CATEGORICAL HOMOTOPY THEORY WITH A VIEW TOWARDS FACTORISATION HOMOLOGY

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VERSION 0.1

Abstract. These notes were written for a talk given in the K-Theory seminar at CUNY, about categorical homotopy theory, higher categories and factorisation homology. The talk was aimed at students with little or no background in higher categories, but some background in category theory.

Contents

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1. Introduction

Factorisation homology $\int_{\_} A$ of an $E_{n,G}$*-algebra* A is a so-called *left Kan-extension* of certain functors of $(\infty, 1)$*-categories*. This generalises singular homology, and (in analogy with the Eilenberg-Steenrod axioms) satisfies the $\otimes$*-excision theorem*.

This talk is meant to give a fast route of formal definitions towards an understanding these concepts. The style of presentation chosen here will see definitions being stated formally but in sometimes slightly condensed formulas. This is an attempt to highlight the "elegance" that (subjectively) derives from these formulas. Unwinding the definitions is fruitful and left to the reader in most cases, and in many cases explicit exercises are suggested. These notes add content and (hopefully helpful) commentary slightly expanding upon the topics discussed during the lecture.

Remark 1.1. There's no claim to originality of the material presented here. Main sources are [Rie14], [Shu06], [Lur09] and [AKMT19]. All of the many errors are my own (but please don't hesitate to contact me if something seems wrong to you!).

2. Categorical homotopy theory

Notation 2.1. We will use $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{M}, \mathcal{V}$ to denote (higher) categories. We use $\mathcal{V}$ often for some monoidal category that we can enrich in (which, in one sentence means, that we abstract "hom sets" to be objects of some category $\mathcal{V}$, instead of objects of the category of sets **Set**. For background about enriched see [KK82]). $\mathcal{M}$ often denotes a category with weak equivalences, which is a "presentation" of an $(\infty, 1)$-category as we will explain. We denote hom functors by $\hom_{\mathcal{C}}(-,-)$, $\mathcal{C}(-,-)$ or simply $\mathcal{C}$. Functor categories will be denoted by both $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ and $\mathcal{C}^{\mathcal{D}}$.

Knowing at least the idea underlying enriched category theory (i.e. replacing hom sets in **Set** with hom objects in a category $\mathcal{V}$) would be useful. The standard reference is [KK82].

2.1. Tensors, cotensors, ends and coends.

Definition 2.2 (tensors, cotensors)**.** We say a $\mathcal{V}$-enriched $\mathcal{C}$ is *tensored* respectively *cotensored* over $\mathcal{V}$ if there are functors

$$- \cdot - : \mathcal{V} \times \mathcal{C} \to \mathcal{C}$$
$$(-)^{-} : \mathcal{C}^{\mathrm{op}} \times \mathcal{V} \to \mathcal{C}^{\mathrm{op}}$$

called tensor (resp. cotensor), such that

$$\mathcal{C}(V \cdot A, B) \cong \mathcal{V}(V, \mathcal{C}(A, B))$$
$$\mathcal{C}(A, B^V) \cong \mathcal{V}(V, \mathcal{C}(A, B))$$

In other words, tensors (resp. cotensors) give adjunctions to hom functors

$$- \cdot A \vdash \mathcal{C}(A, -)$$
$$B^{-} \vdash \mathcal{C}(-, B)$$

Example 2.3. In the case $\mathcal{V} = \mathbf{Set}$, then whenever $\mathcal{C}$ has coproducts it is tensored by setting $S \cdot A = \bigsqcup^{|S|} A$. Similarly, if it has product we can set $B^S = \prod^{|S|} A$.

Remark 2.4. In the decategorified case of finite dimensional Hilbert spaces (i.e. putting Hilbert spaces, which are sets, in place of categories, and the inner product in place of the hom functor), the above universal properties correspond to $\langle rv, w\rangle = r\,\langle v, w\rangle$ and $\langle v, rw\rangle = r\,\langle v, w\rangle$.

Definition 2.5. Assume $\mathcal{B}$ is cotensored. We define the *coend* functor to be the left adjoint

$$\int^{\mathcal{C}} \quad \vdash \quad (-)^{\mathcal{C}(-,-)} \quad : \quad \mathcal{B} \to \mathcal{B}^{\mathcal{C}^{\mathrm{op}} \times \mathcal{C}}$$

(if it exists). Assume $\mathcal{B}$ is tensored. We define the *end* functor to be the right adjoint

$$\mathcal{C}(-,-) \cdot (-) \quad \vdash \quad \int_{\mathcal{C}} \quad : \quad \mathcal{B}^{\mathcal{C}^{\mathrm{op}} \times \mathcal{C}} \to \mathcal{B}$$

(if it exists).

Remark 2.6. In the decategorified case of finite dimensional Hilbert spaces, with functors corresponding to maps of Hilbert spaces (which again form a Hilbert space), and $\times$ corresponding to $\otimes$, we find that the universal properties of ends/coends corresponds to the trace tr, namely for $H : C^* \otimes C \to D$ we have $\langle tr_C(H), v\rangle = \langle H, \mathrm{Id}_C \otimes v\rangle$. The fact that we have both ends and coends reflects that the category **Cat** (unlike $\mathbf{Hilb}^{\mathrm{f.d.}}$) is not self-dual.

Ends and coends are special instances of weighted (co)limits.

Definition 2.7. Let $W : \mathcal{D}^{\mathrm{op}} \to \mathcal{V}$ (resp. $W : \mathcal{D} \to \mathcal{V}$) and $J : \mathcal{D} \to \mathcal{C}$. The W-weight colimits $\mathrm{colim}^W J$ (resp. the W-weighted limit $\lim^W J$) are objects in $\mathcal{C}$ defined by the universal property

$$\mathcal{C}(\mathrm{colim}^W J, b) \cong \mathcal{V}^{\mathcal{D}^op}(W, \mathcal{C}(J-, b))$$
$$\mathcal{C}(a, \lim_W J) \cong cV^{\mathcal{D}}(W, \mathcal{C}(a, J-)$$

Exercise 2.8. Check values coends and ends are weight colimits resp. limits by writing out the natural isomorphism defining the respective adjunction.

Exercise 2.9. Check that for $\mathcal{V} = \mathbf{Set}$ and $W = \star$ the terminal functor, the weighted definitions coincide with the usual colimit and limit.

In fact, every weighted (co)limits can be expressed in term of (co)ends.

Remark 2.10 (Coend coequaliser formula). In the case of $\mathcal{V} = \mathbf{Set}$, there is a another formula for ends and coends in terms of colimits and limits. In the case of coends we have for $H : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ that

$$\int^{\mathcal{C}} H \cong \mathrm{coeq}\left(\coprod_{f : c \to c'} H(c, c') \underset{H(1,f)}{\overset{H(f,1)}{\rightrightarrows}} \coprod_c H(c, c) \right)$$

Remark 2.11. We state four rules of the so-called "(co)end-fu" (see [Lor15])

(i) Natural transformations. Let $F, G : \mathcal{C} \to \mathcal{D}$.

$$\mathrm{Nat}(F, G) \cong \int_{\mathcal{C}} \mathcal{D}(F-, G-)$$

To some extent, this can in fact be taken as the definition of natural transformations (see [KK82]). The next exercise is a consistency check between this claim and Remark 2.10 (in fact, the exercise can be used to prove that remark).

Exercise 2.12. In the case $\mathcal{V} = \mathbf{Set}$ show the formula for $\mathrm{Nat}(F, G)$ using the coequaliser formula from Remark 2.10.

(ii) Hom continuity.

$$\int_{\mathcal{C}} \mathcal{D}(H, d) \cong \mathcal{D}(\int^{\mathcal{C}} H, d)$$
$$\int_{\mathcal{C}} \mathcal{D}(d, H) \cong \mathcal{D}(d, \int_{\mathcal{C}} H)$$

which simply uses that colimits and limits can be commuted with hom-functors (in a variance appropriate sense).

(iii) Fubini.

$$\int_{\mathcal{C}} \int_{\mathcal{B}} H = \int_{\mathcal{C}\times\mathcal{B}} H = \int_{\mathcal{B}} \int_{\mathcal{C}} H$$

(left as an exercise).

(iv) Ninja Yoneda lemma

$$F \cong \int^{\mathcal{C}} \mathcal{D} \cdot F$$
$$F \cong \int_{\mathcal{C}} F^{\mathcal{D}}$$

(proven later). Note that here we condensed notation slightly, for instance, the first line reads as a natural isomorphisms

$$Fd \cong \int^{\mathcal{C}} \mathcal{D}(-, d) \cdot F(-)$$

Remark 2.13 (Weighted limits formula). We can now compute

$$\begin{aligned} \mathcal{C}(\mathrm{colim}^W J, b) &\cong \mathcal{V}^{\mathcal{D}^{op}}(W, \mathcal{C}(J-, b)) \\ &\cong \int_{\mathcal{D}^{\mathrm{op}}} \mathcal{V}(W, \mathcal{C}(J-, b)) \\ &\cong \int_{\mathcal{D}^{\mathrm{op}}} \mathcal{C}(W \cdot J, b) \\ &\cong \mathcal{C}(\int_{\mathcal{D}^{\mathrm{op}}} W \cdot J, b) \end{aligned}$$

where we used the expression of natural transformations as an end, tensoredness, and hom continuity in that order. It follows that

$$\mathrm{colim}^W J \cong \int_{\mathcal{D}^{\mathrm{op}}} W \cdot J$$

Exercise 2.14. Only if you feel like it: In the case $\mathcal{V} = \mathbf{Set}$, using Exercise 2.12 to replace the second step of the previous exercise, show

$$\mathrm{colim}^W J \cong \mathrm{colim}\, J\pi_W$$

where $\pi_W : \mathbf{el}(W) \to \mathcal{D}^{\mathrm{op}}$ is the projection from the *category of elements* of W: objects are tuples $(d \in \mathcal{D}, s \in W(d))$ and morphisms $f : (d, s) \to (d', s)$ are morphisms $f : d' \to d$ such that $W(f)(s) = s'$. π_W maps (d, s) to d.

