Framed combinatorial topology

Christoph Dorn & Christopher L. Douglas

Mar 2023

Mathematical Institute, University of Oxford, Oxford OX2 6GG, United Kingdom
Email addresses: dorn@maths.ox.ac.uk & cdouglas@maths.ox.ac.uk

Contents

Introduction

Framed combinatorial topology is a novel theory describing combinatorial phenomena arising at the intersection of stratified topology, singularity theory, and higher algebra. The theory synthesizes elements of classical combinatorial topology with a new combinatorial approach to framings. The resulting notion of framed combinatorial spaces has unexpectedly good behavior when compared to classical, nonframed combinatorial notions of space. In discussing this behavior and its contrast with that of classical structures, we emphasize two broad themes, 'computability in combinatorial topology' and 'combinatorializability of topological phenomena'. The first theme of computability concerns whether certain combinatorial structures (such as simplicial complexes homeomorphic to spheres) can be algorithmically recognized and classified. The second theme of combinatorializability concerns whether certain topological structures (such as manifolds) can be faithfully represented by a discrete structure. Combining these themes, we will find that in the context of framed combinatorial topology we can overcome a set of fundamental classical obstructions to the computable combinatorial representation of topological phenomena.

We begin this introduction by elaborating the themes of computability and of combinatorializability in, respectively, Section I.1 and Section I.2. We then give a more formal overview of our results in Section I.3, a chapter-by-chapter outline in Section I.4, and an outlook on the larger program and aims of the subject in Section I.5.

I.1   Computability in combinatorial topology

Computability is the ability to solve a 'general problem' by a 'general method', that is, the ability to write a step-by-step procedure which for each specific instance of a problem computes a solution. Combinatorial topology provides, in a sense, a computation-oriented foundation for the study of spaces, by encoding space in discrete structures [RS72] [Bry02]. However, many fundamental problems in combinatorial topology turn out to be computably intractable; such problems include the following [Mar58] [VKF74] [Nab95] [NW03] [Wei04] [Poo14].

1. *Disk recognition.* The statement 'The simplicial complex K is homeomorphic to the n-disk' cannot be computably verified for general finite complexes K. This may be rephrased as saying that we cannot classify topological subdivisions of the n-disk into k simplices (for general $k \in \mathbb{N}$). These uncomputability issues remain in the piecewise linear setting: we cannot classify piecewise linear subdivisions of the n-simplex with some given (general) number of simplices.

2. *Homeomorphism problem.* More generally, it is impossible to algorithmically decide whether two simplicial complexes K and L have homeomorphic, or piecewise linearly homeomorphic, geometric realizations. Similarly, given two embedded, or piecewise linearly embedded, simplicial complexes $K \hookrightarrow \mathbb{R}^n$ and $L \hookrightarrow \mathbb{R}^n$, one cannot in general determine whether the embeddings are ambient homeomorphic, respectively ambient piecewise linearly homeomorphic. This, of course, further generalizes to the impossibility of deciding *stratified* (piecewise linear) homeomorphism.
3. *Manifold classification.* Manifolds turn out to equally be intractible: it cannot be decided whether a given simplicial complex K is a piecewise linear n-manifold, and thus one cannot classify all piecewise linear n-manifolds up to a given number of simplices. In general, no algorithm can 'constructively' produce a list of all (piecewise linear) homeomorphism types of n-manifolds.

One could view these failures of computability as unavoidable imperfections of mathematics as we know it, or one can see them as failures of the classical simplicial methods for the description of topological structures. Adopting the latter viewpoint, one may hope for a form of combinatorial topology with better computability properties, for instance in which one can recognize combinatorial disks, decide the combinatorial homeomorphism problem, and classify combinatorial manifolds.

The first central theme of this book is that, though typical simplicial methods do not provide a computable foundations for combinatorial spaces, there is a different approach, using *framed* combinatorial spaces, that may provide a more suitable basis for computable combinatorial topology. Our theory of 'framed combinatorial topology' differs in two fundamental respects from classical piecewise linear topology: first, we endow simplices and simplicial complexes with a combinatorial framing structure, and second, we generalize the resulting class of 'framed simplicial complexes' to a broader class of 'framed regular cell complexes'. Though classical regular cells are much less tractable even than simplices— indeed even the list of cell shapes is uncomputable—it will turn out that framed regular cells arise as iterated constructible combinatorial bundles and therefore both these cells and their complexes are, remarkably, algorithmically classifiable.

———————————

A classical frame of an m-dimensional vector space is an ordered choice of m linearly independent vectors. We will define a combinatorial frame of an m-simplex to be an ordered choice of m vectors in the spine of the simplex. To make sense of a frame on a simplicial complex, we need a notion of the compatibility of frames along faces shared between simplices. The restriction of a frame of a simplex to a face gives not only information about a frame of the face but also about how that restricted frame embeds in the ambient frame of the simplex; we will be primarily concerned with the resulting notion of embedded framed simplex, and a framed simplicial complex will be a simplicial complex with compatible embedded frames on all its simplices.

Regular cell complexes, that is those complexes whose attaching maps are injective, generalize simplicial complexes by allowing cells of 'polytopic' shapes instead of merely 'triangular' shapes. Regular cells can be identified with the geometric realizations of their face posets [Bjö84] [LW69], and via that identification they obtain piecewise linear simplicial subdivisions. We use that simplicial structure, together with our notion of framed simplicial complexes, to define framings of regular cells and identify a tractable

class of such cells, namely those that are 'flat' in that they admit a framed embedding into euclidean space. A framed regular cell complex, finally, will be a regular cell complex with compatible choices of flat framings on each of its cells.

A space with a homeomorphism to a regular cell complex is 'cellulated', as a space with a homeomorphism to a simplicial complex is 'triangulated'. The fact that cellulated spaces have played a less prominent role than triangulated spaces in classical combinatorial topology is partially due to the aforementioned fundamental computability obstruction: it is impossible to classify all the possible shapes of regular cells, in the sense that one cannot produce a list of all regular cells with a given number of faces, in general; said another way, there is no general algorithm for deciding whether a given poset is the face poset of a regular cell, even though there are only finitely many posets of a given size. Endowing regular cells with a framing overcomes this fundamental issue: framed regular cells, in contrast to their nonframed counterparts, are classifiable. Specifically, given a poset together with a framing of its geometric realization, we can algorithmically recognize whether the poset is the face poset of a framed regular cell. This is possible, at root, because we will discover that flat framed regular cell complexes are the geometric realizations of a novel combinatorial structure, called 'trusses', which are iterated constructible bundles of oriented fence posets.

Framed regular cells strike an unlikely and delicate balance, being simultaneously a class of shapes that is tractable (in that they are algorithmically recognizable, unlike ordinary regular cells) and also a class of shapes that is quite general (unlike ordinary simplices). The generality of the shapes of framed regular cells provides unique combinatorial representatives in a way that is unthinkable with simplicial structures and unknown with any other class of shapes: a flat framed regular cell complex has a computable unique minimal cell structure. Having a computably unique representation of these complexes makes algorithmically decidable almost any question about them; for instance, it follows that framed homeomorphism of these complexes is decidable, in stark contrast to the classical (nonframed simplicial) situation. In this and other related respects, working with framed regular cells and their complexes provides, finally, a computable framework for combinatorial models of spaces.

I.2   Combinatorializability of topological phenomena

Combinatorics is primarily concerned with discrete, and often finite, structures whose constituents can be counted. Topology, by contrast, is primarily concerned with the continuous structure of spaces. The 'combinatorializability' of topological phenomena refers to the ability to faithfully encode continuous objects (spaces, manifolds, continuous maps, bordisms, et cetera) in discrete or finite data structures. This faithful encoding depends both on having a combinatorial representation of the object in question, and on knowing that representation is unique up to some specified combinatorial equivalence relation.

There are by now various known instances of topological phenomena that cannot be faithfully combinatorialized, or even combinatorialized at all, giving an impression of a mysterious and insurmountable divide between topological spaces and any discrete representations of those spaces. A headline instance of this divide is the disproven

'Hauptvermutung' [RCS$^+$96], a conjecture that, roughly speaking, claimed that 'topological isomorphism' (meaning homeomorphism) coincides with 'combinatorial isomorphism' (meaning piecewise linear homeomorphism). This conjecture would in particular imply that combinatorial spaces (that is, geometric realizations of simplicial complexes) that are homeomorphic are also piecewise linear homeomorphic. This intuitive, presumptive claim was eventually disproven [Mil61] by the explicit construction of homeomorphic finite simplicial complexes that are not piecewise linear homeomorphic.

A flurry of results followed in subsequent decades [KS69] [Aki69] [HM74] [KS77] [Fre82] [Don83], quantifying the divide not only between the 'continuous' and the 'combinatorial', but also between the 'combinatorial' and the 'smooth' conceptions of space. Recently, a disproof of the triangulation conjecture [Man16] established an especially stark gap, that in every dimension greater than 4 there exist compact topological manifolds that do not even admit a triangulation. (It will be pertinent later that most instances of the classical topological–combinatorial gap rely on certain infinitary or 'wild' topological constructions.) By contrast, smooth manifolds always admit triangulations and all triangulations of a smooth manifold are 'combinatorially isomorphic'. However, smooth manifolds that are not smoothly isomorphic may nevertheless be combinatorially isomorphic [Mil56], and combinatorial manifolds need not admit any smooth structure [Ker60].

One might dream of a topological foundations or combinatorial framework in which the mismatch between the continuous, combinatorial, and smooth conceptions of space would, at least to some extent, be lessened. One could imagine, for instance, a discrete, perhaps infinitary, combinatorial theory that faithfully represents a delineated class of relevant continuous phenomena, or a discrete, perhaps finitary, combinatorial theory that suitably encodes smooth behavior. Each of these two comparative visions has been pursued, to some but not complete satisfaction: for instance, an 'o-minimal' approach to tame topology provides a method for excluding certain wild topological structures [Gro97] [Shi14], while a 'matroid' perspective aims for a direct combinatorial description of smooth structures [Mac91].

The second central theme of this book is that, in contrast to the classical gap between topological and combinatorial phenomena, in our framed combinatorial setting there is a faithful comparison between framed topological and framed combinatorial phenomena. Furthermore, we expect framed combinatorial structures also faithfully encode all framed smooth phenomena, and therefore will provide an unexpected unification of the continuous, combinatorial, and smooth perspectives on space. The 'framed topological' side of these comparisons will be a class of tame topological structures called 'flat framed stratifications'. These stratifications are 'flat' and 'framed' by an embedding in standard euclidean space, and they are tame in that we insist the stratification admit a refining 'mesh', which is a cellulation by framed regular cells; this cellulation requirement is analogous to working with triangulable spaces and therefore excluding, a priori, certain wild behavior. These mesh cellulations are iterated constructible bundles of stratified 1-manifolds, and will be a precise topological counterpart of the iterated constructible combinatorial structure of trusses mentioned earlier.

The chain of associations, from a flat framed stratification to its mesh cellulation to the corresponding combinatorial truss, does not by itself necessarily ensure a faithful combinatorialization of flat framed stratified topology. As a space can have various inequivalent triangulations, a flat framed stratification could in theory have various inequivalent

meshes (and therefore corresponding trusses)—however, we will prove, crucially, that such a stratification always has a unique coarsest compatible mesh. This uniqueness is an unexpected and stark counterpoint to the classical situation: given two triangulations of a space, traditionally one aims (and fails) to construct a mutual *refinement* and thereby verify their combinatorial equivalence; now instead, given two mesh cellations of a stratification, we construct a canonical mutual *coarsening* and thus establish the desired combinatorial equivalence. The proof of this canonical coarsening relies, of course, on the generality of the shapes of framed regular cells, by contrast with the constrained shapes of classical simplices. This canonical coarsening provides the desired faithful combinatorialization of topological phenomena in flat framed euclidean space; indeed, we will establish the 'flat framed Hauptvermutung', that for flat framed stratifications, framed homeomorphism classes coincide with framed piecewise linear homeomorphism classes.

Regarding the combinatorialization of smooth phenomena, we will conjecture that any smooth manifold can be represented as a flat framed stratification (via a generic embedding in euclidean space) and that the resulting combinatorial representation as a truss faithfully encodes the smooth structure. We will revisit the context and plausibility of this smooth combinatorialization conjecture in the outlook, Section I.5 below.

I.3 Overview

We collect and summarize our main theorems, and along the way further describe and illustrate our core definitions. On overview of the most important notions in our theory will be given in Fig. I.10.

Recall that a framed simplex is an ordinary simplex together with frame, that is a choice of order of its spine vectors. More generally, a framed regular cell is an ordinary regular cell together with a suitably compatible choice of frames on each simplex in the cell's face poset. Though it is impossible to classify regular cells, by contrast framed regular cells are classifiable. The classifying combinatorial structure will be a special case of the notion of 'trusses', which are iterated constructible poset bundles defined as follows.

Definition 1 (Trusses). *A '1-truss' is a fence poset equipped with a total 'frame' order on its elements. An 'n-truss' is a length-n tower of constructible bundles of 1-trusses.*

The notions of '1-truss', their 'constructible bundles', and 'n-truss' are given more precisely in, respectively, Definition 2.1.6, Definition 2.1.54, and Definition 2.3.1. Elements of a 1-truss that are targets of poset arrows are called 'singular' or 'dimension 0', while elements that are sources of poset arrows are called 'regular' or 'dimension 1'. A 1-truss is 'closed' if both its endpoints are singular, and 'open' if both are regular; an n-truss is 'closed' or 'open' if all its fiber 1-trusses are closed or open respectively. In Fig. I.1 we illustrate a closed 2-truss and an open 3-truss; singular elements are red, regular elements are blue, and the frame orders are indicated by green arrows.

A closed truss is called a 'truss block' if its total poset has an initial element; these truss blocks provide the combinatorial correlate of framed regular cells.

Theorem 2 (Classification of framed regular cells). *The category of framed regular cells is equivalent to the category of truss blocks.*
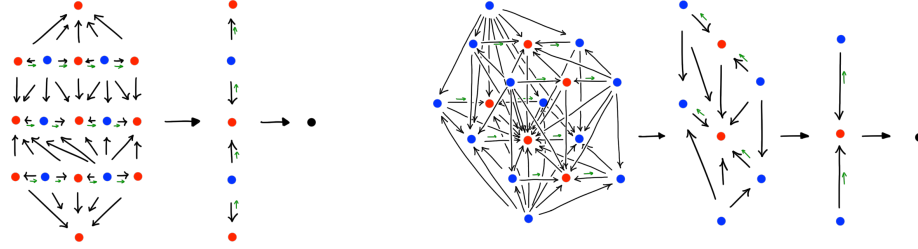
Figure I.1: A closed 2-truss and an open 3-truss.

This result will appear as Theorem 3.1.1. Given a framed regular cell, the total poset of the classifying truss block is the face poset of the cell; by sequentially projecting out frame vectors, this poset determines a tower of 1-truss bundles. In Fig. I.2 we illustrate a few framed regular cells and their corresponding truss blocks.
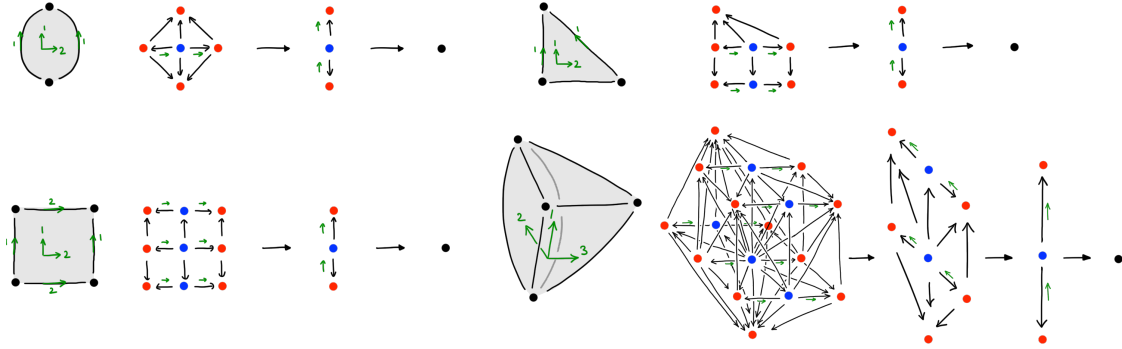
Figure I.2: Framed regular cells and their classifying truss blocks.

The classification of framed regular cells by truss blocks implies a corresponding classification for framed regular cell complexes. As a simplicial set is a presheaf on the category of simplices, similarly a 'truss block set' is a presheaf on the category of truss blocks; a simplicial set is regular if all its simplices embed into its realization, and similarly a truss block set is regular if all its blocks embed into its realization.

Theorem 3 (Classification of framed regular cell complexes). *The category of framed regular cell complexes is equivalent to the category of regular truss block sets.*

This result appears in the main text as Theorem 3.1.3.

––––––––––––––––––––

The face poset of a framed regular cell is a truss block; the geometric realization of a truss block is a framed regular cell. More general trusses also have geometric realizations as 'meshes', which are iterated constructible bundles of stratified lines, analogous to Definition 1, as follows.

Definition 4 (Meshes). *A '1-mesh' is a contractible 1- or 0-manifold, stratified by open intervals and points, and equipped with a framing. An 'n-mesh' is a length-n tower of constructible bundles of 1-meshes.*

The notions of '1-mesh', their 'constructible bundles', and 'n-mesh' are given more precisely in, respectively, in Definition 4.1.7, Definition 4.1.20, and Definition 4.1.46. An n-mesh is 'closed' if its total space is compact, and is 'open' if its total space is an open disc. In Fig. I.3 we illustrate a closed 2-mesh and an open 3-mesh.
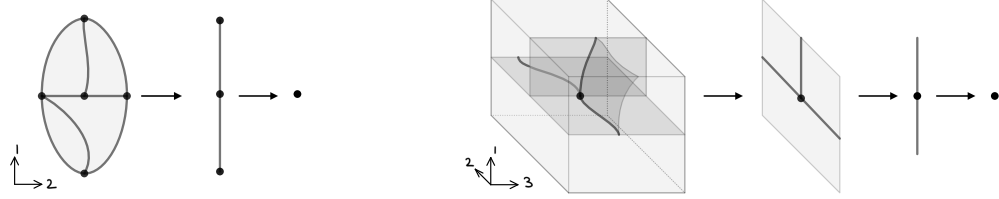
Figure I.3: A closed 2-mesh and an open 3-mesh.

The correspondence of meshes and trusses is at the core of our combinatorialization of topological phenomena.

Theorem 5 (Equivalence of meshes and trusses). *The ∞-category of closed, respectively open, meshes is weakly equivalent to the discrete category of closed, respectively open, trusses.*

This result appears in a more precise form as Theorem 4.2.1. Recall the entrance path poset of a stratified space has an element for each stratum and an arrow indicating when a stratum intersects the closure of another stratum. The above equivalence takes a mesh, a tower of stratified spaces, to the truss given by the tower of corresponding entrance path posets. As an illustration, note that the meshes in Fig. I.3 yield, on application of entrance path posets, the trusses in Fig. I.1.

A closed mesh will be called a mesh cell if it is the closure of a single stratum. Naturally, Theorem 5 restricts to an equivalence of mesh cells and truss blocks. Combining this equivalence with the earlier Theorem 2, relating truss blocks and framed regular cells, yields a correspondence of mesh cells and framed regular cells.

Corollary 6 (Equivalence of mesh cells and framed regular cells). *The ∞-category of mesh cells is weakly equivalent to the discrete category of framed regular cells.*

This result appears as Corollary 4.2.76. Note well that regular cells are at root combinatorial objects and they come with a canonical piecewise linear structure, whereas mesh cells are purely topological objects and a priori have no piecewise linear structure; thus this seemingly innocuous result provides a fundamental bridge between the topological and piecewise linear contexts.

Recall that a basic unsolvable problem of classical combinatorial topology is to classify subdivisions of the n-simplex. By contrast, leveraging the above connection between framed regular cells and mesh cells, we can classify framed subdivisions of framed regular cells.

Theorem 7 (Classification of subdivisions of framed regular cells). *A framed regular cell complex framed subdivides a framed regular cell if and only if, as a framed stratified*

space, the closed mesh corresponding to the regular cell complex refines the closed mesh cell corresponding to the regular cell.

This result appears in more precise form as Corollary 4.2.81. The theorem may appear to translate a piecewise linear classification problem (concerning regular complexes) into a topological classification problem (concerning meshes), but because of the equivalence of meshes and trusses established in Theorem 5, classifying refinements of mesh cells is algorithmically decidable via the combinatorics of trusses. Fig. I.4 illustrates a framed subdivision of a framed regular cell, of the sort classified by the previous result.

Figure I.4: The subdivision of a framed regular cell.

The combinatorial category of closed trusses is isomorphic, via dualizing the truss posets, to the category of open trusses. For instance, under this isomorphism the closed 3-truss in Fig. I.2 corresponds to the open 3-truss in Fig. I.1. By the equivalence of trusses and meshes established in Theorem 5, the self-duality of trusses translates to a self-duality of meshes, as follows.

Corollary 8 (Dualization of meshes). *The ∞-category of closed meshes is weakly equivalent to the ∞-category of open meshes.*

This self-duality appears later as Corollary 4.2.9, and is a crucial advantage meshes have over previously known shape categories. Note that the entrance path poset of the dual n-mesh is dual to the entrance path poset of the original mesh, and so the dimensions of all strata in the mesh are dualized—the collection of geometric shapes and their incidences is completely reorganized. Fig. I.5 illustrates a pair of dual 2-meshes.

Figure I.5: Dualization of meshes.

Meshes, built up from mesh cells and their duals, are a flexible, computationally tractable class of highly structured stratifications; they furthermore provide access to a much broader, almost completely general class of stratifications by considering those stratifications that admit a refinement by a mesh, as follows.

Definition 9 (Flat framed stratifications). *A 'flat n-framed stratification' is a stratification of a subspace of $\mathbb{R}^n$ that admits a refinement by an n-mesh.*

This definition will appear in a more precise form in Definition 5.1.1. In Fig. I.6 we illustrate two flat framed stratifications of an open 4-cube, by depicting three pertinent slices. The first stratification is the classical third Reidemeister move, and the second is the classical swallowtail singularity; that these indeed admit mesh refinements is illustrated in a moment in Fig. I.7 and Fig. I.8.
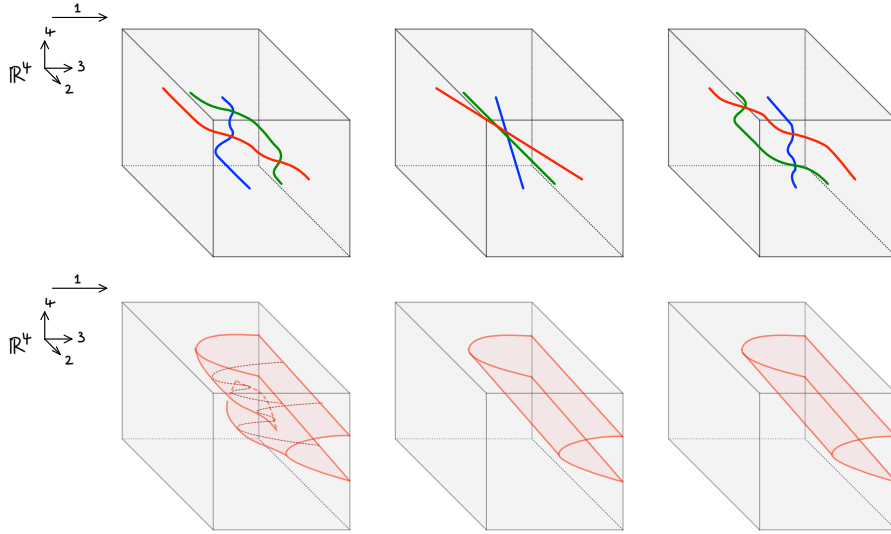
Figure I.6: Two flat framed stratifications of the 4-cube.

At the heart of the computability of our combinatorial model of stratified framed topology is the completely unexpected fact that among the set of all refining meshes of a flat framed stratification, there is always a canonical *coarsest* choice. Heuristically, the coarsest refining mesh of a stratification is built up just from the indispensable critical loci of certain projections of the strata. Needless to say, this situation is in stark contrast to any simplicial model of (stratified) topology, in which two triangulations almost never have a mutual coarsening and typically do not even admit a mutual refinement, preventing any canonical or computable comparison.

Theorem 10 (Coarsest meshes of flat framed stratifications). *Any flat framed stratification has a unique refining mesh that is coarser than any other refining mesh.*

This will be established as Theorem 5.2.20. In Fig. I.7 and Fig. I.8 we depict the coarsest mesh for the third Reidemeister move and for the swallowtail singularity.

A flat framed stratification is refined by a mesh, and a mesh is combinatorialized by a truss; to complete the combinatorialization of flat framed stratifications, we translate the initial stratification into a stratified structure on the truss. A 'stratified poset' is a poset together with a 'stratification map' to another poset (encoding the set of strata and the combinatorial entrance paths between them); a 'stratified truss' is a truss together with a stratification of its total poset. Furthermore, a stratified truss is 'normalized' if it cannot be simplified while preserving the stratification; this property of being normalized corresponds to a mesh being maximally coarsened while still refining a given stratification.
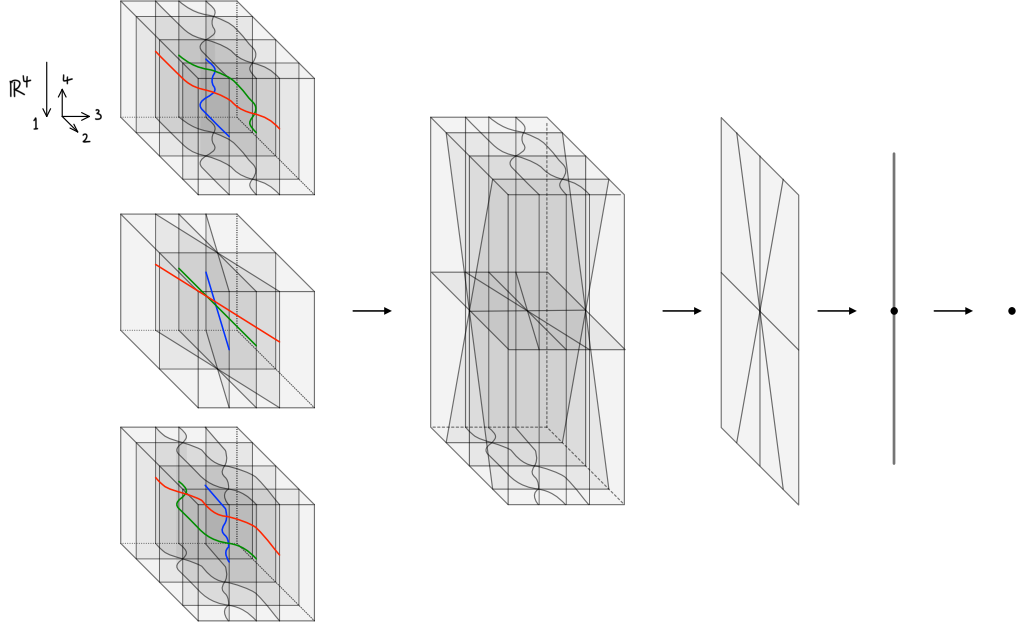
Figure I.7: The coarsest mesh refining the third Reidemeister move.
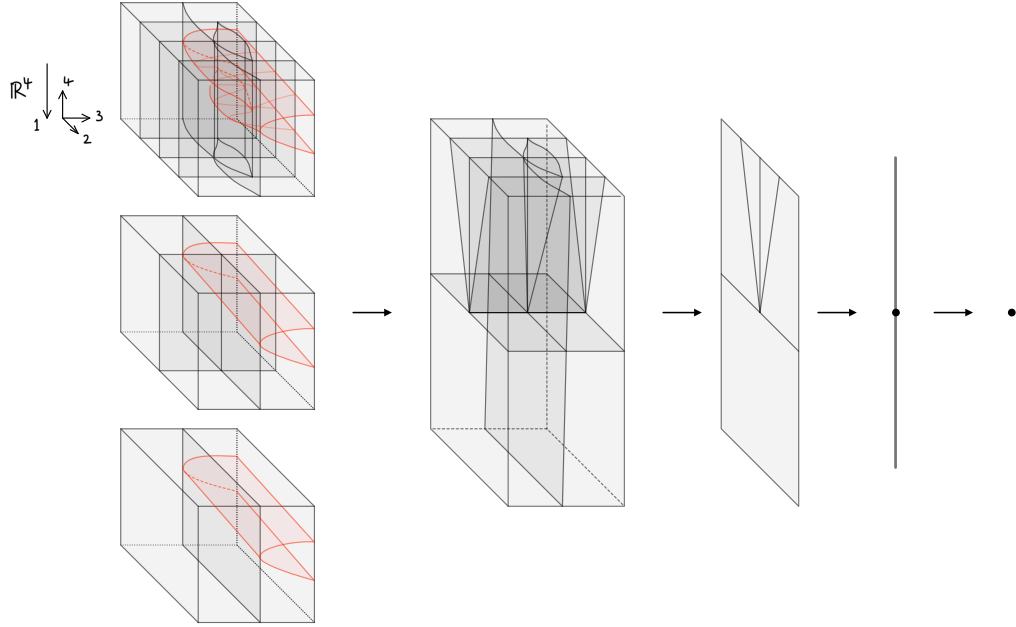
Figure I.8: The coarsest mesh refining the swallowtail singularity.

Theorem 11 (Classification of flat framed stratifications). *Framed stratified homeomorphism classes of flat framed stratifications are in bijective correspondence with isomorphism classes of normalized stratified trusses.*

This will be established as Theorem 5.1.4. Fig. I.9 illustrates a flat framed stratification, its coarsest refining mesh, and the corresponding normalized stratified truss; the stratification on the truss records which strata of the mesh assemble into each stratum of the initial

10

flat framed stratification.
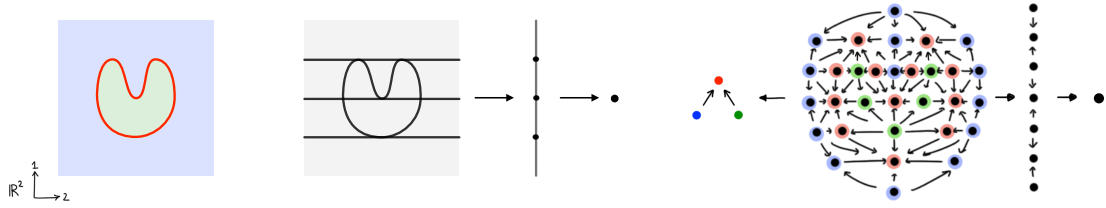
Figure I.9: A flat framed stratification, its coarsest mesh, and its normalized stratified truss.

Flat framed stratifications are intrinsically topological structures, considered up to homeomorphism, while stratified trusses are intrinsically combinatorial or piecewise linear structures, considered up to combinatorial or piecewise linear equivalence—the classification of framed stratifications by stratified trusses thus provides a faithful bridge between the topological and piecewise linear contexts. Recall the classical, false Hauptvermutung, that homeomorphic simplicial complexes are piecewise linear homeomorphic. The failure of correspondence between the topological and the piecewise linear remains even for subspaces (or substratifications) of euclidean space: given two piecewise linear embedded triangulated spaces in euclidean space that are ambient homeomorphic, they need not be ambient piecewise linear homeomorphic. By contrast, in the framed setting we will have a tight correspondence between the topological and piecewise linear, as follows.

Theorem 12 (Flat framed Hauptvermutung). *Flat framed piecewise linear stratifications that are framed stratified homeomorphic are also piecewise linear framed stratified homeomorphic.*

This result will be established later as Corollary 5.1.7. Having a combinatorial or piecewise linear handle on framed stratifications, via stratified trusses, furthermore resolves the fundamental decidability problem for framed homeomorphism.

Theorem 13 (Decidability of framed stratified homeomorphism). *Given two flat framed stratifications, one can algorithmically decide whether they are framed stratified homeomorphic.*

This is recorded later as Corollary 5.1.11.

One would hope to extend the above computable combinatorialization of framed topological phenomena to framed *smooth* phenomena; indeed we expect this is possible, as follows.

Conjecture 14 (Framed homeomorphism implies diffeomorphism). *Given two smooth compact manifolds smoothly embedded in euclidean space, and defining flat framed stratifications there, if they are flat framed homeomorphic then they are diffeomorphic.*

Conjecture 15 (Framed stratifications are dense in smooth embeddings). *Any smooth embedding of a smooth compact manifold into euclidean space has an arbitrarily small perturbation that is a flat framed stratification.*

These conjectures reappear later as Conjecture 5.4.6 and Conjecture 5.4.7. Because flat framed stratifications can be faithfully combinatorialized as stratified trusses, these conjectures together imply that there is a sound and complete combinatorial representation of smooth structures on manifolds.

A summarizing overview of the central notions, as outlined above, is given in Fig. I.10.

Figure I.10: Overview of core notions.

I.4 Outline

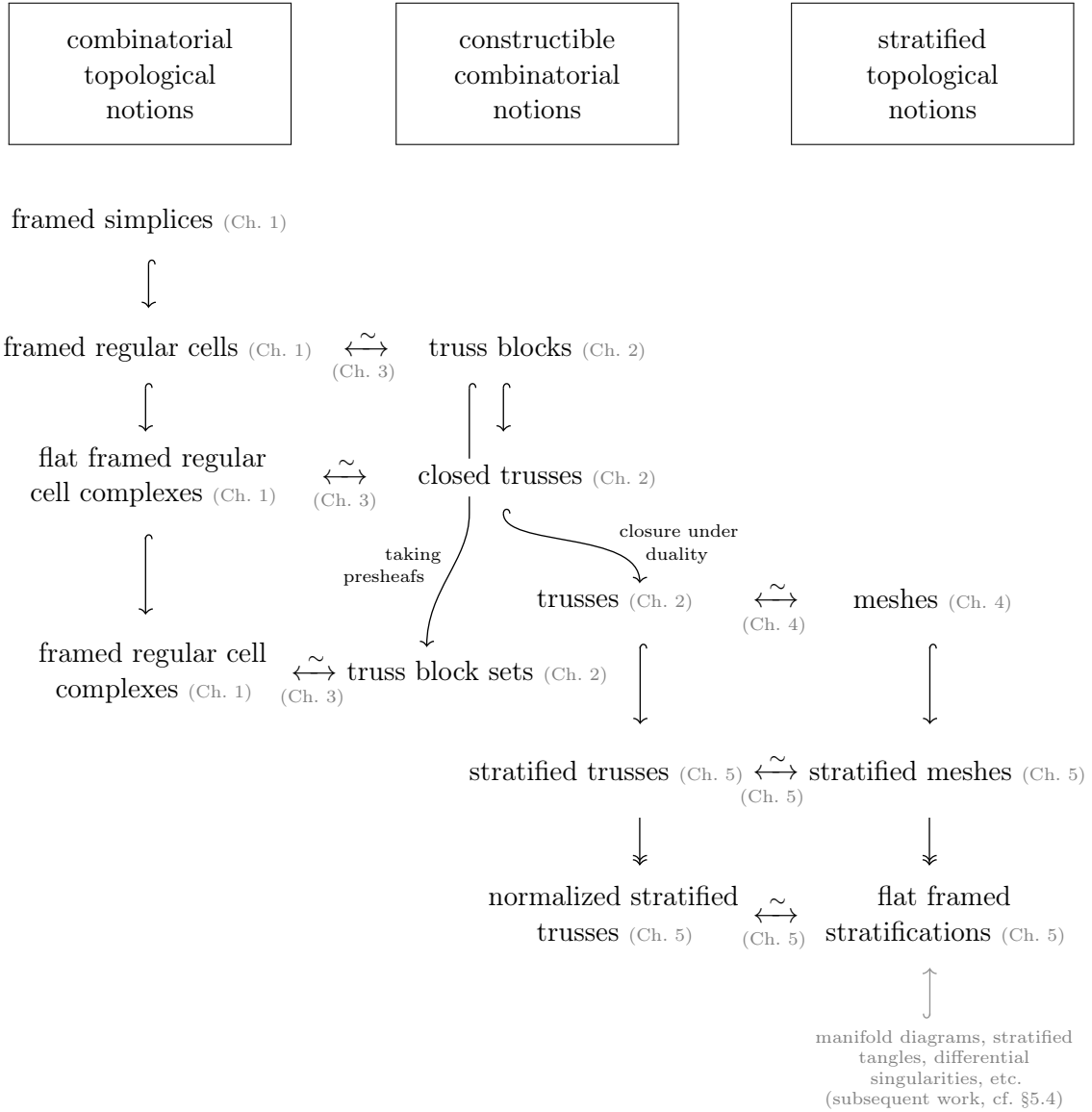Chapter 1 introduces framed combinatorial structures. The first such structure, 'framed simplices' is a combinatorial analog of classical framed vector spaces. A 'framed simplicial

complex' will be a collection of compatibly framed simplices. After recalling the classical combinatorial-topological definition of regular cell complexes, we then further generalize framed simplicial complexes to 'framed regular cell complexes'.

