functoriality of $\otimes$ in $\mathcal{V}$-**CAT** in section 3 this map is clearly 2-natural in $\mathcal{C}$ (we don't know about possible functoriality of $[-,-]$ in $\mathcal{V}$-**CAT**, so we cannot say anything about $\mathcal{A},\mathcal{B}$). Explicitly it maps:

$$\begin{array}{rcl} (G : \mathcal{C} \longrightarrow [\mathcal{A},\mathcal{B}]) & \mapsto & (\mathcal{C}\otimes\mathcal{A} \xrightarrow{G\otimes 1} [\mathcal{A},\mathcal{B}]\otimes\mathcal{A} \xrightarrow{E} \mathcal{B}) \\ (\beta : G \longrightarrow F) & \mapsto & (E(\beta\otimes 1) : P \longrightarrow Q) \end{array} \tag{60}$$

Lemma 5.8. *$\phi_{\mathcal{ABC}}$ is an isomorphism of categories.*

Proof. We need the show that firstly, every $P : \mathcal{C}\otimes\mathcal{A} \longrightarrow \mathcal{B}$ has a unique preimage G, and secondly, every $\alpha : P \longrightarrow Q$ has a unique preimage $\beta : G \longrightarrow F$. Given P, then G is uniquely determined from (60), since by precomposing with $(J^C \otimes 1)l^{-1} : \mathcal{A} \longrightarrow \mathcal{C}\otimes\mathcal{A}$ (cf. def. of partial functors, $GJ^C = J^{GC}$) we get:

$$P(C,-) = E(GC,-) = GC \tag{61}$$

Precomposing with $(1\otimes J^A)r^{-1} : \mathcal{C} \longrightarrow \mathcal{C}\otimes\mathcal{A}$ gives:

$$P(-,A) = E(-,A)G = E_AG \tag{62}$$

The first equation determines G on objects: $GC = P(C,-) \in [\mathcal{A},\mathcal{B}]$. The second yields:

$$\begin{array}{ccc} \mathcal{C}(C,D) & \xrightarrow{G_{CD}} & [\mathcal{A},\mathcal{B}](GC,GD) \\ & \searrow^{P(-,A)_{CD}} & \downarrow E_{A,GCGD} \\ & & \mathcal{B}(GC(A),GD(A)) \end{array} \tag{63}$$

which determines GC by universality of E_A and naturality of $P(-,A)_{CD}$ in A. Again by universality of E_A the $\mathcal{V}$-functor axioms of G are then a consequence of those of $P(-,A)$. This establishes bijectiveness on objects. It also allows us to **construct** H **as follows**:

E being a functor the following is $\mathcal{V}$-natural in all variables:

$$E_{-A,TS} = E_{A,TS} : [\mathcal{A},\mathcal{B}](T,S) \longrightarrow \mathcal{B}(E_AT,E_AS) \tag{64}$$

Now take G, F corresponding to $P : C \otimes \mathcal{A} \longrightarrow \mathcal{B}$ and $Q : \mathcal{D} \otimes \mathcal{A} \longrightarrow \mathcal{B}$ under the bijection we just established. Precompose $E_{A,TS}$ in the last to slots with G, F to get a $\mathcal{V}$-natural

$$\lambda_{ACD} = E_{A,GC\,FD} : [\mathcal{A}, \mathcal{B}](GC, FD) \longrightarrow \mathcal{B}(E_A GC, E_A FD) = \mathcal{B}(P(C, A), Q(D, A)) \tag{65}$$

The left hand side is now an explicit functor being equal to H on objects (by eq. (61)) and making λ_{ACD} natural. Thus by uniquess of H this functor must equal H. We obtained

$$H : \mathcal{C}^{\text{op}} \otimes \mathcal{D} \xrightarrow{G^{\text{op}} \otimes F} [\mathcal{A}, \mathcal{B}]^{\text{op}} \otimes [\mathcal{A}, \mathcal{B}] \xrightarrow{\text{Hom}_{\mathcal{A}\otimes\mathcal{B}}} \hat{\mathcal{V}} \tag{66}$$