Use this to derive Remark 2.10.

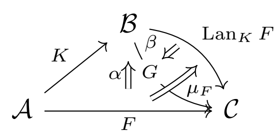
2.2. Kan extensions.

Definition 2.15. Let $K : \mathcal{A} \to \mathcal{B}$. The left and right Kan extension functors Lan_K and Ran_K are defined as adjoint by

$$\mathrm{Lan}_K \vdash - \circ K : \mathrm{Fun}(\mathcal{B}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{A}, \mathcal{C})$$
$$- \circ K \vdash \mathrm{Ran}_K : \mathrm{Fun}(\mathcal{A}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{B}, \mathcal{C})$$

where $- \circ K$ denotes precomposition by K.

Remark 2.16. Spelling the first case out we find for $F \in \mathrm{Fun}(\mathcal{A}, \mathcal{C}), G \in \mathrm{Fun}(\mathcal{B}, \mathcal{C})$ an isomorphism of natural transformations $\mathrm{Fun}(\mathrm{Lan}_K F, G) \cong \mathrm{Fun}(F, GK)$. As for any adjunction this isomorphism is given by acting on some $\beta : \mathrm{Lan}_K F \to G$ with the right adjoint $- \circ K$ and precomposing with the unit $\mu_F : F \to \mathrm{Lan}_K FK$. In other words, any $\alpha : F \to GK$ will factor uniquely that way, which is the more commonly mentioned definition of Kan extensions. The situation is depicted below



Note that by passing to opposite categories (which reverses the direction of natural transformations), left Kan extensions become right Kan extensions and vice versa.

Definition 2.17. We say a right Kan extension of F along K is *pointwise*, if post-composition with any representable functor $\mathcal{C}(c, -)$ yields a right Kan extension (of the $\mathcal{C}(F-, c)$ along $\mathcal{C}(K-, c)$). A left Kan extension is pointwise if its corresponding right Kan extension is pointwise.

Remark 2.18 (A formula). Given a pointwise left Kan extension, we compute, using first the Yoneda lemma and then the property of being pointwise, that

$$\mathcal{C}(\mathrm{Lan}_K Fb, c) \cong \mathbf{Set}^{\mathcal{B}^{\mathrm{op}}}(\mathcal{B}(-, b), \mathcal{C}(\mathrm{Lan}_K F-, c))$$
$$\cong \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}(\mathcal{B}(K-, b), \mathcal{C}(F-, c))$$

We recognise this expresses $\mathrm{Lan}_K Fb$ as a weighted limit (with weight $\mathcal{B}(K-, b)$. Using Remark 2.13, we then find

$$\mathrm{Lan}_K F(b) = \int^{\mathcal{A}} \mathcal{B}(K-, b) \cdot F(-)$$

The isomorphism being natural in b we can also write this as[1]

$$\mathrm{Lan}_K F(-) = \int^{\mathcal{A}} \mathcal{B}(K-, -) \cdot F$$

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[1]To be really explicit about the functor slots in this equation, one could write

$$\mathrm{Lan}_K F(\overset{1}{-}) \cong \int^{\mathcal{A}} \mathcal{B}(K \overset{2}{-}, \overset{1}{-}) \cdot F(\overset{2^*}{-})$$

where the same number across the isomorphism identifies the same slot of isomorphic functors, whereas numbers and their starred version are "bound" by an (co)end.

We further remark that essentially all useful Kan extensions are "pointwise" (see [Rie14, Ch.1]).

Exercise 2.19 (General formula). The same formula holds without the assumption of the Kan extension begin pointwise. The proof can be given using coend-fu and is left to the reader (see e.g. [ML13, X.4.1])

Exercise 2.20 (Right extension case). Similarly, prove $\operatorname{Ran}_K F = \int_{\mathcal{A}} F^{\mathcal{B}(-,K-)}$.

Remark 2.21 (Proof of ninja Yoneda). We can now easily prove the ninja Yoneda lemmas from expanding the observations that $F = \operatorname{Lan}_{\mathrm{id}} F$ and $F = \operatorname{Ran}_{\mathrm{id}} F$ using the above formula.

These formulas, as well as the "(co)end-fu", are useful for deriving general facts about Kan extension. For instance,

Observation 2.22. *If K is fully faithful (as it will be be for factorisation homology) then*

$$(\operatorname{Lan}_K F)K \cong \int^{\mathcal{A}} \mathcal{B}(K-, K-) \cdot F$$
$$\cong \int^{\mathcal{A}} \mathcal{A}(-,-) \cdot F$$
$$\cong F$$

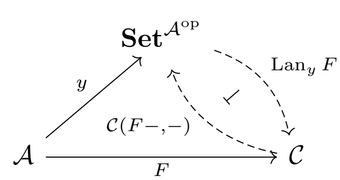
where in the last step we used the ninja Yoneda lemma. In other words, we observe that if K is fully faithful then $\operatorname{Lan}_K F$ *is an actual extension of F in the sense that when "restricting" (i.e. precomposing)* $\operatorname{Lan}_K F$ *to K one recovers F.*

2.3. **Examples.**

2.3.1. *Nerve-realisation paradigm, Dold-Kan correspondence.* Using notation from the previous section, consider the case where $K = y : \mathcal{A} \to \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$ is the Yoneda embedding. We compute (using our weighted limit expression, and then the Yoneda lemma) that

$$\mathcal{C}(\operatorname{Lan}_y Fb, c) \cong \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}(\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}(y-, b), \mathcal{C}(F-, c))$$
$$\cong \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}(b, \mathcal{C}(F-, c))$$

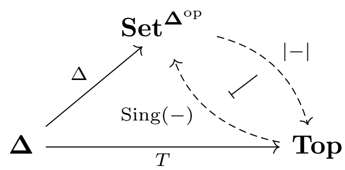
In other words we have an adjunction,



The "yoneda extension" $\operatorname{Lan}_y F$ is also called the *realisation* of presheafs on $\mathcal{A}$ in $\mathcal{C}$, and $\mathcal{C}(F-,-)$ is called the *nerve* of objects of $\mathcal{C}$ in $\mathcal{A}$. F is called *dense* if the nerve is fully faithful.

We will consider special instances of this. Let $\mathbf{\Delta} \subset \mathbf{Cat}$ be the subcategory of finite non-empty total orders. Up to passing to an equivalent (skeletal) category we can assume the objects of $\mathbf{\Delta}$ are $[m]$, $m \in \mathbb{N}$, denoting the total order with $(m+1)$ elements. Set $\mathbf{sSet} = \mathbf{Set}^{\mathbf{\Delta}^{\mathrm{op}}}$, the category of simplicial sets. Denote the Yoneda embedding of $\mathbf{\Delta}$ by Δ.

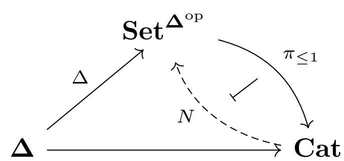
There is an faithful inclusion $T : \Delta \to \mathbf{Top}$, realising m as the topological m-simplex. Passing to the nerve and realisation we find



Usually $|-|$ is called *geometric realisation* (of a simplicial set), and Sing is called *singular complex* (of a topological space).

Exercise 2.23. Use the coequaliser formula to convince yourself that geometric realisation of a simplicial set is actually a reasonable name for this functor.

Similarly, Δ trivially includes into $\mathbf{Cat}$, and we obtain the yoneda extension

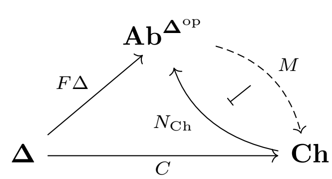


Here, $\pi_{\leq 1}$ is also called *fundamental path category* (of a simplicial set), where as N is called the *nerve* (of a category).

As a final example, consider the following $\mathbf{Ab}$-enriched situation. Note that using the free abelian group functor $F : \mathbf{Set} \to \mathbf{Ab}$, every ordinary ($\mathbf{Set}$-enriched) category can be thought of as being $\mathbf{Ab}$-enriched. Let $\mathbf{Ch}$ be the category of chain complexes (in non-negative degree). Every $[m] \in \mathbf{\Delta}$ can be thought of as a chain complex $C[m]$, whose nth degree is the free group on non-degenerate n-simplices in $[m]$ (that is, elements in $y[m][n]$ not in the image of any $\Delta[m](f)$, $f : [n'] \to [n]$, $n' > n$), and whose nth differential is induced from the alternating sum, over $0 \leq i \leq n$, of maps $\Delta[m](d_i^n)$ (where $d_i^n : [n-1] \to [n]$ is the unique injective map omitting i in its image) by setting degenerate elements equal to zero.

Exercise 2.24. Spell out the definition of $C[m]$, noting in particular that $[m]$ has no non-degenerate n-simplices for $n > m$, and check the chain differentials compose to zero. Check this defines a functor $C : \mathbf{\Delta} \to \mathbf{Ch}$.

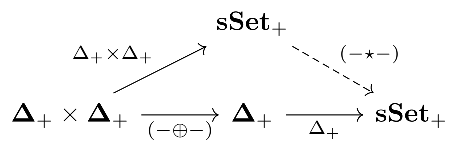
We can now consider the extension of C, and as before obtain an adjunction



In this case the adjunction is actually an adjoint equivalence. This is an instance of the so-called *Dold-Kan correspondence*.