In Chapter 2, we develop our fundamental combinatorial notion of 'trusses', as certain iterated constructible bundles of posets. This development begins with '1-trusses', which are framed fence posets, morphisms between them called '1-truss bordisms', and families of them called '1-truss bundles'. 1-truss bundles over simplices turn out to have an unexpected total order on the top-dimensional simplices in their total posets, and this leads to a crucial method of 'truss induction'. Finally we describe 'n-trusses', as iterated 1-truss bundles, their corresponding 'n-truss bordisms' and 'n-truss bundles', and their elementary constituents 'n-truss blocks'.

Chapter 3 proves the equivalence of the category of truss blocks and the category of framed regular cells, and more generally the equivalence of the category of regular presheaves on truss blocks and the category of framed regular cell complexes, as stated in Theorem 2 and Theorem 3 above. Truss blocks are translated into framed regular cells by an appropriate combinatorial 'gradient' functor; the converse functor, termed 'integration', entails the more technical construction of a tower of 1-truss bundles from the framing information on the regular cell.

In Chapter 4, we introduce our fundamental stratified topological notion of 'meshes', as certain iterated constructible bundles of stratified manifolds. From the outset, meshes appear as a topological analog of the combinatorial structure of trusses, and the notions of '1-mesh', '1-mesh bundle', and 'n-mesh' parallel the corresponding truss notions. Indeed, we prove, as claimed in Theorem 5, that the topologically enriched category of meshes is weakly equivalent to the discrete category of trusses; one direction constructs an entrance path truss combinatorially encoding a mesh, and the other direction produces a classifying mesh geometrically realizing a truss. This fundamental translation allows us to establish the equivalence of mesh cells and framed regular cells (Corollary 6), the classification of subdivisions of framed regular cells (Theorem 7), and the self-duality of meshes (Corollary 8).

Finally, Chapter 5 radically broadens the class of stratifications under investigation, introducing 'flat framed stratifications' as those stratifications that admit a mesh refinement. The core work of the chapter is the proof that every flat framed stratification has a coarsest refining mesh, as claimed in Theorem 10. We leverage that result to establish the combinatorial classification of flat framed stratification in terms of normalized stratified trusses (Theorem 11). We then bridge the topological to piecewise linear chasm, proving the flat framed Hauptvermutung (Theorem 12) that homeomorphic flat framed stratifications are piecewise linear homeomorphic. As a final application we establish the decidability of framed stratified homeomorphism (Theorem 13). In the final portion of the chapter, we will describe a future outlook for framed combinatorial topology, including theories of transversality and manifold diagrams, and of combinatorial tangles and singularities.

Appendix A provides a detailed discussion of classical linear frames, corresponding notions of indframes and proframes (which conceptually underlie the constructions in Chapter 3), as well as their generalizations to the partial and embedded cases of frames (introduced in Chapter 1, and the affine analogs of these structures. Based on the affine notions of framings (and proframings), the appendix also briefly consider the construction of geometric realizations of framed regular cell complex in 'framed spaces'. Appendix B

reviews and elaborates various elementary notions from stratified topology, including the construction of fundamental posets and higher categories of stratifications.
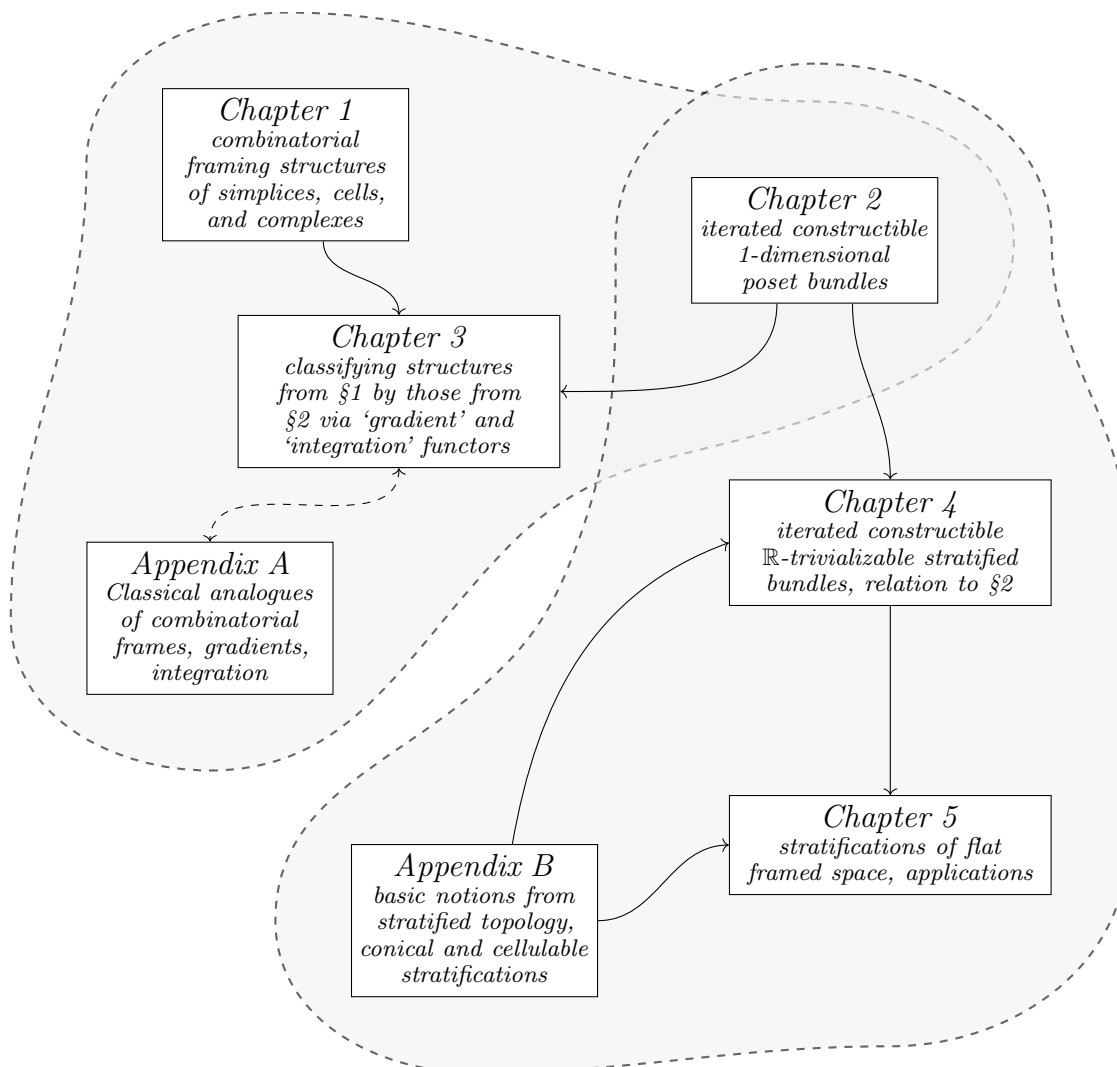
Figure I.11: Overview of chapters, dependencies, and partitioning.

Reader's guide The structure of chapters in this book is not linear: the graph of the main logical dependencies (which ignores some other, less significant interactions between chapters) is shown in Fig. I.11. The figure also indicates that the 'space of chapters' can be covered by two (overlapping) open sets: firstly, Chapter 1, 2, 3 and Appendix A together explore 'affine combinatorial notions of framings', their classifying inductive structure as well as their relation to classical linear algebraic notions. Relatedly, but working in a different direction, Chapter 2, 4, 5 and Appendix B study local 'flat' framed space (whose framings admit trivializations in $\mathbb{R}^n$) and stratifications thereof, such as embedded manifolds $M \hookrightarrow \mathbb{R}^n$. The second part has several immediate applications that we sketch in the next section, and the reader interested in those may and can, for instance, only engage with the second open set of chapters.

14

I.5   Outlook

We briefly summarize our immediate outlook for framed combinatorial topology beyond the present book; a more detailed discussion of these future directions appears in the final Section 5.4.

Flat framed stratifications are already highly, if implicitly, structured by their canonical mesh refinements and the relation of that refinement to the ambient frame on euclidean space. However, the stratification itself need not be in any sort of generic or transverse relation to the ambient frame; for certain theoretical and computational purposes, it is essential to restrict attention to appropriately transverse stratifications. We can define and detect a transverse stratification in purely topological, as opposed to smooth, terms, by insisting that every stratum project by a local homeomorphism to the corresponding stage of the ambient euclidean proframe. We dub these transverse stratifications 'manifold diagrams'; indeed our definition provides a solution to the long-standing search for a formal generalization of 'string diagrams' to all higher dimensions. This notion of manifold diagrams is as powerful as one could hope: first, because of the combinatorializability of flat framed stratifications, manifold diagrams are also completely combinatorializable, into an appropriate notion of transverse trusses; second, using the dualization of meshes, manifold diagrams naturally dualize to a notion of higher cell pasting diagrams that formalizes arbitrary composability structures in higher categories.

Leveraging the theory of manifold diagrams, we can identify a combinatorially tractable class of tame tangles, namely those embeddings of manifolds in euclidean space that admit refinements to manifold diagrams. Of course we expect any embedding of a manifold has an arbitrarily small deformation to a tame tangle, and so nothing is lost by excluding more wild behaviors. Our combinatorial encoding of (tame) tangles immediately provides a novel computational toolkit: we can stratify the space of tangles by algorithmically computable local or global complexity measures, and formalize computable notions of tangle perturbation, simplification, and stability. Having a robust algorithmic approach to tangles is already novel in dimension 4, but indeed our definitions and tools apply in all dimensions and all codimensions. The divergence of our theory from the classical view of tangles becomes especially stark in higher dimensions. A sufficiently small open neighborhood in a tangle is called a 'tangle singularity', or just a 'singularity' for brief. The traditional view has been that singularity classification becomes profoundly unmanageable as the dimension increases: first arise uncountable continuous moduli of distinct singularity types, then the moduli space of singularities itself becomes infinite dimensional, and generally demons abound. By contrast, we see a natural equivalence relation on singularities (not just tame ones, because we expect we can account for wild ones as well via perturbation methods) for which there is a countable, algorithmically computable classification in all dimensions.

That context of manifold diagrams and tangle singularities considered, there arise various open problems and directions for investigation. For instance: classify perturbation-stable singularities. A 'perturbation-stable singularity' is one that cannot be simplified by small deformations, and is therefore in a sense an 'elementary singularity'; this stability condition is straightforward to formalize using the combinatorial complexity measures at our disposal. We expect that the set of isomorphism classes of perturbation-stable singularities in any fixed dimension is *finite*, but the structure of this classification remains

mysterious as the dimension grows. Complementary to singularities, which are the most local sort of tangles, are 'tangle homotopies', or 'homotopies' for brief, which are disconnected tangles encoding the ways manifolds can pass by one another at a distance in euclidean space. As there are distinguished elementary singularities, namely those that are perturbation-stable, similarly there are 'elementary homotopies', namely those that cannot be deformed into a composite of simpler homotopies. Naturally we may then pose the problem: classify elementary homotopies. Again, we expect that the set of isomorphism classes of elementary homotopies in any fixed dimension is *finite*, but a precise classification remains unknown even in relatively low dimensions.

Given a sufficiently generic k-dimensional tangle M^k in euclidean space $\mathbb{R}^n$, the composite map $M^k \hookrightarrow \mathbb{R}^n \to \mathbb{R}^m$ (where the last map is the standard projection in the canonical proframe of euclidean space) should be a prototypical 'm-Morse function', in the sense that all its local singularities and global homotopies would be, in an appropriate sense, elementary. Less precisely than the previous problems, we may ask for the development of a direct definition of m-Morse functions (without reference to tangle embeddings), which retains the combinatorial and computational flavor of our tangles and manifold diagrams, and therefore admits a tractable classification and attendant application to smooth manifold topology. We expect not only that such a combinatorial higher Morse theory exists, but that the resulting combinatorial invariants detect, for instance, all smooth structures on manifolds. The realization of such an expectation depends, most likely, on the validity of our aforementioned conjectures about framed homeomorphism and framed stratifications—indeed they would imply that every combinatorially tame tangle has a canonical smooth structure and that every smooth tangle has such a combinatorially tame representation.

Acknowledgments

Framed combinatorial structures

In this chapter, we introduce notions of framings on classical combinatorial structures. We start by defining frames on simplices in Section 1.1, and introduce core definitions for the 'affine combinatorics' of framed simplices. We then generalize our discussion to a notion of framings on simplicial complexes in Section 1.2, in which framings of complexes will be pieced together from the frames of each individual simplex of the complex. In a yet further generalization, we will then introduce framings on regular cell complexes in Section 1.3. As it turns out, the generalization from 'framed simplices' to 'framed regular cells' is less by choice than it is a crucial step in the theory of framed combinatorial topology, on which many later results will depend.

1.1 Framed simplices

The notion of a frame on a simplex, and later on other combinatorial objects, is of course inspired by and modeled on the classical notion of frames. Classically, a 'trivialization' of an m-dimensional vector space V is specified by a linear isomorphism $V \xrightarrow{\sim} \mathbb{R}^m$. Preimages of standard unit vectors $e_i \in \mathbb{R}^m$ under this trivialization define an ordered list of $(v_1, v_2, ..., v_m)$ in V called a 'frame' of V.

The guiding intuition in the translation of frames in linear algebra into the combinatorics of simplices, is that directed edges of simplices play the role of vectors. However, m-simplices are combinatorially specified by sets of vertices and thus they do not have a distinguished origin. Moreover, their vectors (i.e. directed edges) are 'affine' in that different vectors may start at different points in the simplex. This observation has important implications for the translation of classical intuition of frames in linear algebra into the combinatorics of simplices, and we will highlight this by speaking of the 'affine combinatorics' of framed simplices.

The basic analogy of vectors in a vector space with 'vectors' in a simplex will lead to a notion of 'frames' on a simplex as follows. We say two vectors in a simplex are 'composable' if the endpoint of the first vector is the starting point of the second vector; in this case, their 'composite' is the unique directed edge starting at the starting point of the first vector and ending in the endpoint of the second vector. For instance, given a

3-simplex $S = \{a, b, c, d\}$, the directed edges $d \to b$ and $b \to c$ compose to the directed edge $d \to c$. A 'basis' of an m-simplex is a set of m vectors such that all other vectors (up to reversing their direction) can be written as composites of vectors in the basis. A 'frame' of an m-simplex is an ordered basis.

To further explore the analogy to classical linear frames and trivializations, we can rephrase the notion of simplicial frames as follows. Observe that the elements of any basis of an m-simplex S must be the elements of a chain of m composable vectors in the simplex; we call such a chain a 'spine' of S. A choice of basis therefore determines an identification $S \cong [m]$ of S with the *ordered standard simplex* $[m] = (0 < 1 < ... < m)$ (given by mapping the spine of S to the standard spine of $[m]$). Conversely, any such identification $S \cong [m]$ determines a basis in this way. A frame of an m-simplex S is then an identification $S \cong [m]$ *together* with a choice of order on the set of standard spine vectors $\mathsf{spine}[m]$ of $[m]$. The standard simplex $[m]$ has of course the canonical identity identification with itself, and so we refer to the simplex $[m]$ with an order $\mathcal{F}$ of its spine vector set $\mathsf{spine}[m]$ as a 'framed standard simplex' $([m], \mathcal{F})$.

Framed standard simplices $([m], \mathcal{F})$ *with any choice of frame $\mathcal{F}$* will play the role of euclidean space $\mathbb{R}^m$ *with its standard frame* $\{e_1, e_2, ..., e_n\}$. The fact that there is not only one 'framed standard euclidean space $\mathbb{R}^m$' but S_m-many 'framed standard simplices $([m], \mathcal{F})$' reflects the fact that several affine constellations of standard basis vectors can arise: for instance, for the standard basis vectors e_1 and e_2 in $\mathbb{R}^2$, chaining e_1 with e_2 forms the spine $e_1 \circ e_2$ of a 'standard simplex', whereas chaining e_2 with e_1 forms the spine $e_2 \circ e_1$ of a different 'standard simplex'—see Fig. 1.1. The affine combinatorics of framed simplices accounts for both of these configurations.
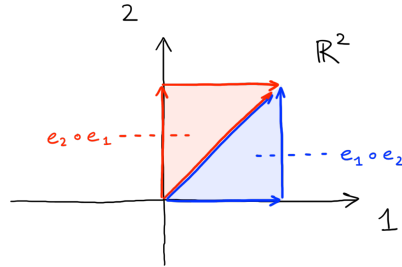
Figure 1.1: Standard vectors spanning distinct standard simplices.

We briefly outline this section. In Section 1.1.1 we begin by describing the combinatorial counterparts of classical linear algebraic notions, and then define 'frames' on simplices together with several generalizations to 'embedded' and 'partial' frames. Note that the combinatorial notions of 'embedded' and 'partial' frames introduced here, may as well be understood in purely classical linear algebraic terms as we explain in Appendix A.

1.1.1 Frames on simplices Before giving the definition of frames on simplices, we specify some basic terminology and notation.

Terminology 1.1.1 (Combinatorial simplices). An 'ordered m-simplex' is a totally ordered poset with $m + 1$ elements. An 'm-simplex' (for clarity also referred to as an 'unordered m-simplex') is a set with $m + 1$ elements. ⌐

Notation 1.1.2 (Category of ordered simplices). We will denote the category of ordered simplices by Δ; its objects are the ordered simplices, and its morphisms are the order-preserving maps. ⌐

Notation 1.1.3 (Category of unordered simplices). We will denote the category of simplices by $\underline{\Delta}$; its objects are the unordered simplices S and its morphisms are all functions. ⌐

Terminology 1.1.4 (Face maps and degeneracy maps). An injective map of (ordered or unordered) simplices is called a 'face map', and a surjective map of simplices is called a 'degeneracy map'. ⌐

Terminology 1.1.5 (Unordering ordered simplices). The 'unordering' functor $(-)^{\mathsf{un}} : \Delta \to \underline{\Delta}$ forgets the order of ordered simplices. ⌐

Notation 1.1.6 (Maps between ordered and unordered simplices). Maps $S \to T$ or $T \to S$ between an unordered simplex S and an ordered simplex T will be parsed as maps $S \to T^{\mathsf{un}}$ resp. $T^{\mathsf{un}} \to S$. ⌐

Terminology 1.1.7 (Standard simplex). The 'ordered standard m-simplex' $[m]$ is the poset $(0 < 1 < \cdots < m)$; considering posets as categories, this simplex is, equivalently, the category $(0 \to 1 \to \cdots \to m)$. ⌐

Since it will be used so frequently, we usually refer to ordered standard m-simplex simply as the 'm-simplex $[m]$'. Every ordered m-simplex S is canonically isomorphic to the standard m-simplex $[m]$. We may therefore work with a skeleton of Δ as follows.

Notation 1.1.8 (The skeleton of standard simplices). Abusing notation, we denote the skeleton of Δ containing only standard simplices $[m]$ for $m \in \mathbb{N}$ again by Δ. ⌐

Terminology 1.1.9 (Unordered standard simplices). The unordering $[m]^{\mathsf{un}}$ of the standard simplex $[m]$ is the 'unordered standard m-simplex' $\{0, 1, ..., m\}$. ⌐

Terminology 1.1.10 (Sets of numerals). The 'set of numerals' or 'numeral set' $\underline{m}$ is the ordered set $\{1 < 2 < \cdots < m\}$. ⌐

Notation 1.1.11 (Category of numeral sets). We will denote the category of numeral sets by ∇; its objects are the numeral sets $\underline{m}$ for $m \in \mathbb{N}$, and its morphisms are the order-preserving functions. ⌐

We next introduce combinatorial notions which mirror ordinary notions from linear and affine algebra. In the most elementary case, this is the correspondence of (nonzero) vectors in a vector space V to (non-degenerate) 1-simplices in a simplex S, as follows.

Terminology 1.1.12 (Simplicial vectors). A 'vector' v in an (ordered or unordered) simplex S is a map $v : [1] \to S$. We assume by default that v is 'non-degenerate' (or 'nonzero'), meaning v is injective, and otherwise say that v is a 'degenerate' (or 'zero') vector. It will be useful to also refer to the set of all zero vectors of S, which we call the 'affine zero vector' of S and denote by 0^{aff}. ⌐

Terminology 1.1.13 (Spine vectors of ordered simplices). A 'spine vector' in an ordered simplex is a non-degenerate vector that cannot written as the composite of more than one non-degenerate vector. The set of spine vectors of the standard simplex is denoted $\mathsf{spine}[m]$; this set $\mathsf{spine}[m] = \{(0 \to 1), (1 \to 2), \dots, (m-1 \to m)\}$ may be canonically

identified with the numeral set $\underline{m}$, by mapping the spine vector $(i - 1 \to i)$ to the numeral element i. ⊔

Note that the realization $|S|$ of any simplex S is naturally carries affine structure (see Appendix A.2 for a brief recollection of affine spaces). Passing to the associated vector spaces of affine spaces, allows to translate 'simplicial vectors' to ordinary 'linear vectors' as follows.

Notation 1.1.14 (Realizations of vectors in a simplex). Given a vector v in S, we can realize v to an affine vector $|v| = \Delta^1 \to |S|$ in the realized simplex $|S|$. Forgetting the vector's base point, we obtain an ordinary vector $\vec{v}$ in the associated vector space $\vec{V}(S)$ of S (that is, $\vec{v} = \mathsf{unbase}\,|v|$ using Notation A.2.3). ⊔

The role of 'linear projections' of vector spaces will be played by 'degeneracies' of simplices. (Indeed, note that any degeneracy $F : S \to T$ of simplices induces a linear projection $\vec{V}(S) \to \vec{V}(T)$ of associated vector spaces, defined to map $\vec{v}$ to $\vec{w}$ if $w = F(v)$.) Degeneracies admit the following 'affine' notion of kernels.

Terminology 1.1.15 (Affine kernels of simplicial degeneracy maps). For a degeneracy $T \to S$, the 'affine kernel' $U = \ker^{\mathrm{aff}}(S \to T)$ is the subset of vectors in S that are mapped to zero vectors in T by the degeneracy. ⊔

We write $S \twoheadrightarrow S/U$ for the simplicial degeneracy of S whose affine kernel is U, and $U \hookrightarrow S$ to indicate that U is an affine kernel. Importantly, note that affine kernels U of S cannot, in general, be canonically expressed as simplicial face maps into S; that is, the degeneracy $S \twoheadrightarrow S/U$ does not have a canonically 'splitting' by a simplicial face $S/U \hookrightarrow S$ (i.e. such that $(S/U \hookrightarrow S \twoheadrightarrow S/U) = \mathrm{id}$).

Analogous to linear embeddings into $\mathbb{R}^n$ being canonically split, the situation slightly improves when the target simplex $T \cong [n]$ is the standard n-simplex as we will now see. In the presence of order (and thus of spine vectors) we first adapt the notion of affine kernels as follows.

Terminology 1.1.16 (Affine kernels for ordered simplices). Given a degeneracy $f : [m] \to [k]$, the 'affine kernel' $\ker^{\mathrm{aff}}(f)$ is the subset of spine vectors in $\mathsf{spine}[m]$ that are mapped to zero vectors by f. ⊔

Note that, in fact, *any* subset $U \subset \mathsf{spine}[m]$ of spine vectors in $[m]$ determines and is determined by a degeneracy $f : [m] \twoheadrightarrow [k]$ with affine kernel $U = \ker^{\mathrm{aff}} f$. This leads to the following notion of affine faces which is 'formally dual' to that of degeneracies.

Terminology 1.1.17 (Affine faces). An 'affine face' $f : [n] \hookrightarrow [m]$ is an ordered map $\mathsf{spine}[n] \hookrightarrow \mathsf{spine}[m]$. Its 'affine image' $\mathrm{im}^{\mathrm{aff}}(f)$ is its image of spine vectors in $\mathsf{spine}[m]$, and its 'affine cokernel' $\mathrm{coker}^{\mathrm{aff}}(f)$ is the complement of that image. ⊔

Remark 1.1.18 (Formal duality of degeneracies and affine faces). There is a correspondence between degeneracies $g : [n] \twoheadrightarrow [m]$ and affine faces $f : [m] \hookrightarrow [n]$ determined by equating their kernels resp. cokernels: $\ker(g) = \mathrm{coker}^{\mathrm{aff}}(f)$. Given a degeneracy $g : [n] \twoheadrightarrow [m]$ we also write $\mathsf{split}(g) : \mathsf{spine}[m] \hookrightarrow \mathsf{spine}[n]$ for the map on spine vectors of its corresponding affine face, and refer to $\mathsf{split}(g)$ as the affine face that 'splits' g. ⊔

Given an unordered k-simplex T, we write $T \hookrightarrow [n]$ to mean a pair consisting of an ordering $T \cong [k]$ and an affine face $[k] \hookrightarrow [n]$. The affine combinatorial notions of

'simplicial projections' (i.e. degeneracies) and of 'simplicial subspaces' (i.e. affine faces) combine to a notion of 'affine simplicial map' as follows.

Terminology 1.1.19 (Affine simplicial maps). An 'affine map' $S \to [n]$ of an m-simplex to the standard simplex $[n]$ is a sequence $S \twoheadrightarrow T \hookrightarrow [n]$ consisting of a degeneracy $S \twoheadrightarrow T$, and identification $T \cong [k]$ (where k is the dimension of T), and an affine face $[k] \hookrightarrow [n]$. The 'affine image' $\mathrm{im}(S \twoheadrightarrow T \hookrightarrow [n])$ of such an affine map is the subset of the spine vector set $\mathsf{spine}[m]$ given by the affine image $\mathrm{im}^{\mathrm{aff}}([k] \hookrightarrow [n])$. ⌐

─────────────

The above combinatorial counterparts to classical linear algebraic notions will allow us to mirror many classical definitions, including that of linear frames, in purely combinatorial terms. Recall a linear frame in an m-dimensional vector space V can be specified by a linear trivialization $V \xrightarrow{\sim} \mathbb{R}^m$. We will, in fact, also be interested in the following generalizations of linear trivializations, which allow for more general types of maps from V to an euclidean space.

Terminology 1.1.20 (Partial, embedded, and embedded partial trivializations). A linear projection $V \twoheadrightarrow \mathbb{R}^k$ will be called a 'k-partial trivialization' (for its relation to 'partial' frames of V, see Appendix A.1). A linear subspace $V \hookrightarrow \mathbb{R}^n$ will be called an 'n-embedded trivialization' (for its relation to 'embedded' frames of V). Yet more generally, a general linear map $V \to \mathbb{R}^n$ with k-dimensional image will be called an 'n-embedded k-partial trivialization' (for its relation to 'embedded partial' frames of V). ⌐

The role of these notions of 'generalized trivializations' will be fundamental later on: when defining framed simplicial complexes in Section 1.2 by patching together global framings from local frames, n-embeddings of frames will allow to compare frames of simplices of any dimensions (serving a similar role to local trivializations of n-dimensional tangential structure of manifolds).

For the translation of linear (partial, embedded and embedded partial) trivializations into combinatorial definition, there now remains only one central difference: namely, instead of just one 'framed standard' model $\mathbb{R}^n$, we will find an S_n-worth of 'framed standard' simplices—this is the set of standard n-simplices with spine order (that is, a total order $\mathcal{F}$ of the spine vector set $\mathsf{spine}[n]$). Taking this difference into account, a summary of the translation of linear trivializations (and their generalizations) into affine combinatorial structures is given in Fig. 1.2. The subsequent sections will give an in-depth discussion of these structures.

1.1.1.1 The definition of framed simplices We introduce frames on simplices, a combinatorial analog of orthonormal frames of euclidean vector spaces, and of linear frames of general vector spaces up to orthoequivalence. The role of frame vectors will be played by the spine vectors of a simplex. As a linear frame is an ordered list of its frame vectors, a simplicial frame will be an ordered collection of spine vectors.

Definition 1.1.21 (Frame on the standard simplex)**.** A **frame $\mathcal{F}$ of the standard m-simplex** $[m]$ is a bijection $\mathcal{F} : \mathsf{spine}[m] \to \underline{m}$ from the set $\mathsf{spine}[m]$ of spine vectors of the simplex to the set of numerals $\underline{m} = \{1, 2, \ldots, m\}$. ⌐

Linear algebra	Affine combinatorics
m-dimensional vector space V	unordered m-simplex S
nonzero vectors v in V	directed edges v in S
the zero vector 0 in V	vertices x in S
vector space projections $V \twoheadrightarrow V'$	simplicial degeneracies $S \twoheadrightarrow S'$
subspaces $(W \hookrightarrow V)$	affine faces $T \dashrightarrow S$
framed standard $\mathbb{R}^m$	standard simplex $[m]$ with spine order
linear trivialization $V \xrightarrow{\sim} \mathbb{R}^m$	isomorphism $S \cong [m]$ with affine image order
partial triv. $V \twoheadrightarrow W \xrightarrow{\sim} \mathbb{R}^k$	degeneracy $S \twoheadrightarrow T \cong [k]$ with aff. image order
n-embedded triv. $V \hookrightarrow \mathbb{R}^n$	affine face $S \dashrightarrow [n]$ with aff. image order
n-embd. partial triv. $V \twoheadrightarrow W \hookrightarrow \mathbb{R}^n$	affine map $S \twoheadrightarrow T \dashrightarrow [n]$ with aff. image order

Figure 1.2: The analogy between notions in linear algebra and notions in affine combinatorics.

We may of course equivalently think of a frame $\mathcal{F}$ in terms of the inverse function $\mathcal{F}^{-1} : \underline{m} \to \mathsf{spine}[m]$ from the set of numerals to the spine, or more concretely as an ordered list $(v_1, v_2, ..., v_m)$ of spine vectors $v_i = \mathcal{F}^{-1}(i)$ of $[m]$. In particular, $\mathcal{F}$ does provide an order on spine vector in $[m]$ as recorded in Fig. 1.2. Following the table further, frames of unordered simplices can now be defined as follows.

Definition 1.1.22 (Frame on a simplex). A **frame** of an m-simplex S is an isomorphism $S \cong [m]$ together with a frame $\mathcal{F}$ on $[m]$. ⌐

We usually denote framed simplices S by tuples $(S \cong [m], \mathcal{F})$. We may also keep the isomorphism $S \cong [m]$ implicit, and simply say that $\mathcal{F}$ is a frame on S.

Example 1.1.23 (Frames on simplices). In Fig. 1.3 we illustrate a few framed m-simplices $(S \cong [m], \mathcal{F})$. The frame $\mathcal{F} : \mathsf{spine}[m] \to \underline{m}$ is indicated in two ways, as follows. Firstly, the spine vector $v \in \mathsf{spine}[m]$ is labeled by its numeral value $\mathcal{F}(v) \in \underline{m}$. Secondly, the labeled spine vectors, thought of as vectors in the linear space spanned by the picture of the simplex, are translated so their sources are coincident, and the resulting labeled 'coordinate frame' is drawn in or near the simplex. ⌐
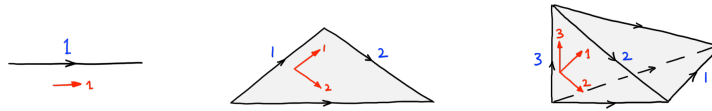

Figure 1.3: Framed simplices.

A linear embedding of a framed simplex into euclidean space may preserve the frame structure in the following sense.

Terminology 1.1.24 (The standard oriented components of euclidean space). For all $i < n$, the image of the linear subspaces $0^{n-i} \times \mathbb{R}^i \hookrightarrow 0^{n-i-1} \times \mathbb{R}^{i+1}$ (with both domain and codomain being subspaces of $\mathbb{R}^n$) has two components ϵ_i^- and ϵ_i^+ given by $0^{n-i-1} \times \mathbb{R}_{<0} \times \mathbb{R}^i$ resp. $0^{n-i-1} \times \mathbb{R}_{>0} \times \mathbb{R}^i$. We call ϵ_i^- and ϵ_i^+ the 'ith negative' resp. 'ith positive standard component' of $\mathbb{R}^n$. ⌐

Definition 1.1.25 (Framed realization of a framed simplex). A **framed realization** of a framed m-simplex $(S \cong [m], \mathcal{F})$ (with frame vectors $v_i = \mathcal{F}^{-1}(i)$) is a linear embedding $r_{\mathcal{F}} : |S| \hookrightarrow \mathbb{R}^m$ of the geometric m-simplex $|S|$ into $\mathbb{R}^m$ such that $\vec{r}_{\mathcal{F}}(\vec{v}_i)$ lies in the ith positive standard component $\epsilon_i^+ \subset \mathbb{R}^m$, for all $i \in \underline{m}$.[1] ⌐

Example 1.1.26 (Framed realization of a framed simplex). In Fig. 1.4 we illustrate framed realizations of two framed 2-simplices. ⌐
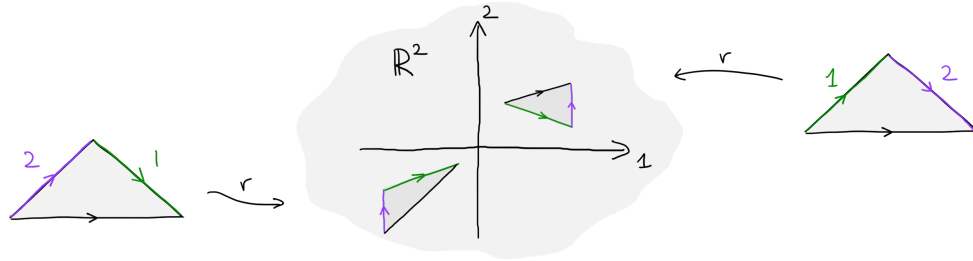
Figure 1.4: Framed realization of framed simplices.

Remark 1.1.27 (Understanding frames in classical linear algebraic terms). The set of all framed realizations of a given framed simplex $(S \cong [m], \mathcal{F})$ describes a certain equivalence class of linear trivializations $|S| \hookrightarrow \mathbb{R}^m$ of the underlying nonframed simplex S (this is formalized in Observation A.2.11). ⌐

———————

We next mention the following generalization of frames to 'partial' frames. Instead of a frame on the 'complete' simplex S, we may consider a frame on S that is defined everywhere but on a subspace $U \hookrightarrow S$.

Definition 1.1.28 (Partial frame on a simplex). A k-**partial frame** on an m-simplex S is a degeneracy $S \twoheadrightarrow T$ together with a frame $(T \cong [k], \mathcal{F})$. ⌐

Note in particular, that the frame $\mathcal{F}$ provides an ordering of the affine image of the affine map $S \twoheadrightarrow T \cong [k]$ (namely, the image is $\mathsf{spine}[k]$), which matches the structure recorded in Fig. 1.2. We usually denote partially framed simplices S by tuples $(S \twoheadrightarrow T \cong [k], \mathcal{F})$ or, more simply, by tuples $(S \twoheadrightarrow [k], \mathcal{F})$ (note, up to canonical ismorphism, T is redundant). Note that in an m-partial frame of an m-simplex $(S \twoheadrightarrow [m], \mathcal{F})$ the degeneracy $S \twoheadrightarrow [m]$ must be an isomorphism, and thus m-partial frames of m-simplices are frames on m-simplices.