We return to our proof. It remains to show bijectiveness on natural transformations. Take $\alpha : P \longrightarrow Q$, we look for β such that $\alpha = E(\beta \otimes 1)$. It follows

$$\alpha_{AC} = (E(\beta \otimes 1_{\mathcal{A}}))_{AC} = E_{GCA\,FCA}(\beta_C \otimes j_A) l^{-1} = E_{A,GCFC}\beta_C = (\beta_C)_A \tag{67}$$

And thus β_C is uniquely defined. We need to show $\mathcal{V}$-naturality of $\beta_C : I \longrightarrow [\mathcal{A}, \mathcal{B}](GC, FC)$ in C. But now we know $[\mathcal{A}, \mathcal{B}](GC, FC) = H(C, C)$ by eq. (66), and since $E_A \beta_C = \alpha_{AC}$ is $\mathcal{V}$-natural in C so is β_C by (57). □

We showed that ϕ is an isomorphism of categories and natural in $\mathcal{C}$, and so we deduce that $([\mathcal{A}, \mathcal{B}], \phi)$ gives a representation of $\mathcal{V}$-**CAT**$(- \otimes \mathcal{A}, \mathcal{B})$. But our 'extra variable representations' Lemma 4.9 applies, and $[\mathcal{A}, \mathcal{B}]$ becomes a 2-functor $[-, -]$ on $\mathcal{V}$-**CAT** ('where it exists'). Actually,

$$[-, -]_{(\mathcal{A}\mathcal{B})(\mathcal{A}'\mathcal{B}')} : \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{A}, \mathcal{A}') \times \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{B}, \mathcal{B}') \longrightarrow \mathcal{V}\text{-}\mathbf{CAT}([\mathcal{A}, \mathcal{B}], [\mathcal{A}', \mathcal{B}'])$$

can be easily determined in this case (at least, more easily than in the formula of Lemma 4.9, but based on the same idea), since the counit now explicitly depends on $[-, -]$ (see [2]).

We make a small digression: Let's take the *perspective of section 3.2*. That is, we ask for a representation $\omega_{\mathcal{C}} : \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{C}, \mathcal{D}) \cong \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{C} \otimes \mathcal{A}, \mathcal{B})$ instead of an explicit construction of $[\mathcal{A}, \mathcal{B}]$. We then have

Lemma 5.9 (functor $\mathcal{V}$-categories by adjunctions). *Given such representation $(\mathcal{D}, \omega)$ with counit E', if $\mathcal{V}$ has a initial object 0 then $\mathcal{D} \cong [\mathcal{A}, \mathcal{B}]$.*

Proof. The counit $E' : I \longrightarrow \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{D} \otimes \mathcal{A}, \mathcal{B})$ can be regarded as $\mathcal{V}$-functor $D \otimes \mathcal{A} \longrightarrow \mathcal{B}$. By Cor. 4.7 'Yoneda for representables' we can then express ω by E' as $\omega = E'(- \otimes 1)$. We first construct a bijection O on objects, by using $\mathcal{I}$ in place of $\mathcal{C}$

$$\begin{array}{rcccccc} O : \text{obj}\mathcal{D} & \cong & V\mathcal{V}\text{-}\mathbf{CAT}(\mathcal{I}, \mathcal{D}) & \cong & V\mathcal{V}\text{-}\mathbf{CAT}(\mathcal{I} \otimes \mathcal{A}, \mathcal{B}) & \cong & \text{obj}[\mathcal{A}, \mathcal{B}] \\ T & \mapsto & (G : 1 \mapsto T) & \mapsto & \omega(G) = E'(G \otimes 1) & \mapsto & E'(T, -) \end{array}$$