Remark 2.25. The latter is a fundamental bridge between algebra and homotopy theory: While chain complexes seem like something "fully algebraic and discrete", we can now see that through there correspondence with simplicial abelian groups, and thus in particular their map to simplicial sets, and thus (via geometric realisation) to spaces, they can be seen to inherit a "continuous" or "homotopical" structure from the latter. Indeed, weak homotopy equivalences correspond to quasi-isomorphism of chain complexes under this translation. The fact that chain complexes come (rather naturally) equipped with this homotopical structure, is a reason for their big role in classical homological algebra.

2.3.2. *Day convolution.* The process of *Day convolution* is transferring monoidal structure from a category to its presheaf category, and can be defined as a left Kan extension as well. We give the following example (which can be easily generalised to other situations): set $\mathbf{\Delta}_+ \subset \mathbf{Cat}$ the subcategory of finite (possibly empty) total orders. There is a strict monoidal structure on $\mathbf{\Delta}_+$ given by "ordered union" $[m] \oplus [n] = [m+n+1]$ on objects which can be easily seen to extend to morphisms in $\mathbf{\Delta}_+$. Consider the left Kan extension of $\Delta_+ \oplus$ along $\Delta_+ \times \Delta_+$ as follows



Exercise 2.26. Using the coequaliser formula give an explicit description of the simplices in $K \star J$ for (augmented) simplicial sets $K, J \in \mathbf{sSet}_+ = \mathbf{Set}^{\Delta_+}$. Show $K \star \Delta_+[-1] = \Delta_+[-1] \star K = K$.

The functor $(- \star -)$ is also called the *join* and can be see to endow $\mathbf{sSet}_+$ with a monoidal structure.

The join further has the following special properties. It also yields a bifunctor on $\mathbf{sSet}$ (since, setting $K[-1] =$ for $K \in \mathbf{sSet}$, we have a functor $\mathbf{sSet} \to \mathbf{sSet}_+$). Together with the observation that there are natural maps $K = \Delta_+[-1] \star K \to J \star K$ and similarly $K \to K \star J$, this induces functors

$$(- \star K), (K \star -) : \mathbf{sSet} \to K/\mathbf{sSet}$$

which admit adjoints $(- \star K) \vdash (-)_{/-} : (F : K \to X) \mapsto X_{/F}$, called the *slice*, and similarly $(K \star -) \vdash (-)_{-/}$, called the *coslice*.

Exercise 2.27. Using the adjunction isomorphism (that is, $\mathrm{Map}(\Delta[m], X_{/F}) \cong ...$) find an explicit description of slice and coslice.

2.3.3. *Derived functors, homotopy colimits.* Inching closer to the topic of higher categories, we introduce the notion of *relative categories*, as a category $\mathcal{M}$ with a class of morphisms $\mathcal{W}$ in $\mathcal{M}$ called *weak equivalences* and closed under composition and containing all identities. Frequently, stronger assumptions are put on $\mathcal{W}$ such as in the theory of model categories (or similarly, in the theory of "categories with weak equivalences" or "homotopical categories"). The underlying idea remains the same: a relative category describes a category equipped with a "homotopy theory" in that the class of weak equivalences describes those morphisms that are "homotopically invertible". It is reasonable to ask functors to preserve weak equivalences. If they do, they are called *homotopical.* If they don't then a canonical approximation by homotopical functors is given Kan extensions as follows.

Definition 2.28. Let $(\mathcal{M}, \mathcal{W})$ be a relative category. Its *homotopy category* $\gamma_{\mathcal{M}} : \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$ is defined by the following universal property: Precomposition $- \circ \gamma_{\mathcal{M}} : \mathrm{Fun}(\mathrm{Ho}(\mathcal{M}), \mathcal{A}) \to \mathrm{Fun}(\mathcal{M}, \mathcal{A})$ is an equivalence of categories when restricting the codomain to "homotopical" functors $\mathrm{Fun}_\sim(\mathcal{M}, \mathcal{A})$ mapping weak equivalences to isomorphisms.

Let $(\mathcal{M}_i, \mathcal{W}_i)$ $(i \in \{1, 2\})$ be relative categories. Let $F : \mathcal{M}_1 \to \mathcal{M}_2$. The *total left derived functor* of F is the *right* Kan extension $\mathbf{L}F$ of $\gamma_{\mathcal{M}_2} F$ along $\gamma_{\mathcal{M}_1}$

$$\begin{array}{ccc} \mathcal{M}_1 & \xrightarrow{F} & \mathcal{M}_2 \\ & \overset{\mathbb{L}F}{\dashrightarrow} & \\ \gamma_{\mathcal{M}_1}\downarrow & & \downarrow\gamma_{\mathcal{M}_2} \\ \mathrm{Ho}(\mathcal{M}_1) & \underset{\mathbf{L}F}{\dashrightarrow} & \mathrm{Ho}(\mathcal{M}_2) \end{array}$$

Often what we are interested is instead the *left derived functor* $\mathbb{L}F$ which is a homotopical functor $\mathcal{M}_1 \to \mathcal{M}_2$ such that $\gamma_{\mathcal{M}_2}\mathbb{L}F$ (by initiality of $\gamma_{\mathcal{M}_1}$) induces a total left derived functor. Similar definitions can be given for (total) right derived functors defined using *left* Kan extensions and denoted by $\mathbb{R}F$ (resp. $\mathbf{R}F$).

Exercise 2.29. Think of an explicit construction of $\mathrm{Ho}(\mathcal{M})$ (or have a look at [Rie14, 2.1.6]).

Exercise 2.30. Check that left (and right) derived functors are unique up to natural isomorphism.

Remark 2.31 (Homotopy colimits). Given a relative category $(\mathcal{M}, \mathcal{W})$, then the functor category $\mathcal{M}^{\mathcal{D}}$ inherits a notion of weak equivalence: namely we define the class of weak equivalence $\mathcal{W}_{\mathcal{D}}$ to consists of those natural transformation which are componentwise weak equivalences. (We remark that if $(\mathcal{M}, \mathcal{W})$ and $\mathcal{D}$ has more structure then $(\mathcal{M}^{\mathcal{D}}, \mathcal{W}_{\mathcal{D}})$ can be given more structure too, e.g. a model structure such as the Reedy model structure, which we met in last week's talk. However, the definition of weak equivalence $\mathcal{W}_{\mathcal{D}}$ remains the same).

In general the functor $\mathrm{colim} : \mathcal{M}^{\mathcal{D}} \to \mathcal{M}$, taking a $\mathcal{D}$ shaped diagram to its colimit, is not homotopical. Its left derived functor

$$\mathrm{hocolim} := \mathbb{L}\,\mathrm{colim}$$

is usually called the *homotopy colimit*.

Remark 2.32 (Computation of derived functors). A *left deformation* Q of $\mathcal{M}$ is an element in $\mathcal{M}^{\mathcal{M}}$ with a weak equivalence $\delta : Q \xrightarrow{\sim} \mathrm{id}$ (it is a right deformation if $\delta : \mathrm{id} \xrightarrow{\sim} Q$). Given a functor $F : \mathcal{M}_1 \to \mathcal{M}_2$ of relative categories, if Q is a deformation of $\mathcal{M}_1$ such that FQ is homotopical, then it can be shown that

$$\mathbb{L}F = FQ$$

Indeed, using the definition of the homotopy category, it suffices to check that $\gamma_{\mathcal{M}_2} FQ$ admits a universal natural transformation μ to $\gamma_{\mathcal{M}_2} F$, such that for $G \in \mathrm{Fun}_\sim(\mathcal{M}_1, \mathrm{Ho}(\mathcal{M}_2))$, any $\beta : G \to \gamma_{\mathcal{M}_2} F$ uniquely factors through μ.

Exercise 2.33. Show this (or see [Rie14, 2.2]).

Remark 2.34 (Classical derived functors). Write R-**Mod** for the category of left R-module. This embeds $\deg_0 : R\text{-}\mathbf{Mod} \hookrightarrow \mathbf{Ch}_R$ into R-module chaing complexes as chain complexes concentrated in degree zero. In particular, any additive (i.e. **Ab**-enriched) functor on $F : R\text{-}\mathbf{Mod} \to S\text{-}\mathbf{Mod}$ induces a functor on $F_\bullet : \mathbf{Ch}_R \to \mathbf{Ch}_S$ by acting degree-wise. But using our previous observation about the Dold-Kan correspondence, we know $\mathbf{Ch}_R$ (unlike R-**Mod**) has homotopical structure, with weak equivalences given by quasi-isomorphisms. These are not preserved by functors $F_\bullet$ in general, but chain homotopy equivalences are. There is a deformation $Q : \mathbf{Ch}_R \to \mathbf{Ch}_R$ turning quasi-isomorphic chains into chain homotopy equivalent chains ("projective resolution"), and thus for any F, $\mathbb{L}F_\bullet = F_\bullet Q$. Classically, the nth derived functor of F is then obtained by passing to nth homology $H_n FQ$. For example, for $F = (- \otimes_R N)$ we get the nth Tor functor $\mathrm{Tor}_n^R(-, N)$.

2.3.4. *Bar construction.* Our final example of Kan extensions is the two-sided bar construction. Given $G : \mathcal{D}^{\mathrm{op}} \to \mathcal{B}$, $F : \mathcal{D} \to \mathcal{C}$ and $\otimes\colon \mathcal{B} \times \mathcal{C} \to \mathcal{C}$ we define the *two-sided bar construction* $B(G, \mathcal{D}, F)$ as the left Kan extension

$$\begin{array}{ccccc} & & \mathbf{\Delta}^{\mathrm{op}} & & \\ & \overset{\pi_{N\mathcal{D}}}{\nearrow} & & \overset{B(G,\mathcal{D},F)}{\searrow} & \\ \mathrm{el}N\mathcal{D} & \xrightarrow[(\mathrm{tgt},\mathrm{src})]{} & \mathcal{D}^{\mathrm{op}} \times \mathcal{D} & \xrightarrow[G\otimes F]{} & \mathcal{C} \end{array}$$

Recall $\mathrm{el}N\mathcal{D}$ is the category of elements of the nerve $N\mathcal{D}$ of $\mathcal{D}$. Objects are strings of morphisms in $\mathcal{D}$, and the functors tgt and src associate to a string its target respectively source object.