Terminology 1.1.29 (Unframed subspace). The 'unframed subspace' of a k-partially framed simplex $(S \twoheadrightarrow [k], \mathcal{F})$ is the affine kernel $U = \mathrm{ker}^{\mathrm{aff}}(S \twoheadrightarrow [k])$. ⌐

———————

[1]Technically, by 'linear embedding' $|S| \hookrightarrow \mathbb{R}^m$ we mean an 'affine embedding' in the sense of Remark A.2.1.

Example 1.1.30 (Partial frames on simplices). In Fig. 1.5 we illustrate a few k-partially framed m-simplices $(S \twoheadrightarrow [k], \mathcal{F})$: we depict degeneracies $S \twoheadrightarrow [k]$ by highlighting their unframed subspace (in orange) and illustrate the framed simplices $([k], \mathcal{F})$ as in Example 1.1.23. Note that the degeneracy $S \twoheadrightarrow [k]$ can equivalently be expressed as a partial order on S (indicated in green), and the frame $\mathcal{F}$ may then be indicated labeling vectors v in S with $i \in \underline{k}$ whenever $w = (S \twoheadrightarrow T)(v) \in \mathsf{spine}[k]$ and $\mathcal{F}(w) = i$ (indicated in purple). ⌐

Figure 1.5: Partially framed simplices.

A linear embedding of a partially framed simplex into euclidean space may preserve the frame structure in the following sense.

Definition 1.1.31 (Framed realization of a partially framed simplex). Consider a k-partially framed m-simplex $(S \twoheadrightarrow [k], \mathcal{F})$ with unframed subspace $U = \ker^{\mathrm{aff}}(S \twoheadrightarrow T)$. A **framed realization** of $([m], \mathcal{F})$ is a linear map $r_{\mathcal{F}} : \Delta^m \to \mathbb{R}^k$ of the standard geometric m-simplex Δ^m to $\mathbb{R}^k$ such that $\vec{r}_{\mathcal{F}}(\vec{v}) = 0 \in \mathbb{R}^k$, for $v \in U$, and such that $\vec{r}_{\mathcal{F}}(\vec{v}) \in \epsilon_i^+ \subset \mathbb{R}^k$ whenever $w = (S \twoheadrightarrow T)(v) \in \mathsf{spine}[k]$ and $\mathcal{F}(w) = i$. ⌐

Example 1.1.32 (Framed realization of a partially framed simplex). In Fig. 1.6 we illustrate the framed realizations of a 1-partially framed 2-simplex and of a 2-partially framed 3-simplex. ⌐
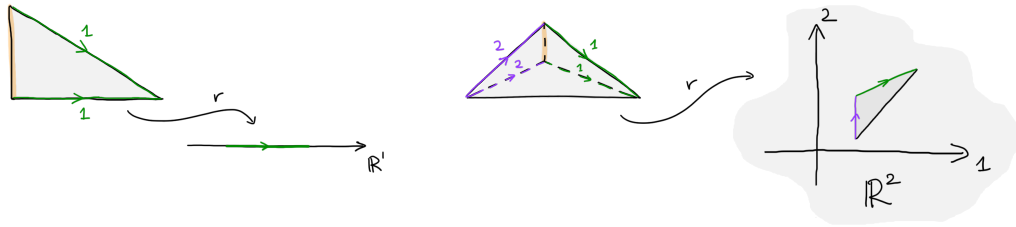
Figure 1.6: Framed realization of partially framed simplices.

Remark 1.1.33 (Understanding partial frames in classical affine algebraic terms). The set of all framed realizations of a given k-partial framed simplex $(S \twoheadrightarrow [k], \mathcal{F})$ describes a certain equivalence class of linear trivializations $|S| \to \mathbb{R}^k$ of the underlying nonframed simplex S (this is formalized in Observation A.2.14). ⌐

1.1.1.2 Embedded frames We introduce a combinatorial analog of the notion of linear embedded trivializations. Recall, linear embedded trivializations are linear vector space inclusions $V \hookrightarrow \mathbb{R}^n$ which, if V is euclidean, up to orthoequivalence are represented by orthonormal n-embedded frames on V, that is, by ordered lists of vectors $(v_1, v_2, ..., v_n)$ of which exactly $\dim(V)$ are nonzero and form an orthonormal basis of V.

Definition 1.1.34 (Embedded frame on the standard simplex). An n**-embedded frame** $\mathcal{F}$ **of the standard** m**-simplex** $[m]$ is an injective function $\mathcal{F} : \mathsf{spine}[m] \hookrightarrow \underline{n}$ from the spine of the simplex into the set of numerals $\{1, 2, \ldots, n\}$. ⌐

We may of course equivalently think of an n-embedded frame $\mathcal{F}$ of $[m]$ in terms of the partial inverse function $\mathcal{F}^{-1} : \underline{n} \to \mathsf{spine}[m]$ from the set of numerals to the spine, or more concretely as an ordered list $(v_1, v_2, ..., v_n)$ where $v_i = 0^{\mathrm{aff}}$ if $\mathcal{F}^{-1}$ is undefined on i, and $v_i = \mathcal{F}^{-1}(i)$ otherwise. Note that an m-embedded framed m-simplex is the same as a framed m-simplex as previously defined. For general simplices, we introduce the following.

Definition 1.1.35 (Embedded frame on a simplex). An n**-embedded frame** of an m-simplex S is an isomorphism $S \cong [m]$ together with an n-embedded frame $\mathcal{F}$ on $[m]$. ⌐

We usually denote n-embedded framed simplices S by tuples $(S \cong [n], \mathcal{F})$. In line with Fig. 1.2, we remark that there is the following equivalent definition of embedded frames.

Remark 1.1.36 (Embedded frames via simplicial subspaces). An n-embedded frame $\mathcal{F}$ of an m-simplex S is equivalently given by an affine face $S \cong [m] \hookrightarrow [n]$ together with an ordering of the image of that affine face. ⌐

Example 1.1.37 (Embedded frames on simplices). In Fig. 1.7 we illustrate a few n-embedded framed m-simplices $(S \cong [m], \mathcal{F})$. As before, the frame $\mathcal{F} : \mathsf{spine}[m] \hookrightarrow \underline{n}$ of $[m]$ is indicated in two ways: the spine vector $v \in \mathsf{spine}[m]$ is labeled by its numeral value $\mathcal{F}(v) \in \underline{n}$, and the m labeled spine vectors are translated into a labeled coordinate frame. ⌐
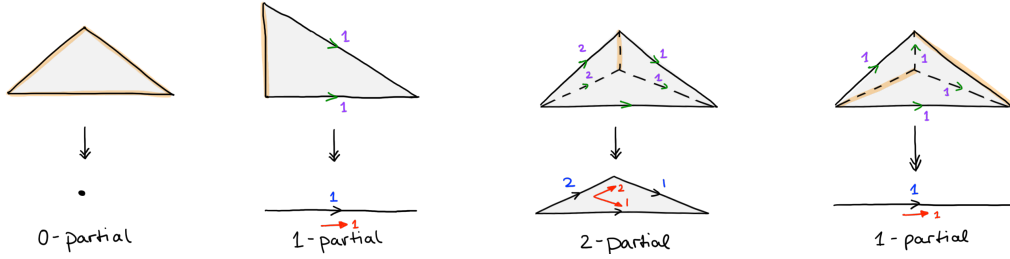

Figure 1.7: Embedded framed simplices.

A linear embedding of a framed simplex into euclidean space may preserve the frame structure in the following sense.

Definition 1.1.38 (Framed realization of an embedded framed simplex). A **framed realization** of an n-embedded framed m-simplex $(S \cong [m], \mathcal{F})$ (with nonzero frame vectors $v_i = \mathcal{F}^{-1}(i)$, $i \in \mathrm{im}(\mathcal{F})$) is a linear embedding $r_{\mathcal{F}} : |S| \hookrightarrow \mathbb{R}^n$ of the geometric m-simplex $|S|$ into $\mathbb{R}^n$ such that $\vec{r}_{\mathcal{F}}(\vec{v}_i) \in \epsilon_i^+ \subset \mathbb{R}^m$, for all $i \in \mathrm{im}(\mathcal{F})$. ⌐

Example 1.1.39 (Framed realization of embedded framed simplices)**.** In Fig. 1.8 we illustrate framed realizations of two 3-embedded framed simplices; note that the framed realizations of the 3-embedded framed 1-simplex (shown on the right) must yield affine vectors that, after translating their base points to the origin, lie in the component ϵ_i^+ (which consists of vectors in the $\langle e_2, e_3 \rangle$-plane with strictly positive 2-component). ⌐⌐
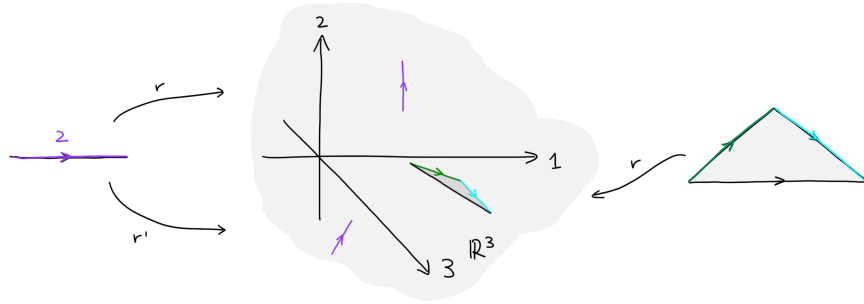
Figure 1.8: Framed realization of embedded framed simplices.

Remark 1.1.40 (Understanding embedded frames in classical affine algebraic terms)*.* The set of all framed realizations of a given n-embedded framed simplex $(S \twoheadrightarrow [m], \mathcal{F})$ describes a certain equivalence class of linear trivializations $|S| \hookrightarrow \mathbb{R}^n$ of the underlying nonframed simplex S (this is formalized in Observation A.2.11). ⌐⌐

While we will be interested ultimately only in non-partial embedded frames, we also mention the following generalization of embedded frames to the partial case. Instead of an embedded frame on the 'complete' simplex S, we may consider an embedded frame on S that is defined everywhere but on a subspace $U \hookrightarrow S$. Following Fig. 1.2 embedded partial frames of unordered simplices can be defined as follows.

Definition 1.1.41 (Embedded partial frame on a simplex)**.** An n**-embedded** k**-partial frame** on an m-simplex S is a degeneracy $S \twoheadrightarrow T$ together with an n-embedded frame $(T \cong [k], \mathcal{F})$. ⌐⌐

We usually omit the (canonically determined) simplex T, and denote n-embedded k-partially framed simplices S by tuples $(S \twoheadrightarrow [k], \mathcal{F})$ (where $\mathcal{F}$ is an n-embedded frame of the standard simplex $[k]$). Note, in line with Fig. 1.2, we may equivalently think of the structure of an embedded partially framed simplex as an affine map $S \twoheadrightarrow T \hookrightarrow [k]$ together with an ordering of its affine image (see Remark 1.1.36).

Terminology 1.1.42 (Unframed subspace)*.* The 'unframed subspace' of an n-embedded partially framed simplex $(S \twoheadrightarrow [k], \mathcal{F})$ is the affine kernel $U = \ker^{\mathrm{aff}}(S \to T)$. ⌐⌐

Example 1.1.43 (Embedded partial frames on simplices)**.** In Fig. 1.9 we illustrate a few n-embedded k-partially framed m-simplices $(S \twoheadrightarrow [k], \mathcal{F})$: we depict degeneracies $S \twoheadrightarrow [k]$ by highlighting their unframed subspace (in orange) and illustrate the framed simplices $([k], \mathcal{F})$ as in Example 1.1.37. Note that the degeneracy $S \twoheadrightarrow [k]$ can equivalently be expressed as a partial order on S (indicated in green), and the frame $\mathcal{F}$ may then be indicated labeling vectors v in S with $i \in \underline{n}$ whenever $w = (S \twoheadrightarrow T)(v) \in \mathsf{spine}[k]$ and $\mathcal{F}(w) = i$ (indicated in purple). ⌐⌐
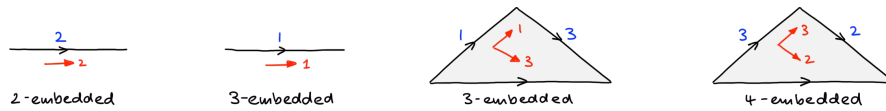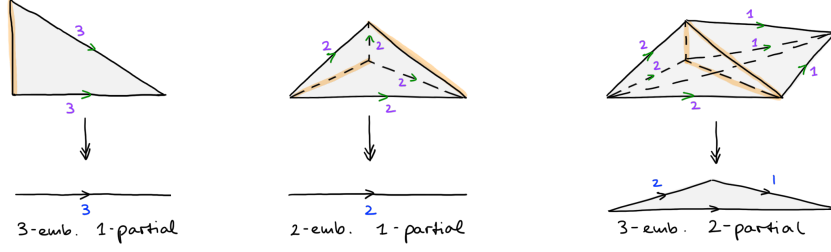
Figure 1.9: Embedded partially framed simplices.

A linear embedding of a embedded partially framed simplex into euclidean space may preserve the frame structure in the following sense.

Definition 1.1.44 (Framed realization of a embedded partially framed simplex)**.** Consider a embedded partially framed m-simplex $(S \twoheadrightarrow [k], \mathcal{F})$ with unframed subspace $U = \ker^{\mathrm{aff}}(S \twoheadrightarrow T)$. A **framed realization** of $([m], \mathcal{F})$ is a linear embedding $r_{\mathcal{F}} : \Delta^m \hookrightarrow \mathbb{R}^m$ of the standard geometric m-simplex Δ^m into $\mathbb{R}^m$ such that $\vec{r}_{\mathcal{F}}(\vec{v}) = 0 \in \mathbb{R}^k$, for $v \in U$, and such that $\vec{r}_{\mathcal{F}}(\vec{v}) \in \epsilon_i^+ \subset \mathbb{R}^k$ if $w = (S \twoheadrightarrow T)(v) \in \mathsf{spine}[k]$ and $\mathcal{F}(w) = i$. ⌐

Example 1.1.45 (Framed realization of an embedded partially framed simplex)**.** In Fig. 1.10 we illustrate framed realizations of a 2-embedded 1-partial frame of the 3-simplex and of a 3-embedded 2-partial frame of the 4-simplex (for the latter example, we think of the central $\mathbb{R}^2$ plane as being embedded in $\mathbb{R}^3$ as indicated by the grey coordinate axis). ⌐
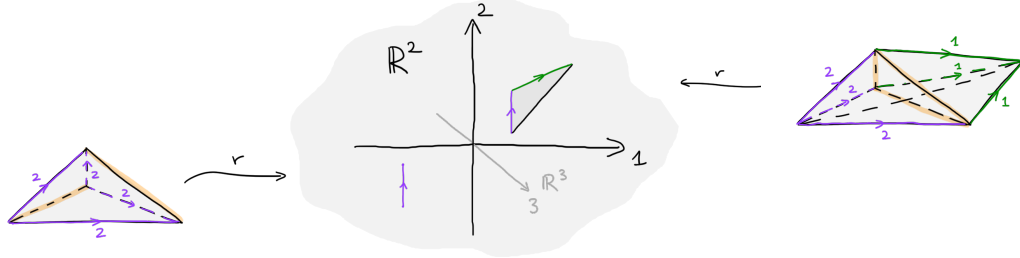
Figure 1.10: Framed realization of an embedded partially framed simplex.

Remark 1.1.46 (Understanding embedded partial frames in classical affine algebraic terms)*.* The set of all framed realizations of a given n-embedded k-partial framed simplex $(S \twoheadrightarrow [k], \mathcal{F})$ describes a certain equivalence class of linear trivializations $|S| \to \mathbb{R}^n$ of the underlying nonframed simplex S (this is formalized in Observation A.2.14). ⌐

1.1.2 Mapping frames on simplices

1.1.2.1 Restricting frames A linear trivialization $V \xrightarrow{\sim} \mathbb{R}^n$ restricts on any linear subspace $W \hookrightarrow V$ to a linear n-embedded trivialization $W \hookrightarrow \mathbb{R}^n$. We discuss the combinatorial analog of this process, that is, how frames and embedded frames of simplices restrict to embedded frames on simplicial faces. In geometric terms, our definition can be expressed as follows.

Remark 1.1.47 (Restricting frames in linear algebraic terms). Consider an n-embedded framed m-simplex $(S \cong [m], \mathcal{F})$, and a simplicial j-face $f : T \hookrightarrow S$. Pick any framed realization $r : |S| \hookrightarrow \mathbb{R}^n$. The 'restriction' of the n-embedded frame of S to T is the unique n-embedded frame of T which is framed realized by the linear embedding $r \circ |f| : |T| \hookrightarrow |S| \hookrightarrow \mathbb{R}^n$. ⌐

Describing the process of frame restriction in purely combinatorial terms, without reference to the affine framed structure of $\mathbb{R}^n$, is more subtle. To properly account for the combinatorial situation, we introduce a combinatorial notion of 'non-orthogonality' of simplicial vectors that we refer to as 'kinship'.

Definition 1.1.48 (Akin simplicial vectors). The vectors $v = (a \to b)$ and $w = (c \to d)$ in the simplex $[m] \equiv (0 \to 1 \to \cdots \to m)$ are **akin**, denoted $v \pm w$, if there is a vector u that is a factor of both, i.e. such that $v = \tilde{v} \circ u \circ \tilde{\tilde{v}}$ and $w = \tilde{w} \circ u \circ \tilde{\tilde{w}}$ for some possibly degenerate vectors $\tilde{v}, \tilde{\tilde{v}}, \tilde{w}, \tilde{\tilde{w}}$. ⌐

Note that, like the relation of non-orthogonality of linear vectors in a euclidean space, the 'kinship' relation between simplicial vectors is reflexive and symmetric but not transitive.

Example 1.1.49 (Kinship of vectors). In Fig. 1.11 we illustrate the kinship of vectors in the 3-simplex. To emphasize the informal conceptual relationship of this notion with the geometry of non-orthogonality, the simplex is drawn with its three spine vectors (highlighted in red, green, and blue) being orthogonal in the ambient euclidean 3-space. The three red vectors are akin, the four green vectors are akin, and the three blue vectors are akin; no other vectors are akin. Note that indeed these kinship relations are precisely the non-orthogonal vectors of this geometric 3-simplex. ⌐
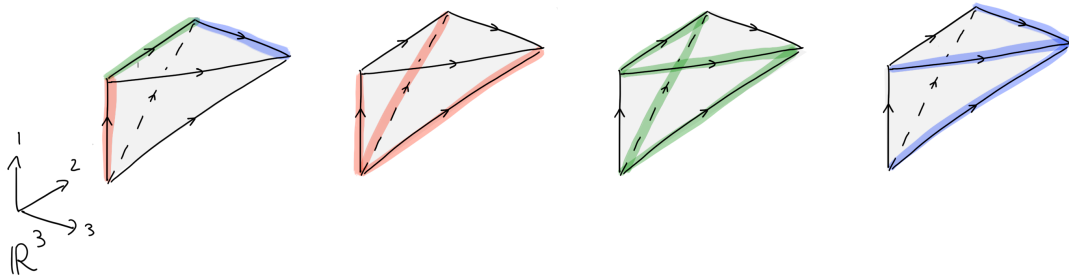
Figure 1.11: Kinship of simplicial vectors.

Using the notion of kinship, we can describe the restriction of a frame to any vector of the standard simplex, and subsequently to any face of the standard simplex. The frame order on the spine of the simplex plays a paramount role in this process: the frame label of a general vector will be the lowest numeral among the frame labels of the spine vectors akin to the given vector.

Construction 1.1.50 (Frame restriction to simplicial vectors). Given an n-embedded frame $\mathcal{F} : \mathsf{spine}[m] \hookrightarrow \underline{n}$ of the simplex $[m]$, and a vector $v : [1] \to [m]$ of that simplex, the 'restriction' $\mathcal{F}|_v : \mathsf{spine}[1] \hookrightarrow \underline{n}$ of the frame to the vector is the n-embedded frame of the simplex $[1]$ whose single label is the minimal frame label of the spine vectors akin to the vector v, i.e. $\mathcal{F}|_v(0 \to 1) = \min\{\mathcal{F}(w) \mid w \pm v\}$. ⌐

This restriction procedure produces a plethora of combinatorial arrangements quite distinct from any permutation of its application to the standard frame on the simplex.

Example 1.1.51 (Frame restriction to simplicial vectors of the standard simplex). In Fig. 1.12 we illustrate various embedded framed 3-simplices along with the corresponding embedded framed restrictions to their 1-faces. ⌐
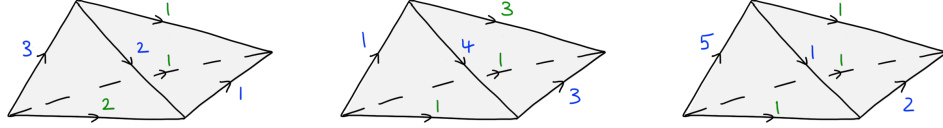
Figure 1.12: Restriction of embedded frames to vectors of a simplex.

The restriction of an embedded frame to any j-face of the simplex is determined directly by the restrictions to the 1-faces, as follows.

Construction 1.1.52 (Frame restriction to simplicial faces of the standard simplex). Let $\mathcal{F} : \mathsf{spine}[m] \hookrightarrow \underline{n}$ be an n-embedded frame of the simplex $[m]$, and let $f : [j] \to [m]$ be a j-face of that simplex. The 'restriction' $\mathcal{F}|_f : \mathsf{spine}[j] \hookrightarrow \underline{n}$ of the frame to the j-face is the n-embedded frame whose label value on the spine vector $v : [1] \to [j]$ is the numeral $\mathcal{F}|_{f \circ v} \in \underline{n}$. ⌐

The restriction of frames on standard simplices canonically carries over to the case of general simplices.

Notation 1.1.53 (Restricting trivializations to faces). Given a degeneracy $S \twoheadrightarrow [k]$ from an m-simplex S to a standard simplex, and a j-face $f : T \hookrightarrow S$, then we denote by $(S \twoheadrightarrow [k])|_f : T \cong [j]$ the unique degeneracy as well as (abusing notation) by $f : [j] \hookrightarrow [k]$ the unique face, which together factor the composite $(S \twoheadrightarrow [k]) \circ f$ as $f \circ (S \twoheadrightarrow [k])|_f$. (We call $(S \twoheadrightarrow [k])|_f$ the 'restricted degeneracy', or the 'restricted isomorphism' if $S \twoheadrightarrow [k]$ is an isomorphism, and $f : [j] \hookrightarrow [k]$ the 'induced standard face'). ⌐

Definition 1.1.54 (Frame restrictions to simplicial faces of simplices). The **frame restriction** of an n-embedded framed m-simplex $(S \cong [m], \mathcal{F})$ to a simplicial j-face $f : T \hookrightarrow S$ is the n-embedded frame $(T \cong [j], \mathcal{F}|_f)$ of T where $T \cong [j]$ equals the restricted isomorphism $(S \cong [k])|_f$ and the n-embedded frame $\mathcal{F}|_f$ is obtained by restricting $\mathcal{F}$ to the induced standard face $f : [j] \hookrightarrow [m]$. ⌐

Example 1.1.55 (Frame restriction to simplicial faces). In Fig. 1.13 we depict an embedded framed 3-simplex along with the restriction of its frame to various faces. ⌐

One may of course similarly define frame restrictions of embedded *partial* frames of simplices.

Definition 1.1.56 (Frame restrictions to simplicial faces of simplices). The **frame restriction** of an n-embedded k-partially framed m-simplex $(S \twoheadrightarrow [k], \mathcal{F})$ to a j-face $f : T \hookrightarrow S$ is the n-embedded partial frame $(T \twoheadrightarrow [l], \mathcal{F}|_f)$ of T where $T \twoheadrightarrow [l]$ equals the restricted degeneracy $(S \cong [k])|_f$ and the n-embedded frame $\mathcal{F}|_f$ is obtained by restricting $\mathcal{F}$ to the induced standard face $f : [l] \hookrightarrow [k]$. ⌐
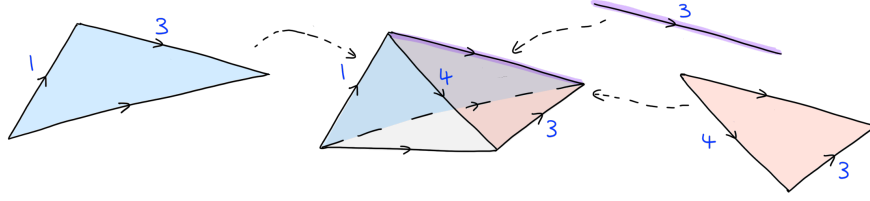
Figure 1.13: Restriction of an embedded frame to faces of a simplex.

The defining property of such restrictions is, in geometric terms, again the following: pick any framed realization $r : |S| \hookrightarrow \mathbb{R}^n$. The frame restriction of the given n-embedded partial frame of S to the face $f : T \hookrightarrow S$ is the unique n-embedded partial frame of T induced by the simplicial embedded partial trivialization $r \circ |f| : |T| \hookrightarrow |S| \hookrightarrow \mathbb{R}^n$.

1.1.2.2   Framed maps As previously described subspaces $W \hookrightarrow V$ of embedded trivialized vector spaces $V \hookrightarrow \mathbb{R}^n$ inherit themselves an embedded trivialization $W \hookrightarrow \mathbb{R}^n$ by restricting the trivialization of V. The inclusion $(W \hookrightarrow \mathbb{R}^n) \hookrightarrow (V \hookrightarrow \mathbb{R}^n)$ is a prototypical 'trivialization inclusion'. Conversely, one can define a 'trivialization projection' $(V \hookrightarrow \mathbb{R}^n) \twoheadrightarrow (W \hookrightarrow \mathbb{R}^n)$ to be a projection $V \twoheadrightarrow W$ that splits a framed trivialized inclusion. Combinatorially, this can be translated as follows.

Definition 1.1.57 (Framed faces)**.** Given n-embedded framed simplices $(S \cong [l], \mathcal{F})$ and $(T \cong [m], \mathcal{G})$, **framed face** $F : (S \cong [l], \mathcal{F}) \hookrightarrow (T \cong [m], \mathcal{G})$ is a face $F : [l] \hookrightarrow [m]$ such that $\mathcal{F} = \mathcal{G}|_F$. ⌐

Definition 1.1.58 (Framed degeneracies)**.** Given n-embedded framed simplices $(S \cong [l], \mathcal{F})$ and $(T \cong [m], \mathcal{G})$, a **framed degeneracy** $F : (S \cong [l], \mathcal{F}) \twoheadrightarrow (T \cong [m], \mathcal{G})$ is a degeneracy $F : [l] \twoheadrightarrow [m]$ such that $\mathcal{G} = \mathcal{F} \circ \mathsf{split}(F)$ (see Remark 1.1.18). ⌐

Framed faces and framed degeneracies are the monomorphisms resp. epimorphisms in a category of 'framed simplices and framed maps'; such framed maps may be uniformly described as follows.

Definition 1.1.59 (Framed map of framed simplices)**.** Given n-embedded framed simplices $(S \cong [l], \mathcal{F})$ and $(T \cong [m], \mathcal{G})$, a **framed map** $F : (S \cong [l], \mathcal{F}) \to (T \cong [m], \mathcal{G})$ is a simplicial map $F : [l] \to [m]$ such that for every vector $v : [1] \to [l]$ in the simplex $[l]$, either its frame label is preserved, i.e. $\mathcal{F}|_v = \mathcal{G}|_{F \circ v}$, or the vector is degenerated, i.e. $F \circ v : [1] \to [m]$ is constant. ⌐

Note, injective framed maps are exactly framed faces, and a surjective framed maps are exactly framed degeneracies.

Notation 1.1.60 (Categories of embedded framed framed simplices)*.* Denote by FrSimp_n the category of n-embedded framed simplices and their framed maps. (Note that the objects of this category are simplices of dimension necessarily at most n.) ⌐

Observation 1.1.61 ((Epi,mono)-factorization of framed maps)*.* As a map of simplices factors as a degeneracy map (epimorphism) followed by a face map (monomorphism), similarly any framed map of framed simplices factors as a framed degeneracy map followed by a framed face map. ⌐

Example 1.1.62 (Framed and non-framed maps). In Fig. 1.14 we illustrate four maps between 2-embedded framed simplices; in each case the map is a face map (with highlighted image), an identity, or a standard simplicial degeneracy (with highlighted affine kernel). The first map is framed, as the one frame label is preserved. The second map is not framed, as it preserves the frame label of neither spine vector. The third map is framed, as the first spine vector is degenerated and the other two vectors have their frame labels preserved. The fourth map is not framed, even though every spine vector is degenerated or has its frame label preserved, because the frame label of the non-spine vector is not preserved. ⌐
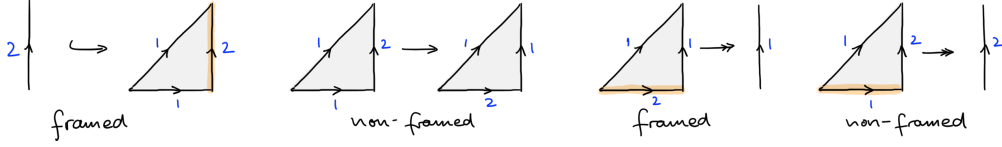
Figure 1.14: Framed and non-framed maps of framed simplices.

Framed maps of n-embedded simplices can also be understood in geometric terms.

Remark 1.1.63 (Framed maps in linear algebraic terms). Consider n-embedded framed simplices $(S \cong [l], \mathcal{F})$ and $(T \cong [m], \mathcal{G})$ and a simplicial map $F : S \to T$. For framed realizations $r : |S| \hookrightarrow \mathbb{R}^n$ and $q : |T| \hookrightarrow \mathbb{R}^n$, the map F is framed if for any vector v in S with $\vec{r}(\vec{v}) \in \epsilon_i^+$ we have $\vec{q} \circ \vec{F}(\vec{v}) \in \epsilon_i^+ \cup \{0\}$. ⌐

Example 1.1.64 (Framed maps in linear algebraic terms). In Fig. 1.15, for the four framed maps of 2-embedded framed simplices $F : (S \cong [l], \mathcal{F}) \to (T \cong [m], \mathcal{G})$ that were given in Fig. 1.14, we depict framed realizations $r : |S| \hookrightarrow \mathbb{R}^2$ and $q : |T| \hookrightarrow \mathbb{R}^2$ of their domain and codomains. Underneath, we depict the linear action $\vec{F} : \vec{V}(S) \to \vec{V}(T)$ on vectors v in S (we depict vectors by their images under the embeddings $\vec{V}(S) \hookrightarrow \mathbb{R}^2$ resp. $\vec{V}(T) \hookrightarrow \mathbb{R}^2$). ⌐
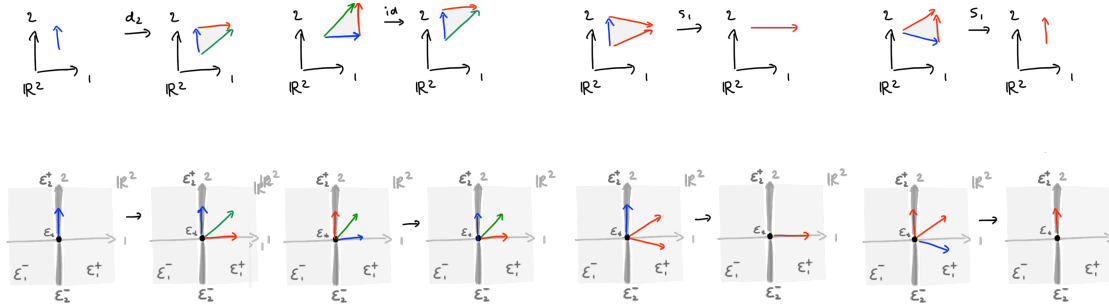
Figure 1.15: Framed and non-framed maps via framed realization.

Note that the geometric description of framed maps of embedded framed simplices immediately generalizes to the case of embedded partially framed simplices. In combinatorial terms, this may be phrased as follows.

Definition 1.1.65 (Framed maps of embedded partially framed simplices)**.** Given n-embedded partially framed simplices $(S \twoheadrightarrow [j], \mathcal{F})$ and $(T \twoheadrightarrow [k], \mathcal{G})$, a **framed map** $F : (S \twoheadrightarrow [j], \mathcal{F}) \to (T \twoheadrightarrow [k], \mathcal{G})$ is a simplicial map $F : S \to T$ that 'descends' to a framed map of n-embedded simplices $F_n : ([j], \mathcal{F}) \to ([k], \mathcal{G})$, that is, $F_n : [k] \to [j]$ commutes with F and the degeneracies $S \twoheadrightarrow [k]$ and $T \twoheadrightarrow [j]$. $\quad\quad \sqsubset$

Note that any framed map of n-embedded partially framed simplices $F : (S \twoheadrightarrow [j], \mathcal{F}) \to (T \twoheadrightarrow [k], \mathcal{G})$ descends to a *unique* framed map of n-embedded framed simplices $([j], \mathcal{F}) \to ([k], \mathcal{G})$ in this way, and we always denote the latter map by F_n.

Notation 1.1.66 (Category of embedded partially framed simplices)**.** Denote by $\mathsf{PartFrSimp}_n$ the category of n-embedded partially framed simplices and their framed maps. $\quad\quad \sqsubset$

Framed maps either preserve the frame label of a vector or degenerate that vector to zero; there is a more general notion of 'subframed map' in which vectors may degenerate not just to the zero vector but to any vector with more specialized frame label. We first describe this geometric viewpoint, and then give a purely combinatorial definition of subframed maps.

Remark 1.1.67 (Subframed maps in linear algebraic terms)**.** Consider n-embedded framed simplices $(S \cong [l], \mathcal{F})$ and $(T \cong [m], \mathcal{G})$ and a simplicial map $F : S \to T$. For framed realizations $r : |S| \hookrightarrow \mathbb{R}^n$ and $q : |T| \hookrightarrow \mathbb{R}^n$, we say F is 'subframed' if for any vector v in S with $\vec{r}(\vec{v}) \in \epsilon_i^+$ we have $\vec{q} \circ \vec{F}(\vec{v}) \in \overline{\epsilon}_i^+$ (where $\overline{\epsilon}_i^+$ is the closure of the ith positive component ϵ_i^+ of $\mathbb{R}^n$; note that this closure in particular contains $0 \in \mathbb{R}^n$, see Remark 1.1.63). $\quad\quad \sqsubset$

Note that subframed maps may in particular send vectors from a positive component ϵ_i^+ into a negative component $\epsilon_j^- \subset \overline{\epsilon}_i^+$ (where $j > i$).

**Definition 1.1.68** (Subframed map)**.** Given n-embedded framed simplices $(S \cong [l], \mathcal{F})$ and $(T \cong [m], \mathcal{G})$, a **subframed map** $F : (S \cong [l], \mathcal{F}) \to (T \cong [m], \mathcal{G})$ is a simplicial map $F : S \to T$ such that for every vector $v : [1] \hookrightarrow S$ in the simplex S, either its frame label is preserved, i.e. $\mathcal{F}|_v = \mathcal{G}|_{F \circ v}$, specialized, i.e. $\mathcal{F}|_v < \mathcal{G}|_{F \circ v}$, or the vector is degenerated, i.e. $F \circ v : [1] \to [m]$ is constant. $\quad\quad \sqsubset$

The definition of subframed maps extends, as with that of framed maps, to the embedded partially framed case, by insisting that a vector without a frame label is either mapped to zero or to again a vector without a frame label.