Now let $\mathcal{C} = \mathcal{C}_Y : \text{obj}\mathcal{C}_Y = \{1, 2\}, \mathcal{C}_Y(1, 2) = Y, \mathcal{C}(i, i) = I, \mathcal{C}_Y(2, 1) = 0$ (with canonical composition maps and identities). With this we find another bijection N:

$$\begin{aligned} \{f : Y \longrightarrow \mathcal{D}(T_1, T_2)\} &\cong \{G \in \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{C}_Y, \mathcal{D}); G_{12} = f,\ Gi = T_i\} \\ &\cong \{P \in \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{C}_Y \otimes \mathcal{A}, \mathcal{B}); P = \omega(G)\} \\ &\cong \{\alpha_A : Y \longrightarrow \mathcal{B}(S_1 A, S_2 A); S_i = P(i, -),\ \alpha_A = P(-, A)_{12}\} \end{aligned}$$

Now $P = \omega(G)$ means $S_i = P(i, -) = E'(Gi, -) = O(Gi) = O(T_i)$ and also $P(-, A)_{12} = E'(-, A)_{T_1 T_2} G_{12} = E'(-, A)_{T_1 T_2} f$ (using eq. (61), (62)). Thus $D(T, S)$ is an end of $\mathcal{B}(TA, SA)$ (identifying T and $O(T)$) with counit $E'(-, A)_{TS}$: The unique factorizing f for a natural family α is given by $N^{-1}(\alpha)$. □

Finally, a remark on **closedness of $\mathcal{V}$-CAT**. We have seen that $[\mathcal{A}, \mathcal{B}]$ exists for all $\mathcal{B}$ for example if $\mathcal{A}$ is small. In these cases, $- \otimes \mathcal{A}$ has right adjoint $[\mathcal{A}, -]$, where the construction above gave us both the adjunction ϕ (for fixed $\mathcal{A}$) and 2-functoriality of $[\mathcal{A}, -]$. Thus we can refine the previous statement from section 3 as follows:

$\mathcal{V}$-**CAT** *is a symmetric monoidal (partially) closed 2-category.*

5.3 The strong Yoneda Lemma

We now have the toolset to strengthen the weak Yoneda Lemma. Recall that q_0 was our 'change of curried variable' adjunction. In particular for $F : \mathcal{A} \longrightarrow \mathcal{V}$:

$$F_{KA} : \mathcal{A}(K, A) \longrightarrow [FK, FA] \quad \Rightarrow \quad q_0(F_{KA}) : FK \longrightarrow [\mathcal{A}(K, A), FA]$$

Lemma 5.10 (strong Yoneda). *Let $F : \mathcal{A} \longrightarrow \mathcal{V}$ and $K \in \mathcal{A}$. Then the $\mathcal{V}$-natural family $\theta_A := q_0(F_{KA}) : FK \longrightarrow [\mathcal{A}(K, A), FA]$ is an end. As a direct consequence there exists a (factorizing) isomorphism $\theta : FK \cong [\mathcal{A}, \mathcal{V}](\mathcal{A}(K, -), F)$*

Proof. Consider the chain of mutual implications:

$$\begin{aligned}
& \alpha_A : Y \longrightarrow [\mathcal{A}(K, A), FA] \text{ is } \mathcal{V}\text{-natural in } A \\
\Leftrightarrow\ & \overline{\alpha_A} = q_0(\alpha_A) : \mathcal{A}(K, A) \longrightarrow [Y, FA] \text{ is } \mathcal{V}\text{-natural in } A \\
\Leftrightarrow\ & \overline{\alpha_A} = [\eta, 1] F_{KA} \text{ for unique } \eta : Y \longrightarrow FK \\
\Leftrightarrow\ & \alpha_A = q_0(\overline{\alpha_A}) = q_0(F_{KA})\eta = \theta_A \eta \text{ for unique } \eta
\end{aligned}$$

where in the second implication we used 'Yoneda for representables' 4.7, and in the last calculation we borrowed eq. (49). ☐

The weak Yoneda lemma can now be rederived as the bijection underlying θ (recall the underlying set of $[\mathcal{A}, \mathcal{V}](\mathcal{A}(K, -), F)$ is the set of natural transformations $\mathcal{A}(K, -) \longrightarrow F$). But note that we used the weak Yoneda Lemma in the above proof.