Exercise 2.35. Using the coequaliser formular obtain an explicit expression for the values of $B(G, \mathcal{D}, F)$ on $\mathbf{\Delta}^{\mathrm{op}}$ (or see [Rie14, 4.2]). If $\mathcal{V} = \mathcal{M} = \mathbf{Set}$, and $*$ is the terminal functor to $\mathbf{Set}$, then show $B(*, \mathcal{D}, *) = N\mathcal{D}$.

Remark 2.36 (The free theory of left and right modules). The two-sided bar construction has a slightly more general incarnation: Given a monad M (as an endomorphism in a bicategory $\mathcal{B}$) and left and right modules F and G for M then $B(G, M, F)$ describes the functor from the theory $\mathbf{I}$ of "a monad I_m with left module I_f and right module I_g" to $\mathcal{B}$ mapping I_g, I_m, I_f to G, M, F, but restricted to the hom category $\mathrm{hom}_{\mathbf{I}}(I_f, I_g)$. The latter category is indeed $\mathbf{\Delta}^{\mathrm{op}}$. There are nice string diagrams that can be drawn to visualise this, see e.g. [Bae07, Lect. 24]. For instance, $B(G, M, F)_n$ is the object GM^nF of that hom category.

A connection to the construction given here is based on the observation that an ordinary category $\mathcal{D}$ is itself a monad $M = \mathcal{D}$ in the bicategory of spans of sets $\mathrm{Span}(\mathbf{Set})$ while profunctors (and thus functors by postcomposition with hom functors) are bimodules in that bicategory. For instance, composition being given by pullback in the bicategory of spans, the term M^n appearing above translates as $M^n = \mathrm{mor}(\mathcal{D}) \times_{\mathrm{obj}(\mathcal{D})} \ldots \times_{\mathrm{obj}(\mathcal{D})} \mathrm{mor}(\mathcal{D})$ back into the explicit description using strings of morphism of length n that can be derived from the definition given here (see previous exercise).

We will revisit the explanation of the idea underlying the bar construction in Remark 2.41.

Definition 2.37. The *functor tensor* of functors F, G as above is defined by

$$G \otimes_{\mathcal{D}} F = \int^{\mathcal{D}} G \otimes F$$

We remark that there is a dual notion of functor cotensor, depending on the presence of a "cotensor" of $\mathcal{M}$ over $\mathcal{V}$.

Remark 2.38. Since the the inclusion of the two arrows $[1] \rightrightarrows [0]$ into $\mathbf{\Delta}^{\mathrm{op}}$ is *final* (i.e. colimits are preserved when precomposing with this functor), one can see that

$$\mathrm{colim}_{\mathbf{\Delta}^{\mathrm{op}}} B(G, \mathcal{D}, F) = G \otimes_{\mathcal{D}} F$$

by rewriting the left colimit as a coequaliser and using the coequaliser formula on the right.

Exercise 2.39. Let R be a one object **Ab**-enriched category, i.e. a ring. let $N : R \to \mathbf{Ab}$, $M : R^{\mathrm{op}} \to \mathbf{Ab}$ be **Ab**-functors, making them right resp. left modules. Consider their underlying **Set** functors and use the monoidal structure $\otimes_{\mathbb{Z}}$ on **Ab**, to compute the functor tensor

$$N \otimes_R M = \int^R N \otimes_{\mathbb{Z}} M$$

using the coequaliser formula. Show that it coincides with the usual tensor product.

Examples 2.40. Using previous results, we find the following range of examples. The formula for weighted colimits gives

$$\mathrm{colim}^W J = W \otimes_{\mathcal{D}} J$$

In particular,

$$\mathrm{colim}\, J = * \otimes_{\mathcal{D}} J$$

Our ninja Yoneda lemmas can now be stated as

$$F = \mathcal{D} \otimes_{\mathcal{D}} F$$

making $\mathcal{D}$ into "free modules" for the functor tensor. More generally for Kan extensions we find,

$$\mathrm{Lan}_K F = \mathcal{B}(K-, -) \otimes_{\mathcal{A}} F$$

In particular, geometric realisation, can be written as for $X : \Delta^{\mathrm{op}} \to \mathcal{M}$ and $\mathcal{M}$ tensored over **sSet**)

$$|X| = \Delta \otimes_{\mathbf{\Delta}^{\mathrm{op}}} X$$

A functor tensor is a generalisation of the usual tensor, "using the same coequaliser" formula. The two-sided-cobar construction is a "coherent" (or "cofibrant") resolution of the functor tensor as we will explained again informally in the next remark.

Remark 2.41 (Resolutions and free monoids). $\mathbf{\Delta}^{\mathrm{op}}$ is the free (monoidal) category containing a strict monoid object (see [Bae07, Lect. 24]). A general purpose of free structures is to generate all possible composites of their generating elements, and thus to provide a data structure to record equalities/coherences between these composites. In this (informal) sense $\mathbf{\Delta}^{\mathrm{op}}$, is for instance the right category to encode higher coherences of monoids, and in particular monoidal categories (monoids in $\mathbf{Cat}^{\times}$) can for instance be defined like (pseudo-)functors

$$M : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Cat}^{\times}$$

(together with some conditions of natural isomorphisms $M[n] \cong M[1]^n$, analogous to the so-called "Segal conditions"). This is in contrast to defining a monoidal category simply by

$$M : \mathbf{\Delta}^{\mathrm{op}}_{\leq [2]} \to \mathbf{Cat}^{\times}$$

$\mathbf{\Delta}^{\mathrm{op}}_{\leq[2]}$ (the full subcategory of $\mathbf{\Delta}^{\mathrm{op}}$ with objects $[0],[1],[2]$) can represent data for identities and a monoid multiplication $(-\otimes-):[2]\to[1]$. But it does not contain not all possible "freely generated" compositions $[k]\to[l]$ of $\otimes$ with itself and how these compositions cohere — a simplest example of a coherence is for instance associativity, stating $(-\otimes-)\otimes- = -\otimes(-\otimes-)$.

We can now also revisit our understanding of the bar construction. $B(G,\mathcal{D},F)$ not only contains "simple, unary" actions $B(G,\mathcal{D},F)_1$ of the monad $\mathcal{D}$ on its modules F and G, but also "composite" actions (and their coherences) $B(G,\mathcal{D},F)_{n>1}$, and these are indexed by (a hom category in) the free theory of a monad with modules $\mathbf{I}$, see Remark 2.36, analogous to $\otimes$-composites and their coherences of monoidal categories being indexed in Δ^{op}.

It (now hopefully expectedly) turns out, instead of quotienting out (higher) paths of (higher) actions as we did with the coequaliser, we should just be gluing them together using our geometric realisation construction. In particular, instead of the coequaliser

$$\operatorname{colim} F = \operatorname{colim} B(*,\mathcal{D},F)$$

we should consider the geometric realisation

$$|B(*,\mathcal{D},F)|$$

Indeed, we have the following.

Remark 2.42 (Deformation for colim). Let us assume we know what a simplicial (i.e. **sSet**-enriched) model category is, and that $\mathcal{M}$ is an instance of that definition, together with a "cofibrant replacement functor" $Q:\mathcal{M}\to\mathcal{M}$ (As an example of the previous definition the reader may take $\mathcal{M}=\mathbf{Top}$, which obtains **sSet**-enrichment using the singular complex functor Sing defined earlier). Let $F:\mathcal{D}\to\mathcal{M}$ be a diagram. Then

$$\operatorname{hocolim} F = |B(*,\mathcal{D},QF)|$$

or for a pointwise cofibrant F,

$$\operatorname{hocolim} F = |B(*,\mathcal{D},F)|$$

We can further reformulate (by commuting colimits)

$$\begin{aligned}|B(*,\mathcal{D},QF)| &= |B(*\otimes_{\mathcal{D}}\mathcal{D},\mathcal{D},QF)|\\ &= *\otimes_{\mathcal{D}}|B(\mathcal{D},\mathcal{D},QF)|\\ &= \operatorname{colim}|B(\mathcal{D},\mathcal{D},QF)|\end{aligned}$$

which means $|B(\mathcal{D},\mathcal{D},Q-)|:\mathcal{M}^{\mathcal{D}}\to\mathcal{M}^{\mathcal{D}}$ is a deformation for colim, in the sense previously defined.

Exercise 2.43 (Homotopy colimits as enriched weighted limits). Firstly, observe the geometric realisation $|X|$ of a bisimplicial set $X:\mathbf{\Delta}^{\mathrm{op}}\to\mathbf{sSet}$ is its diagonal $|X|_n = X_{n,n}$. Secondly For $\mathcal{D}$ a **Set**-enriched category, verify $B_n(*,\mathcal{D},\mathcal{D}(d,-)) = N(d/\mathcal{D})_n$, or, varying d and n, $B_\bullet(*,\mathcal{D},\mathcal{D}) = N(-/\mathcal{D})$. Thirdly, note, the constant functors $\mathrm{const}_{\_}:\mathbf{Set}\to\mathbf{sSet}$ turn sets into discrete simplicial sets, and allow to think of ordinary categories as **sSet**-enriched categories. This implies $B_\bullet(*,\mathcal{D},\mathcal{D})$ for discrete **sSet**-enriched $\mathcal{D}$ yield the (horizontally discrete) bisimplicial set $N(-/\mathcal{D})_{\bullet,\bullet}$. Combining this verify (or see [Rie14, 4.1]) the computation that

$$|B_\bullet(*,\mathcal{D},\mathcal{D})| = |N(-/\mathcal{D})_{\bullet,\bullet}| = N(-/\mathcal{D})_\bullet : \mathcal{D}\to\mathbf{sSet}$$

Further verify that (commuting colimits)

$$|B_\bullet(*,\mathcal{D},\mathcal{D})| \otimes_{\mathcal{D}} F = |B_\bullet(*,\mathcal{D},\mathcal{D}\otimes_{\mathcal{D}} F)| = |B_\bullet(*,\mathcal{D},F)|$$

Deduce that

$$\operatorname{hocolim} F = N(-/\mathcal{D}) \otimes_{\mathcal{D}} F$$

which we recognise as a (enriched) weighted limit, yielding in turn the expression

$$\mathcal{M}(\operatorname{hocolim} F, m) \cong \mathbf{sSet}^{\mathcal{D}^{\mathrm{op}}}(N(-/\mathcal{D}), \mathcal{M}(F-, m))$$

Definition 2.44. A 0-simplex of $\mathbf{sSet}^{\mathcal{D}^{\mathrm{op}}}(N(-/\mathcal{D}), \mathcal{M}(F-, m))$ will be called a *homotopy coherent cocone* over F in $\mathcal{M}$.