Example 1.1.69 (Subframed maps)**.** In the first column of Fig. 1.16 we illustrate two subframed maps of 2-embedded framed 1-simplices $F : (S \cong [1], \mathcal{F}) \to (T \cong [1], \mathcal{F})$, labeled 'id' and '$-$id' respectively. In the second column, we pick framed realizations $r : |S| \hookrightarrow \mathbb{R}^2$ and $q : |T| \hookrightarrow \mathbb{R}^2$, and illustrated in green the image $r(v)$ and in red the image $q(w)$ of the unique nonzero frame vectors v and w in S resp. T (as based vectors in $\mathbb{R}^2$); for both maps, we also depict the image $q \circ F(v)$ (in the same picture as $q(w)$). In the third column, we similarly depict the unbased action of F as a linear map $\vec{F} : \vec{S} \to \vec{T}$ on the corresponding linear vectors $\vec{v}$ and $\vec{w}$ (shown embedded in $\mathbb{R}^2$ via $\vec{r} : \vec{V}(S) \hookrightarrow \mathbb{R}^2$ resp. $\vec{q} : \vec{V}(T) \hookrightarrow \mathbb{R}^2$). In particular, note that for the second map the vector $\vec{v} \in \epsilon_1^+$ to a vector in ϵ_2^-. $\quad\quad \sqsubset$
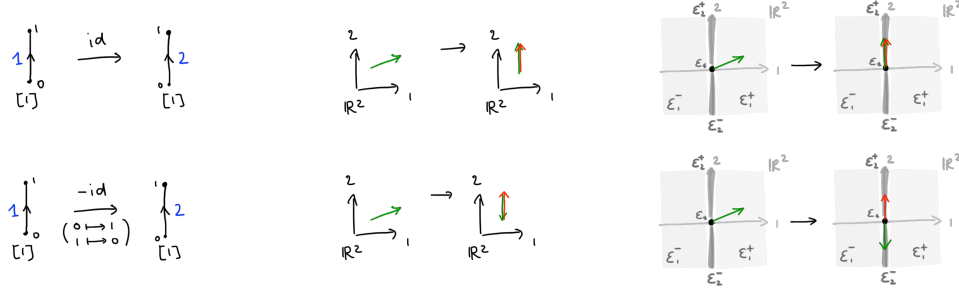
Figure 1.16: Subframed maps and their geometric interpretation.

1.2 Framed simplicial complexes

Our goal in this section will be to introduce a combinatorial notion of 'framings' on simplicial complexes. This is a globalization of the notion of frames on simplices introduced in the previous section: just as manifolds are spaces that are locally modeled on euclidean space, simplicial complexes are modeled on simplices; and just as framed manifolds are locally modeled on framed euclidean space (i.e. endowed with a continuous choice of frames in each tangent spaces), framed simplicial complexes will be modeled on framed simplices. The notion of framed simplicial complexes has two important features which distinguish the combinatorial approach to framings from the classical geometric approach to framings of manifolds.

Firstly, framings of simplicial complexes are not 'local linear' structures but 'piecewise affine' structures in the following sense. As previously discussed, simplices are not infinitesimal geometric objects but are extended affine spaces; frames of simplices are, correspondingly, 'affine frames', i.e. frames not based at any specific point of the simplex but defined 'up to translation'. We will define framings on simplicial complexes by piecing together affine frames of each of their simplices, and in this sense framings will be 'piecewise affine'. This stands in contrast to the classical 'tangential' notion of framings which defines framings locally, i.e. by picking a frame in the tangent space of each point.

Secondly, simplicial complexes are naturally 'singular spaces' and generally not manifolds. As a consequence, framed simplicial complex will in fact provide a combinatorial model of 'framed singular spaces' and not just of classical 'framed manifolds'. Classically, 'singular spaces' are gluings of 'manifold strata', and singular spaces themselves need not be manifolds. The question of framing singular spaces is subtle since the usual machinery of tangent spaces relies on local euclidean trivializations; these need not exist everywhere in singular spaces. We will not attempt to geometrically define 'framed singular spaces', but instead focus on leveraging the tools of affine combinatorics: namely, in combinatorial terms, individual open simplices will play to role of ('pieced') manifold strata, and the question of how framings can transition between strata of different dimension will find an answer using the notion of 'n-embedded' frames developed in the preceding section.

Note 'framed singular spaces' arise naturally in familiar situations: on the left in Fig. 1.17, we depict a 'Morse saddle point' given by a function from the 2-disk to $\mathbb{R}^1$. Up to choosing a Riemannian structure, this defines a gradient vector field (indicated by red arrows) everywhere *except* at the central critical point; if we regard this point as a 0-dimensional stratum with 0-dimensional frame (and its complement as a 2-dimensional

stratum with 1-dimensional frame given by the gradient vector field), then this is an example of a 'framed singular space'. Combinatorially, this can be modelled by a sequence

Figure 1.17: A '1-framed singular space' modelled by a partial 1-framed simplicial complex.

of surjective maps of simplicial complexes as shown on the right. Note that restricting this sequence to any simplex, yields a 1-embedded partially framed 2-simplex as previously defined, and thus, by passing to gradient frames, a 1-embedded partially framed simplex (indicated by directing edges in red)[2]. The resulting simplicial complex, now with each simplex endowed with the structure of a 1-embedded partial frame, yields our first example of a (partial) 1-framed simplicial complex, and it models the '1-framed singular space' to its left.

We outline this section. We introduce a combinatorial notion of framings in Section 1.2.1, which will be a straight-forward generalization of our earlier notions of framed simplices. In Section 1.2.2 we will then introduce a combinatorial notion of (locally) flat framings, which provide an important class of 'uniquely integrable' framings; local flatness yields a condition for combinatorial framings to be 'locally trivializable' and thus 'free of singularities', which is therefore analogous to classical tangential framings of manifolds as discussed above.

1.2.1 Framings on simplicial complexes

1.2.1.1 Simplicial complexes Before giving the definition of framings on simplicial complexes, we specify some basic terminology and notation.

Terminology 1.2.1 (Simplicial sets). A 'simplicial set X' is a presheaf $X : \Delta^{\mathrm{op}} \to \mathsf{Set}$ on the simplex category Δ. The 'category of simplicial sets' is the category of presheaves on Δ, and is usually denoted by SSet.   ⌐

We will tacitly Yoneda embed $\Delta \to \mathsf{SSet}$ and, abusing notation, use the simplex $[k] \in \Delta$ to also denote the representable simplicial set that it defines under this embedding (this is sometimes denoted by $\Delta[k]$ in the literature). The set of maps $x : [k] \to X$ provides the set of k-simplices in a simplicial set X.

Terminology 1.2.2 (Non-degenerate simplices). A simplex $x : [k] \to X$ in a simplicial set X is 'non-degenerate' if there is no non-identity degeneracy map $d : [k] \to [j]$ through which x factors; otherwise, x is 'degenerate'.   ⌐

In piecewise linear topology one commonly uses the following weaker notion in place of simplicial sets.

---

[2]Following previous notation, each directed edge should also carry the frame label '1', see Example 1.1.30.

Terminology 1.2.3 (Simplicial complexes). A 'simplicial complex' consists of a set of vertices $K[0]$ together with a list of subsets $K[1], K[2]...$ of the vertex powerset $\wp K[0]$ with the property that elements in $K[i]$ are sets of cardinality $(i+1)$ and any subset of an element in $K[i]$ is an element in some $K[j]$, $j \leq i$. Elements of $K[i]$ are also called i-simplices of K. Maps of simplicial complexes $K \to L$, called 'simplicial maps', are maps of vertex sets $f : K[0] \to L[0]$ whose image on each i-simplex $x \in K[i]$ yields a j-simplex $f(x) \in L[j]$ (for some j). The category of simplicial complexes and their simplicial maps will be denoted by SimpCplx. ⨆

Just as simplicial sets, simplicial complexes may be characterized as (certain) presheaves as follows.

Remark 1.2.4 (Simplicial complexes as presheaves). Recall the category $\underline{\Delta}$ of unordered simplices (see Notation 1.1.3). Every simplicial complex K gives rise to a presheaf $K : \underline{\Delta}^{\mathrm{op}} \to \mathsf{Set}$ by defining $K(S)$ to be the set of functions $x : S \to K[0]$ whose image $\mathrm{im}(x)$ lies in some $K[j]$, and defining $K(f : S' \to S)$ to act by precomposition with f. This construction gives rise to a full and faithful embedding of SimpCplx into the category $\mathrm{PSh}(\underline{\Delta})$ of presheaves on $\underline{\Delta}$. ⨆

Again, we will tacitly Yoneda embedded simplices in $\underline{\Delta}$ into presheaves $\mathrm{PSh}(\underline{\Delta})$. Note that the Yoneda embedding $\underline{\Delta} \to \mathrm{PSh}(\underline{\Delta})$ lands in the subcategory SimpCplx $\hookrightarrow \mathrm{PSh}(\underline{\Delta})$, making simplices in particular simplicial complexes.

Recall the 'unordering' functor $(-)^{\mathsf{un}} : \Delta \to \underline{\Delta}$ which forgets orders (see Terminology 1.1.5). This extends to simplicial sets as follows.

Terminology 1.2.5 (Unordering functor). The 'unordering functor' $(-)^{\mathsf{un}} : \mathsf{SSet} \to \mathrm{PSh}(\underline{\Delta})$ forgets the order of vertices in each simplex: formally, this can be defined as the left adjoint to precomposing presheaves with $(-)^{\mathsf{un}} : \Delta \to \underline{\Delta}$. ⨆

Rourke and Sanderson introduce 'ordered simplicial complexes' for simplicial complex with an order on their objects [RS71, §1]; we will use the same term for the following more general notion, which requires each simplex in a simplicial complex to be (consistently) ordered, but this 'local' order need not induce a 'global' order on the set of objects.

Terminology 1.2.6 (Ordered simplicial complexes). A simplicial set X is called an 'ordered simplicial complex' if its unordering X^{un} is an ordinary simplicial complex. ⨆

Ordered simplicial complexes can be more explicitly described: a simplicial set X is an ordered simplicial complex if each non-degenerate simplex in K can be uniquely identified by its set of vertices (as a subset of $K[0]$).

Remark 1.2.7 (Non-degenerate simplices inject into complexes). A simplex $x \in K[m]$ in an ordered simplicial complex is non-degenerate if and only if the presheaf map $x : [m] \to K$ is an injection. A similar observation holds in the case of simplicial complexes. ⨆

Terminology 1.2.8 (The category of ordered simplicial complexes). The 'category of ordered simplicial complexes' SimpCplx$^{\mathrm{ord}}$ is the full subcategory of SSet consisting of ordered simplicial complexes. Its morphisms will be called 'ordered simplicial maps'. ⨆

Terminology 1.2.9 (Ordering unordered simplicial complexes and their maps). An 'ordering' of a simplicial complex K (resp. of a simplicial map $F : K \to L$) is a choice of a preimage K^{ord} of K (resp. a preimage $F^{\mathrm{ord}} : K^{\mathrm{ord}} \to L^{\mathrm{ord}}$ of F) under the unordering functor.

For fixed orderings K^{ord} and L^{ord} of simplicial complexes K and L, a simplicial map $F : K \to L$ is said to be 'order-preserving' if it has a (necessarily unique) ordering $F^{\mathsf{ord}} : K^{\mathsf{ord}} \to L^{\mathsf{ord}}$. ⌟

The standard ordering of the unordered standard simplex $[m]^{\mathsf{un}}$ is the (ordered) standard simplex $[m]$.

1.2.1.2   The definition of framings We now introduce framings on ordinary simplicial complexes. Recall the definition of embedded frames $\mathcal{F}$ of m-simplices S from Definition 1.1.34 which endows S with an isomorphism $S \cong [m]$ together with an n-embedded frame $\mathcal{F}$ of $[m]$ (given by an injection $\mathcal{F} : \mathsf{spine}[m] \hookrightarrow \underline{n}$). Recall also, for a j-face $f : T \hookrightarrow S$, such an n-embedded frame of S restricts to an n-embedded frame of T defined by the restricted isomorphism $(S \cong [m])|_f : T \cong [j]$ and the restricted n-embedded frame $\mathcal{F}|_f$ (see Definition 1.1.54).

Definition 1.2.10 (Framings of simplicial complexes)**.** An n-**framing** $(\alpha, \mathcal{F})$ of a simplicial complex K endows each m-simplex $x : S \hookrightarrow K$ with an n-embedded frame $(\alpha_x : S \cong [m], \mathcal{F}_x)$ such that, for any j-face $f : T \hookrightarrow S$, the restriction of the chosen frame of x to the face f coincides with the chosen frame of $x \circ f$; that is, $\alpha_{x \circ f} = \alpha_x|_f$ and $\mathcal{F}_{x \circ f} = \mathcal{F}_x|_f$. ⌟

The following observation will simplify the data of framings.

Observation 1.2.11 (Isomorphism data of framings is an ordering). An n-framing $(\alpha, \mathcal{F})$ of a simplicial complex K gives rise to an ordering of K; indeed, for each m-simplex $x : S \hookrightarrow K$ in K, an order on the vertices of S is determined by the isomorphism $\alpha_x : S \cong [m]$ and together (since choices of α_x are compatible with faces) these orderings of vertices of simplices determine an ordering of K itself. This has an inverse: any ordering of K restricts to an ordering of each simplex $x : S \hookrightarrow K$, and thus yields (compatible) isomorphisms $S \cong [m]$. ⌟

We therefore obtain the following equivalent way of phrasing the notion of n-framings.

Remark 1.2.12 (Framings of simplicial complexes by orderings). To define an n-framing $\mathcal{F}$ of a simplicial complex K we may equivalently specify an ordering of K together with an n-embedded frame $\mathcal{F}_x$ of $[m]$ for each (order-preserving) m-simplex $x : [m] \hookrightarrow K$, such that, for any face $f : [k] \hookrightarrow [m]$, we have $\mathcal{F}_{x \circ f} = \mathcal{F}_x|_f$. ⌟

We will henceforth adopt this more concise reformulation of n-framings in terms of orderings (as recorded by Remark 1.2.12). We refer to the pair $(K, \mathcal{F})$, of a simplicial complex and an n-framing on it, as an 'n-framed simplicial complex'. Note that the ordering of K itself will be implicit in our notation, and we adopt the following convention.

Convention 1.2.13 (Keeping orderings implicit). Given a framed simplicial complex $(K, \mathcal{F})$, when considering K as an 'ordered simplicial complex' (for instance, when considering maps $[m] \to K$) we will always assume this ordering to be the ordering of K provided by the n-framing $\mathcal{F}$. ⌟

Note that a simplicial complex K cannot contain simplices of dimension greater than n in order for it to admit an n-framing.

Example 1.2.14 (Framings and non-framings on simplicial complexes)**.** In Fig. 1.18, on top, we depict three simplicial complexes consisting of one, two and three 2-simplices respectively. Underneath each we depict an ordering, underneath which we depict several choices of frames for individual simplices (to depict these we use coordinate frame notation as introduced in Example 1.1.37). Not all choices lead to framings: for the indicated simplices in blue and in red the compatibility of frame choices with face restrictions is not met. ⌐
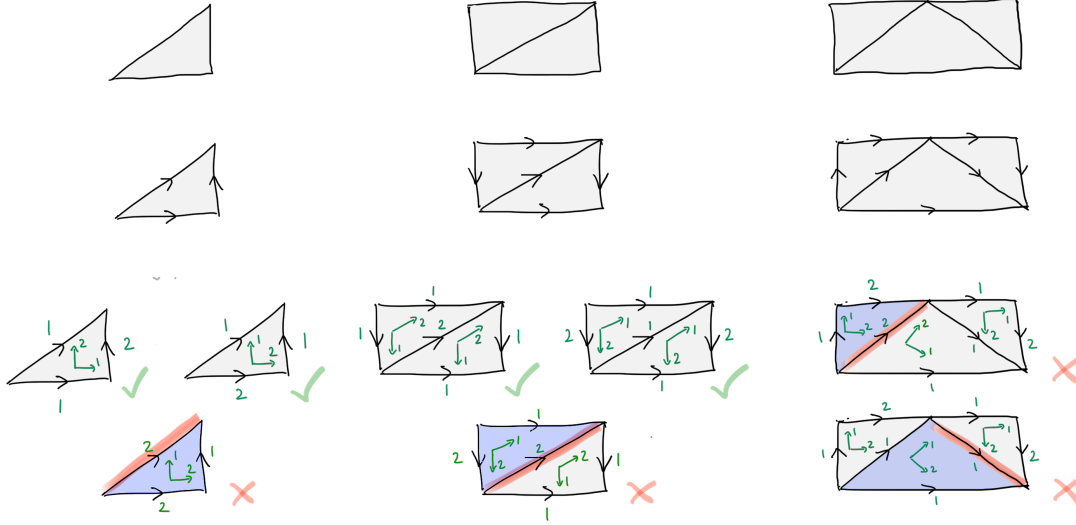
Figure 1.18: Simplicial complexes with orderings, framings and non-framings.

Remark 1.2.15 (1-Skeletal notation)**.** Instead of depicting frames $\mathcal{F}_x$ separately for each simplex x in K (as in the previous example) it suffices to give frame labels of 1-simplices: indeed, each simplex frame is determined the frames of its spine, and the compatibility condition in the definition of framed simplicial complexes ensures that if two simplices share a spine vector that vector must carry the same label in both of their frames.[3] ⌐

Example 1.2.16 (More framings on simplicial complexes)**.** We depict three framed simplicial complexes in Fig. 1.19. Observe the difference between the second and third example: the second example realizes to the annulus $S^1 \times D^1$ while the their example realizes to the Mobius band. Note that the framing of the Mobius band, in a sense, is 'singular' as it 'flips' when traversing the band; we will later describe this as a failure of *local flatness* of the framing (see Remark 1.2.37). ⌐

Definition 1.2.17 (Restrictions)**.** Given an n-framed simplicial complex $(K, \mathcal{F})$ and a simplicial subcomplex $L \hookrightarrow K$, the **restriction** $\mathcal{F}|_L$ **of** $\mathcal{F}$ **to** L is the n-framing of L obtained by restricting the ordering of K to L, and setting $(\mathcal{F}|_L)_x = \mathcal{F}_x$ for simplices $x : [k] \hookrightarrow L \hookrightarrow K$. ⌐

---

[3]The converse is not true: we cannot 'only' label 1-simplices in a simplicial complex with frame labels in $\underline{n}$ to define an n-framing—we must also check that such a labeling defines valid frames for each k-simplex in the complex, $k > 1$.
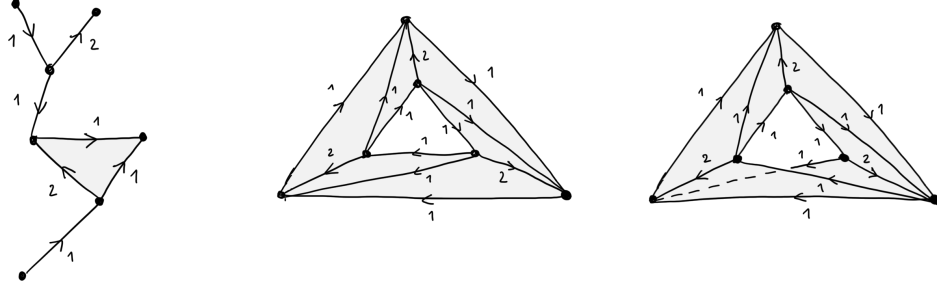
Figure 1.19: Framed simplicial complexes depicted using 1-skeletal notation.

———————————

Framings of simplicial complex can similarly be defined in the 'partial' case.

Definition 1.2.18 (Partial framings of simplicial complexes). A **partial n-framing** $(\alpha, \mathcal{F})$ of a simplicial complex K endows each m-simplex $x : S \hookrightarrow K$ with an n-embedded partial frame $(\alpha_x : S \twoheadrightarrow [k], \mathcal{F}_x)$ such that, for any j-face $f : T \hookrightarrow S$, the restriction of the chosen n-embedded partial frame of x to the face f coincides with the chosen n-embedded partial frame of $x \circ f$; that is, $\alpha_{x \circ f} = \alpha_x|_f$ and $\mathcal{F}_{x \circ f} = \mathcal{F}_x|_f$. ⌐

Note that a 'partial n-framed simplicial complex' $(K, \mathcal{F})$ may now have m-simplices of any dimension m. Note also that Observation 1.2.11 no longer holds: a partial framing need not determine an ordering on a simplicial complex. Going forward, we will be mainly interested in the non-partial case of framings; nonetheless, all subsequent definitions have analogs in the partial case as well.

Example 1.2.19 (A familiar partial framing). An example of a partial 1-framing was depicted and discussed earlier in Fig. 1.17. ⌐

———————————

We next define maps of framed simplicial complexes. This straight-forwardly generalizes the notion of framed maps of framed simplices (see Definition 1.1.59).

Definition 1.2.20 (Framed maps of framed simplicial complexes). Consider n-framed simplicial complexes $(K, \mathcal{F})$ and $(L, \mathcal{G})$. A **framed simplicial map** $F : (K, \mathcal{F}) \to (L, \mathcal{G})$ (or simply, a 'framed map') is a simplicial map $F : K \to L$ that restricts on all simplices $x : [k] \hookrightarrow K$ and $y = \mathrm{im}(F \circ x) : [l] \hookrightarrow L$ to a framed map $F : ([k], \mathcal{F}_x) \to ([l], \mathcal{G}_y)$ of n-embedded framed simplices. ⌐

In particular, note that any framed map $F : (K, \mathcal{F}) \to (L, \mathcal{G})$ is order-preserving. When referring to the ordering of F, we will speak of the 'ordered simplicial map' $F : K \to L$.

Notation 1.2.21 (The category of framed simplicial complexes). The category of n-framed simplicial complexes and framed maps will be denoted by $\mathsf{FrSimpCplx}_n$. ⌐

Definition 1.2.22 (Unframing framed simplicial complexes). The **unframing** functor $\mathsf{Unframe} : \mathsf{FrSimpCplx}_n \to \mathsf{SimpCplx}$ takes a framed simplicial complex $(K, \mathcal{F})$ to the simplicial complex K, and a framed map $F : (K, \mathcal{F}) \to (L, \mathcal{G})$ to the simplicial map $F : K \to L$. ⌐

Remark 1.2.23 (Subframed maps of framed simplicial complexes). The notion of 'subframed maps of framed simplices' described in Definition 1.1.68 also generalizes to framed simplicial complexes. A 'subframed maps of framed simplicial complexes' $F : (K, \mathcal{F}) \to (L, \mathcal{G})$ is a simplicial map $F : K \to L$ that restricts on all simplices $x : [k] \hookrightarrow K$ and $y = \text{im}(F \circ x) :$ $[l] \hookrightarrow L$ to a subframed map $F : ([k]^{\mathsf{un}} \cong [k], \mathcal{F}_x) \to ([l]^{\mathsf{un}} \cong [l], \mathcal{G}_y)$. Note, unlike framed maps, subframed maps need not be order-preserving. ⌐

1.2.2   Flatness In this section we discuss 'flat framings': intuitively, flatness can be thought of as requiring framed spaces to 'trivialize' as framed subspaces of standard framed euclidean space. We start by generalizing the definition of framed realizations of n-embedded framed simplices to the case of framed simplicial complexes (see Definition 1.1.38).

Definition 1.2.24 (Framed realization). Given an n-framed simplicial complex $(K, \mathcal{F})$, a linear embedding $r : |K| \hookrightarrow \mathbb{R}^n$ (that is, an embedding that is linear on each simplex) is called a **framed realization** of K if for each simplex $x : [m] \hookrightarrow K$ the restriction $r \circ |x|$ is a framed realization of $([m], \mathcal{F}|_x)$. ⌐

Remark 1.2.25 (Flat framings are determined by their framed realizations). Since any framed realization $r : |S| \hookrightarrow \mathbb{R}^n$ of an n-embedded framed simplex $(S \cong [m], \mathcal{F})$ determines both the isomorphism $S \cong [m]$ and the frame $\mathcal{F}$ on $[m]$ uniquely, it follows that any framed realization of $r : |K| \hookrightarrow \mathbb{R}^n$ of an n-framed simplicial complex $(K, \mathcal{F})$ determines the framing $\mathcal{F}$ of K uniquely. We may therefore depict framed simplicial complexes by their framed realizations: this is illustrated in Fig. 1.20, where we depict a framed realization $r : |K| \hookrightarrow \mathbb{R}^2$ on the left, together with the induced 2-framing $\mathcal{F}$ on the right. ⌐
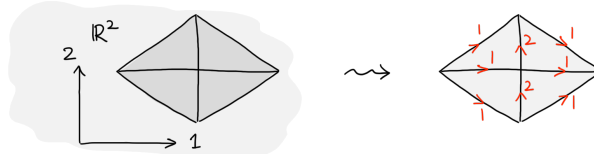
Figure 1.20: Flat framings are determined by their framed realizations.

We will be interested in framed realizations that are sufficiently regular in that they have bounded images that interplay nicely with the framing of euclidean space. The appropriate notion of 'framed bounds' is spelled out in the following definition.

Terminology 1.2.26 (Framed half-spaces). A map $\gamma : \mathbb{R}^n \to \mathbb{R}$ is said to be 'k-frame dependent' $(1 \leq k \leq n)$ if it only depends on the first $(k-1)$ coordinates of $\mathbb{R}^n$. Such a function defines the 'k-framed upper half-space' $\mathbb{R}^n_\gamma$ which contains points $x \in \mathbb{R}^n$ iff $x_k \geq \gamma(x)$. One similarly defines the 'k-framed lower half-space' $\mathbb{R}^n_{\leq \gamma}$. ⌐

Definition 1.2.27 (Framed bounded subspaces). A **framed bounded subspace** S of $\mathbb{R}^n$ is a compact subspace of the form

$$S = \bigcap_{i \leq n} (\mathbb{R}^n_{\geq \gamma_i^-} \cap \mathbb{R}^n_{\leq \gamma_i^+})$$

where $\gamma_i^\pm$ are i-frame dependent such that $\gamma_i^- \leq \gamma_i^+$. We call γ_i^- resp. γ_i^- the 'lower' resp. 'upper ith bound' of S. ⌐

Example 1.2.28 (Framed half-spaces and bounded subspaces). In Fig. 1.21 we depict a set of framed half-spaces with bounds $\gamma_1^\pm$ respectively $\gamma_2^\pm$. Intersecting these half-spaces yields the shown framed bounded subspace of $\mathbb{R}^2$. ⌐
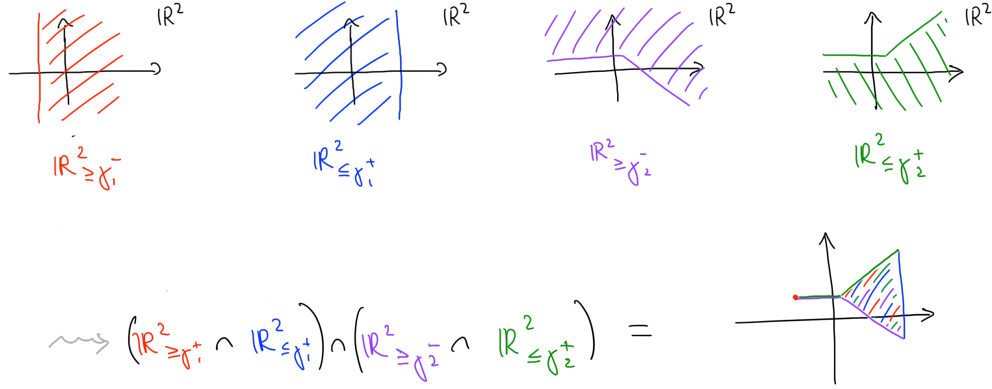
Figure 1.21: Framed half-spaces bounding a framed bounded subspace of $\mathbb{R}^2$.

We may combine the notions of framed realizations and framed bounds into the following terminology.

Terminology 1.2.29 (Bounded framed realization). A framed realization of a framed simplicial complex is called a 'framed bounded realization' if its image is framed bounded. ⌐

Definition 1.2.30 (Flat framings). A framed simplicial complex $(K, \mathcal{F})$ is **flat** if it admits a framed bounded realization $r : |K| \hookrightarrow \mathbb{R}^n$. ⌐

Notation 1.2.31 (Category of flat framings). Denote the full subcategory of $\mathsf{FrSimpCplx}_n$ consisting of flat framings by $\mathsf{FlatFrSimpCplx}_n$. ⌐

Example 1.2.32 (Flat framings). In the upper row of Fig. 1.22 we depict examples of flat framings. The first framing admits a framed bounded realization with bounds as given in Fig. 1.21; one checks that the other framings similarly admit framed bounded realizations. In the lower row of Fig. 1.22 we depict non-examples of flat framings; the first framing fails to be flat since no framed realization into $\mathbb{R}^1$ exists. The second example embeds in $\mathbb{R}^2$ but its image is not bounded continuously in the sense of Definition 1.2.27. The third framing does not admit a framed realization in $\mathbb{R}^2$ (note that directions of the left and right vertical 1-simplices are 'reversed'). The fourth example again fails to admit a framed bounded realization; while the last example fails to admit a framed realization in $\mathbb{R}^3$. ⌐

———————————

We may further impose flatness *locally*. This can be used to recover classical intuition about framings of manifolds as we will see.

Terminology 1.2.33 (Stars). Recall the ordinary notion of 'stars' $\mathsf{star}(x)$ of vertices x in simplicial complexes U: this is defined as the minimal subcomplex $\mathsf{star}(x)$ of U containing all simplices that have x as a vertex. ⌐
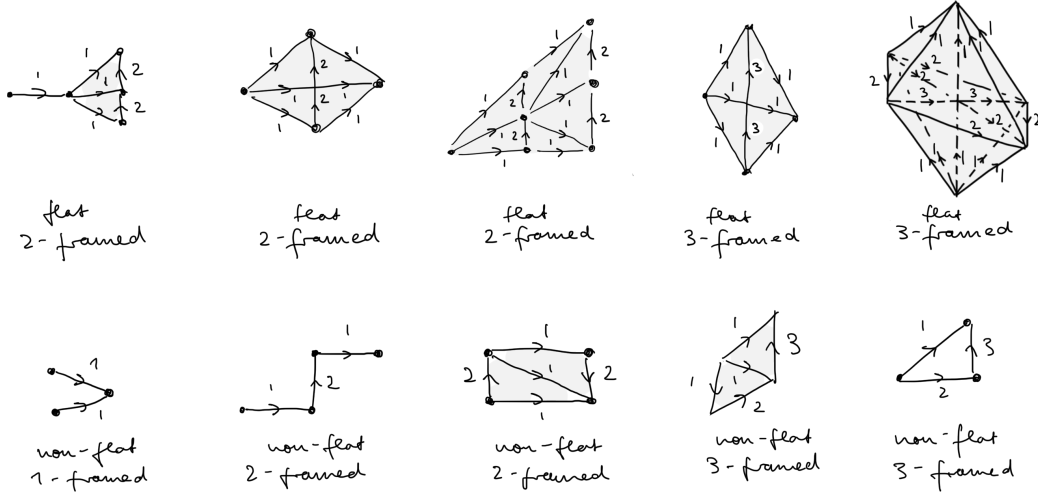
Figure 1.22: Flat and non-flat framed simplicial complexes.

Definition 1.2.34 (Local flatness). We say an n-framing $(K, \mathcal{F})$ of simplicial complex K is locally flat if for each vertex $x \in K$ the restricted n-framing $(\mathsf{star}(x), \mathcal{F}|_{\mathsf{star}(x)})$ is flat. ⌋

Remark 1.2.35 (Flatness implies local flatness). It is true that each flat framing is in particular locally flat. ⌋

Example 1.2.36 (Locally flat framed triangulated 1-manifolds). In Fig. 1.23 on the left we depict two locally flat 1-framed simplicial complexes $(K, \mathcal{F})$ each triangulating a connected 1-manifold. On the right we depict two framed simplicial complex that are not locally flat framed. ⌋
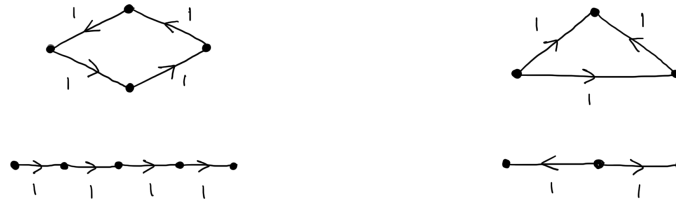
Figure 1.23: Examples and non-examples of locally flat framed triangulated 1-manifolds.

We can now revisit our earlier example of the Mobius band, adding the following insight.

Remark 1.2.37 (Frameability of Mobius band revisited). Recall the last two framed simplicial complexes from Fig. 1.19: the first example is locally flat, while the second (the 'Mobius band') isn't. In fact, the Mobius band is not 'locally flat 2-frameable', in the sense that no triangulation admits a locally flat framing. ⌋

1.3 Framed regular cell complexes

In the final section of this chapter, we will introduce n-framings on regular cell complexes. Our goal will be the definition of a category of 'n-framed regular cell complexes' $\mathsf{FrCellCplx}_n$,

fitting into the following diagram of categories (in which vertical arrows are fully faithful embeddings of categories, while horizontal arrows forget framing structures)

$$\begin{array}{ccc} \mathsf{FrSimpCplx}_n & \xrightarrow{\text{Unframe}} & \mathsf{SimpCplx} \\ \downarrow & & \downarrow \\ \mathsf{FrCellCplx}_n & \xrightarrow{\text{Unframe}} & \mathsf{CellCplx} \end{array} \quad .$$

Regular cell complexes, as we recall in Section 1.3.1, are complexes of cells of general 'polytopic shape': in Fig. 1.24 we illustrate shapes of regular cells in dimension 2 and 3. Despite their generality, regular cell complexes, unlike their non-regular counterpart, are
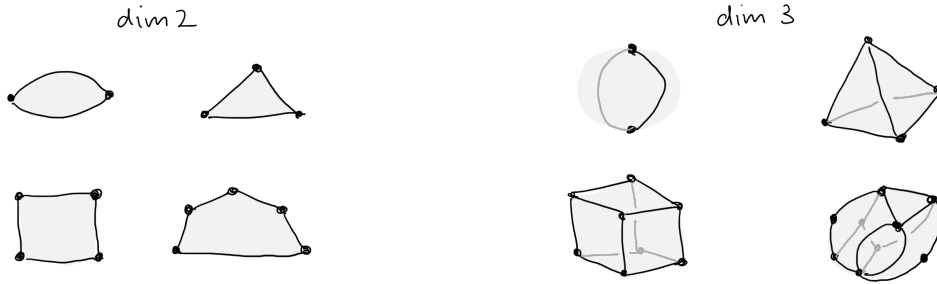
Figure 1.24: An illustration of regular 2-cells and 3-cells.

combinatorializable. The fundamental property of regular cell complexes that enables this combinatorialization is the 'homotopical triviality' of cell attachments (see Remark 1.3.5). This entails that one can describe a class of so-called 'cellular' posets, which are exactly the entrance path posets of regular cell complexes; in fact, geometric realizations of cellular posets recover the homeomorphism type of their corresponding regular cell complexes. The resulting translation between regular cell complexes and cellular posets provides the claimed combinatorialization of regular cell complexes. Crucially however, the combinatorialization is (computably) *intractable* in the following sense: given a poset, there can be no general algorithm to determine if that poset is cellular. In particular, it is impossible to algorithmically write down a list classifying 'all the shapes' of regular cells (such as those in Fig. 1.24) up to some general bound in, say, the number boundary cells. By *framing* regular cells this intractibility will find a natural resolution.

The definition of framed regular cells will directly rely on our previous work on framings of simplicial complexes. Namely, an n-framed regular cell will be an n-framing of the simplicial complex that underlies the cellular poset of the cell (with the additional condition that the framing is flat on the cell itself as well as on each subcell in its boundary). Our approach therefore combines the following two ingredients: firstly, use to the correspondence of regular cells and cellular posets to endow regular cells with canonical 'piecewise linear' structure; secondly, require simplicial framings on cells to be flat. The second condition assures that framings of cells are 'trivial' (in the sense that the framed cell is framed realizable as a subspace of $\mathbb{R}^n$) which, heuristically, reflects that framed regular cells, just like framed simplices, will play the role of 'small', and therefore 'trivializable' framed pieces from which larger framed spaces will be built.