In analogy to the set case we want to establish **$\mathcal{V}$-naturality in the two remaining 'free variables' of the Yoneda isomorphism θ**: The $\mathcal{V}$-functor F and the $\mathcal{A}$-object K. We have to take into account possible non-existence of $[\mathcal{A}, \mathcal{V}]$ as a $\mathcal{V}$-category, in which case F is not a valid variable to test $\mathcal{V}$-naturality. As before, we introduce an extravariable functor $P : \mathcal{C} \otimes \mathcal{A} \longrightarrow \mathcal{V}$. Then define as in the strong Yoneda Lemma

$$\theta_{A,CK} := q_0(P(C, -)_{AK}) \tag{68}$$

This is $\mathcal{V}$-natural in A, C, K since $P(C, -)_{AK}$ is. But our induced isomorphism θ_{CK} satisfies $E_A \theta_{CK} = \theta_{A,CK}$ and is thus itself natural in C, K (by (57)). We obtained a family of isomorphisms:

$$\theta_{CK} : P(C, K) \cong [\mathcal{A}, \mathcal{V}](\mathcal{A}(K, -), P(C, -)) = H(K, C) \tag{69}$$

natural in K and C. (The functor H is defined as before.)

Now let us assume $[\mathcal{A},\mathcal{V}]$ does exists, so we can speak about $\mathcal{V}$-naturality in the variable $F \in [\mathcal{A},\mathcal{V}]$ and the functor H is given to us explicitely. To adapt the previous result 'parametrize F by itself': $P = E$, $E(F,-) = F$. To find H as in (66) we need to determine $\phi^{-1}(E)$ and $\phi^{-1}(\mathcal{A}(-,-))$. Clearly $\phi(1) = E$.

Define $Y = \phi^{-1}(\mathcal{A}(-,-)) : \mathcal{A}^{\text{op}} \longrightarrow [\mathcal{A},\mathcal{V}]$ and call it the **Yoneda embedding**. By (61) we have $YK = \mathcal{A}(K,-)$ on objects. By (62) we derive that Y is **full and faithful**:

$$\begin{array}{ccc} \mathcal{A}^{\text{op}}(K,L) & \xrightarrow{Y_{KL}} & [\mathcal{A},\mathcal{V}](\mathcal{A}(K,-),\mathcal{A}(L,-)) \\ & \searrow^{\mathcal{A}(-,A)_{KL}} & \downarrow E_A \\ & & [\mathcal{A}(K,A),\mathcal{A}(L,A)] \end{array} \tag{70}$$

But $\mathcal{A}(-,A)_{KL} = q_0(\mathcal{A}(L,-)_{KA}) = \theta_A$ (the first equality expresses $M^{\text{op}} = Mc$ in curried form). And thus $Y_{KL} = \theta$ by universality of E_A. So Y_{KL} is iso.

Now H is by (66) given as $H = \text{Hom}_{[\mathcal{A},\mathcal{V}]}(Y^{\text{op}} \otimes 1)$. Plugging everything into (69) we can finally state:

$$\theta_{FK} : E(F,K) = FK \cong [\mathcal{A},\mathcal{V}](YK,F) = H(F,K) \tag{71}$$

is natural in K and F!