We will now turn our attention to a different perspective on higher categories, yielding another notion of homotopy coherent (co)cone and compare the two.

3. ∞-Category theory

3.1. **General remarks on higher categories.** The idea of higher category theory has several origins. One is of topological nature: If we think of a groupoid (a category in which all morphisms are isomorphisms) as the fundamental groupoid of a space containing points and paths, then a higher groupoid should contain points, paths, paths of paths ("homotopies") etc. The correspondence of (the algebraic notion of) higher groupoids and (the topological notion of) spaces is called homotopy hypothesis, and is a guiding paradigm in the development of higher category theory.

Unlike in higher groupoids (also call ∞-groupoid), in a $(\infty,1)$-category not all paths need to be invertible but are instead "directed" (however all k-paths, that is, paths of $(k-1)$-paths with 1-paths being ordinary paths, are invertible for $k > 1$). More generally in an n-category (or (∞,n)-category), we not require k-paths to be invertible up to dimensions $k \leq n$.

The difficulty of higher categories lies in finding a description of how composites of higher paths cohere. There are several models of $(\infty,1)$-categories and more generally higher categories.

(i) *Space enrichment*: One idea underlying some of these models is to use (Set-theoretic or combinatorial) models of topological spaces to describe the hom spaces of $(\infty,1)$-categories. Examples are Kan complex-enriched categories, **Top**-enriched categories or (in a broader sense of the idea) quasicategories.
(ii) *Contractibility*: Another idea is to use the idea of "contractibility" that says that all disk-like composites cohere as long as they compose to the same shape. This idea can be adapted to give definitions of (∞,n)-categories as well. Examples as θ_n-spaces, n-fold Segal spaces, and Batanin-Leinster type definitions of n-categories[2].

Remark 3.1 (Classification of coherences). Neither space enrichment nor contractibility give us a good handle of classifying coherences: the former, "hides" coherences in the theory of spaces, the latter only describes them explicitly in the simple case of disk-like coherence, with general coherence being complex amalgamations of these simpler disk-like ones. However, most interesting coherences (such as those hidden in the homotopy groups of sphere by the homotopy hyptohesis) are of this complex

[2]A different between the latter and the former however is, that the latter specifies compositions uniquely making it a so-called "algebraic" model, whereas in the former compositions are only given up to contractible choice making it a so-called "non-algebraic" model"

nature. Furthermore, in low dimensions a classification of these coherences is known and includes for instance the Reidemeister moves or the Yang-Baxter equation. A research programme aiming at a direct description and classification of interesting coherences in general dimensions was started in the speaker's thesis [Dor18].

We will focus our attention on quasicategories. As pointed out in the previous remark, the business of coherences is "hidden away" in the theory of Kan complexes (to be defined below). It should not come as a surprise that as a result of hiding away higher coherences $(\infty, 1)$-category ends up to be much more similar to (enriched) 1-category theory than to, e.g., 2-category theory. This also justifies why we've spent so much time on 1-category theory in the previous section.

3.2. **Quasicategories.**

Definition 3.2. A Kan fibration is a map of simplicial sets $f : X \to Y \in \mathbf{sSet}$ satisfying the lifting property (i.e. the dashed arrow always exists)

$$\begin{array}{ccc} \Lambda^i[n] & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \Delta[n] & \longrightarrow & Y \end{array}$$

where $\Lambda^i[n]$, $0 \leq i \leq n$, is the ith *horn* of $\Delta[n]$, a sub-simplicial set obtained from $\Delta[n]$ by removing the unique non-degenerate n-simplex from it, as well as the unique $(n-1)$-simplex not containing the 0-simplex i.

Kan complexes are a combinatorial model for topological spaces.

Exercise 3.3. Show that every singular complex of a topological space is a Kan complex.

Definition 3.4. A *quasicategory* $\mathcal{C}$ is a simplicial set such that the following lifts always exist

$$\begin{array}{ccc} \Lambda^i[n] & \longrightarrow & \mathcal{C} \\ \downarrow & & \\ \Delta[n] & & \end{array}$$

for $0 < i < n$. Let $c_1, c_2 \in \mathcal{C}$ be 0-simplices (called objects). We define the hom space $\mathcal{C}(c_1, c_2)$ as the pullback

$$\begin{array}{ccc} \mathcal{C}(c_1, c_2) & \longrightarrow & \mathcal{C}_{/c_2} \\ \downarrow & & \downarrow \\ c_1 & \longrightarrow & \mathcal{C} \end{array}$$

(here we are using the slice construction defined earlier).

The intuition is as follows: m-simplices in $\mathcal{C}$ are m-paths, $m > 0$, and objects for $m = 0$. For $m = 1$ these are directed, having a source object and a target objects. For $m > 1$, the distinction between source and target need not be made anymore (and would be an arbitrary choice albeit a possible one), since all m-path are "invertible" up to a higher path. This is a consequence of the horn-filling condition.

Remark 3.5 (Recovering a quasicategory from a space enriched category). Every **sSet**-enriched category $\mathcal{C} \in \mathbf{sSetCat}$ can be translated into a quasicategory $N_\Delta(\mathcal{C}) \in \mathbf{sSet}$. For this, not that $[n] \in \mathbf{\Delta}$ naturally obtains the structure of an **sSet**-category $S[n]$, if we set hom simplicial sets $S[n](i,j)$ to be the nerve of the poset of all string of morphims $i \to ... \to j$ in $[n]$ ordered by inclusion (note that strings of morphisms in n are determined by the subset objects they contain, which is why we can chose to order by inclusion). By Kan extension we also obtain

$$S = \mathrm{Lan}_{\Delta \subset \mathbf{Cat}} S : \mathbf{Cat} \to \mathbf{sSetCat}$$

both this and $S : \Delta \to \mathbf{sSetCat}$ are called *simplicial thickening*. Kan extending S along Δ we also obtain (using the nerve realisation paradigm)

$$C[-] \vdash N_\Delta :$$

with N_Δ called the *coherent nerve*. If $\mathcal{C}$ is Kan-complex enriched (that is its hom spaces are actually Kan complexes) then $N_\Delta(\mathcal{C})$ is a quasicategory. In fact, more technically, Kan-complex enriched categories are the fibrant objects in the Bergner model structure on **sSetCat**, and $C[-] \vdash N_\Delta$ is a Quillen equivalence when putting Joyal model structure on **sSet** in which quasicategories are fibrant. For general $\mathcal{C}$ one thus needs to apply fibrant replacement $Q\mathcal{C}$ before recovering a quasicategory via $N_\Delta(Q\mathcal{C})$.

Definition 3.6 (Functors and universe)**.** Functors $F : \mathcal{C} \to \mathcal{D}$ of quasicategories are maps of their underlying simplical set. The functor space $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is the maximal Kan complex containing the the internal hom $\mathbf{sSet}(\mathcal{C}, \mathcal{D}) \in \mathbf{sSet}$ (recall **sSet** is cartesian closed, thus $\mathbf{sSet}(\Delta[m], \mathbf{sSet}(\mathcal{C}, \mathcal{D})) \cong \mathbf{sSet}(\Delta[m] \times \mathbf{sSet}(\mathcal{C} \times \Delta[m], \mathcal{D})$ gives you an idea of what the m-simplex in $\mathbf{sSet}(\mathcal{C}, \mathcal{D})$ are).

We define the Kan complex-enriched category $\mathbf{Cat}_\infty^\Delta$ to have as objects quasicategories $\mathcal{C}$, and as hom spaces functor spaces $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. The *quasicategory of quasicategories* $\mathbf{Cat}_\infty$ is defined as $N_\Delta(\mathbf{Cat}_\infty^\Delta)$.

Most of the usual operations of 1-categories (such as the Yoneda lemma, notion of fullness/faithfulness, or simply pre- and post-composition inducing functors on hom spaces) can be recovered along similar lines. The reader is invites to invite to think about these notions themselves. Details can be found in [Lur09].

3.3. **Adjunctions, Kan extensions and colimits.**

We now "lift" the our earlier definitions into the context of quasicategories (however, our formulation will be so general, that they likely apply for other models of $(\infty, 1)$-categories too).

Definition 3.7 (Adjunctions)**.** See [Lur09, 5.2.2]. Given $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, together with a natural transformation $\mu : \mathrm{id} \to GF$ such that for all $c \in \mathcal{C}$, $d \in \mathcal{D}$, the map $\mathcal{D}(Fc, d) \xrightarrow{G(-)\mu_c} \mathcal{C}(c, Gd)$ is a weak equivalence (an isomorphism in the homotopy category of spaces), then we say $F \vdash G$ are *adjoint* functors.