In contrast to the case of nonframed regular cells, framed regular cells can now be computably recognized and classified. The classification will be constructed in Chapter 3.

At the same time, the generality of framed regular cells, as opposed to mere framed simplices, will be at the very heart of fundamental results in framed combinatorial topology— for instance, working with framed cells will enable the construction of 'canonical cellulations' of flat framed stratifications, as we will explain in Chapter 5 (see also Example 1.3.57 in this section). This will highlight that the passage from framed simplices to framed regular cells is not 'by choice', but of central importance in framed combinatorial topology.

We outline this section. After recalling regular cell complexes and related notions in Section 1.3.1, we will first introduce the notion of framings on regular cells in Section 1.3.2. In Section 1.3.3 we will then obtain a notion of 'framed space' by defining framings on regular cell complexes by compatibly endowing each cell with a framing. We will further discuss an important special case of framed regular cell complexes that are themselves flat, which will later resurface in the construction of canonical cellulation of flat framed stratifications.

1.3.1 Regular cell complexes We start by recalling definitions and facts about regular cell complex. Our subsequent goal will be the introduction of a 'combinatorial' category CellCplx of regular cell complexes, generalizing that of simplicial complexes SimpCplx.

1.3.1.1 Regular cell complexes as cellular posets

Definition 1.3.1 (Regular cell complexes). A **regular cell complex** is a stratification whose strata are open disks (also called 'open cells') while closures of strata are closed disks (also called 'closed cells'). ⌐

Terminology 1.3.2 (Cellulations). In analogy to the notion of 'triangulation', we speak of 'cellulation' when decomposing a space into a regular cell complex. (More generally, the term is used in the context of cellulable stratifications, see Appendix B.3.) ⌐

A simplicial complex is a simple type of regular cell complex in which each n-dimensional cell has exactly $(n+1)$ faces of dimension $(n-1)$.

Definition 1.3.3 (Entrance path posets of regular cell complexes). The **entrance path poset** Entr X of a regular cell complex X is the poset whose objects are the open cells x in X, with an arrow $x \to y$ whenever the closure $\overline{x}$ contains y. ⌐

Entrance path posets of regular cell complexes are graded by dimension, that is, they admit a functor dim : Entr $X \to \mathbb{N}^{\mathrm{op}}$ with discrete preimages, mapping each cell to its dimension. Cells of a regular cell complex X which are minimal elements in Entr X will be called 'facets'—these are exactly cells which are not contained in any other cell's boundary.

It will further be useful to think of regular cell complexes as stratified (i.e. 'singular') spaces.

Definition 1.3.4 (Regular cell complexes as stratified spaces). The **cell stratification** cell X of a regular cell complex X is the stratification of the space underlying X whose strata are the open cells of X. ⌐

Note that the definition of entrance path posets of regular cell complexes is consistent with that of stratified spaces (see Definition B.1.6), that is, Entr $X =$ Entr (cell X). In particular, the characteristic function cell $X : X \to$ Entr X (see Definition B.1.13) takes

points in the open cell x of X to the object $x \in \mathsf{Entr}\, X$. As an aside, we remark that regular cell complex have the following properties as stratifications.

Remark 1.3.5 (The homotopical triviality of cell attachments). Regular cell complexes are, as stratified spaces, *conically* stratified (see Proposition B.3.20). They thus have 'entrance path ∞-categories' (see Definition B.3.9) which is a higher categorical analog of entrance path posets. Importantly, reflecting the 'homotopical triviality' of cell attachments, the entrance path ∞-categories of regular cell complexes are in fact 0-truncated, and thus equivalent to their entrance path posets (see Observation B.3.22). ⌐

Understanding regular cell complexes as stratifications means they inherit the following notion of maps. Recall a stratified map is a map that maps strata into strata; equivalently, this means stratified maps factor through characteristic maps by a map of entrance path posets (see Definition B.2.1).

Definition 1.3.6 (Maps of regular cell complexes). A **map of regular cell complexes** $F : X \to Y$ is a stratified map $F : \mathsf{cell}\, X \to \mathsf{cell}\, Y$. ⌐

Notation 1.3.7 (Regular cell complexes as stratified spaces). We henceforth notationally identify X and $\mathsf{cell}\, X$, thinking of regular cell complexes in terms of their cell stratifications. ⌐

Remark 1.3.8 (Functoriality of entrance path posets). Note that entrance path poset construction is functorial: for any map of regular cell complexes $F : X \to Y$ we obtain a poset map $\mathsf{Entr}(F) : \mathsf{Entr}(X) \to \mathsf{Entr}(Y)$, mapping a cell of X to the cell in Y that contains the image $F(x)$. ⌐

While 'entrance path posets' extract posets from regular cell complexes (and more generally, from stratifications) we can, conversely, turn posets into stratifications by the following notion of 'classifying stratifications'.

Terminology 1.3.9 (Upper and strict upper closures). Given a poset P and an element $x \in P$, then the 'strict upper closure' $P^{>x}$ of x in P is the full subposet with objects $y \in P$ with $y > x$. Similarly, the 'upper closure' $P^{\geq x}$ is the full subposet of objects $y \in P$ with $y \geq x$. ⌐

Terminology 1.3.10 (Classifying spaces of posets). Recall, the 'classifying space' $|P|$ of a poset P is the geometric realization of the nerve NP of P. (Abusing terminology, we also refer to $|P|$ itself as the 'geometric realization' of P, and say P 'realizes' to $|P|$.) ⌐

Terminology 1.3.11 (Classifying stratification of posets). Given a poset P, the 'classifying stratification' $\mathsf{CStr}\, P$ of P (also called the 'stratified realization' of P) is the stratification of $|P|$ whose strata are given subspaces $\mathsf{str}(x) := \left| P^{\geq x} \right| \setminus \left| P^{>x} \right|$ for $x \in P$; its characteristic function, denoted by $\mathsf{CStr}\, P : |P| \to P$, takes points in $\mathsf{str}(x)$ to x. ⌐

Terminology 1.3.12 (Classifying stratified maps of posets maps). Given a poset map $F : P \to Q$, the 'classifying stratified map' $\mathsf{CStr}\, F : \mathsf{CStr}\, P \to \mathsf{CStr}\, Q$ maps vertices $p \in |P|$ to vertices $F(p) \in |Q|$, and linearly extends this mapping to all other simplices in $|P|$. ⌐

An alternative phrasing of classifying stratifications, based on an explicit definition of classifying spaces, can be found in Construction B.1.41 (for maps, see also Construction B.2.14).

Example 1.3.13 (Classifying stratifications)**.** We illustrate three classifying stratifications in Fig. 1.25: from left to right, we depict the classifying stratifications $\mathsf{CStr}\,P$ of the 1-simplex $P = [1]$, the 2-simplex $P = [2]$ and the product of two 1-simplices $P = [1] \times [1]$. ⌐
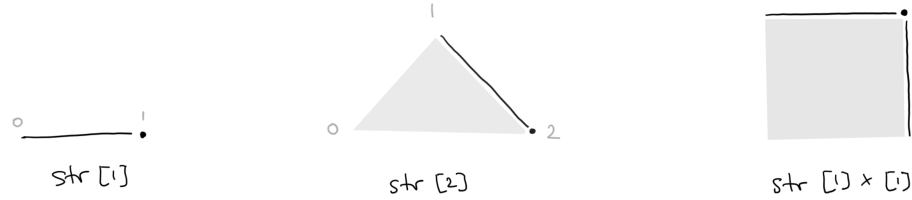
Figure 1.25: Classifying stratifications of posets.

The class of posets that can be obtained as entrance path posets of regular cell complexes will be described by so-called 'cellular posets'

Definition 1.3.14 (Cellular posets)**.** A poset $(X, \leq)$ is called **cellular** if the realization $|X^{>x}|$ of the strict upper closure of any $x \in P$ is homeomorphic to a sphere. ⌐

Example 1.3.15 (Cellular and non-cellular posets)**.** In Fig. 1.26 we depict cellular posets, as well as posets which fail to be cellular. Note, in particular, even if upper closures $P^{\geq x}$ realizes to topological balls it need not be the case that strict upper closures $P^{>x}$ realize to spheres. ⌐
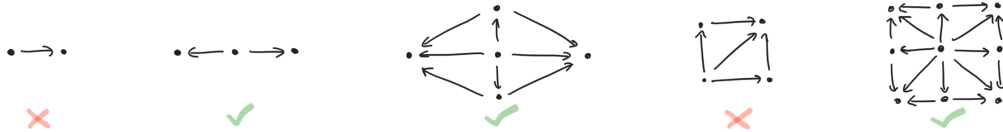
Figure 1.26: Cellular and non-cellular posets.

Centrally, classifying stratifications of cellular posets are exactly regular cell complexes as recorded in the following result.

Proposition 1.3.16 (Regular cell complexes are classifying stratifications of cellular posets)**.** *Regular cell complexes are exactly classifying stratifications of cellular posets, in the following sense.*
1. *The classifying stratification of a cellular poset is a regular cell complex.*
2. *The entrance path poset of a regular cell complex is a cellular poset.*
3. *Every regular cell complex X is stratified homeomorphic the classifying stratification of its entrance path poset, that is, $X \cong \mathsf{CStr}\,\mathsf{Entr}X$.[4]*
4. *Every cellular poset X is canonically homeomorphic to the entrance path poset of its classifying stratification, that is, $X \cong \mathsf{Entr}\,\mathsf{CStr}\,X$.*

---
[4]The isomorphism is 'canonical up to homotopy'.

Proof. Statement (1) and (3) are discussed in [Bjö84, §3] (see also [LW69]). Statement (2) and (4) follow from the definitions. □

The correspondence of regular cell complexes (up to stratified homeomorphism) and cellular posets provides the *combinatorialization* of regular cell complexes by cellular posets, as promised earlier.

1.3.1.2   The category of combinatorial regular cell complexes Let us discuss how to extend the combinatorialization to include maps between regular cell complexes resp. between cellular poset. This well lead us to introduce a class of so-called 'cellular' maps.

Terminology 1.3.17 (Closure preservation for stratifications). A stratified map is said to be 'closure preserving' if it maps closures of strata onto closures of strata.   ⌐

Definition 1.3.18 (Cellular maps of regular cell complexes)**.** A **cellular map of regular cell complexes** $F : X \to Y$ is a map of regular cell complexes that is closure preserving; that is, for each open cell x in X we have $F(\overline{x}) = \overline{y}$, where $y = \mathsf{Entr}(F)(x)$ is the open cell into which x is mapped.   ⌐

Notation 1.3.19 (The category of regular cell complexes and cellular maps). Denote by CellStrat the category of regular cell complexes and their cellular maps.   ⌐

Let us next discuss cellular maps for cellular posets.

Terminology 1.3.20 (Closure preservation for posets). A map of posets $F : P \to Q$ is 'upper-closure preserving' if for each $x \in P$ there exists $y \in Q$ such that the image $FP^{\geq x}$ equals $Q^{\geq y}$.   ⌐

Definition 1.3.21 (Cellular maps of cellular posets)**.** A **cellular map of cellular posets** P and Q is a map of posets that is upper-closure preserving.   ⌐

Notation 1.3.22 (The category of cellular posets and cellular maps). Denote by CellPos the category of cellular posets and their cellular maps.   ⌐

Example 1.3.23 (Cellular and non-cellular maps)**.** In Fig. 1.27 we depict cellular and non-cellular maps of regular cell complexes as well of their entrance path posets, which are cellular posets. In each case we indicate the mapping by coloring images and preimages in the same color.   ⌐

The two definitions of cellular maps can be related by the following functors.

Observation 1.3.24 (Entrance path and classifying functors). The entrance path poset construction (see Remark 1.3.8), as well as the classifying stratification construction (see Terminology 1.3.11 and Terminology 1.3.12), yield functors

$$\mathsf{CellStrat} \underset{\mathsf{CStr}}{\overset{\mathsf{Entr}}{\rightleftarrows}} \mathsf{CellPos} \quad .$$

        ⌐

The central observation about cellular maps is the following.
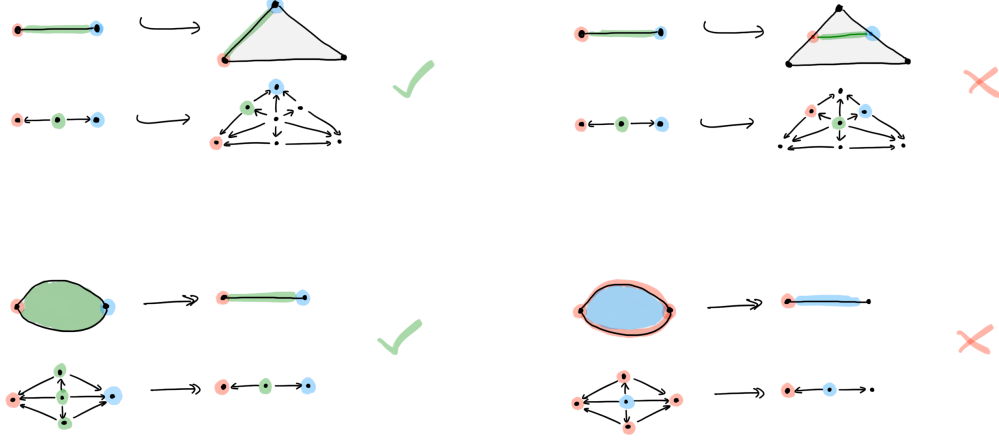
Figure 1.27: Cellular and non-cellular maps.

Remark 1.3.25 (The equivalence of regular cell complexes and cellular posets). For sufficiently nice spaces, the entrance path poset construction in fact yields a functor of *topological* categories Entr : $\mathcal{S}trat \to \mathcal{P}os$ (see Construction B.2.21). Denote the topological subcategory of $\mathcal{S}trat$ given by regular cell complexes and their cellular maps by $\mathcal{C}ellStrat$. Similarly, one can consider the subcategory of $\mathcal{P}os$ given by cellular posets and cellular maps: due to the cellularity condition on maps, this subcategory in fact has discrete hom spaces, recovering the 'ordinary' category CellPos. The entrance path poset functor now restricts to a functor Entr : $\mathcal{C}ellStrat \to$ CellPos. The functor provides a 'weak equivalence' of topologically enriched categories (the 'weak inverse' of Entr is given by the classifying stratification functor CStr). The category $\mathcal{C}ellStrat$ of regular cell complexes and their cellular maps is, in this sense, 'combinatorializable'.⌐

While we not prove the preceding remark (nor will we use it in any form), it motivates that CellPos is in fact a good model for the 'combinatorial' category of regular cell complexes. We record this as a definition.

Definition 1.3.26 (The combinatorial category of regular cell complexes)**.** The **'combinatorial' category of regular cell complexes** CellCplx is the category of cellular posets and their cellular maps.⌐

Note that the adjective 'combinatorial' is simply meant to highlight the combinatorial nature of the category's objects. We often refer to objects in CellCplx as 'combinatorial regular cell complexes', which emphasizes that objects can be interpreted both in combinatorial terms (as cellular posets) and a stratified-topological terms (as regular cell complexes obtained via the classifying stratification construction).

We will be particularly interested in 'combinatorial regular cells' as recorded by the following terminology. Recall, the 'depth' of an object x in a poset measures the length k of maximal chains $x_0 \to x_1 \to ... \to x_k$ in the poset starting at that object, $x = x_0$.

Terminology 1.3.27 (Regular cells). A cellular poset $X \in$ CellCplx is called a 'combinatorial regular k-cell' (or simply, a 'regular k-cell' if no confusion arises) if X has an initial object of depth k. We often denote initial elements of cells by $\bot$.⌐

Equivalently, in stratified topological terms, a regular cell complex X is a regular k-cell if it is the closure of a single k-dimensional cell in the complex.

———————————

We next show that the 'combinatorial' category of simplicial complexes SimpCplx fully faithfully embeds in the 'combinatorial' category of regular cell complexes CellCplx.

Observation 1.3.28 (Simplicial complexes embed in regular cell complexes)*. Given a simplicial complex K, its 'entrance path poset' Entr K is the poset whose objects of Entr K are simplices x in K with an arrow $x \to y$ whenever the simplex y is a face of the simplex x. For a simplicial map of unordered simplicial complexes $F : K \to L$, we obtain a map of entrance path posets Entr F : Entr $K \to$ Entr L mapping a simplex x in K to the simplex Fx in L. This yields the functor

$$\mathsf{Entr} : \mathsf{SimpCplx} \to \mathsf{CellCplx}$$

which, in fact, is a fully faithful embedding of categories. Note if we were to allow non-cellular poset maps in CellCplx the latter claim would fail to hold. ⌐

———————————

To end this section, let us briefly address the discrepancy between 'topology' and 'PL topology' which is, in fact, also visible at the level of 'combinatorial regular cell complexes'. For this, we introduce the following PL analog of the definition of cellular posets.

Definition 1.3.29 (PL cellular posets)**. A poset $(X, \leq)$ is called **PL cellular** if the realization $|X^{>x}|$ of the strict upper closure of any $x \in P$ is PL homeomorphic to a PL sphere. ⌐

Remark 1.3.30 (Cellular is not always PL cellular)*. Note that, while 'PL cellular' always implies 'cellular' the converse is in general not true. Indeed, there exist triangulations of the sphere which are not PL spheres (see [Edw80], [Bry02, Thm. 9.1]). Adjoining a new minimal element to the entrance path poset of such a triangulation yields a poset that is cellular but not PL cellular. In contrast, we will later on find that in the framed setting the adjectives 'cellular' and 'PL cellular' can be used interchangeably (see Remark 1.3.42). ⌐

Terminology 1.3.31 (The combinatorial category of regular cell complexes)*. The '(combinatorial) category of regular PL cell complexes' $\mathsf{CellCplx}^{\mathrm{PL}}$ is the category of cellular posets and their cellular maps. ⌐

Remark 1.3.32 (The discrepancy of PL and TOP and its resolution in the framed case)*. By the preceding remark, we have that $\mathsf{CellCplx}^{\mathrm{PL}} \subsetneq \mathsf{CellCplx}$. While our notation therefore needs to distinguish the PL from the topological case here, we will find that 'framed' analogs $\mathsf{FrCellCplx}_n^{\mathrm{PL}}$ and $\mathsf{FrCellCplx}_n$ of these categories are in fact the same category, and we shall therefore not distinguish them notationally (see Remark 1.3.42). ⌐

1.3.2 Framings on regular cells In this section we define framings on regular cells. This will combine our discussion of framings on simplices and simplicial complex, with the notion of 'combinatorial regular cell complexes', and their category CellCplx, as discussed in detail in the previous section. The following (abuse of) notation will reduce the amount of symbols needed in this and subsequent sections.

Notation 1.3.33 (Structures related to cellular posets). Consider a cellular poset $X \in$ CellCplx (i.e. a 'combinatorial regular cell complex').

1. The regular cell complex CStr X classifying X will, abusing notation, usually be referred to simply as the 'regular cell complex X'.

2. The simplicial complex NX^{un} obtained by unordering the nerve of X will, abusing notation, be referred to as the '(underlying) simplicial complex X'.

The abuse of notation similarly applies to maps of cellular posets $F : X \to Y$ (and thus to subposets $X \hookrightarrow Y$), which context-dependently may be used to denote maps of corresponding regular cell complexes and of underlying simplicial complexes. ⌐

In Fig. 1.28 we illustrate a cellular poset X, together with its corresponding regular cell complex X and its underlying simplicial complex X.
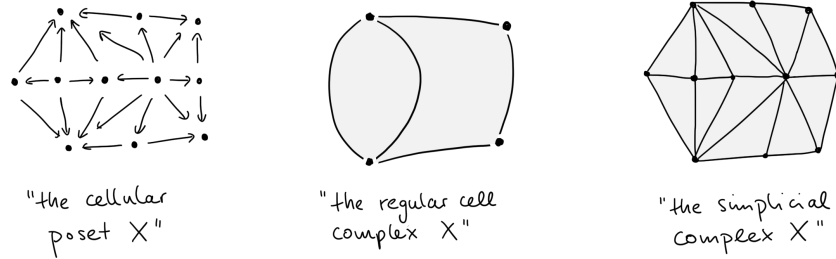
Figure 1.28: A cellular poset, together with its corresponding regular cell complex and underlying simplicial complex.

1.3.2.1   The definition of framed cells We now define framings on regular cells. Recall, a regular cell is a cellular poset with an initial element (see Terminology 1.3.27).

Definition 1.3.34 (Framed regular cells). An n**-framing** $\mathcal{F}$ **of a regular cell** X is an n-framing $\mathcal{F}$ of the simplicial complex X such that, for each $x \in X$, the framing restricts on the upper closure $X^{\geq x}$ of x to a flat framing $\mathcal{F}|_{X^{\geq x}}$ of the simplicial subcomplex $X^{\geq x} \hookrightarrow X$. ⌐

We will refer to the pair $(X, \mathcal{F})$, of a regular cell X together with an n-framing $\mathcal{F}$ on it, as an 'n-framed regular cell'.

Notation 1.3.35 (The ordered simplicial complex X). Given an n-framed regular cell $(X, \mathcal{F})$, note that the n-framing $\mathcal{F}$ endows the underlying simplicial complex X with an ordering: this yields, abusing notation, the 'ordered simplicial complex X' (which generally differs from the 'nerve of X'). ⌐

Notation 1.3.36 (Restricting framings to subcells). Given an n-framed regular cell $(X, \mathcal{F})$ and $x \in X$, we abbreviate the restricted n-framing $\mathcal{F}|_{X^{\geq x}}$ by $\mathcal{F}|_x$. ⌐

Example 1.3.37 (2-Framed regular cells). In Fig. 1.29 we illustrate several examples of 2-framed regular k-cell $(X, \mathcal{F})$, for $k = 1$ and 2: in each case we depict three pieces of data, namely, the regular cell complex X in the top row, the underlying simplicial complex X in the second row, and the ordered simplicial complex X together with the framing $\mathcal{F}$ (indicated by frame labels) in the third row. In the fourth row, we exploit the flatness of $\mathcal{F}$ to provide an alternative illustration of the third row by framed realizations in $\mathbb{R}^2$ (see Remark 1.2.25). ⌐
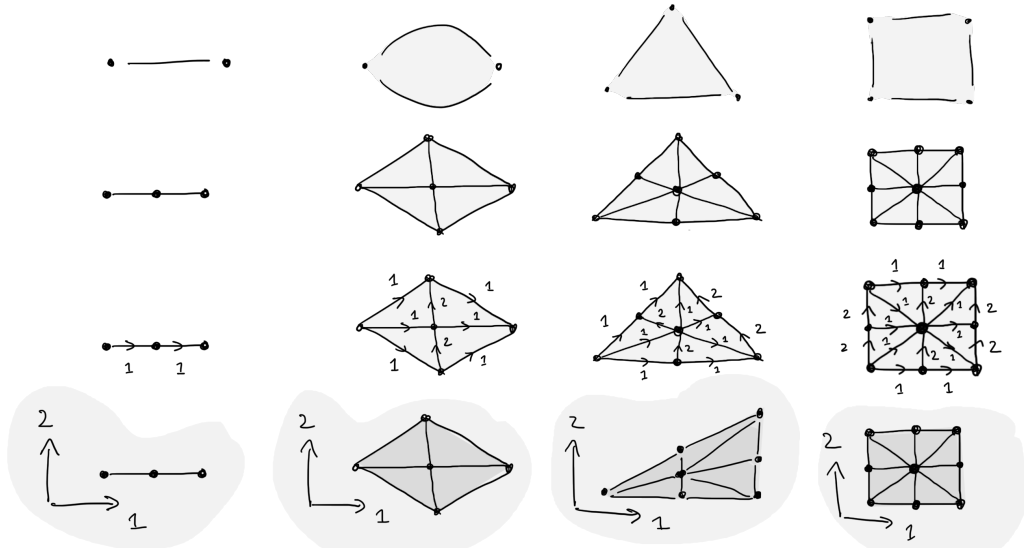
Figure 1.29: Data of 2-framed regular cells.

Usually, depictions of regular cell complexes X, of their underlying simplicial complexes, and of n-framings $\mathcal{F}$ on them, will be condensed into a single picture as the next remark explains.

Remark 1.3.38 (Condensing cell structure, simplicial structure and framings). We usually depict n-framed regular cells $(X, \mathcal{F})$ by embedding the regular cell X in $\mathbb{R}^n$: this, by passing to the underlying simplicial complex X and then deriving an n-framing from the resulting framed realization in $\mathbb{R}^n$, determines all data of $(X, \mathcal{F})$. We illustrate this with further examples in Fig. 1.30: note that the first row re-illustrates exactly the four examples given in Fig. 1.29. ⌐⌐
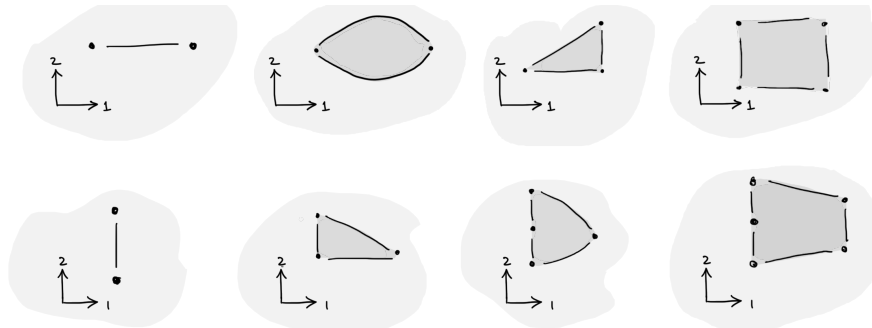
Figure 1.30: 2-Framed regular cells depicted via framed realizations.

Example 1.3.39 (Non-examples of framed regular cells). In Fig. 1.31 we illustrate three non-examples of n-framings on regular k-cells X: in each case, we separately depict the regular cell X (as a cell complex) on top, together with a framing $\mathcal{F}$ on X (as a simplicial complex) underneath it. In the leftmost example, the chosen framing fails to be flat. In the second example, the framing is flat, but fails to be flat when restricted to either of

the two 1-cells. Conversely, the third framing is flat when restricted to any subcell in the boundary but not on the 2-cell itself. ⌐
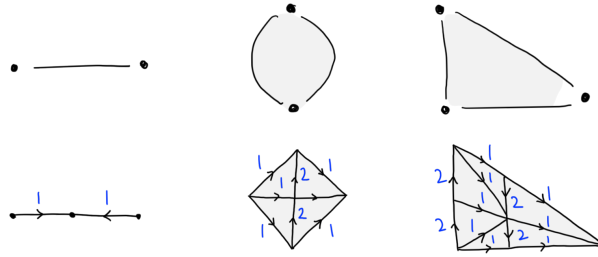
Figure 1.31: Non-examples of framed regular cells.

Example 1.3.40 (The simplest 3-framed regular 3-cell). In Fig. 1.32 on the left, we depict a framed regular 3-cell $(X, \mathcal{F})$. For clarity, we also separately depict the 3-framing $\mathcal{F}$ on the underlying simplicial complex X on the right. ⌐

Figure 1.32: The simplest 3-framed regular 3-cell.

The previous example is the simplest example of a 3-framed regular 3-cell (indeed, its underlying regular cell complex is the simplest regular cell complex of the 3-ball). In general, 3-framed regular 3-cells can be of various shapes as the next example illustrates.

Example 1.3.41 (More 3-framed regular 3-cells). We depict few more 3-framed regular 3-cells in Fig. 1.33. Note that cells in the boundary of these 3-cells yields various 3-framed k-cells for $k < 3$. ⌐

Having introduced the notion of framed regular cells, let us briefly revisit two central punchlines. Firstly, recall our earlier Remark 1.3.32 about the 'discrepancy of topological cells and PL cells'. In the framed setting the discrepancy disappears.

Remark 1.3.42 (Unity of framed cells and PL cells). Topological and PL cells coincide in the framed setting, that is, given a framed regular cell $(X, \mathcal{F})$ then the framing forces X to not only be a regular cell, but in fact a regular *PL* cell: this will be shown in Corollary 3.4.10. There is consequently no need to consider notions of 'framed cells' and 'framed regular PL cell' separately, as both these notions coincide. ⌐

Secondly, framed regular cells can (unlike nonframed regular cells) be classified.
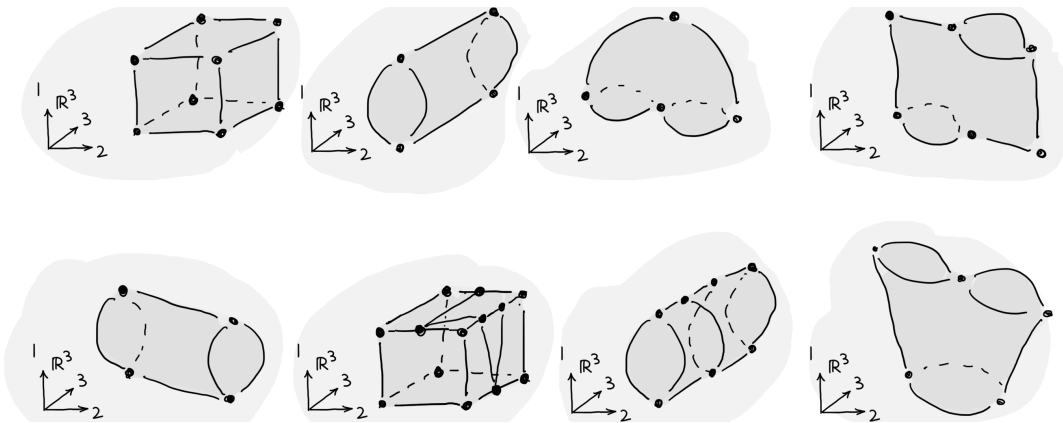
Figure 1.33: 3-framed regular 3-cells.

Remark 1.3.43 (Tractability of framed regular cells). We can recognize framed regular cell among framed posets (here, a 'framed poset' is simply a poset together with a framing of its underlying simplicial complex). We will show this by providing a 'constructive' classification of framed regular cells in Chapter 3. ⌐

In this way, framed regular cells provide a novel class of general shapes that is computationally tractable.

Remark 1.3.44 (Framed regular cells with degenerate faces). A yet more general, and still tractable class of shapes can be obtained by allowing faces of framed regular cells to 'degenerate'. While these shapes naturally appear when considering general presheafs on framed regular cells, we will not consider them further here. ⌐

1.3.2.2 Framed cellular maps We next define framed maps of framed regular cells. Recall, in the case of framed simplices, we defined framed maps as maps that preserve frames on each 'vector'; the setup crucially relied on vectors in m-simplices being generated by their m spine vectors (which provided a 'basis' for the affine space of the simplex). Unfortunately, there is a priori no good analog for 'spine vectors' in the context of regular cells in general; however, for *framed* regular cells, we may recover a notion of 'final frame vectors' for any given framed regular cell as explained below. This turns out to be just enough to define framed maps of framed regular cell (and indeed, by constructing final frame vectors inductively one can recover a sort of 'spine' for framed regular cells after all).

Terminology 1.3.45 (Final frame vectors of framed regular cells). Given an n-framed regular k-cell $(X, \mathcal{F})$ with initial element $\bot \in X$, a 'final frame vector' v is a 1-simplex in the ordered simplicial complex X containing $\bot$ as a vertex and whose frame label $\mathcal{F}_v \in \underline{n}$ is maximal among all frame labels of 1-simplices in X. ⌐

Notation 1.3.46 (Final frame vectors). Given an n-framed regular k-cell $(X, \mathcal{F})$ and an object $x \in X$, we denote by $\mathsf{vec}(x) \subset X$ the subcomplex of the ordered simplicial complex X spanned by the final frame vectors of the cell $(X^{\geq x}, \mathcal{F}|_x)$. ⌐

Example 1.3.47 (Final frame vectors of framed regular cells)**.** In Fig. 1.34, for several framed regular cells $(X, \mathcal{F})$ (which reproduce selected cells from both Fig. 1.30 and

Fig. 1.33), we depict the final frame vectors by red edges; in the first row, we also depict underneath each cell its corresponding ordered simplicial complex (but leave these complexes implicit in the second row). Note that, in each case, there are in fact exactly two final frame vectors: an edge that ends in the vertex $\perp$, and an edge that starts at the vertex $\perp$. We further indicate, by green and blue edges, the final frame vectors of a selection of subcells in the given cells boundary. ⌐
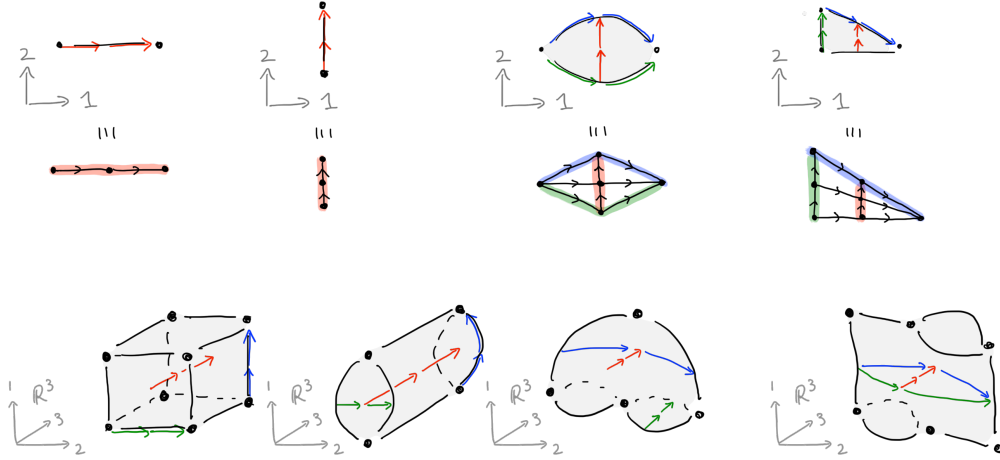
Figure 1.34: Final frame vectors of framed regular cells.

Remark 1.3.48 (Final frame vectors 'form a vector'). The observation in the preceding example generalizes: any framed cell $(X, \mathcal{F})$ has exactly two final frame vectors (if X is non-trivial, that is, $\dim(X) > 0$). The subcomplex $\mathsf{vec}(\perp)$ of the ordered simplicial complex X is isomorphic to the linear simplicial complex ($\bullet \rightarrowtail \bullet \rightarrowtail \bullet$), while the subposet $\mathsf{vec}(\perp)$ of the poset X (determined by the subcomplex $\mathsf{vec}(\perp) \subset X$) is isomorphic to the entrance path poset ($\bullet \leftarrow \bullet \rightarrow \bullet$) of a 1-simplex. In this sense the final frame vectors $\mathsf{vec}(\perp)$ of a framed regular cell may be regarded as forming a 'single directed 1-simplex'. We will revisit and prove this claim in Corollary 3.3.23. ⌐

The definition of framed maps of framed regular cells now mirrors the definition of framed maps of framed simplices (see Definition 1.1.59) as follows.