There is, of course, an extra variable version of the strong Yoneda Lemma, which is a nice application of Fubini. Consider $F : \mathcal{B}^{\text{op}} \otimes \mathcal{A} \longrightarrow \mathcal{V}$ and $K : \mathcal{B} \longrightarrow \mathcal{A}$. Yoneda gives us:

$$\theta_{CKB} : F(C,KB) \cong [\mathcal{A},\mathcal{V}](\mathcal{A}(KB,-),F(C,-)) = H(KB,C) \tag{72}$$

natural in C and B. Now we can 'integrate' on both sides, writing out H as integral itself, and apply Fubini to obtain:

$$\int_B F(B,KB) \cong [\mathcal{B}^{\text{op}} \otimes \mathcal{A},\mathcal{V}](\mathcal{A}(K-,-),F(-,-)) \tag{73}$$

The elements of the left hand side are $\mathcal{V}$-natural families $\eta_B : 1 \longrightarrow F(B,KB)$, whereas elements of the right hand side are natural transformations $\alpha_{AB} : \mathcal{A}(KB,A) \longrightarrow F(B,A)$. So the underlying bijection recovers our 'weak extra variable Yoneda' Corollary.

5.4 Note on existence of functor categories

Throughout the essay we assume that obj$\mathcal{A}$ is a set. The notions small and non-small are used with respect to a chosen (Grothendieck) universe U, i.e. $\mathcal{A}$ being small means obj$\mathcal{A} \in U$. In the following we will speak of U-small for clarity, and Set-small/large will mean the difference of sets and and proper classes: When we spoke of **Cat** and **CAT** we made the difference between the Set-small **Cat** (category of U-small categories) and the large **CAT** (category of Set-small categories). Similarly for $\mathcal{V}$-**CAT**. Notions for completeness were also used with respect to our universe U. Thus, for instance $\mathcal{V}$-**Cat** is actually closed while $\mathcal{V}$-**CAT** was called partially closed.

The problem of possible non-existence of functor categories $[\mathcal{A},\mathcal{B}]$ can be addressed as follows. It turns out that one can always embed $\mathcal{V} \subset \mathcal{V}'$ (limit preserving) such that $[\mathcal{A},\mathcal{B}]$ is at least a $\mathcal{V}'$-category. As an example we consider $\mathcal{V} = U$-**Set**. Then if $\mathcal{A}$ is not $U-$small $[\mathcal{A},\mathcal{B}]$ might not be a U-**Set**-category. But employing the 'axiom of universes' we can find a larger universe U', $U \subset U'$ such that ob$\mathcal{A} \in U'$. Then $\mathcal{V} \subset \mathcal{V}' = U'$-**Set** and $[\mathcal{A}, \mathcal{B}]$ is certainly a $\mathcal{V}'$-category. There is more to say about this (see [2]).

6 Conclusion and Outlook: Generalized Enrichment

6.1 Conclusion

There might be less technical and more straight forward introductions to the basic notions of enrichment than this essay. And the advanced reader might possibly get annoyed by the level of detail in which some verifications were presented. But the tools that were developed in this essay, the techniques presented when it came to internalizing hom sets and categories, and additional categorical notions like ends and extraordinary $\mathcal{V}$-naturality that were included, are very powerful (and might of course have important generalizations which the author is not aware of). So let us recap what we did in this essay.

Starting from the definition of monoidal categories we defined $\mathcal{V}$-enriched categories in a straightforward fashion. This already made us rediscover many different concepts from (strict) 2-categories to generalized metric spaces. Symmetry was shown to be a property that allowed $\mathcal{V}$-**CAT** to obtain a symmetric monoidal (2-)structure itself, leading for instance to the inductive definition of (strict) $(n+1)$-categories by enriching in n-**Cat**. It also provided us with an involution operation $(\,\cdot\,)^{\mathrm{op}}$. We pinpoint closedness to be the property that allowed us to internalize Hom sets of $\mathcal{V}$ such that $\mathcal{V} \in \mathcal{V}$-**CAT**. Then using the powerful tools of ordinary and extraordinary $\mathcal{V}$-naturality (which were explained to have the 'same roots'), we were easily able to established the statement of a Yoneda bijection. As an application we recovered many results about adjunctions from **CAT** in $\mathcal{V}$-**CAT**.

To strengthen Yoneda, we wanted to lift the bijection to an isomorphism. For this we needed to lift functor categories to functor $\mathcal{V}$-categories. We did this with the notion of ends, but reconciled our construction with the earlier notion of closedness. The Yoneda bijection and it's corollaries could then be easily lifted to corresponding $\mathcal{V}$-versions.