We remark that a slightly more elegant definition is possible: any functor $F : \mathcal{C} \to \mathcal{D}$ can be regarded as a bundle "classified" by a functor $\mathcal{D} \to \mathbf{Prof}$ to profunctors. If $\mathcal{D} = \Delta[1]$, then $\mathcal{D} \to \mathbf{Prof}$ picks out a single profunctor $R : \mathcal{C}_0 \nrightarrow \mathcal{C}_1$ (setting $\mathcal{C}_i = F^{-1}(i)$). If this is both covariantly ($R \cong \mathcal{C}_1(G_0-, -)$) and contravariantly ($Riso\mathcal{C}_0(-, G_1-)$) represented then $G_0 \vdash G_1$. This represented condition for R can be translated to properties of F: To represent an adjunction F needs to be both a *coCartesian* and *Cartesian fibration* (see [Lur09]).

Definition 3.8 (Kan extensions)**.** See [Lur09, 4.4.2]. Let $K : \mathcal{A} \to \mathcal{B}$, and $\mathcal{C}$ be (functors of) quasicategories. The *left Kan extension functor* (if it exists) is the left adjoint

$$\mathrm{Lan}_K \vdash - \circ K : \mathrm{Fun}(\mathcal{B}, \mathcal{C}) \to Fun(\mathcal{A}, \mathcal{C})$$

Definition 3.9 (Colimits and limits)**.** See [Lur09, 1.2.13]. $c \in \mathcal{C}$ is called *initial* if $\mathcal{C}_{c/} \to \mathcal{C}$ is a trivial Kan fibration (that is, a Kan fibration whose geometric realisation is a weak equivalence). Now let $F : \mathcal{D} \to \mathcal{C}$. The *($\infty$-)colimit* $\mathrm{colim}(F)$ is an initial object of $\mathcal{C}_{F/}$.

Exercise 3.10. Dualise the above discussion, writing down definition of limits and right Kan extensions.

3.4. **Monoidal structures and tensor products.** We give sketchy definitions, built upon our work in the 1-categorical context, of monoidal structures and (functor) tensors. In particular recall Remark 2.41 that Δ^{op} is the free (monoidal) category containing a monoid, and that all other monoids (in particular, monoidal categories which are monoids in **Cat**) can be described as functors from Δ^{op}. The correct $(\infty, 1)$-categorical analogue is the following.

Definition 3.11 (Symmetric monoidal $(\infty, 1)$-categories)**.** See [Lur12, 2.0.0]. A monoidal quasicategory is a functor

$$\mathcal{V} : N(\Delta^{\mathrm{op}}) \to \mathbf{Cat}_\infty$$

satisfying the *Segal conditions*: The natural map $\mathcal{V}[n] \to \mathcal{V}[1]^n$ is a weak equivalence.
Similarly, a *symmetric monoidal quasicategory* is a functor

$$\mathcal{V} : N(\mathrm{Fin}_*) \to \mathbf{Cat}_\infty$$

(where Fin_* is the category of finite pointed sets) satisfying again the natural Segal condition.

In the similar spirit, recalling Remark 2.36, we sketch the following.

Definition 3.12 (Tensor products)**.** See [Lur12, 4.4.2]. The bar construction $B(G, \mathcal{D}, F)$ for appropriate functors G, F of quasicategories, can be defined analogous to the 1-categorical case, yielding a functor (i.e. map of simplicial sets) $B(G, \mathcal{D}, F) : N(\Delta^{\mathrm{op}}) \to \mathcal{M}$. We define the functor tensor

$$G \otimes_{\mathcal{D}} F = \mathrm{colim}\, B(G, \mathcal{D}, F)$$

Note, that the colimit now refers to the ∞-colimit.

In particular, if $\mathcal{D}$ is some algebraic theory (such as a higher categorical version of a ring R, like in one of our earlier examples) this will be a good candidate to define the tensor product of "higher" modules G and F over $\mathcal{D}$. (The point here is simply that: The bar construction is a the right construction for a coherent tensor product, as explained with more detail in Remark 2.36. A formal construction in the language of ∞-operads is given in [Lur12]).

3.5. **Comparison to 1-categorical presentations.** We have now seen two different approaches to higher categories:

(i) *Presentations of higher categories*: 1-categorical structures with "weak equivalences", e.g. relative categories
(ii) *Higher categories*: Structures with "higher morphisms", e.g. quasicategories

Having seen these two approaches, we mention a couple of natural questions.

- *How can we assure that these theories do the same job?* There are techniques to pass back and forth between them and showing they are equivalent. The latter mostly by introducing model structures and constructing Quillen equivalences, which then guarantee exactly that at least *at the level of homotopy categories* we will obtain equivalent theories. A caveat (one of several!) we should mention is that model categories (unlike relative categories) do not present all $(\infty, 1)$-categories (but only a [possibly improper] subset of the co/complete ones).
- *Is it even reasonable to assume that every quasicategory, and thus every ∞-groupoid, can be presented by the simplistic structure of relative categories?* The underlying idea of passing from presentations to higher categories is a "localisation" construction that treats weak equivalence as if they were we (weakly) invertible. In fact, for instance [BK12] manage to package this into a nerve construction N_ξ (which unlike our nerve here contains additional subdivisions allowing us exactly to walk backwards along weak equivalence). The left adjoint "realisation" K_ξ to N_ξ, realises a (bi)simplicial set X by gluing together relative categories from these subdivided simplices with weak equivalences according the gluing of (bi)simplices in X.
- *How does our definition of derived functors defined by Kan extension translate to higher categories?* There is no real analogue of derived functors in higher categories. Derived functor are "homotopical corrections" to non-homotopical functors. But in higher categories, every functor is already homotopical (e.g. every functor of quasicategories preserves weak equivalences). Nonetheless, we can ask whether a given homotopical corrected functor represents a functor of higher categories. For instance, does hocolim present ∞-colim? A answer in the case of quasicategories is given in [Lur09, 4.2.4]. Another is suggested in the exercise below. In general, both for derived functors and for novel "corresponding" definitions stated in higher categories we have to verify that they satisfy the desired universal properties, such as our definition of hocolim and ∞-colim above (however, derived functors do of course inherit a universal property from being define by Kan extensions, in which sense they are a "closest" homotopical approximation to the orignal universal property. Also see [Rie17, 6.4.12]).

Exercise 3.13 (Comparing notions of colimits). Let $\mathcal{M} = \mathbf{Top}$, and $F : \mathcal{D} \to \mathbf{Top}$ a pushout diagram in $\mathbf{Top}$. Using Exercise 2.43, a "homotopy coherent cocone C" under X" is an object

$$C \in \mathbf{sSet}^{\mathcal{D}^{\mathrm{op}}}(N(-/\mathcal{D}), \mathbf{Top}(F-, X))$$

Compare this with a "cocone under X" in the sense of ∞-colimit of quasicategories: for the latter note (recalling S, N_Δ)

$$\mathbf{sSet}(N(\mathcal{D}), N_\Delta(\mathbf{Top})) \cong \mathbf{sSetCat}(S(\mathcal{D}), \mathbf{Top})$$

A pushout diagram $F : N(\mathcal{D}) \to N_\Delta(\mathbf{Top})$ thus corresponds to a functor $F : S(\mathcal{D}) \to \mathbf{Top}$, and a $(\infty, 1)$-*cocone with summit* m in $N_\Delta(\mathbf{Top})_{F/}$ corresponds to a functor $C : S(\mathcal{D} \star *) \to \mathbf{Top}$ with $C(*) = m$ and $C = F$ on $S(\mathcal{D})$.

Convince yourself that data needed to specify cocones is the same, and thus that hocolim and ∞-colim coincide in the sense sketched above.

4. Factorisation homology

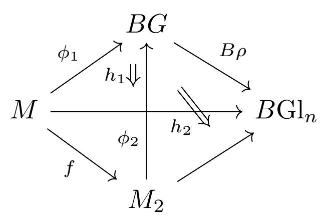
4.1. ***G*-structures.** Recall, given a group G with $\rho : G \to \mathrm{Gl}_n(\mathbb{R})$, a G-structure on a manifold M consists of a map $\phi : M \to BG$ (BG denoting the classifying space of G) together with a homotopy $h : B(\rho)\phi \xrightarrow{\sim} \tau$ where $\tau : M \to B\mathrm{Gl}_n$ is the classifying map of the tangent bundle of M.

Let $\mathbf{Mfld}_n$ denote the $(\infty, 1)$-category of n-manifolds (without boundary) with hom spaces being space of embeddings with compact-open topology (in other words, $\mathbf{Mfld}_n$ is defined as a $\mathbf{Top}$-enriched category). We define the $(\infty, 1)$-category of n-manifolds $\mathbf{Mfld}_{n,G}$ with G structure by the pullback (in $\mathbf{Cat}_\infty$)

$$\begin{array}{ccc} \mathbf{Mfld}_{n,G} & \longrightarrow & \mathbf{Top}/_{BG} \\ \big\downarrow & \lrcorner & \big\downarrow {\scriptstyle \mathbf{Top}/_{B(\rho)}} \\ \mathbf{Mfld}_n & \xrightarrow{\tau} & \mathbf{Top}/_{B\mathrm{Gl}_n} \end{array}$$

A similar definition can be given for $\mathbf{Disk}_{n,G}$ based on $i : \mathbf{Disk}_n \subset \mathbf{Mfld}_n$, the full subcategory of finite disjoint unions of $\mathbb{R}^n$.