Definition 1.3.49 (Framed maps of framed regular cells)**.** Given n-framed regular cells $(X, \mathcal{F})$ and $(Y, \mathcal{G})$, a **framed cellular map** $F : (X, \mathcal{F}) \to (Y, \mathcal{G})$ is a cellular map of cellular posets $F : X \to Y$, such that for all $x \in X$, either F preserves the framing of final frame vectors $\mathsf{vec}\, x$, that is, F restricts to a framed simplicial isomorphism $(\mathsf{vec}\, x, \mathcal{F}|_{\mathsf{vec}\, x}) \cong (F(\mathsf{vec}\, x), \mathcal{G}|_{F(\mathsf{vec}\, x)})$, or the final frame vectors are degenerated by F, i.e. $F(\mathsf{vec}\, x) \subset Y$ is a point. ⌐

Example 1.3.50 (Framed cellular maps)**.** In the upper row of Fig. 1.35 we illustrate examples of framed maps of 2-framed regular cells; in each case we highlight image and preimage cells in the same color. Underneath each map we also depict the corresponding maps of simplicial complex as well as the orderings of those complexes. In the lower row of Fig. 1.35 we similarly depict non-examples of framed maps. While all three depicted maps are cellular they fail preserve frame vectors in the required sense; the first fails to

preserve final frame vectors of one of the two blue edges; the second fails to preserve final frame vectors of the blue 2-cells; the last fails to preserve the final frame vectors of the left red edge. ⌐
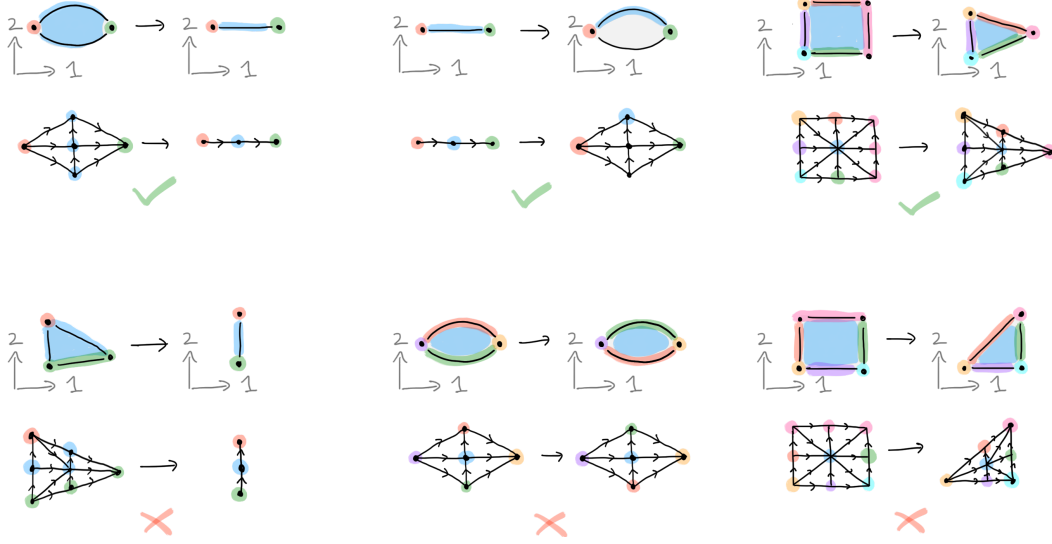
Figure 1.35: Examples and non-examples of framed cellular maps.

Remark 1.3.51 (Framed maps of cell are subframed on simplices). Note that in framed maps $F : (X, \mathcal{F}) \to (Y, \mathcal{G})$ of framed regular cells, need not descend to framed maps of framed simplicial complexes $F : (X, \mathcal{F}) \to (Y, \mathcal{G})$, but they do descend to *subframed* maps of framed simplicial complexes $F : (X, \mathcal{F}) \to (Y, \mathcal{G})$ (see Remark 1.2.23); that is, F need not preserve the simplicial ordering and may 'specialize' frame labels of vectors. This is illustrated in Fig. 1.36. ⌐

Figure 1.36: A framed cellular map need not induce a framed simplicial map, but does induce a subframed simplicial map.

Notation 1.3.52 (The category of framed regular cells). The category of n-framed regular cells and their framed cellular maps will be denoted by FrCell_n. ⌐

1.3.3   Framings on regular cell complexes We now define framings of regular cell complexes, generalizing the notions of framed regular cells introduced in the previous section. In fact, this generalization only requires minor modifications of previous definitions, replacing 'cells' by 'cell complexes' and requiring '(pro)framed maps' to be '(pro)framed maps on each cell'.

1.3.3.1   The definition of framings on cell complexes Framings of regular cell complexes are framings of their simplicial complexes in which 'all cells are flat'.

Definition 1.3.53 (Framings on cell complexes). Given a regular cell complex X, an *n*-**framing** $\mathcal{F}$ of X is an *n*-framing of the simplicial complex X such that, for all $x \in X$, the framing restricts on the upper closure $X^{\geq x}$ to a flat framing $\mathcal{F}|_{X^{\geq x}}$ of the simplicial complex $X^{\geq x}$. ⌐

We will refer to the pair $(X, \mathcal{F})$, of a regular cell complex X together with an *n*-framing $\mathcal{F}$ on it, as an '*n*-framed regular cell complex'. As before, we will denote framings restricted to subcells $X^{\geq x}$ by $\mathcal{F}|_x$ (see Notation 1.3.36).

Terminology 1.3.54 (Flat and locally flat framed regular cell complexes). An *n*-framed regular cell complex $(X, \mathcal{F})$ is called 'flat' (resp. 'locally flat') if the framing $\mathcal{F}$ of the simplicial complex X is flat (resp. 'locally flat'). ⌐

Example 1.3.55 (Non-flat framed regular cell complexes). Framed regular cell complexes can be thought of as 'gluings' of framed regular cells: we illustrate this in two instances in Fig. 1.37: in both cases we glue two 2-framed regular cells along their boundary is shown; the resulting frame structure of the cell complex is indicated by moving ambient frames 'inside' their respective cells. Note that in the first case, the resulting cell complex realizes to a contractible space, while in the second case it realizes to the 2-sphere. Neither of the framed regular cell complexes is complexes flat. ⌐
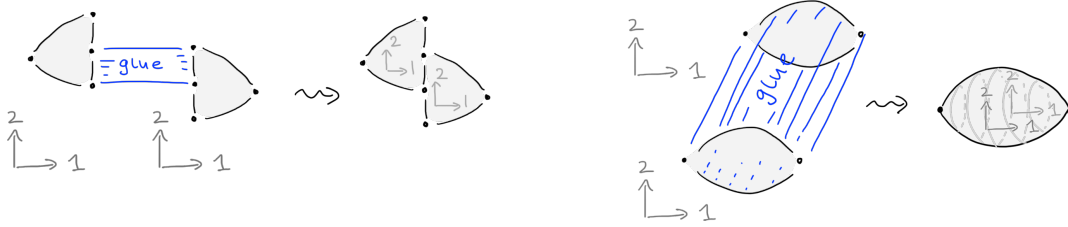
Figure 1.37: Framed regular cell complexes obtained by gluing framed regular cells.

Example 1.3.56 (Locally flat cellulations of 1-manifolds). In Fig. 1.38 on the left we depict several locally flat 1-framed regular cell complexes $(X, \mathcal{F})$ each cellulating a connected 1-manifold. On the right we depict two 1-framed regular cell complexes that are not locally flat framed. ⌐

Figure 1.38: Locally flat and non-locally flat 1-framed cellulated 1-manifolds.

Example 1.3.57 (Flat cellulation of the Hopf circle). We motivate a slightly more complicated example. Consider the 'Hopf circle' embedding of the circle S^1 into $\mathbb{R}^3$, whose image projects along $\pi_3 : \mathbb{R}^3 \to \mathbb{R}^2$ to the figure eight, as shown in Fig. 1.39 below. Take any compact 3-cube $I^3 \subset \mathbb{R}^3$ containing the image of the Hopf circle embedding
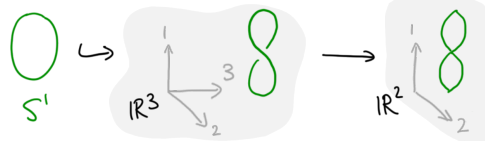
Figure 1.39: The Hopf circle embedded in $\mathbb{R}^3$.

$S^1 \hookrightarrow \mathbb{R}^3$. There exist a 'canonical framed cellulation' of I^3 by a flat 3-framed regular cell complex containing the circle embedding in its 1-skeleton (here, the term 'cellulation' is the cellular analog to the notion of 'triangulation' by simplices). This complex is shown in Fig. 1.40: it consists of eight 3-framed 3-cells glued together as indicated (note all framings are determined by the same flat ambient 3-frame, which is indicated only once); we highlighted 0- and 1-cells cellulating the Hopf circle in blue. We will formally show
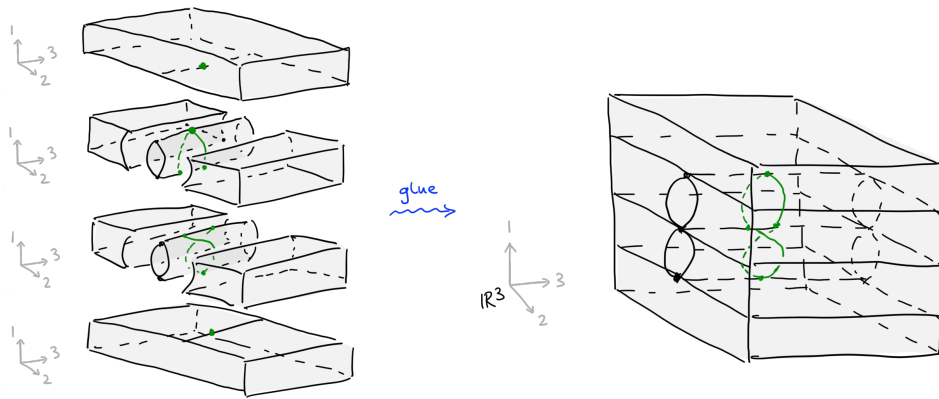
Figure 1.40: Cellulation the Hopf circle embedding into eight 3-framed 3-cells.

that this complex is canonical in the sense that any other cellulation by framed regular cells must subdivide it—the precise statement for the case of the Hopf circle embedding will be revisited in Remark 5.2.27.

Example 1.3.58 (Dualizing framed cell complexes)**.** In Fig. 1.41 we depict another flat 3-framed regular cell complex made up of four framed 3-cell: this is the 'geometric dual' to the framed cell complex given in the previous example. The two complexes can be related
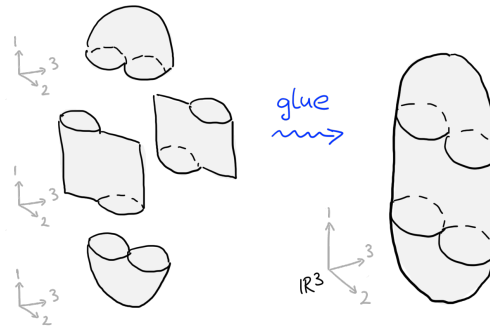
Figure 1.41: Dualizing the cellulation of the Hopf circle.

by a 'dualization operation' which switches the roles of dimension and codimension for each cell—we will reconsider this example more formally in Remark 4.2.87. ⌐

_____

We next define framed maps of framed regular cell complexes; the definition mirrors the definition of framed simplicial maps (see Definition 1.2.20).

Definition 1.3.59 (Framed maps of framed regular cell complexes). Consider n-framed regular cell complexes $(X, \mathcal{F})$ and $(Y, \mathcal{G})$. A **framed cellular map** $F : (X, \mathcal{F}) \to (Y, \mathcal{G})$ is a cellular map $F : X \to Y$ that restricts on all cells $x \in X$ to a framed cellular map $F : (X^{\geq x}, \mathcal{F}|_x) \to (Y^{\geq Fx}, \mathcal{G}|_{Fx})$. ⌐

We often refer to 'framed cellular maps' simply as 'framed maps'.

Notation 1.3.60 (The category of framed regular cell complexes). The category of n-framed regular cell complexes and their framed maps will be denoted by $\mathsf{FrCellCplx}_n$. The full subcategory of n-framed regular cell complexes that are flat will be denoted by $\mathsf{FlatFrCellCplx}_n$. ⌐

1.3.3.2   Comparison to framed simplicial complexes   Finally, let us compare framed regular cell complexes to framed simplicial complexes. Recall, we set out to construct a fully faithful embedding $\mathsf{FrSimpCplx}_n \hookrightarrow \mathsf{FrCellCplx}_n$ which descends to the ordinary (nonframed) embedding $\mathsf{SimpCplx} \hookrightarrow \mathsf{CellCplx}$ via functors that forget framings. The desired embedding will be given by 'framed' version of the entrance path poset construction, which we outline as follows.

Construction 1.3.61 (Framed entrance path posets). Given an n-embedded framed simplicial complex $(K, \mathcal{F})$, we construct a framed regular cell complex $(\mathsf{Entr}\, K, \mathsf{Entr}^{\mathsf{fr}}\, \mathcal{F})$, referred to as the 'framed entrance path poset' of $(K, \mathcal{F})$, where $\mathsf{Entr}\, K$ is the entrance path poset of K and $\mathsf{Entr}^{\mathsf{fr}}\, \mathcal{F}$ is an n-framing inductively constructed as follows.

First assume $K = S$ is an m-simplex. For $m > 0$, given n-embedded framed m-simplex $(S \cong [m], \mathcal{F})$, consider any framed realization $r : |S| \hookrightarrow \mathbb{R}^n$. Then $\mathsf{Entr}^{\mathsf{fr}}\, \mathcal{F}$ is the unique framing of $\mathsf{Entr}\, K$ such that there exists a framed realization $e : |\mathsf{Entr}\, K| \hookrightarrow \mathbb{R}^n$ which factors through r by a map $|\mathsf{Entr}\, K| \to |K|$ that realizes the barycentric subdivision of K.

Next assume K is any simplicial complex. Then is the framing that, on each simplex $x : [m] \hookrightarrow K$, restricts on $\mathsf{Entr}\, x : \mathsf{Entr}[m] \hookrightarrow \mathsf{Entr}\, K$ to the framing $(\mathsf{Entr}^{\mathsf{fr}}\, \mathcal{F})\big|_{\mathsf{Entr}\, x} = \mathsf{Entr}^{\mathsf{fr}}\, (\mathcal{F}|_x)$. ⌐

Construction 1.3.62 (The framed entrance path poset functor). The **framed entrance path poset functor**

$$\mathsf{Entr}^{\mathsf{fr}} : \mathsf{FrSimpCplx}_n \to \mathsf{FrCellCplx}_n$$

takes n-embedded framed simplicial complex $(K, \mathcal{F})$ to their framed entrance path posets $(\mathsf{Entr}\, K, \mathsf{Entr}^{\mathsf{fr}}\, \mathcal{F})$, and framed simplicial maps $F : (K, \mathcal{F}) \to (L, \mathcal{G})$ to the framed cellular map determined by the cellular map $\mathsf{Entr}\, F : \mathsf{Entr}\, K \to \mathsf{Entr}\, L$. ⌐

We omit a detailed verification of Construction 1.3.61 and Construction 1.3.62. The punchline is recorded in the following observation.

Observation 1.3.63 (Framed simplicial complexes are framed regular cell complexes). The functor $\mathsf{Entr}^{\mathsf{fr}} : \mathsf{FrSimpCplx}_n \to \mathsf{FrCellCplx}_n$ is a fully faithful embedding of categories. ⌐⌐

In this sense, we may think of framed simplicial complexes as a special case of framed regular cell complexes, and the of latter notion as being a generalization of the former notion.

Constructible framed combinatorics: trusses

In this chapter we develop the basic theory of 'trusses': trusses are combinatorial structures that can be obtained by iteratively building towers of 'constructible combinatorial bundles'. Trusses both underlie the tractable classification of framed regular cell complexes (as discussed in Chapter 3) while also providing a link between framed combinatorics and framed topology (as discussed in Chapter 4 and Chapter 5). Several ideas in the theory of trusses conceptually root in, and improve upon, notions developed in [Dor18].

We will start in Section 2.1 by introducing 1-dimensional trusses, or '1-trusses' as well as their 'bordism' and 'bundles'. Subsequently, in Section 2.2, we describe an important ordering property of bundles of 1-trusses, which we will refer to as 'truss induction'. This in turn will allow us to describe the theory of trusses in general dimension n as discussed in Section 2.3.

2.1 1-Trusses, 1-truss bordisms, and 1-truss bundles

1-Trusses are '1-dimensional' combinatorial objects. As we will illustrate, their 1-dimensionality has a concrete geometric interpretation: 1-trusses can be obtained by taking the '*framed* entrance path posets' of '*framed* stratified intervals'. Here, the notion of 'framed' entrance path poset generalizes the usual notion of entrance path posets of stratifications to the case of (appropriately) 'framed' stratifications. While both these notions will be formalized only later in Chapter 4 in the context of 'constructible framed topology', thinking of 1-trusses as modelling stratified intervals will be nonetheless useful as a guiding intuition, and we will illustrate this intuition with several examples throughout this section.

We outline the section. In Section 2.1.1 we introduce the category of 1-trusses and their maps. We will discuss several (often geometrically motivated) notions and constructions for 1-trusses. In particular, we define an involutive 'dualization' functor on 1-trusses. In geometric terms, this will correspond to dualizing the dimension of each stratum in a stratification. In Section 2.1.2 we then introduce a notion of 1-truss

bordisms. In geometric terms, the notion models the behavior of 'stratified intervals in $\mathbb{R}$-parametrized families'. Understanding 1-truss bordisms, will in turn allow us to define 1-trusses bundles in Section 2.1.3, which then provide a combinatorial model of general 'constructible stratified line bundles'. 1-Truss bundles play a central role: by iterating such combinatorial constructible stratified line bundles we will later build combinatorial models of 'n-dimensional' stratified space, yielding the notion of 'n-trusses'.

2.1.1   1-Trusses

2.1.1.1   1-Trusses as framed fences Recall the classical combinatorial notion of fences (1-trusses will be a strengthening of this notion). The definition of fences finds a concise expression via the notion of a classifying space of a category: recall this is the space realizing the simplicial set given by the nerve of a category.

Definition 2.1.1 (Fence)**.** A **fence** X is a category of countable size, whose classifying space is a connected k-manifold where $k \in \{0, 1\}$.
1. If F realizes to a point we say F is **trivial**.
2. If F realizes to an interval we say F is **linear**.
3. If F realizes to a circle we say F is **circular**.   ⌐

We call a fence X a 'finite fence' if it is finite as a category. For convenience, we often extend both the class of linear and circular fences to include the trivial fence as well (and speak of the 'trivial linear' resp. 'trivial circular' fence in this case). Note that a fence cannot contain composable non-identity morphisms.

Example 2.1.2 (Fences)**.** Finite fences of different types are shown in Fig. 2.1; in each case we indicate a fence X by its classifying space $|X|$ (where directions of morphisms in X are recorded by directing edges of $|X|$).   ⌐

trivial   Linear   circular
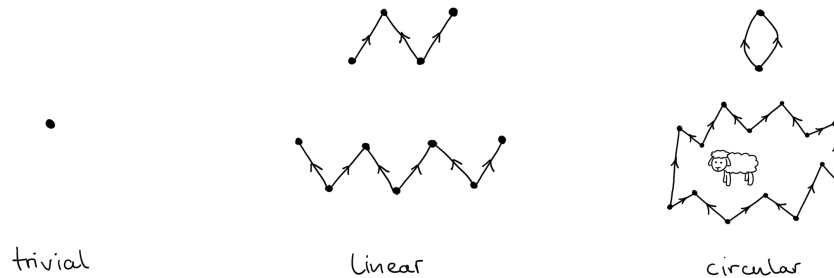
Figure 2.1: Fences of different types.

Observation 2.1.3 (Regular cell complexes). Given a finite fence X, then the classifying space $|X|$ of X naturally has the structure of a regular cell complex (whose 0-cells realize objects of X, and whose 1-cells realize morphisms of X).   ⌐

Observation 2.1.4 (Height maps). Every non-trivial fence X comes equipped with 'height map' $X \to [1]$ given by the unique functor whose preimages are discrete categories (the map 'folds the fence onto a single fence post'). Note that for trivial fences, the choice of height map $X \to [1]$ is ambiguous!   ⌐

Conversely, regular cell complexes with height maps yield fences as follows.

Observation 2.1.5 (Non-trivial fences are regular cell complexes with height map). Any regular cell complex X cellulating a connected 1-manifold, together with a cellwise bijective map $X \to |[1]|$, determines a non-trivial fence whose classifying space is X and whose height map realizes to $X \to |[1]|$. ⌐

The notion of 1-trusses strengthens that of fences in two ways: firstly, 1-trusses contain the data of a 'framing' of the 1-manifolds that non-trivial fences cellulate (technically, this uses the combinatorial notion of locally flat 1-framings of regular cell complexes, see Definition 1.3.53 and Example 1.3.56); secondly, 1-trusses resolve the ambiguity of height maps for trivial fences by introducing them as additional structure (inverting variance, this now will be recorded by a map to $[1]^{\mathrm{op}}$, which we call the 'dimension map').

Definition 2.1.6 (General 1-trusses). A **1-truss** $(T, \dim, \mathcal{F})$ is a finite fence T, together with a poset map $\dim : T \to [1]^{\mathrm{op}}$ as well as a locally flat 1-framing $\mathcal{F}$ of the corresponding regular cell complex. A 1-truss is **trivial** resp. **linear** resp. **circular** whenever its underlying fence is. ⌐

Example 2.1.7 (General 1-trusses). 1-Trusses of different types are shown in Fig. 2.2: in each case we depict the underlying fence as before, we depict the dimension map by coloring preimages of 0 in red, and preimages of 1 in blue, and we depict the locally flat 1-framing by indicating the 1-framing of each cell by a purple frame vector. ⌐
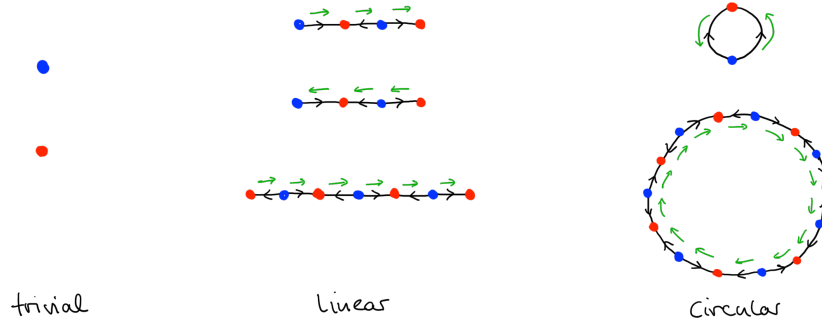
Figure 2.2: 1-Trusses of different types.

Going forward, we now make the following simplification: we will henceforth focus solely on the case of linear trusses. While much of the theory developed here, including notions of 'higher-dimensional' n-trusses, does generalize to the case of general 1-trusses as well, our main interest and applications will ultimately lie with the linear case. We therefore adopt the following convention.

Convention 2.1.8 (Linear 1-truss by default). We will use the term '1-truss' to refer to 'linear 1-trusses' unless otherwise noted. ⌐

In the linear case, we may further reformulate the above definition in purely order-theoretic terms. First, note that if a fence T is linear (which we take to include the trivial case), then it must be a poset.[5] We usually denote the resulting poset order of linear fences T either by $(T, \to)$ or $(T, \trianglelefteq)$ (the latter notation being convenient for expressing

---

[5] Every preorder (resp. poset) $(X, \leq)$ is equivalently a category with object set X and a single morphism $x \to y$ whenever $x \leq y$; and every map of preorders is a functor of corresponding categories.

'strictly less/greater' relations using the symbols '$\lhd/\rhd$'). Secondly, a locally flat 1-framing $\mathcal{F}$ of a linear fence T may be equivalently recorded by a total order, which we usually denote by $\preceq$. This leads us to the following definition of 1-trusses (i.e. linear 1-trusses).[6]

Definition 2.1.9 (1-Trusses)**.** A (linear) **1-truss** $(T, \trianglelefteq, \dim, \preceq)$ is a finite non-empty set T together with the following structure.

1. a partial order $\trianglelefteq$ called the 'face order' of T,
2. a poset map $\dim : (T, \trianglelefteq) \to [1]^{\mathrm{op}}$ called the 'dimension map' of T,
3. a total order $(T, \preceq)$ called the 'frame order' of T, in which a succeeds b or b succeeds a iff either $a \lhd b$ or $b \rhd a$. ⌐

Notation 2.1.10 (1-Trusses)*.* When working with 1-trusses, we will usually keep face orders, dimension maps as well as frame orders implicit; that is, we will abbreviate 1-trusses $(T, \trianglelefteq, \dim, \preceq)$ simply by T. ⌐

The fact that we refer to $\trianglelefteq$ as the 'face order' of T reflects the relation of 1-trusses and (framed) stratified intervals that we alluded to earlier: under this relation, elements a of 1-trusses translate to open 1-cells of dimension $\dim(a)$, and arrows $a \lhd b$ translate to 'b being a face of a'. The next example illustrates this 'geometric translation' of 1-trusses.[7]

Example 2.1.11 (Translating 1-trusses to stratified intervals)**.** In the upper row of Fig. 2.3 we depict two 1-trusses T. In each case, we depict the face order by black arrows, we color preimages of 1 of the dimension map in blue and preimages of 0 in red, and indicate the frame order by a purple coordinate axis. Underneath each 1-truss, we depict a stratified interval: note that each object $x \in T$ corresponds to a stratum of dimension $\dim(x)$, and the face order $(T, \trianglelefteq)$ can be recovered as the entrance path poset of the stratified interval (see Definition B.1.6 for a definition of 'entrance path posets'). ⌐

Figure 2.3: 1-Trusses and their translation to stratified intervals.

Terminology 2.1.12 (Singular and regular objects)*.* An element $p \in T$ of a 1-truss T is called 'singular' if $\dim(p) = 0$ and 'regular' if $\dim(p) = 1$. We denote the subset of singular objects T by $\mathsf{sing}(T)$ resp. the subset of regular objects by $\mathsf{reg}(T)$. ⌐

Remark 2.1.13 (Induced orders on singular/regular objects)*.* Note that, endowed with the face order both $(\mathsf{sing}(T), \trianglelefteq)$ and $(\mathsf{reg}(T), \trianglelefteq)$ are discrete orders, and endowed instead with the frame order both $(\mathsf{sing}(T), \preceq)$ and $(\mathsf{reg}(T), \preceq)$ are total orders. ⌐

2.1.1.2 Maps of 1-trusses We next define maps between 1-trusses. In order to simultaneously keep track of face and frame orders in the definition of 1-trusses, the following abstraction will be useful.

---

[6]Note, in particular, that this definition is then no longer dependent on any notions from Chapter 1.
[7]Note that the resulting translation of 1-trusses to stratified intervals is fundamentally different to our earlier translation of fences to regular cell complexes.

Definition 2.1.14 (Diposets and their maps). A **diposet** $(X, \trianglelefteq, \preceq)$ is a set X with two orders $\trianglelefteq$ and $\preceq$. A **diposet map** $F : (X, \trianglelefteq, \preceq) \to (Y, \trianglelefteq, \preceq)$ is a map of sets $F : X \to Y$ that separately respects both orders, i.e. induces poset maps $F : (X, \trianglelefteq) \to (Y, \trianglelefteq)$ and $F : (X, \preceq) \to (Y, \preceq)$. ⌐

Definition 2.1.15 (1-Truss maps). A **map of 1-trusses** $T \to S$ is a diposet map $(T, \trianglelefteq, \preceq) \to (S, \trianglelefteq, \preceq)$. ⌐

Note that the preceding definition of 1-truss maps does not impose any conditions on how maps interact with the dimension data of trusses; there are several sensible ways to impose such conditions.

Definition 2.1.16 (Regular, singular, and balanced maps). Let $F : T \to S$ be a map of 1-trusses.
1. We call F **singular** if it maps singular objects of T to singular objects of S. That is, for all $x \in T$, $\dim(x) \geq \dim(Fx)$.
2. We call F **regular** if it maps regular objects of T to regular objects of S. That is, for all $x \in T$, $\dim(x) \leq \dim(Fx)$.
3. We call F **balanced** if it is both regular and singular. That is, for all $x \in T$, $\dim(x) = \dim(Fx)$. ⌐

Example 2.1.17 (Maps of 1-trusses). In Fig. 2.4 we depict examples of singular, regular and balanced maps of 1-trusses. ⌐
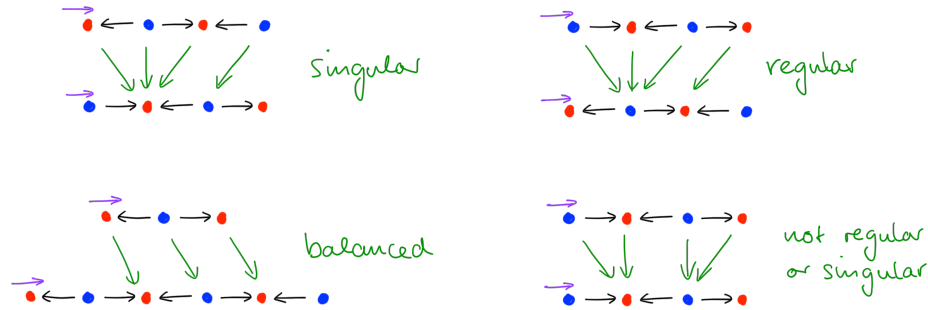
Figure 2.4: Four different types of maps of 1-trusses.

Notation 2.1.18 (Category of 1-trusses). The category of 1-trusses and their maps is denoted by Trs_1. The wide subcategory containing only regular respectively singular respectively balanced maps will be denoted by $\mathsf{Trs}_1^{\mathsf{r}}$ respectively $\mathsf{Trs}_1^{\mathsf{s}}$ respectively $\mathsf{Trs}_1^{\mathsf{rs}}$. ⌐

To avoid identifying the trivial truss T with $\dim(a) = 0$ (where $a \in T$ is the single element of T) with the trivial truss T with $\dim(a) = 1$ we will default to the following notion of isomorphism.

Terminology 2.1.19 (Balanced isomorphism by default). Unless otherwise noted, the term 'isomorphism of 1-trusses' will by refer to isomorphisms in the category $\mathsf{Trs}_1^{\mathsf{rs}}$, that is, to balanced bijective truss maps. ⌐

Remark 2.1.20 (Balanced isomorphisms are unique). Observe that if two trusses are balanced isomorphic, then the balanced isomorphism must be unique. There is thus no

harm in thinking about 1-trusses 'skeletally', i.e. in terms of their balanced isomorphism classes.[8] ⌐

We now distinguish balanced isomorphism classes by their 'endpoint type'.

Definition 2.1.21 (Endpoints of 1-trusses)**.** Given a truss $(T, \trianglelefteq, \dim, \preceq)$ we denote by $\mathsf{end}_- T$ the minimal element of $(T, \preceq)$, called the **lower endpoint**, and by $\mathsf{end}_+ T$ the maximal element, called the **upper endpoint** of T. ⌐

Terminology 2.1.22 (Endpoint types of 1-trusses)*.* Let T be a 1-truss. Then T falls in exactly one of the following six cases of isomorphism classes in $\mathsf{Trs}_1^{\mathsf{rs}}$.
1. If T has a single regular element, then T is the **trivial open** 1-truss denoted by $\bar{\bar{\mathbb{T}}}_0$.
2. If T has a single singular element, then T is the **trivial closed** 1-truss denoted by $\mathring{\mathring{\mathbb{T}}}_0$.

For the next cases, T has more than one element.
3. If both endpoints $\mathsf{end}_\pm T$ of T are regular, then T is **open**. If T has $2k+1$ elements it is denoted by $\mathring{\mathring{\mathbb{T}}}_k$.
4. If both endpoints $\mathsf{end}_\pm T$ of T are singular, then T is **closed**. If T has $2k+1$ elements it is denoted by $\bar{\bar{\mathbb{T}}}_k$.
5. If $\mathsf{end}_- T$ is regular and $\mathsf{end}_+ T$ singular, then T is **half-open half-closed**. If T has $2k$ elements it is denoted by $\mathring{\bar{\mathbb{T}}}_k$.
6. If $\mathsf{end}_- T$ is singular and $\mathsf{end}_+ T$ regular, then T is **half-closed half-open**. If T has $2k$ elements it is denoted by $\bar{\mathring{\mathbb{T}}}_k$. ⌐

Example 2.1.23 (Types of 1-trusses)**.** In Fig. 2.5 we illustrate each type of 1-truss with an example. ⌐

Figure 2.5: Types of 1-trusses.

We will be mainly be interested in the case of closed (and, dually, open) trusses, because of their relation to framed cellular geometry. We therefore introduce the following notation for their corresponding categories.

Notation 2.1.24 (Open and closed 1-trusses)*.* The subcategory of Trs_1 containing only open trusses (including the trivial open truss) and regular maps will be denoted by $\mathring{\mathsf{Trs}}_1$, and the subcategory containing only closed trusses (including the trivial closed truss) and singular maps by $\bar{\mathsf{Trs}}_1$. ⌐

---

[8]In fact, note that (not necessarily balanced) isomorphisms between any two trusses are unique as well if they exists; the reason for working with balanced isomorphism classes roots in their relation to '1-truss bordism isomorphism classes', as discussed later in Remark 2.1.47.

Observation 2.1.25 (Maps that are balanced). Note that any regular map of open trusses is balanced and, likewise, any singular map of closed trusses is balanced. In other words, the categories $\mathring{\mathsf{Trs}}_1$ and $\mathring{\mathsf{Trs}}_1^{\mathsf{rs}}$ (resp. $\bar{\mathsf{Trs}}_1$ and $\bar{\mathsf{Trs}}_1^{\mathsf{rs}}$) are identical. ⌟

2.1.1.3 Dualization of 1-trusses Trusses admit a natural dualization operation, which maps closed trusses to open trusses and vice-versa.

Construction 2.1.26 (Dualization of 1-trusses). The **dualization functor**

$$\dagger : \mathsf{Trs}_1 \cong \mathsf{Trs}_1$$

is an involutive functor defined as follows. Given a 1-truss $T \equiv (T, \trianglelefteq, \dim, \preceq)$ its dual is the 1-truss $T^\dagger \equiv (T, \trianglelefteq^{\mathrm{op}}, \dim^{\mathrm{op}}, \preceq)$: that is, face order of $T^\dagger$ is opposite to that of T; its dimension map is the opposite of the dimension map of T (post-composed with the identification $[1] \cong [1]^{\mathrm{op}}$); its frame order is equal to that of T. Similarly, the dual map $F : T^\dagger \to S^\dagger$ of a 1-truss map $F : T \to S$ is the map that equals F as a map of underlying sets. ⌟

Example 2.1.27 (Dualization). In Fig. 2.5, 1-trusses in the left column dualize to 1-trusses shown in the right column; while in Fig. 2.4 the depicted regular map dualizes to the depicted singular map. ⌟

Observation 2.1.28 (Dualization of dimension). Given a 1-truss T and an element $p \in T$, then p is regular (resp. singular) in T if and only if p is singular (resp. regular) in $T^\dagger$. In particular, if $F : T \to S$ is a map of trusses, then $F^\dagger$ is regular (resp. singular) if and only if $F^\dagger$ is singular (resp. regular). ⌟

Observation 2.1.29 (Dualization of boundary types). The preceding observation entails that the dualization functor restricts to isomorphisms $\mathring{\mathsf{Trs}}_1 \cong \bar{\mathsf{Trs}}_1$. ⌟

2.1.2 1-Truss bordisms We now introduce 1-truss bordisms. While 1-trusses geometrically translate into (framed) stratified intervals as previously illustrated, 1-truss bordisms model changes that can occur in continuous families of such intervals: these families can be formalized in terms of (framed constructible) stratified bundles[9] over the stratified 1-simplex $\{1\} \subset [0, 1]$, which is illustrated in two instances in Fig. 2.6. In the first example, the generic fiber is a stratified open interval with two point strata; when reaching the special fiber, a third point stratum spontaneously appears. In the second example, the two point strata in the generic fiber converge into a single point stratum in the special fiber. In both cases, we also indicate the corresponding 'poset bundle' obtained by passing to entrance path posets. The notion of 1-truss bordisms will provide a unified combinatorial description of the 'deformations between fibers' that may appear in such entrance path poset bundles (while appropriately keeping track of framings and dimension data of 1-trusses). The description of these fiber deformations plays a most central role: it will shortly enable us to define 1-truss bundles, which in turn, 'by iteration', will later on lead us to the definition of n-truss bundles as towers of 1-truss bundles. Note

---

[9]Recall, stratified bundles generalize fiber bundles in that they allow (stratified) fibers to change when passing between strata in the base, see Definition B.2.24. The precise 'framed constructible' variation alluded to here is introduced in Definition 4.1.20 under the name of '1-mesh bundles'.
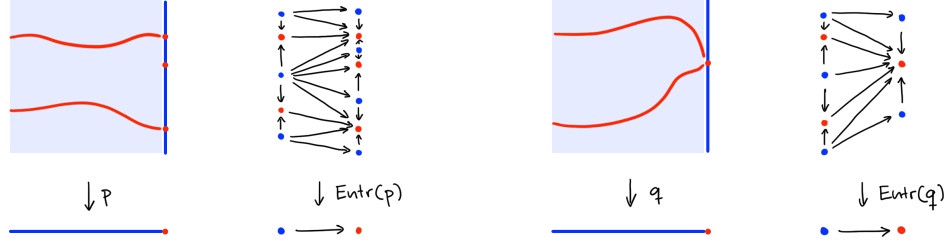
Figure 2.6: Continuous families of open stratified intervals as stratified bundles over the stratified 1-simplex.

that our chosen terminology hints at a relation of 'truss bordisms' to 'classical *stratified bordisms*'.[10]

2.1.2.1 Recollection on Boolean profunctors To streamline the definition of 1-truss bordisms it is convenient to encode their general structure in terms of Boolean profunctors between preorders. Recall, the category Bool of booleans has two objects 'true' (also written '1') and 'false' (also written '0'), with a single non-identity arrow from 'false' to 'true'. It will be useful to think of preorders as Bool-enriched categories (which translate back to categories via the fully faithful inclusion Bool $\hookrightarrow$ Set).