There is much more to say about enrichement in monoidal categories and a lot of further theory (indexed limits, Kan extensions) can be found in [2]. But there is also no reason to restrict our idea of enrichment to the case of monoidal categories, though it is the best-known case. A different (and very general) perspective will be the topic of the following final section.

6.2 Outlook: Generalized enrichment

This is a lightweight and rough outlook to a generalized theory of enrichment and it is included 'just for fun'. Everything here is from Tom Leinster [5] and [6].

As sketched in the introduction, monoidal structure in a category allows us to handle multivariable operations (like composition M) via tensoring. However, more generally we could allow multivariable arrows in our category to represent these operations. That is, arrows can have as domain a list of n objects instead of just one. This leads to the idea of (classical) operads and multicategories. We can sketchily formalize the idea: Our **multicategory** C should consist of a class of objects C_0 and for each n a class of n-ary arrows $C(a_1, ..., a_n; a)$ (depicted e.g. as arrows starting at $a_1, ..., a_n$, joining into one and ending at a). There should be identities $1_a \in C(a,a)$ and composition. Associativity and identity axioms should be required accordingly.

Examples 6.1. Every (strict) monoidal category $\mathcal{V}$ can be seen to have a underlying multicategory. We just define $\mathcal{V}(a_1, ..., a_n; a)$ to be $\mathcal{V}(a_1 \otimes ... \otimes a_n, a)$ and composition as composition in $\mathcal{V}$ by tensoring the inputs. (The condition of strictness can actually be dropped, see [5] Chapter 2 and 3).

We want to *generalize* these multicategories. To give a small motivation let us take a different perspective on categories in the sense of internal category theory. Underlying every ordinary category there is a graph of arrows and objects. So a graph can be represented by a diagram

$$\begin{array}{ccc} & C_1 & \\ {\scriptstyle\text{dom}}\swarrow & & \searrow{\scriptstyle\text{cod}} \\ C_0 & & C_0 \end{array} \tag{74}$$

in **Set**, where C_0 is regarded as set of objects, C_1 as set of arrows. To make it a category we need additional structure, namely a composition map $M : C_1 \times_{C_0} C_1 \longrightarrow C_1$ and identities $j : C_0 \longrightarrow C_1$ satisfying axioms. Here $C_1 \times_{C_0} C_1$ denotes the pullback of the diagram $C_1 \xrightarrow{\text{dom}} C_0 \xleftarrow{\text{cod}} C_1$.

In analogy to letting Hom objects live in other categories $\mathcal{V}$, we can allow this structure to live in other categories $\mathcal{E}$. We then speak of $\mathcal{E}$-graphs and $\mathcal{E}$-categories and we can define maps between them so that they form categories. There is a forgetful functor from $\mathcal{E}$-categories to $\mathcal{E}$-graphs. In the case $\mathcal{E} = \textbf{Set}$ this has as left adjoint the 'free category' functor. We obtain a monad **fc** on graphs. So far, all this is seems reasonable.

Examples 6.2. Taking $\mathcal{E}$ to be **Top**, lets us discover the notion of 'topological categories': Object set and arrow set obtain a topology. For instance a topological space X gives rise to a category $\Pi_1 X$ where objects are points (with topology of X) and arrows are homotopy classes of paths with fixed enpoints (with topology $X^{[0,1]}/\sim$).

How would we define enrichment in $\mathcal{V}$ from this perspective? We could take $I(C_0)$ to be a full graph (i.e. each arrow representing an ordered pair of objects). Then specifying a map from arrows of $I(C_0)$ to objects of $\mathcal{V}$ can serve as definition of Hom objects; a map, however, which does *not* take place at the same categorical level (arrows to objects). We would need to specify axioms (M1), (M2) by hand. This is not very satisfactory.