Unwinding definitions one finds that that a morphism $f : M_1 \to M_2$ not only consists of a map of the underlying manifolds but also filler for each blank face as well as the interior of the following tetrahedron



Remark 4.1 (hom space of G-structure disk). Pulling back the above pullback at the object $\mathbb{R}^n \simeq *$ we obtain a pullback diagram of one-object $(\infty, 1)$-categories, i.e. delooped spaces. This pullback diagram of spaces allows us to determine $\mathbf{Mfld}_{n,G}(\mathbb{R}^n, \mathbb{R}^n)$ (see [AKMT19]) and from the top arrow one obtains a homotopy equivalence

$$\mathbf{Mfld}_{n,G}(\mathbb{R}^n, \mathbb{R}^n) \xrightarrow{\sim} \mathbf{Top}/_{BG}(*, *) \simeq \Omega BG \simeq G$$

4.2. **Definition and axiomatisation.** Note that using disjoint union $\bigsqcup$ as a monoidal product, both $\mathbf{Disk}_{n,G}$ and $\mathbf{Mfld}_{n,G}$ obtain symmetric monoidal structure (as $(\infty, 1)$-categories).

Definition 4.2. Let $\mathcal{C}^\otimes$ be a symmetric monoidal $(\infty, 1)$-category. A (symmetric monoidal) functor $A : \mathbf{Disk}_{n,G} \to \mathcal{C}^\otimes$ is called an $E_{n,G}$-algebra.

Factorisation homology $\int_{\_} A$ (with coefficients in A) is the left Kan extension

$$\begin{array}{ccc} & \mathbf{Mfld}_{n,G} & \\ {\scriptstyle i} \nearrow & & \searrow {\scriptstyle \int_{\_} A} \\ \mathbf{Disk}_{n,G} & \xrightarrow[A]{\qquad\qquad} & \mathcal{C}^\otimes \end{array}$$

We make the following observations

(i) The inclusion $\mathbf{Disk}_{n,G} \subset \mathbf{Mfld}_{n,G}$ is fully faithful. Thus, as observed previously in the 1-categorical case (and assuming an analogous result for the $(\infty, 1)$-categorical case), we have

$$\int_D A \xrightarrow{\sim} A(D)$$

for $D \in \mathbf{Disk}_{n,G}$

(ii) The inclusion $\mathbf{Disk}_{n,G} \subset \mathbf{Mfld}_{n,G}$ is dense in the sense previously defined. This means each manifold can be written as a canonical colimit of disks.

(iii) Since $\mathbf{Mfld}_{n,G}(\mathbb{R}^n, \mathbb{R}^n) \cong G$ we have that $A_0 \equiv A(\mathbb{R}^n)$ (abusing notation here!) has a G action (by functoriality of A).

The easiest case is that of $G = *$, in which case G-structure is also called *framing*. In this case, we also simply speak of E_n-algebra. E_n-algebras (and $E_{n,G}$-algebras) obtain their algebraic structure from disk embeddings. E.g. the embedding of the binary union

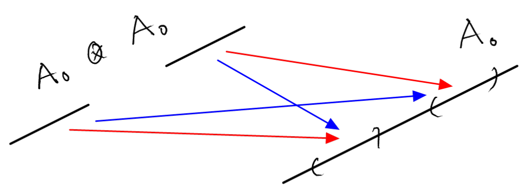
$$\mathbb{R}^n \bigcup \mathbb{R}^n \hookrightarrow \mathbb{R}^n$$

induces a multiplication operation

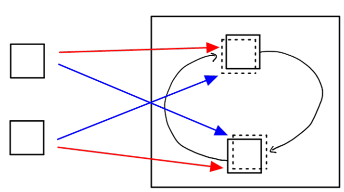
$$\begin{aligned} m_A : A_0 \otimes A_0 \cong A(\mathbb{R}^n \bigcup \mathbb{R}^n) &\to A(\mathbb{R}^n) = A_0 \\ (a, b) \qquad &\mapsto m_A(a, b) \end{aligned}$$

Note that this requires to order the two components of $\mathbb{R}^n \bigcup \mathbb{R}^n$, a choice which when changed will lead to the *opposite algebra* structure A^{op}, that is, $m_A(a, b) = m_{A^{\mathrm{op}}}(b, a)$. Also note that

(i) For $n = 1$, we have $m_A(a, b) \neq m_A(b, a)$ in general. We can visualise this as the fact that the blue and red inclusions of 1-disks $\mathbb{R} \sqcup \mathbb{R} \hookrightarrow \mathbb{R}$ below are non-homotopic



(ii) For $n = 2$, we have $m_A(a, b) \xrightarrow{\sim} m_A(b, a)$ by the usual Eckmann-Hilton argument, visualised below



however, there are two such homotopies of embeddings: above we've visualised a homotopy by clockwise rotation, but counter-clockwise rotation works just as well. These two homotopy in turn need not be homotopic themselves. In particular $m_A(a, b) \xrightarrow{\sim} m_A(b, a) \xrightarrow{\sim} m_A(a, b)$ need not be the identity — a divergence from the classical strict commutativity.

(iii) For $n = 3$, the two rotational homotopies become homotopic due two the presence of the third dimension ($\pi_1 S^2 = 0$). However they are so by two different homotopies, which again are not homotopic. This pattern continues for higher n.

Thus the higher the n, the "more commutativity" is present in an E_n-algebra.

We will now come to the main theorem about factorisation homology. Recall a collar gluing of n-manifolds is a manifold obtained as the gluing $X_1 \sqcup_{W\times\mathbb{R}} X_2$ of manifolds X_1, X_2 along a submanifold $W \times \mathbb{R} \subset X_i$.

Definition 4.3 ($\otimes$-excisiveness)**.** A monoidal functor $F : \mathbf{Mfld}_{n,G}^{\sqcup} \to \mathcal{C}^{\otimes}$ is $\otimes$-*excisive* if for each collar gluing we have

$$F(X_1 \sqcup_{W\times\mathbb{R}} X_2) = F(X_1) \otimes_{F(W\times\mathbb{R})} F(X_2)$$

Let $Fun_{\otimes\text{-exc}}(\mathbf{Mfld}_{n,G}), \mathcal{C}^{\otimes})$ denote the full subcategory of $\otimes$-excisive functors in $\mathrm{Fun}(\mathbf{Mfld}_{n,G}), \mathcal{C}^{\otimes})$

Implicit in the previous definition is the natural $E_{1,G}$-algebra structure of $F(W \times \mathbb{R})$ and the natural module structure of $F(X_i)$ for this algebra. The tensor product is then constructed as the (∞-)colimit of an appropriate bar construction as explained earlier. We will revisit these structure in an example in a moment.

Theorem 4.4 (Aximatic characterisation of factorisation homology)**.** *The factorisation homology functor $\int_{\_} A$ is $\otimes$-excisive, and the adjunction*

$$\int_{\_} - \vdash (- \circ i) : \mathrm{Fun}_{\otimes}(\mathbf{Disk}_{n,G}, \mathcal{C}^{\otimes}) \to Fun_{\otimes\text{-exc}}(\mathbf{Mfld}_{n,G}), \mathcal{C}^{\otimes})$$

is an adjoint equivalence.

Trivial proof sketch. For $\otimes$-excisiveness of $\int_{\_} A$, dissect the definition of the tensor product in terms of the bar construction in dimension 1, and deduce the statement. By definition of left Kan extensions, we already have an adjunction, and thus units and counits. It remains to show that those are indeed natural equivalences.

In other words all $\otimes$-excisive functor from G-manifolds to $\mathcal{C}^{\otimes}$ actually arise as factorisation homology from an $E_{n,G}$-algebra. This is analogous to the following characterisation of singular homology: let $\mathrm{Fun}_{\otimes\text{-exc}}(\mathbf{Top}_{CW}, \mathbf{Ch}^{\oplus}_{\mathrm{proj}})$ denote functors from the topological category of spaces homotopy equivalent to finite CW complexes to the topological category of projective chain complexes, that are "excisive", not with respect to collar gluings, but general pushouts of cofibrations. Then we have the following:

Theorem 4.5 (Reformulating the Eilenberg-Steenrod axioms)**.** *There is an equivalence of categories*

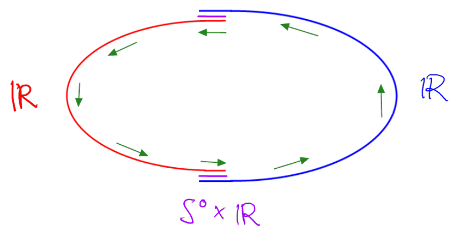
$$\mathrm{Fun}_{\otimes\text{-exc}}(\mathbf{Top}_{CW}, \mathbf{Ch}^{\oplus}_{\mathrm{proj}}) \underset{C_*(-;-)}{\overset{\mathrm{ev}_*}{\rightleftarrows}} \mathbf{Ch}^{\oplus}_{\mathrm{proj}}$$

where $V \to C_*(-;V)$ denotes singular homology with coefficients in V.

How exactly this is a reformulation of the Eilenberg-Steenrod axioms we will discuss again in the following examples. In particular we will mention how factorisation homology is a generalisation of singular homology.