Definition 2.1.30 (Boolean profunctors and (co)functoriality)**.** Given two preorders X and Y, a **boolean profunctor** $R : X \nrightarrow Y$ is a functor $R : X^{\mathrm{op}} \times Y \to$ Bool. R is called **functorial** if it is of the form $\mathrm{Hom}_Y(f-,-)$ (for a functor $f : X \to Y$) and **cofunctorial** if it is of the form $\mathrm{Hom}_X(-, f-)$ (for a functor $f : Y \to X$).[11] ⌐

Remark 2.1.31 (Composition of boolean profunctors). Given Boolean profunctors $R : X \nrightarrow Y$ and $S : Y \nrightarrow Z$, then their composition $S \circ R : X \nrightarrow Z$ is determined by setting $Z(x \in X, z \in Z)$ to be true if and only if there is an element $y \in Y$ so that both $R(x, y)$ and $S(y, z)$ are true. ⌐

Notation 2.1.32 (Category of boolean profunctors). Preorders and their boolean profunctors form a category denoted by BoolProf. ⌐

Definition 2.1.33 (Underlying relations of boolean profunctors)**.** The **underlying relation functor** rel : BoolProf $\to$ Rel takes preorders to their object sets, and Boolean profunctors $R : X \nrightarrow Y$ to the relation $R^{-1}(1) \subset X \times Y$. ⌐

Remark 2.1.34 (Boolean profunctors are 'functorial relations'). Given preorder X and Y, a Boolean profunctor R between them may be completely encoded by its underlying relation rel $R \subset X \times Y$; in other words, the functor rel is faithful. However, the functor is not full: the relations $R \subset X \times Y$ in the image of rel are exactly the 'functorial relations', namely those for which $\{a \to a', R(a', b)\} \Rightarrow R(a, b)$ and $\{b' \to b, R(a, b')\} \Rightarrow R(a, b)$. ⌐

---

[10]Most visually, this becomes apparent with later examples of (higher-dimensional) trusses underlying stratifications such as those in Fig. 5.17.

[11]Frequently, (co)functorial profunctors are also called '(co)representable profunctors'; however, we find this terminology badly chosen since the notion does not relate to an actual representability condition. Moreover, the terminology does not 'decategorify' well to the case of relations, where one often speaks of '(co)functional' relations as mentioned in Remark 2.1.35.

Remark 2.1.35 (Boolean profunctors of discrete preorders and (co)functionality). A boolean profunctor $R : X \nrightarrow Y$ between discrete preorders is a relation between the corresponding sets in the ordinary sense. In this case, '(co)functorial profunctors' are usually called 'entire (co)functional relations'—keeping 'entireness' implicit, we will simply speak of (co)functional relations. ⌐

Terminology 2.1.36 (Non-emptiness). We say a Boolean profunctor $R : X \nrightarrow Y$ is **non-empty** if there is a choice of $a \in X$, $b \in Y$ for which $R(a, b)$ holds. ⌐

Terminology 2.1.37 (Restrictions of profunctors). Given a boolean profunctors $R : X \nrightarrow Y$ and subpreorders $U \hookrightarrow X$, $V \hookrightarrow Y$, then R restricts to boolean profunctor between U and V which, abusing notation, we denote again by $R : U \nrightarrow V$. ⌐

A final helpful notion for the definition of 1-truss bordisms is that of 'bimonotonicity' of relations: bimotone relations of preoders do not transpose any two order-related elements.

Definition 2.1.38 (Transpositions and bimonotonicity)**.** Consider a relation $R \subset X \times Y$ between preorders X and Y (that is, R is an ordinary relation between the object sets of X and Y). A pair consisting of a non-identity arrow $x \to x'$ in X and a non-identity arrow $y \to y'$ in Y is called a **transposition of** R if both $R(x, y')$ and $R(x', y)$ are true. If the relation R has no transpositions it is said to be **bimonotone**. ⌐

Example 2.1.39 (Transpositions and bimonotonicity)**.** In Fig. 2.7 we depicted relations $R, S \subset X \times Y$ between posets $X \cong Y \cong [1]$. The relation R has a transposition while S has no transpositions and thus is bimonotone. ⌐

Figure 2.7: Transpositions and bimonotonicity of relations.

2.1.2.2   1-Truss bordisms as bimonotone bifunctional profunctors We finally turn to a definition of 1-truss bordisms. Recall that singular (resp. regular) objects in trusses T yield discrete subposets $(\mathsf{sing}\ T, \trianglelefteq)$ (resp. $(\mathsf{reg}\ T, \trianglelefteq)$) of the face order $(T, \trianglelefteq)$ (see Remark 2.1.13).

Definition 2.1.40 (1-truss bordisms of 1-trusses)**.** A **1-truss bordism** $R : T \nrightarrow S$ of 1-trusses T and S is a non-empty boolean profunctor $R : (T, \trianglelefteq) \nrightarrow (S, \trianglelefteq)$ of the face orders of T and S subject to the following conditions.
  1. *Bifunctionality*: The restriction $R : (\mathsf{sing}\ T, \trianglelefteq) \nrightarrow (\mathsf{sing}\ S, \trianglelefteq)$ is functional, and the restriction $R : (\mathsf{reg}\ T, \trianglelefteq) \nrightarrow (\mathsf{reg}\ S, \trianglelefteq)$ is cofunctional.
  2. *Bimonotonicity*: The relation $\mathsf{rel}\ R$ of frame orders $(T, \preceq)$ and $(S, \preceq)$ is bimonotone. ⌐

Remark 2.1.41 (Unwinding bifunctionality). The bifunctionality condition can be made more explicit: it requires that for each singular object $a \in \text{sing } T$ there is a unique singular object $\text{sing}_R(a) \in \text{sing } T$ such that $R(a, \text{sing}_R(a))$, and for each regular object $d \in \text{reg } T$ there is a unique regular object $\text{reg}^R(d) \in \text{reg } T$ such that $R(\text{reg}^R(d), d)$. The resulting function $\text{sing}_R : (\text{sing } T, \trianglelefteq) \to (\text{sing } S, \trianglelefteq)$ is called the **singular function** of R. Similarly, the function $\text{reg}^R : (\text{reg } S, \trianglelefteq) \to (\text{reg } T, \trianglelefteq)$ is called the **regular function** of R. ⌐

Example 2.1.42 (A 1-truss bordism)**.** In Fig. 2.8 we depict a 1-truss bordism $R : T \nrightarrow S$. The domain 1-truss T is drawn on the left, the codomain 1-truss S is drawn on the right, and elements of the underlying relation of R are indicated by black lines (the 'edges' of R) between objects of T and S. The fact that our choice of R is bimonotone, i.e. has no transpositions, is witnessed by there being no crossings among its edges. The fact that R is bifunctional is witnessed by the left-to-right function highlighted in red (providing the singular function $\text{sing}_R : \text{sing}(T) \to \text{sing}(S)$) and the right-to-left function highlighted in blue (providing the regular function $\text{reg}^R : \text{reg}(S) \to \text{reg}(T)$). ⌐
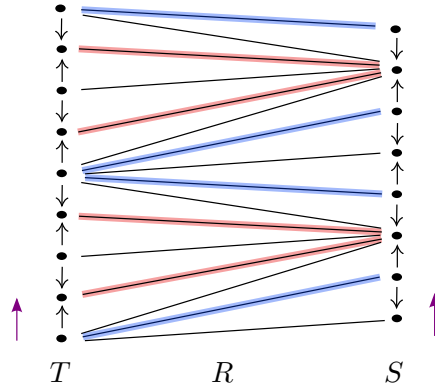
Figure 2.8: A 1-truss bordism $R : T \nrightarrow S$.

Remark 2.1.43 (Relation to 1-truss maps). The relationship of 1-truss bordisms to 1-truss maps (as introduced in Definition 2.1.15) is analogous to the relationship of profunctors and functors. In particular, while every 1-truss map descends to a functor of underlying face orders, every 1-truss bordism will descend to a profunctor of underlying face orders. ⌐

2.1.2.3   Composition and dualization of 1-truss bordisms The fact that 1-truss bordisms organize into a category follows from the following observation.

Observation 2.1.44 (Bimonotonicity and bifunctionality compose). The properties of 'bimonotonicity' and 'bifunctionality' in Definition 2.1.40 are preserved under composition of Boolean profunctors. Indeed, given 1-truss bordisms, $R : T \nrightarrow T'$ and $R' : T' \nrightarrow T''$ then the Boolean profunctor $R' \circ R : T \nrightarrow T''$ is again bifunctional (its singular function is $\text{sing}_{R'} \circ \text{sing}_R$, and its regular function is $\text{reg}^{R'} \circ \text{reg}^R$), as well as bimonotone (any transposition of $R' \circ R$ with respect to the frame orders of T and T'' would imply a transposition in at least on of R or R'). Therefore, the Boolean profunctor $R' \circ R : T \nrightarrow T''$ in fact defines another 1-truss bordism, which we can take to be the composite of R and R'. ⌐

Note further that 'identity 1-truss bordisms' are given by (Bool enriched) hom functors $\mathrm{Hom}_{(T,\trianglelefteq)}(-,-)$, for any 1-truss $(T,\trianglelefteq,\dim,\preceq)$. Thus, 1-truss bordisms are morphisms of the following category.

Notation 2.1.45 (The category of 1-truss bordisms). The **category of 1-truss bordisms** (whose objects are 1-trusses and whose morphisms are 1-truss bordisms) will be denoted by TBord^1. The wide subcategories containing only open respectively closed 1-trusses will be denoted by $\mathring{\mathsf{T}}\mathsf{Bord}^1$ respectively by $\bar{\mathsf{T}}\mathsf{Bord}^1$. ⌐

Observation 2.1.46 (The terminal and initial 1-trusses). The terminal object of TBord^1 is the trivial closed 1-truss $\bar{\mathbb{T}}_0$. The unique bordism $R : T \twoheadrightarrow \bar{\mathbb{T}}_0$ can be defined by setting $R(a,0)$ to hold for all $a \in T$.

The initial object of TBord^1 is the trivial open 1-truss $\mathring{\mathbb{T}}_0$. The unique bordism $R : \mathring{\mathbb{T}}_0 \twoheadrightarrow T$ can be defined by setting $R(0,a)$ to hold for all $a \in T$. ⌐

Remark 2.1.47 (Invertible 1-truss bordisms are unique). Given two 1-trusses, any invertible 1-truss bordism between them is unique if one exists. There is thus no harm in working 'skeletally', i.e. working with 1-trusses up to invertible bordisms. Further the classes of 1-trusses up to invertible 1-truss bordisms are exactly the classes of 1-trusses up to balanced 1-truss isomorphisms (see Remark 2.1.20). ⌐

Recall the dualization functor † on 1-trusses (and their maps) from Construction 2.1.26. A similar dualization construction applies to the category of 1-truss bordism as follows.

Construction 2.1.48 (Dualization of 1-truss bordisms). Let $R : T \twoheadrightarrow S$ be a 1-truss bordism. We define its **dual** $R^\dagger : S^\dagger \twoheadrightarrow T^\dagger$ to be the 1-truss bordism determined by transposing the underlying relation of R; that is, $R^\dagger$ is given by the Boolean profunctor

$$(S,\trianglelefteq^{\mathrm{op}})^{\mathrm{op}} \times (T,\trianglelefteq^{\mathrm{op}}) \cong (T,\trianglelefteq)^{\mathrm{op}} \times (S,\trianglelefteq) \xrightarrow{R} \mathsf{Bool}$$

One verifies that $R^\dagger$ satisfies both bimonotonicity and bifunctionality. Dualization gives rise to an involutive isomorphism of categories

$$\dagger : \mathsf{TBord}^1 \cong (\mathsf{TBord}^1)^{\mathrm{op}}$$

As before this restricts to an equivalence $\dagger : \mathring{\mathsf{T}}\mathsf{Bord}^1 \cong (\bar{\mathsf{T}}\mathsf{Bord}^1)^{\mathrm{op}}$. ⌐

2.1.2.4 Properties of 1-truss bordisms We discuss several immediate properties of 1-truss bordisms.

Lemma 2.1.49 (Dimension monotonicity). *Given a 1-truss bordism $R : T \twoheadrightarrow S$ and elements $a \in T$, $b \in S$ such that $R(a,b)$ holds, then $\dim(b) \leq \dim(a)$.*

Proof. We show that if $R(a,b)$ holds and $\dim(a) = 0$, then we must have $\dim(b) = 0$. In other words, R never relates singular elements of T to regular elements in S. Arguing by contradiction, assume $R(a,b)$ holds for singular a and regular b. Since frame orders are total, and since b is regular, we either have $b \prec \mathsf{sing}_R(a)$ or $\mathsf{sing}_R(a) \prec b$. Assume the former case holds (the latter case is symmetric). By bimonotonicity we must have $\mathsf{reg}^R(b) \prec a$. Then there exists a face order arrow $a - 1 \triangleleft a$ in $(T,\trianglelefteq)$ ('$a - 1$' denotes the predecessor of a in the total order $(T,\preceq)$). By profunctoriality of R this implies that $R(a - 1, \mathsf{sing}_R(a))$ holds. But, since we assumed $R(a,b)$ and since $b \prec \mathsf{sing}_R(a)$, this contradicts bimonotonicity. □

Recall our definition and notation for endpoints of 1-trusses (see Definition 2.1.21).

Lemma 2.1.50 (Endpoint preservation)**.** *Any 1-truss bordism $R : T \twoheadrightarrow S$ must relate lower (resp. upper) endpoints of T to lower (resp. upper) endpoints of S.*

Proof. We argue in the case of lower endpoints $\mathsf{end}_- T$, $\mathsf{end}_- S$ (the case of upper endpoints is symmetric). First, assume $\mathsf{end}_- S$ is regular. Set $a := \mathsf{reg}^R(\mathsf{end}_- S)$. We must have $a = \mathsf{end}_- T$ which shows the claim: indeed, if $a \neq \mathsf{end}_- T$ then $R(a-1, \mathsf{sing}_R(a-1))$ would hold, which would contradict bimonotonicity (since $\mathsf{end}_- S \prec \mathsf{sing}_R(a-1)$).

Next, assume $\mathsf{end}_- S$ is singular. If $\mathsf{end}_- T$ is singular then a dual argument to the first case shows $\mathsf{sing}_R(\mathsf{end}_- T) = \mathsf{end}_- S$ as claimed. If $\mathsf{end}_- T$ is regular then we argue as follows. If both T and S contain a single object, then $R(\mathsf{end}_- T, \mathsf{end}_- S)$ holds since R is assumed non-empty. Otherwise, we can assume that either T or S contain at least two objects. We argue in the former case (the argument for the latter case is similar). Since T has at least two elements, its lower endpoint has a successor $a = \mathsf{end}_- T + 1$. Set $b := \mathsf{sing}_R(a)$. If $b = \mathsf{end}_- S$, then $R(\mathsf{end}_- T, \mathsf{end}_- S)$ holds as claimed, since $\mathsf{end}_- T \lhd a$ and since R is profunctorial. On the other hand, if $b \succ \mathsf{end}_- S$, then we must have $\mathsf{reg}^R(\mathsf{end}_- S + 1) = \mathsf{end}_- T$ by bimonotonicity. Since $\mathsf{end}_- S \rhd \mathsf{end}_- S + 1$, profunctoriality of R implies again that $R(\mathsf{end}_- T, \mathsf{end}_- S)$ holds as claimed. $\square$

As we've seen every 1-truss bordism $R : T \twoheadrightarrow S$ determines a 'singular function' $\mathsf{sing}_R : (\mathsf{sing}(T), \preceq) \to (\mathsf{sing}(S), \preceq)$ and a 'regular function' $\mathsf{reg}^R : (\mathsf{reg}(S), \preceq) \to (\mathsf{reg}(T), \preceq)$. By the previous Lemma 2.1.50, it follows that sing_R 'preserves singular endpoints' in the following sense: whenever a (lower resp. upper) endpoint $\mathsf{end}_\pm T$ is a singular in T, then the image $\mathsf{sing}_R(\mathsf{end}_\pm) = \mathsf{end}_\pm S$ is a (lower resp. upper) endpoint in S, which is necessarily singular itself. Dually, reg^R 'preserves regular endpoints' in that, whenever a (lower resp. upper) endpoint $\mathsf{end}_\pm S$ is regular in S, then the image $\mathsf{reg}^R(\mathsf{end}_\pm S) = \mathsf{end}_\pm T$ is a (lower resp. upper) endpoint in T, which is necessarily a regular itself. In fact, frame order preserving maps of singular resp. regular elements in trusses, which preserve singular resp. regular endpoints in this sense, determine 1-truss bordisms as the next result records.

Lemma 2.1.51 (Singular and regular determined bordisms)**.** *Let T and S be 1-trusses.*

> Singular determined*: Given a function $f : (\mathsf{sing}(T), \preceq) \to (\mathsf{sing}(S), \preceq)$ that preserves singular endpoints, there is a unique 1-truss bordism $R : T \twoheadrightarrow S$ with singular function $\mathsf{sing}_R = f$.*

> Regular determined*: Given a function $f : (\mathsf{reg}(S), \preceq) \to (\mathsf{reg}(T), \preceq)$ that preserves regular endpoints, there is a unique 1-truss bordism $R : T \twoheadrightarrow S$ with regular function $\mathsf{reg}^R = f$.*

Proof. For the first case, define $R(a, b)$ to hold if and only if either (1) the object a is singular and $b = \mathsf{sing}_R(a)$, or (2) the object a is regular and both $b \leq \mathsf{sing}_R(a+1)$ (whenever $a + 1 \in T$) and $b \geq \mathsf{sing}_R(a-1)$ (whenever $a - 1 \in T$). For the second case, define $R(a, b)$ to hold if and only if either (1) the object b is regular and $a = \mathsf{reg}^R(b)$, or (2) the object b is singular and both $a \leq \mathsf{reg}^R(b+1)$ (when $b + 1 \in S$) and $a \geq \mathsf{reg}^R(b-1)$ (when $b - 1 \in S$). $\square$

We say a relation (or Boolean profunctor) $R : T \twoheadrightarrow S$ 'fully matches objects' if for each $a \in T$ there exists $a' \in S$ with an $R(a, a')$ and for each $b \in S$ there exists $b' \in T$ with $R(b', b)$. The description of 1-truss bordisms in the previous lemma (and its proof) then has the following two corollaries.

Corollary 2.1.52 (Truss bordisms fully match objects). *Any 1-truss bordism $R : T \twoheadrightarrow S$ fully matches objects.* $\square$

Corollary 2.1.53 (Dependence of full matching and functionality). *Consider 1-trusses T and S, and a boolean profunctor $R : (T, \trianglelefteq) \twoheadrightarrow (S, \trianglelefteq)$ such that $\mathsf{rel}\, R \subset (T, \preceq) \times (S, \preceq)$ is bimonotone and fully matches objects. If either $R(u, v)$ is functional on singular objects or cofunctional on regular objects, then R is a 1-truss bordism.*

Proof. The assumptions on R imply that R satisfies exactly the conditions in one of the two cases of Lemma 2.1.51. $\square$

2.1.3   1-Truss bundles We now define 1-truss bundles: these are bundles over posets whose fibers are 1-trusses, and whose fiber transitions are described by 1-truss bordisms.

2.1.3.1   1-Truss bundles as collections of 1-truss bordisms

Definition 2.1.54 (1-Truss bundle). Consider posets $(B, \rightarrow)$ and $(T, \trianglelefteq)$, equipped with a functor $\dim : (T, \trianglelefteq) \rightarrow [1]^{\mathrm{op}}$ and a second order $(T, \preceq)$. A **1-truss bundle** is a map of diposets $p : (T, \trianglelefteq, \preceq) \rightarrow (B, \rightarrow, =)$ (where '$=$' is the discrete order) satisfying the following.

1. For every object x in B, the datum $(T, \trianglelefteq, \dim, \preceq)$ restricts on the fiber over x to a 1-truss $p^{-1}(x) \equiv (p^{-1}(x), \trianglelefteq, \dim, \preceq)$.
2. For every arrow $x \rightarrow y$ in B, the fiber $p^{-1}(x \rightarrow y)$ is a 1-truss bordism, in the sense that the relation

$$a \sim b \text{ iff } (a \trianglelefteq b \text{ in } (T, \trianglelefteq) \text{ s.t. } p(a \trianglelefteq b) = x \rightarrow y)$$

is the underlying relation of a 1-truss bordism $p^{-1}(x) \twoheadrightarrow p^{-1}(y)$.

We call $(T, \trianglelefteq)$ the **total poset** and $(B, \leq)$ the **base poset** of the 1-truss bundle p. We call $\trianglelefteq$ the **face order**, and $\preceq$ the **frame order** of the bundle. ⌐

Notation 2.1.55 (Face, dimension, and frame structure). When working with 1-truss bundles, we will usually keep face orders, dimension maps, frame orders as well as base poset orders implicit; that is, we will denote 1-truss bundles simply by maps $p : T \rightarrow B$. When needed, we will use either arrows '$\rightarrow$' or the relation symbol '$\trianglelefteq$' for face orders; 'dim' for dimension maps $\dim : (T, \trianglelefteq^{\mathrm{op}}) \rightarrow [1]$; the relation symbol '$\preceq$' for frame orders; and arrows '$\rightarrow$' for base poset orders. ⌐

Remark 2.1.56 (Frame orders are trivial across fibers). Observe that, given a 1-truss bundle $p : T \rightarrow B$, then objects $a, b \in T$ are related in the frame order of p if and only if a and b live in the same fiber of p: indeed, no objects between different fibers can be frame order related since $p : (T, \preceq) \rightarrow (B, =)$ is a poset map, and frame orders are total on each fiber by the condition that fibers are 1-trusses. ⌐

Example 2.1.57 (1-Truss bundle)**.** We illustrate a 1-truss bundle $p : T \to B$ in Fig. 2.9 (we indicate total frame orders on each fiber, obtained by restricting the frame order $\preceq$ to that fiber, by a green coordinate axis). Note that we chose all such axis to point in the same ('upwards') direction. Flipping this direction of frame orders on all fibers (i.e. letting axis point downwards) would lead to another valid example. ⌐
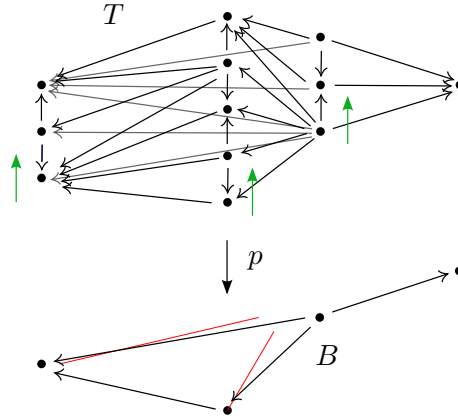
Figure 2.9: A 1-truss bundle $p : T \to B$.

Remark 2.1.58 ($\mathbb{Z}_2$ action on frame orders)**.** As observed in the preceding example, $\mathbb{Z}_2$ acts on the frame order of 1-truss bundles by 'flipping' the direction of total the frame order on every fiber. Most examples in this chapter do not change in nature under this action. Therefore, we usually depict frame orders of bundles only 'up to' this $\mathbb{Z}_2$ action; namely, we will parallely align all fibers of the bundle (as in Fig. 2.9) and assume all fiber frame orders run in the 'same direction', but we will usually not fix this direction. ⌐

Terminology 2.1.59 (Open and closed bundles)**.** A 1-truss bundle in which all fibers are open (resp. closed) 1-trusses is called an 'open' (resp. 'closed') 1-truss bundle. ⌐

Terminology 2.1.60 (Singular and regular objects in bundles)**.** Given a 1-truss bundle $p : T \to B$ we call an element $a \in T$ 'singular' if $\dim(a) = 0$ and 'regular' if $\dim(a) = 1$. We denote by $\mathsf{sing}(T)$ respectively by $\mathsf{reg}(T)$ the full subposets of $(T, \trianglelefteq)$ containing all singular respectively all regular objects. ⌐

We mention two 'lifting' properties of 1-truss bundles.

Observation 2.1.61 (Underlying discrete (op)fibrations)**.** Recall the standard notion of discrete opfibrations: a functor $F : \mathsf{C} \to \mathsf{D}$ is called a discrete opfibration if for every object c in C and every morphism of the form $g : F(c) \to d$ in D there is a unique morphisms $h : c \to c'$ such that $F(h) = g$. Dually, one defines a discrete fibration $F : \mathsf{C} \to \mathsf{D}$ to be a discrete opfibration $F^{\mathrm{op}} : \mathsf{C}^{\mathrm{op}} \to \mathsf{D}^{\mathrm{op}}$ after dualization. From the bifunctionality of 1-truss bordism it follows that 1-truss bundles $p : T \to B$ restrict on singular objects to a discrete opfibration $p : \mathsf{sing}(T) \to B$ and, dually, on regular objects to a discrete fibration $p : \mathsf{reg}(T) \to B$. ⌐

Observation 2.1.62 (Lifts in 1-truss bundles)**.** Consider a 1-truss bundle $p : T \to B$. For an object a in T and an arrow $x \to y$ in B with $p(a) = x$ there exists at least one object a' in T with an arrow $a \to a'$ and such that $p(a \to a') = (x \to y)$. Conversely, for b

in T and $x \to y$ in B with $p(b) = y$, there exists at least one b' in T with $b' \to b$ and $p(b' \to b) = (x \to y)$. Both claims follow from Corollary 2.1.52. ⎓

We remark about the following generalization of the definition given here.

Remark 2.1.63 (1-Truss bundles over categories). In Definition 2.1.54 we only consider 1-truss bundles over posets B. However, the definition works just as well for bundles over categories B, by considering functors $p : \mathsf{T} \to \mathsf{B}$ and requiring 1-truss structures on fibers over objects and 1-truss bordism structure on fibers over morphisms as above. ⎓

2.1.3.2 Maps of 1-truss bundles Using the notion of maps of 1-trusses 'fiberwise', we obtained the following notion of maps of 1-truss bundles.

Definition 2.1.64 (Maps of 1-truss bundles). For 1-truss bundles $p : T \to B$ and $q : S \to C$, a **1-truss bundle map** $F : p \to q$ is a diposet map $F : T \to S$ such that $q \circ F$ factors through p by a (necessarily unique) map G; that is, the following square commutes:

$$
\begin{array}{ccc}
T & \xrightarrow{\ F\ } & S \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\
B & \xrightarrow{\ G\ } & C
\end{array}
$$

In the case where $G = \mathrm{id}_B$ we say the 1-truss bundle map F is **base preserving**. ⎓

Note that a 1-truss bundle map $F : p \to q$ restricts on each fiber of p to a 1-truss map: that is, for each $x \in B$, the bundle map F restricts to a 1-truss map $F : p^{-1}(x) \to q^{-1}G(x)$.

Terminology 2.1.65 (Singular, regular and balanced 1-truss bundle maps). Consider a 1-truss bundle map $F : p \to q$. If F maps $\mathsf{sing}(T)$ to $\mathsf{sing}(S)$ we call F a 'singular' bundle map; similarly, if it maps $\mathsf{reg}(T)$ to $\mathsf{reg}(S)$ we call it a 'regular' bundle map. If F is both singular and regular, than we call it 'balanced'. ⎓

Note that a 1-truss bundle map $F : p \to q$ is singular if and only if $\dim F(x) \leq \dim x$ for all $x \in T$, and regular if and only if $\dim F(x) \geq \dim x$ for all $x \in T$.

Notation 2.1.66 (Category of 1-truss bundles and their maps). The category of 1-truss bundles and their maps is denoted by TrsBun_1. The full subcategories of open truss bundles with regular maps resp. closed truss bundles with singular maps are denoted by $\mathring{\mathsf{TrsBun}}_1^{\mathsf{r}}$ resp. $\bar{\mathsf{TrsBun}}_1^{\mathsf{s}}$. ⎓

Notation 2.1.67 (Categories of 1-truss bundles over a fixed base). The subcategory of the category of 1-truss bundles TrsBun_1 of bundles over a fixed poset B and base preserving maps will be denoted by $\mathsf{Trs}_1(B)$. Similarly one defined $\mathring{\mathsf{Trs}}_1(B)$ resp. $\bar{\mathsf{Trs}}_1(B)$. ⎓

Definition 2.1.68 (Restrictions of 1-truss bundles). Given a truss bundle $p : T \to B$ and a subposet $C \hookrightarrow B$, the **restriction** of p to C is the 1-truss bundle $p|_C : T|_C \to C$ obtained by restricting p to $T|_C = p^{-1}(C) \hookrightarrow T$ (and all data on T is restricted accordingly). This induces the restriction functor $-|_C : \mathsf{TrsBun}_1(B) \to \mathsf{TrsBun}_1(C)$ ⎓

2.1.3.3   Classification and totalization for 1-truss bundles 1-Truss bundles admit bundle classification and totalization constructions; their classifying category is the category of 1-truss bordisms TBord^1. For instance, in the simple case of the base poset being the 1-simplex $[1]$, observe that 1-truss bundles $p : T \to [1]$ are in correspondence with 1-truss bordisms and thus with functors $\mathsf{F} : [1] \to \mathsf{TBord}^1$. We now describe a more general correspondence of 1-truss bundles $p : T \to B$ over a general base poset B with functors $\mathsf{F} : B \to \mathsf{TBord}^1$ from B into the category of 1-truss bordisms.

Construction 2.1.69 (Classifying functors)**.** Let $p : T \to B$ be a 1-truss bundle over a poset B. We construct the **classifying functor** $\chi_p : B \to \mathsf{TBord}^1$ as follows. For each object x, the image $\chi_p(x)$ is the 1-truss $p^{-1}(x)$ and for each arrow $x \to y$, the image $\chi_p(x \to y)$ is the 1-truss bordism $p^{-1}(x \to y)$ (as given in Definition 2.1.54). The functoriality of χ_p follows from $(T, \trianglelefteq)$ being a poset and by the definition of Boolean profunctor composition. ⌐

Conversely, one constructs 'total bundles' as follows.

Construction 2.1.70 (Totalizing bundles)**.** Given a poset B, consider a functor $\mathsf{F} : B \to \mathsf{TBord}^1$. We construct the **total 1-truss bundle** $\pi_{\mathsf{F}} : \mathrm{Tot}\mathsf{F} \to B$ as follows. The 'total poset' $(\mathrm{Tot}\mathsf{F}, \trianglelefteq)$ has as elements the pairs $(x \in B, a \in \mathsf{F}(x))$. Morphisms $(x, a) \trianglelefteq (y, b)$ are given whenever the 1-truss bordism $\mathsf{F}(x \to y)$ from the 1-truss $\mathsf{F}(x)$ to the 1-truss $\mathsf{F}(y)$ relates the elements a and b. The frame order $(\mathrm{Tot}\mathsf{F}, \preceq)$ is defined to relate $(x, a) \preceq (x, b)$ whenever $a \preceq b$ in $\mathsf{F}(x)$. The dimension functor $\dim : (T, \trianglelefteq^{\mathrm{op}}) \to [1]$ is equally defined fiberwise; the fact that this extends to a functor on all of $(T, \trianglelefteq^{\mathrm{op}})$ follows since truss bordisms are 'dimension monotonic' by Lemma 2.1.49. ⌐

In order to promote the preceding constructions to functors, we introduce the following notion of 'bundle concordance'.

Definition 2.1.71 (Concordance of 1-truss bundles)**.** For a poset B, a **1-truss bundle concordance** $u : p \Rightarrow q$ between 1-truss bundles $p : T \to B$ and $q : S \to B$ is a 1-truss bundle $u : U \to B \times [1]$ such that $r|_{B \times \{0\}} = p$ and $r|_{B \times \{1\}} = q$. ⌐

If the base B is trivial, a 1-truss bundle concordance is simply a 1-truss bordism.

Remark 2.1.72 (Invertible concordance are unique)**.** Generalizing Remark 2.1.47, observe that, given two 1-truss bundles, any invertible concordance between them is unique if one exists. There is thus no harm in working 'skeletally', i.e. working with 1-truss bundles up to invertible concordances. Further, the classes of 1-truss bundles up to invertible concordance are exactly the classes of 1-truss bundles up to base preserving balanced 1-truss bundle isomorphisms. ⌐

Note that, given a 1-truss bundle concordance $u : p \Rightarrow q$, its classifying functor χ_u is a functor $B \times [1] \to \mathsf{TBord}^1$ restricting on $B \times \{0\}$ to χ_p and on $B \times \{1\}$ to χ_q. Equivalently, χ_u is thus a natural transformation of classifying functors $\chi_p \Rightarrow \chi_q$. We will refer to χ_u as the 'classifying natural transformation' of u.

Definition 2.1.73 (Categories of 1-truss bundle concordances)**.** For a poset B, define the **category of concordances of 1-truss bundles over** B, denoted by $\mathsf{TrsConc}_1(B)$, to have 1-truss bundles over B as objects, and 1-truss bundle concordances as morphisms.

Composition of two truss concordances $u : p \Rightarrow q$ and $v : q \Rightarrow r$ is determined by the condition that $\chi_{u \circ v} = \chi_v \circ \chi_u$.[12]

With these definitions in place, we obtain the following equivalence of categories.

Construction 2.1.74 (Classification and totalization functors)**.** Given a poset, we define the **classification functor**

$$\chi_- : \mathsf{TrsConc}_1(B) \to \mathrm{Fun}(B, \mathsf{TBord}^1)$$

to map 1-truss bundles $p : T \to B$ to their classifying functor χ_p, and concordances $u : p \Rightarrow q$ to their classifying natural transformations $\chi_u : \chi_p \Rightarrow \chi_q$. Conversely, we define the **totalization functor**

$$\pi_- : \mathrm{Fun}(B, \mathsf{TBord}^1) \to \mathsf{TrsConc}_1(B)$$

to take functors $\mathsf{F} : B \to \mathsf{TBord}^1$ to their total 1-truss bundles π_{F}, and natural transformations $\alpha : B \times [1] \to \mathsf{TBord}^1$ to their total 1-truss bundles π_α.

Observation 2.1.75 (Classification and totalization are inverse)*.* Classification and totalization functors provide an equivalence of categories.

Example 2.1.76 (The composition of 1-truss bordisms as a truss bundle over the 2-simplex)**.** Two composable 1-truss bordisms $R_1 : T_0 \to T_1$ and $R_2 : T_1 \to T_2$ together with their composite $R_2 \circ R_1 : T_0 \to T_2$ evidently define a functor $[2] \to \mathsf{TBord}^1$ from the poset $[2]$ to the category of 1-truss bordisms; by the above construction this situation can be equivalently considered as a 1-truss bundle over the 2-simplex. For example, in Fig. 2.10 we illustrate truss bordisms R_1 and R_2, along with the corresponding bundle $p : T \to [2]$ on the right.