Let us come back to multicategories first: If we want to apply the above construction to multicategories we apparently need to modify the 'domain part' of our diagram, to allow multi-object domains. We finally define a generalized multicategory, called **a T-multicategory**, to be a diagram

$$\begin{array}{ccc} & C_1 & \\ {\scriptstyle\text{dom}}\swarrow & & \searrow{\scriptstyle\text{cod}} \\ T(C_0) & & C_0 \end{array} \tag{75}$$

in a category $\mathcal{E}$ with monad T (with convenient properties), and functions for composition and identities looking much like our functions above together with corresponding axioms (a detailed definition can be found in [5] Chapter 4). The diagram on its own gives us again the notion of a T-graph.

How is this a generalization of multicategories? We recover classical multicategories if we choose T to be the free monoid monad on $\mathcal{E} = \mathbf{Set}$. Even more, we recover our ordinary category diagram above if we take the identity monad on $\mathcal{E} = \mathbf{Set}$. So our new notion encompasses classical categories and multicategories as special cases.

As before we have a forgetful functor from T-**Multicat** to T-**Graph** (both names should be clear in meaning). Under 'suitability assumptions' (see [5] Chapter 6.5) this has a left adjoint 'free T-multicategory' functor and the adjunction is monadic, giving rise to a monad T^+ on $\mathcal{E}^+ = T$-**Graph**. Thus we can speak of T^+-multicategories. Now, take $I(C_0)$ to be the (unique) 'indiscrete' T-multicategory corresponding to the graph

$$T(C_0) \xleftarrow{\pi_1} T(C_0) \times C_0 \xrightarrow{\pi_2} C_0 \tag{76}$$

$I(C_0)$ is an algebra of our free T-mulicategory monad T^+, i.e. there is $\alpha : T^+(I(C_0)) \longrightarrow I(C_0)$. So this in turn let's us construct a T^+-multicategory $(I(C_0))^+$ corresponding to the graph

$$T^+(I(C_0)) \xleftarrow{1} T^+(I(C_0)) \xrightarrow{\alpha} I(C_0) \tag{77}$$

We are in the position to state the central and final definition of this section:

Definition 6.3 ([5], Chapter 6.8)**.** Given a T^+-multicategory $\mathcal{V}$, a $\mathcal{V}$-enriched T-multicategory consist of an object $C_0 \in \mathcal{E}$ and a map $(I(C_0))^+ \longrightarrow \mathcal{V}$ of T^+-multicategories.

And how does this generalize *our theory* of enrichment in a monoidal category? Well, in the easiest case of T being the identity monad on $\mathcal{E} = \mathbf{Set}$, T^+ becomes the **fc** monad on the category of graphs. T^+-categories are then called **fc**-multicategories. These are 2-dimensional structures encompassing for instance bicategories ('weakly' enriched **Cat**-categories). And just as strict monoidal categories are one-object (strict) 2-categories, weak monoidal categories are one-object bicategories.

Examples 6.4. Given a topological space X we obtain a bicategory $\Pi_2 X$ as follows: Objects (0-cells) are points, 1-cells are paths between points, 2-cells our homotopy classes of path homotopies with fixed endpoints. Note that composition of paths is then only associative up to a 2-cell isomorphism. Then, the full subcategory determined by one point x lets us recover exactly our initial example 2.3 e): paths (loops) become objects, concatenation becomes tensor product.

We now divine the solution to our above problem of a strange mapping between arrows to objects: monoidal categories should be regarded as 2-dimensional structure, objects *becoming* 1-cells, the tensor map becoming composition of these. It sounds reasonable that lifting objects to arrows might enable us to bundle the parts of our classical definition of enrichment into the a single map, as the map in the above definition.

Ultimately working out the terms and notions involved in the above definition in the case of T^+ being **fc**, and then choosing a monoidal category to be $\mathcal{V}$ (more precisely, it's corresponding **fc**-category) will make us indeed recover the notion of enrichment that this essay started out with. So we made the circle.

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