4.3. **Examples.**

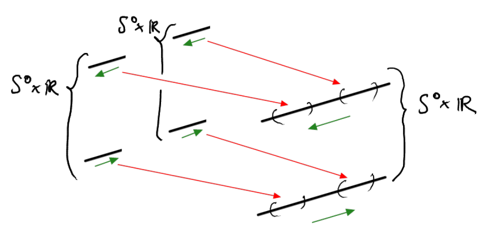
4.3.1. $n = 1$*: Hochschild chains.* Our first example is meant to illustrate the tensor product $X_1 \otimes_{W\times\mathbb{R}} X_2$ from Definition 4.3 in the case of the trivial structure group, that is, the case of framed manifolds. In dimension $n = 1$, manifolds are unions of open disks and circles, and the notion of framing and orientation can be seen to coincide (thus we assume our 1-manifolds and their embeddings to be oriented from now). The circle however can be written as a collar gluing $S^1 = \mathbb{R} \bigsqcup_{S^0\times\mathbb{R}} \mathbb{R}$, illustrated in the following



As a consequence of the excision theorem we derive

$$\int_{S^1} A = A_0 \otimes_{\int_{S^0\times\mathbb{R}} A} A_0$$

$\int_{S^0\times\mathbb{R}} A$ has itself E_1 algebra structure (and the argument applies to all objects of the form $\int_{W\times\mathbb{R}} A$ as used in the $\otimes$-excision theorem), which can be seen as follows: recall that algebra multiplication operations on $A_0 = A(\mathbb{R})$ (resp. their coherences) are the images under A of embeddings of disks (resp. their (higher) homotopies). Taking the product of these embeddings with S^0 (or more generally, with W) shows that the image of $A(S^0 \times \mathbb{R})$ carries the same algebra structure. For instance, multiplication operations on $A(S^0 \times \mathbb{R})$ are induced by embeddings such as



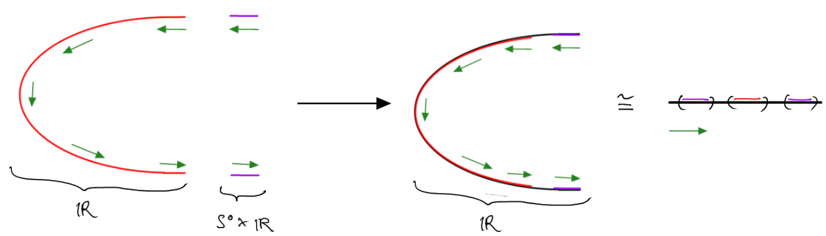
Of course, $S^0 \times \mathbb{R}$ is just two copies of disks. However, note that the orientation (marked by arrows in all previous pictures) of the collar $S^0 \times \mathbb{R} \subset S^1$, force that the algebra structures on the two disks are *opposite* to each other. Monoidality of A implies that we have (up to a global "op")

$$A(S^0 \times \mathbb{R}) = A_0^{\mathrm{op}} \otimes A_0$$

The action of $A_0^{\mathrm{op}} \otimes A_0$ on the two A_0 in the above collar gluing (or more generally the $\int_{X_i} A$ in the excision theorem) can then be understood as follows. By nature of being a collar there are natural embeddings $S^0 \times \mathbb{R} \sqcup \mathbb{R} \hookrightarrow \mathbb{R}$ (or more generally, $(W\times\mathbb{R})\sqcup X_i \hookrightarrow X_i$) which by monoidality of A induce a map $(A_0^{\mathrm{op}} \otimes A_0) \otimes A_0 \to A_0$, or more generally

$$\int_{W\times\mathbb{R}} A \otimes \int_{X_i} A \to \int_{X_i} A$$

In summary, the first summand has E_1-algebra structure by embeddings it into itself, and the second summand obtains suitable module structure by similar embeddings. In fact, for our simple example it is quite straight-forward to understand this module structure: A is a right $(A_0^{\mathrm{op}} \otimes A_0)$-module if the latter acts by $(a \otimes b) \cdot c = abc$. Switching left and right multiplication it is also a left $(A_0^{\mathrm{op}} \otimes A_0)$-module. We can derive and visualise this module structure simply by writing out the above map $S^0 \times \mathbb{R} \sqcup \mathbb{R} \hookrightarrow \mathbb{R}$



and applying A to it, obtaining the according map $A_0 \times A_0 \times A_0 \to A_0$ for a ternary multiplication, that is, left and right components act by left and right multiplication on the middle component. The fact that left multiplication is a right A^{op}-action and right multiplication a right A-action can also be visualised in this way, which is left to the listener.

So far we haven't chosen any specific $\mathcal{C}^{\otimes}$. If we chose $\mathcal{C}^{\otimes} = N(\mathrm{Vect}^{\otimes})$ (recall that every 1-category can be turned into a quasicategory by applying the nerve, and such a quasicategory is called *1-truncated*), then the colimit of the bar construction becomes the ordinary colimit over a simplicial objects, which as mentioned before can be computed as the usual "functor tensor coequaliser" defining the tensor product over an algebra. The reader can verify that for this choice of $\mathcal{C}$ we have

$$\int_{S^1} A = A_0 \bigotimes_{A_0^{\mathrm{op}} \otimes A_0} A_0 = A_0/[A_0, A_0]$$

called the cocentre.

Next choose $\mathcal{C}^{\otimes} = \mathbf{Ch}_R^{\otimes}$ to be the $(\infty, 1)$-category of cochain complexes of R-modules in non-negative degree. This is now understood as a (true, non-truncated) $(\infty, 1)$-category for instance by using (a dual version of) the Dold-Kan correspondence. A more direct description as a quasicategory is also possible, see e.g. [AKMT19]. Then,

$$\int_{S^1} A = A_0 \bigotimes_{A_0^{\mathrm{op}} \otimes A_0} A_0 = HC(A_0)$$

is usually called the Hochschild chain complex of the E_1-algebra A_0. The history of this homological gadget is quite interesting, and the presentation above in terms of a tensor product of higher algebras is a recent one. One immediate consequence of having phrased this definition in the language of factorisation homology is that $HC(A_0)$ obtains an action of $S^1 \hookrightarrow \mathbf{Mfld}_1(S^1, S^1)$.

4.3.2. *Translating $\otimes$-excision into Mayer-Vietoris.* Next lets consider chain complexes $\mathcal{C}^{\otimes} = \mathbf{Ch}_{\mathbb{Z}}^{\oplus}$ as a 1-truncated $(\infty, 1)$-category. $\oplus$ acts degree-wise. Every chain complex $V \in \mathbf{Ch}_{\mathbb{Z}}^{\oplus}$ is an E_n-algebra for arbitrary n, simply by addition. Let us define the chain complex

$$C_*(-;V) = \int_- V$$

and call it "formal singular chains with coefficients in V". Now let $X_1 \sqcup_{W\times\mathbb{R}} X_2$ be a collar gluing. Write $C_*(X_1 \cup X_2) = C_*(X_1 \sqcup_{X_1\cap X_2} X_2; V)$. The $\otimes$-excision theorem now gives

$$C_*(X_1 \cup X_2) = C_*(X_1;V) \bigotimes_{C_*(X_1\cap X_2;V)} C_*(X_2;V)$$

As before, the tensor product reduces to the usual tensor product. The module structure can be shown to translate to addition as a consequence of the E_n-algebra structure on $C_*(\mathbb{R}^n;V) = V$ being given by addition. To compute the above tensor product we are thus led to the coequaliser

$$= \operatorname{coeq}\left(C_*(X_1;V) \oplus C_*(X_1\cap X_2;V) \oplus C_*(X_2;V) \underset{(-,-+-)}{\overset{(-+-,-)}{\rightrightarrows}} C_*(X_1;V) \oplus C_*(X_2;V) \right)$$

which in turn exhibits $C_*(X_1 \cup X_2)$ as the cokernel of the map $(-x,x) : C_*(X_1 \cap X_2;V) \to C_*(X_1;V) \oplus C_*(X_2;V)$. This map itself is injective, leading a short exact sequence

$$0 \to C_*(X_1\cap X_2;V) \xrightarrow{(-x,x)} C_*(X_1;V) \oplus C_*(X_2;V) \to C_*(X_1\cup X_2) \to 0$$

This can be seen to be the Mayer-Vietoris "axiom" over singular homology, which can be taken to determine singular cohomology to some degree[3]. We conclude that $C_*(X;V)$ are actual singular chains in X with coefficients in V. In this sense factorisation homology generalises singular homology, and $\otimes$-excision translates to the Mayer-Vietoris axiom.

Remark 4.6. The domain of spaces to which factorisation homology applies can be extended to include CW-complexes by embedding them into high-dimensional Euclidean space, and thickening them to obtain homotopy equivalent manifolds.

4.3.3. *more E_n-algebras.* Here are some more basic examples of E_n-algebras in nature and their properties, essentially copied over from [AKMT19], and left to the listener for verification.

(i) E_1-algebras in (the 1-category) $\mathbf{Set}^\times$ are monoids
(ii) E_n-algebras, $n \geq 2$, in $\mathbf{Set}^\times$ are commutative monoids
(iii) E_2-algebra in (the $(2,1)$-category) $\mathbf{Cat}^\times$ are braided monoidal categories.
(iv) $E_{2,\mathrm{or}}$-algebra in $\mathbf{Cat}^\times$ are balanced monoidal categories
(v) E_n-algebras, $n \geq 3$, in $\mathbf{Cat}^\times$ are symmetric monoidal categories
(vi) E_n-algebras in $\mathbf{Top}^\times$ include n-fold loop spaces $\Omega^n X$ and configuration spaces $\mathrm{Conf}(\mathbb{R}^n)$ consisting of a finite number of points distributed in $\mathbb{R}^n$ (that is, $\mathrm{Conf}(\mathbb{R}^n) = \bigsqcup_n \mathrm{Map}(*^{\sqcup n}, \mathbb{R}^n)$). Showing these to spaces have E_n algebra structure is a good exercise to get going with E_n-algebras. In fact, $\mathrm{Conf}(\mathbb{R}^n)$ is the free E_n-algebra on one generator in $\mathbf{Top}^\times$.

[3]see https://mathoverflow.net/questions/97621/mayer-vietoris-implies-excision

5. Epilogue

Factorisation homology is a powerful conceptual tool to generate new invariants from algebraic data on local models. It is a very active area of research, and generalisations for instance to the setting of "stratified spaces" have been given. This is important, because it allows us to talk about more general spaces than manifolds and also brings us in the realm of (∞, n)-category theory, with possible new proofs of important conjectures such as the cobordism hypothesis on the horizon. This paragraph is however the opposite of comprehensive, and a much better overview of the scope of this theory can be found e.g. in [AKMT19] and [AF15]. The purpose of this talk was merely to give a coherent (pun intended!) story leading up to the definition of factorisation homology, which could serve as an inspiration to explore more topics from higher category theory of homotopical algebra for the listener.

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