The classification and totalization constructions described here for 1-truss bundles, are of course analogous to many of the classical combinatorial instances of these constructions (see for instance [Bén00], [Str01]). In particular, they generalize to 1-truss bundles over categories, as the following remark records.

Remark 2.1.77 (Classifying 1-truss bundles over categories)*.* Recall from Remark 2.1.63 that we can define 1-truss bundles $p : T \to B$ over a base *category* B as well. The above classification/totalization construction immediately carries over to this case, and shows that 1-truss bundles over B are (up to bundle concordance isomorphism) precisely classified by functors $B \to \mathsf{TBord}^1$ (up to natural isomorphism).

2.1.3.4   Pullback, dualization, and suspension of 1-truss bundles With notions of classification and totalization at hand, we now describe several further constructions on 1-truss bundles. Firstly, 1-truss bundles may be pulled back along poset maps into their base as follows.

---

[12]Explicitly, working skeletally (that is, identifying 1-truss bundles up to invertible concordances, see Remark 2.1.72), this can be defined by setting $v \circ u = \pi_{\chi_v \circ \chi_u}$.
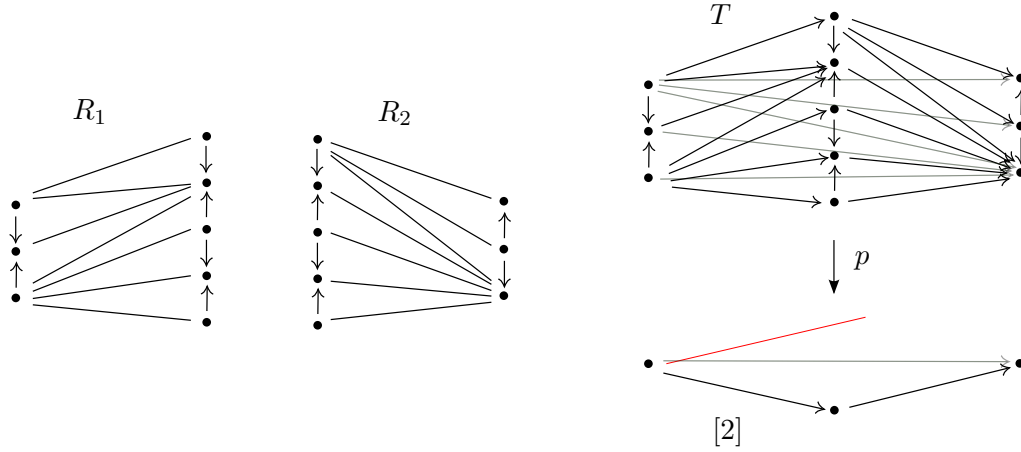
Figure 2.10: 1-Truss bordisms composition as a bundle over the 2-simplex.

Definition 2.1.78 (Pullback 1-truss bundles)**.** Let $p : T \to B$ be a 1-truss bundle with classifying map $\chi_p : B \to \mathsf{TBord}^1$. Consider a map of posets $F : C \to B$. We define the **pullback** $F^*p : F^*T \to C$ **of** p **along** F to be the total 1-truss bundle $\pi_{\chi_p F}$ of the composite functor $\chi_p \circ F : C \to \mathsf{TBord}^1$. The canonical 1-truss bundle map $F^*p \to p$, which is fiberwise given by isomorphisms $(F^*p)^{-1}y \cong p^{-1}F(y)$, will be denoted by $\mathrm{Tot} F$. ⌟

Notation 2.1.79 (Pullback of 1-truss bundles). A pullback of 1-truss bundles will often be indicated by a 'pullback square'

$$
\begin{array}{ccc}
S & \xrightarrow{\mathrm{Tot} F} & T \\
{\scriptstyle q}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\
C & \xrightarrow{\ F\ } & B
\end{array}
$$

where the 1-truss bundle q is the pullback of the 1-truss bundle p along the poset map F. Note, while '1-truss bundles' and 'poset maps' do not live in the same category, we may forget part of the 1-truss bundle structure and think of p and q as poset maps instead: this then makes the above a pullback in the category of posets and poset maps. ⌟

Remark 2.1.80 (Pullbacks generalize restrictions).) In the special case where $F : C \hookrightarrow B$ is a subposet of B, the pullback bundle F^*p equals the restriction bundle $p|_C$. ⌟

Example 2.1.81 (Pullback 1-truss bundles)**.** We depict the canonical pullback map $(\mathrm{Tot} F, F) : F^*p \to p$ of a 1-truss bundle pullback in Fig. 2.11 (in fact, this is a restriction in the sense of Remark 2.1.80)). ⌟

Next, we generalize our dualization construction for 1-trusses to the case of 1-truss bundles; this simply 'dualizes all fibers' of a given bundles.

Definition 2.1.82 (Dual 1-truss bundles)**.** Given a 1-truss bundle $p : T \to B$ over B, its **dual bundle** $(p^\dagger : T^\dagger \to B^{\mathrm{op}})$ is the 1-truss bundle over B^{op} whose total poset $(T, \trianglelefteq^{\mathrm{op}})$ is
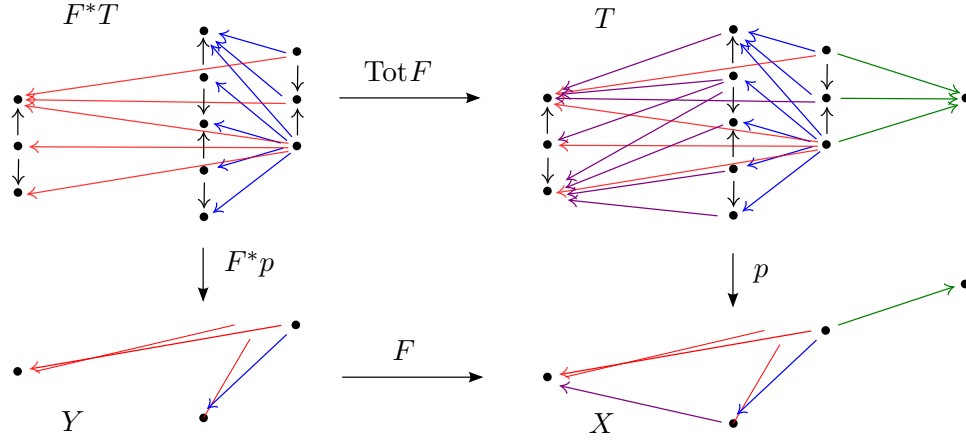
Figure 2.11: The pullback of a 1-truss bundle p along a base poset map F.

opposite to that of p, whose dimension functor is the composite $(T, \trianglelefteq^{\mathrm{op}}) \xrightarrow{\dim^{\mathrm{op}}} [1] \cong [1]^{\mathrm{op}}$ (where dim is the dimension functor of p), and whose frame order $(T, \preceq)$ is equal to that of p. ⌙

Remark 2.1.83 (Dualization via classifying functors). Using the classification and totalization equivalence, given a 1-truss bundle p, its dual $p^\dagger$ may equivalently be defined to have classifying functor

$$\left(\chi_{p^\dagger} : B^{\mathrm{op}} \to \mathsf{TBord}^1\right) = \left(B \xrightarrow{\chi_T} \mathsf{TBord}^1 \xrightarrow{\dagger} (\mathsf{TBord}^1)^{\mathrm{op}}\right)^{\mathrm{op}}.$$

(The functor $\dagger : \mathsf{TBord}^1 \to (\mathsf{TBord}^1)^{\mathrm{op}}$ was defined in Construction 2.1.48.) ⌙

Definition 2.1.84 (Dual 1-truss bundle maps). Given a 1-truss bundle map $F : (p : T \to B) \to (q : S \to C)$ one defines the **dual bundle map** $F^\dagger : p^\dagger \to q^\dagger$ by setting $F^\dagger = F^{\mathrm{op}} : (T, \trianglelefteq^{\mathrm{op}}) \to (S, \trianglelefteq^{\mathrm{op}})$. ⌙

Note that dualization maps open 1-truss bundles to closed 1-truss bundles and vice-versa, as well as singular bundle maps to regular ones and vice versa.

Definition 2.1.85 (Dual 1-truss bundle concordances). Given a 1-truss bundle concordance $u : p \Rightarrow q$ one defines the **dual bundle concordance** $u^\dagger : p^\dagger \Rightarrow q^\dagger$ by setting $u^\dagger$ to be the dual bundle of u. ⌙

The preceding constructions now yield the following functors.

Observation 2.1.86 (Dualization functors on 1-truss bundles). Generalizing the dualization functor on the category of 1-trusses, we obtain a dualization functor

$$\dagger : \mathsf{TrsBun}_1 \cong \mathsf{TrsBun}_1.$$

This recovers our earlier Construction 2.1.26 if we restrict to $\mathsf{TrsBun}_1(*) \hookrightarrow \mathsf{TrsBun}_1$.

Similarly, generalizing the dualization functor on the category of 1-truss bordisms, we obtain, for any poset B, a dualization functor

$$\dagger : \mathsf{TrsConc}_1(B) \cong \mathsf{TrsConc}_1(B)^{\mathrm{op}}.$$

This recovers Construction 2.1.48 if we set $B = *$. ⌐

As a final elementary construction, we now introduce 'suspensions' of 1-truss bundles: this adds new initial and final elements to a given truss bundle. We start with the case of posets.

Construction 2.1.87 (Suspension of posets)**.** Let X be a poset. Its **suspension** ΣX is the poset obtained from X by adjoining two elements $\top$ and $\bot$ and arrows $x \to \top$ and $\bot \to x$ for each $x \in X$. ⌐

Construction 2.1.88 (Suspension of 1-truss bundles)**.** Let $p : T \to B$ be a 1-truss bundle. The suspension bundle $\Sigma p : \Sigma T \to \Sigma B$ is the 1-truss bundle
1. whose base poset is the suspension ΣB of B,
2. whose total poset $(\Sigma T, \trianglelefteq)$ is the suspension $\Sigma(T, \trianglelefteq)$ of $(T, \trianglelefteq)$,
3. whose dimension functor restricts on $T \hookrightarrow \Sigma T$ to the dimension functor of p while mapping the new initial object $\bot$ in ΣT to 0 and terminal object $\top$ in ΣT to 1,
4. whose frame order $(\Sigma T, \preceq)$ relates elements if and only if they are already related in $(T, \preceq)$. ⌐

Remark 2.1.89 (Suspension bundles via classifying functors)**.** Given a 1-truss bundle p, its suspension bundle Σp is classified by the unique map $\chi_{\Sigma p} : \Sigma B \to \mathsf{TBord}^1$ which restricts on $B \hookrightarrow \Sigma B$ to χ_p, maps $\bot$ to the initial object $\mathring{\mathbb{T}}_0$ in TBord^1, and map $\top$ to the terminal object $\bar{\mathbb{T}}_0$ in TBord^1 (see Observation 2.1.46). ⌐

Example 2.1.90 (Suspension bundles)**.** In Fig. 2.12 we depict a 1-truss bundle $p : T \to B$ on the left together with its suspension bundle $\Sigma p : \Sigma T \to \Sigma B$ on the right (for clarity we colored arrows in the total poset and in the base poset in corresponding colors). ⌐
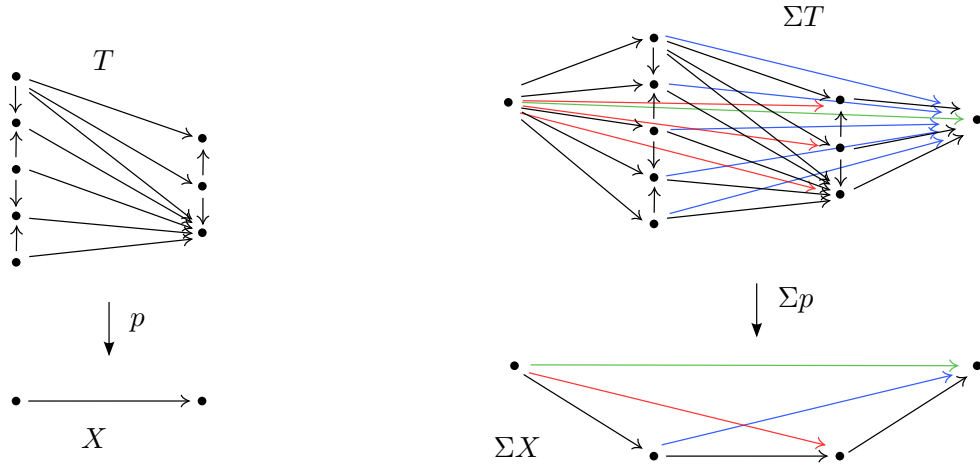
Figure 2.12: Suspension bundle Σp of a 1-truss bundle p.

2.2 Truss induction along simplices in bundles

We study simplices in total posets of 1-truss bundles and find natural linear orders on subsets of k-simplices that lie over the same base simplex; we will refer to this linear order

as the 'scaffold order' of the truss bundle. This observation will provide a useful tool in later combinatorial proofs, by allowing us to induct along a progression of simplices in the total poset of a bundle as determined by the scaffold order; we refer to this general principle as 'truss induction'.

2.2.1 Sections and spacers We first introduce the following distinction of simplexes in 1-truss bundles.

2.2.1.1 The definition of sections and spacers Recall that a k-simplex $K : [k] \to P$ in a poset P is called non-degenerate if it is injective on objects; otherwise we say it is degenerate.

Definition 2.2.1 (Sections)**.** For a 1-truss bundle $p : T \to B$, a k**-section** K **in** p is a non-degenerate k-simplex $K : [k] \hookrightarrow (T, \trianglelefteq)$ such that the composite map $pK : [k] \to B$ is a non-degenerate simplex in B. ⌐⌐

Definition 2.2.2 (Spacers)**.** For a 1-truss bundle $p : T \to B$, a k**-spacer** K **in** p is a non-degenerate k-simplex $K : [k] \hookrightarrow (T, \trianglelefteq)$ such that $pK : [k] \to B$ is a degenerate simplex in B. ⌐⌐

Terminology 2.2.3 (Simplices in bundles). Given a 1-truss bundle $p : T \to B$, we often say 'a simplex in p' to mean a non-degenerate simplex $[k] \to (T, \trianglelefteq)$. ⌐⌐

Note that every simplex in 1-truss bundle p is either a section or a spacer.

Terminology 2.2.4 (Base projections). Let $K : [k] \hookrightarrow T$ be a k-simplex in a 1-truss bundle $p : T \to B$. We denote by $\mathrm{im}(pK) : [m] \hookrightarrow B$ the unique non-degenerate simplex in B whose image equals that of $pK : [k] \hookrightarrow B$, and call it the 'base projection' of K. ⌐⌐

In the special where the base poset b is itself the m-simplex $B = [m]$, it further makes sense to ask for simplices in truss bundles to project to the 'entire' base.

Terminology 2.2.5 (Base-surjectivity). A k-simplex $K : [k] \hookrightarrow T$ in a 1-truss bundle $p : T \to B$ is called 'base-surjective' if $\mathrm{im}(pK) = [m]$. ⌐⌐

Example 2.2.6 (Base-surjective sections and spacers in 1-truss bundles)**.** In Fig. 2.13 we depict simplices in a 1-truss bundle $p : T \to B$: on the left we depict a selection of sections in p, on the right we depict a selection of spacers in p. All simplices are base-surjective. ⌐⌐

In the case of base-surjective simplices we may simplify their visualization by solely depicting their spine as explained by the following remark.

Remark 2.2.7 (Spine-only notation). On the left in Fig. 2.14 we depict base-surjective sections (in blue and red) and spacers (in green and orange) of a 1-truss bundle over the 2-simplex $[2]$. On the right we depict the same simplices in the same bundle using a more convenient representation: namely, to depict $p : T \to [m]$, it suffices depict 1-truss bordisms lying over the spine of $[m]$ (note that this fully determines p), and similarly, to depict simplices in p, it suffices to depict their spines T. ⌐⌐

The case of 1-truss bundles over simplices and their base-surjective simplices is of particular importance: indeed, any simplex in any 1-truss bundle canonically factors as a base surjective simplex and a bundle inclusion as follows.
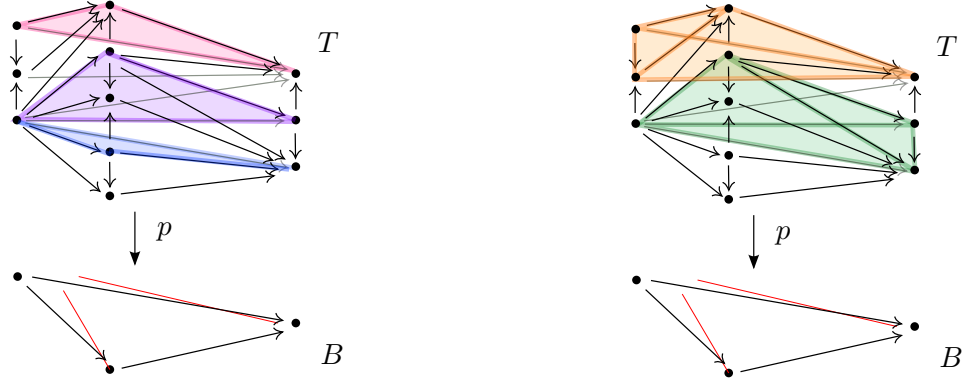
Figure 2.13: Sections and spacers in a 1-truss bundle $p : T \to B$.
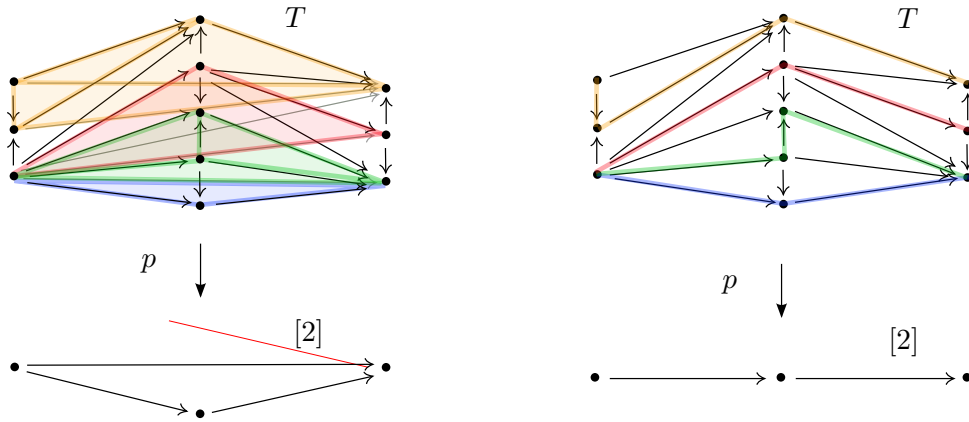
Figure 2.14: Base-surjective simplices may be represented by their spine which lies in the spine of the bundle.

Remark 2.2.8 (Factoring simplices through base-surjective simplices). Let $K : [k] \hookrightarrow T$ be a k-simplex in a 1-truss bundle $p : T \to B$. Abbreviate its base projection $\mathrm{im}(pK) : [m] \hookrightarrow B$ by $F : [m] \hookrightarrow B$. Then $K : [k] \hookrightarrow T$ uniquely factors through the bundle pullback map $\mathrm{Tot}\, F : F^* p \hookrightarrow p$ as a k-simplex $F^* K : [k] \hookrightarrow F^* T$. Note $F^* K$ is base-surjective. In fact, the mapping $K \mapsto F^* K$ yields a bijection between simplices in the bundle p whose base projection is F and base-surjective simplices in the pullback bundle $F^* p$. ⌐

In the rest of this section, we will almost exclusively work with bundles over simplices, and assume base-surjectivity by default.

Convention 2.2.9 (Base-surjectivity by default). All sections and spacers will be assumed to be base-surjective unless otherwise noted. ⌐

2.2.1.2 The spines of sections and spacers Sections and spacers admit concrete combinatorial descriptions of their respective spines, as we will now explain. The fundamental observation is as follows.

Observation 2.2.10 (Spines of simplices in truss bundles). consider a 1-truss bundle $p : T \to [m]$ over the m-simplex and a k-simplex $K : [k] \to T$ in p. Recall, if $(i, a) \to (j, b)$ is a morphism in T, then we must have $\dim(a) \geq \dim(b)$ (see Lemma 2.1.49). In particular,

for each spine vector $(i - 1 \to i)$ in $[k]$ we must have $\dim K(i-1) \geq \dim K(i)$. As a consequence, simplices in truss bundles fall into one of three categories.

1. The chain of morphisms $K(i - 1 \to i)$ contains only regular objects $K(i)$ (i.e. $\dim K(i) = 1$). Since all non-identity arrows in fibers of p run between regular and singular objects, K must be a section simplex; we call K a 'purely regular' section in this case.

2. The chain of morphisms $K(i - 1 \to i)$ contains only singular objects $K(i)$ (i.e. $\dim K(i) = 0$). Similar to the previous case, K must be a section simplex; we call it a 'purely singular' section in this case.

3. There is a unique morphism $K(j - 1 \to j)$ whose domain is regular and whose codomain is singular. The index j is called the 'transition index' of K. All objects $K(i < j)$ are regular, and all objects $K(i \geq j)$ are singular. ⌐

The next two remarks spell out the preceding observation separately in the cases of sections resp. spacers. Both remarks use the following 'tuple' notation to keep track of base projections of total poset elements in truss bundles.

Notation 2.2.11 (Pair notation for 1-truss bundles). Given a 1-truss bundle $p : T \to B$, we will sometimes redundantly denote objects $a \in T$ by pairs (x, a): here, $x \in B$ is the object $p(a)$ in the base poset B, and thus $a \in p^{-1}(x)$ is lives in the 1-truss fiber over x. ⌐

Remark 2.2.12 (Spines of sections). If K is a section simplex in a 1-truss bundle $p : T \to [m]$, then the spine vectors $K(i \to i + 1)$ of K must be a chain of morphisms in T of the form

$$(0, a_0) \to (1, a_1) \to ... \to (j - 1, a_{j-1}) \to (j, b_j) \to (j + 1, b_{j+1}) \to ... \to (m, b_m)$$

where each a_i is regular, each b_j is singular and j is a unique index $0 \leq j \leq m + 1$. Corresponding to the first two cases of Observation 2.2.10 note the following: if $j = 0$ then K is purely singular, and conversely, if $j = m + 1$ then K is purely regular. In either of these boundary cases, we again refer to j as the 'transition index' of K. ⌐

Remark 2.2.13 (Spines of spacers). If L is a spacer in $p : T \to [m]$, then the spine vectors $L(i \to i + 1)$ of L must be a chain of morphisms in T of the form

$$(0, a_0) \to (1, a_1) \to ... \to (j, a_j) \to (j, b_j) \to (j + 1, b_{j+1}) \to ... \to (m, b_m)$$

where each a_i is regular, each b_j is singular and j is a unique index $0 \leq j \leq m$. Note that in particular we must have $\dim(K) = m + 1$, that is, all filler simplices in bundles over the m-simplex are of dimension $(m+1)$. Corresponding to the third case in Observation 2.2.10, the index j is called the transition index of K. Note that the fiber $p^{-1}(j)$ over j contains *two* vertices $L(i)$ of L (a regular and a singular one), and that every spacer in particular has at least one regular and one singular vertex. ⌐

The boundary cases observed with sections (in which all elements are either regular or singular) can be resolved by passing to suspensions. Recall the construction of suspension bundles from Construction 2.1.88. The following notation will be useful.

Notation 2.2.14 (Suspending simplices). For numeric convenience, we will usually identify the suspension $\Sigma[m]$ of the m-simplex $[m]$ with the poset $(-1 \to 0 \to 1 \to ... \to (m-1) \to m \to m + 1)$. ⌐

Construction 2.2.15 (Suspending sections)**.** Let $p : T \to [m]$ be a 1-truss bundle over the m-simplex $[m]$, and consider a section $K : [m] \to T$ in p. Construct the suspended section $\Sigma K : \Sigma[m] \to \Sigma T$ by setting $\Sigma K(\bot) = \bot$ and $\Sigma K(\top) = \top$ (where $\bot$ respectively $\top$ denote newly adjoined initial respectively terminal objects); equivalently, using our numeric notational conventions (see Notation 2.2.11 and Notation 2.2.14), this may be written as $\Sigma K(-1) = (-1, 0)$ and $\Sigma K(m+1) = (m+1, 0)$. Observe that the mapping $K \mapsto \Sigma K$ establishes a 1-to-1 correspondence between sections in p and in Σp: indeed, the inverse mapping is obtained by restricting sections $\Sigma[m] \to \Sigma T$ along $[m] \hookrightarrow \Sigma[m]$ and $T \hookrightarrow \Sigma T$. ⌐

Note since ΣK adjoint a new regular object to the start of K and a new singular object to the end of K it follows that ΣK always contains at least one regular and one singular object (even if K doesn't). In other words, suspended sections are never purely regular or purely singular.

We now characterize sections and spacers in terms of certain morphism. We start with the case of sections in a 1-truss bundle p, which we will show correspond to so-called 'jump morphism' in the suspension bundle Σp.

Definition 2.2.16 (Jump morphisms)**.** Let $p : T \to [m]$ be a 1-truss bundle over the m-simplex. A **jump morphism** f in p is a morphism $(T, \trianglelefteq)$ whose domain $\mathrm{dom}(f)$ is regular, whose codomain $\mathrm{cod}(f)$ is singular and whose base projection pf is a spine vector in $[i]$ (that is, pf is of the form $i - 1 \to i$ for $0 < i \leq m$). ⌐

Construction 2.2.17 (Correspondence of sections in p and jump morphisms in Σp)**.** Let $p : T \to [m]$ be a 1-truss bundle over the m-simplex $[m]$. Every section $K : [m] \to T$ in p has a 'associated jump morphism' f in Σp, given by $f = \Sigma K(j - 1 \to j)$ where j is the transition index of K. Conversely, let f be a jump morphism in Σp lying over a spine vector $j - 1 \to j$ in $\Sigma[m]$ (use Notation 2.2.14 for objects in $\Sigma[m]$). Then f has an 'associated section' $K : [m] \to T$ constructed by setting
1. for $i < j$, $K(i) = \mathsf{reg}^{\chi_p(i \to j-1)}(\mathrm{dom}\, f)$,
2. for $i \geq j$, $K(i) = \mathsf{sing}_{\chi_p(j \to i)}(\mathrm{cod}\, f)$.
The constructions are mutually inverse, thus providing a correspondence between sections in K and jump morphisms in Σp. ⌐

Example 2.2.18 (Correspondence of sections and jump morphisms)**.** In Fig. 2.15 we depict a bundle $p : T \to [2]$, together with its suspension Σp (highlighted in purple). We indicate regular elements of T in blue and singular elements in red. We colored the spines of several sections in P (using the 'spine-only' notation discussed in Remark 2.2.7); in each case we then mark their associated jump morphism in Σp by a big dot of the same color. ⌐

Next, we establish a similar correspondence between spacers and so-called 'fiber morphisms' defined as follows.

Definition 2.2.19 (Fiber morphisms)**.** Let $p : T \to [m]$ be a 1-truss bundle over the m-simplex. A **fiber morphism** f in p is a morphism f in $(T, \trianglelefteq)$ whose domain $\mathrm{dom}(f)$ is regular, whose codomain $\mathrm{cod}(f)$ is singular, and whose base projection pf is an identity morphism in $[m]$. ⌐
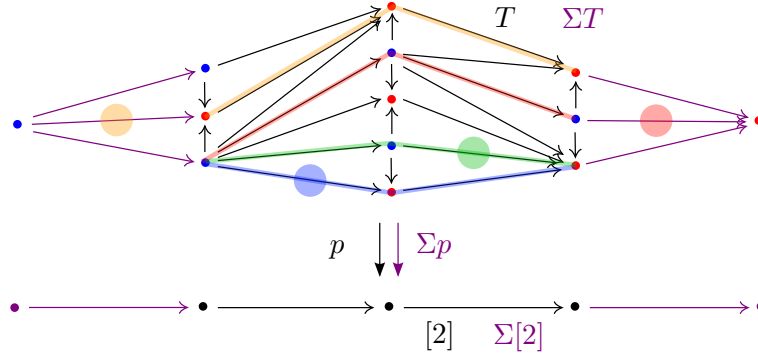
Figure 2.15: Sections and their associated jump morphisms.

Construction 2.2.20 (Correspondence of spacers and fiber morphisms). Let $p : T \to [m]$ be a 1-truss bundle over the m-simplex $[m]$. Every spacer $L : [m + 1] \to T$ in p has a 'associated fiber morphism' f in p, given by $f = L(j - 1 \to j)$ where j is the transition index of L. Conversely, let f be a fiber morphism in Σp lying the object j in $[m]$. Then f has an 'associated spacer' $L : [m + 1] \to T$ constructed by setting

1. for $i \leq j$, $L(i) = \mathsf{reg}^{\chi_p(i \to j)}(\mathrm{dom}\, f)$,
2. for $i > j$, $L(i) = \mathsf{sing}_{\chi_p(j \to i)}(\mathrm{cod}\, f)$.

The constructions are mutually inverse, thus providing a correspondence between spacers in K and fiber morphisms in p. ⌐

Example 2.2.21 (Correspondence of spacers and fiber morphisms). In Fig. 2.16 we colored the spines of several spacers in a bundle $p : T \to [2]$; we then mark their associated jump morphisms in p by a big dot of the same color. ⌐
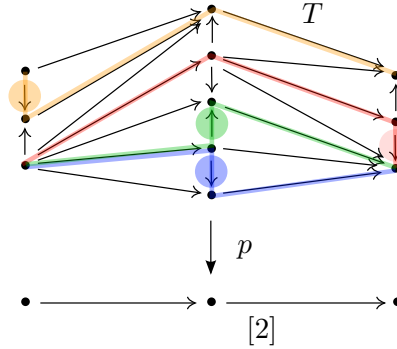
Figure 2.16: Spacers and their associated fiber morphisms (indicated by dots with corresponding color).

2.2.2   The scaffold order We now construct canonical linear orders on the sets of sections and spacers in a 1-truss bundle over a simplex.

Notation 2.2.22 (Set of sections and spacers). Given a 1-truss bundle $p : T \to [m]$, we denote its sets of sections and spacers as follows.

$$\Gamma_p = \{\text{sections } K : [m] \to T \text{ of } p : T \to [m] \}$$

$$\Psi_p = \{\text{spacers } L : [m+1] \to T \text{ of } p : T \to [m]\} \qquad \lrcorner$$

2.2.2.1 The case of sections We first construct a total order on the set of sections Γ_p of a 1-truss bundle $p : T \to [m]$ over the m-simplex: we call this order the 'scaffold order of sections'. Using the correspondence of sections in p and jump morphisms in the suspension Σp (see Construction 2.2.17) one may easily visualize the scaffold order as follows, and in fact, our formal construction of the scaffold order will directly rely on the idea underlying this visualization.

Example 2.2.23 (Scaffold orders)**.** In Fig. 2.17 we depict *all* sections in a 1-truss bundle $p : T \to [2]$ over the 2-simplex (by coloring their spine). We also depict their corresponding jump morphisms in Σp (by correspondingly colored dots). The scaffold order on sections Γ_p, and thus on jump morphisms, is the total order indicated by a directed path (in red) passing through all jump morphisms in order. This directed path is uniquely determined as follows: it is (up to reversal) the unique directed path that passes through all jump morphisms of the suspended bundle once while intersecting exactly one fiber morphism in between any two consecutive jump morphisms. A slightly more complex example, using
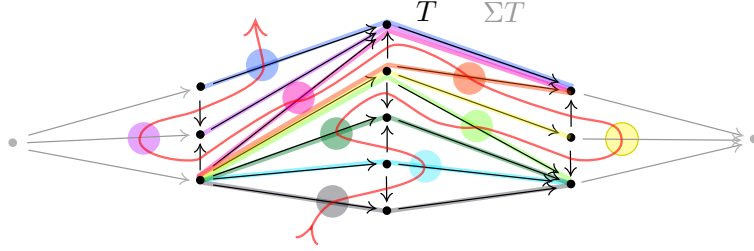
Figure 2.17: The scaffold order on sections in a 1-truss bundle over a 2-simplex indicated by a directed path through corresponding jump morphisms.

the same directed path notation to illustrate the scaffold order of jump morphisms (and thus of sections), is shown in Fig. 2.18: this depicts the scaffold order of sections in a bundle $p : T \to [3]$ over the 3-simplex. $\qquad \lrcorner$
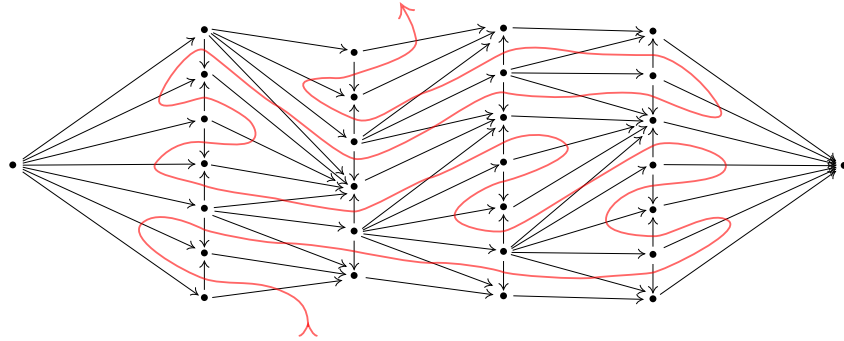
Figure 2.18: The scaffold order on sections in a 1-truss bundle over a 3-simplex indicated by a directed path.

We now formally construct the scaffold order on sections Γ_p. We first introduce a 'scaffold norm'; this is a function which associates to each section $K \in \Gamma_p$ a natural

number. We will show that, in fact, the scaffold norm is an injective function $\Gamma_p \hookrightarrow \mathbb{N}$; endowing $\mathbb{N}$ with the standard order of natural numbers, this will then induce the scaffold order on Γ_p as required.

Definition 2.2.24 (Scaffold norm of sections)**.** Consider a 1-truss bundle $p : T \to [m]$ and its set of sections Γ_p. The **scaffold norm** $\langle - \rangle$ is the function

$$\langle - \rangle : \Gamma_p \to \mathbb{N}$$
$$K \mapsto \sum_{i \in [m]} \mathsf{num}(K(i))$$

where $\mathsf{num} : (T, \preceq) \to (\mathbb{N}, \leq)$ takes elements $a \in T$ to numbers $i - 1 \in \mathbb{N}$ iff a is the ith element in the frame ordered fiber over $p(a)$.[13] ⌟

Observation 2.2.25 (Suspension preserves scaffold norm)**.** Recall the suspension operation on sections defined in Construction 2.2.15. Observe that taking suspensions $\Sigma : \Gamma_p \to \Gamma_{\Sigma p}$ preserves scaffold norm, that is, $\langle K \rangle = \langle \Sigma K \rangle$ for $K \in \Gamma_p$. ⌟

In order to describe the image of the scaffold norm, we first construct its 'bottom' and 'top sections'.

Construction 2.2.26 (Minimal and maximal values of the scaffold norm)**.** Let $p : T \to [m]$ be a 1-truss bundle over the m-simplex $[m]$. We construct sections $\mathsf{K}_p^-, \mathsf{K}_p^+ : [m] \to T$ of p, called the **bottom** respectively **top** sections of p, which are minimal respectively maximal sections with respect to the scaffold norm. Namely, we define $\mathsf{K}_p^\pm$ to map $i \in [m]$, to the lower respectively upper endpoints of the fiber over i, that is, $\mathsf{K}_p^\pm(i) = \mathsf{end}_\pm p^{-1}(i)$. The fact that this defines valid sections follows since 1-truss bordisms relate endpoints (see Lemma 2.1.50). Further, observe that the scaffold norm indeed attains its minimal and maximal value on (and only on) $\mathsf{K}_p^\pm(i)$: namely, its minimal value is $\langle \mathsf{K}_p^- \rangle = 0$ and its maximal values is $\langle \mathsf{K}_p^+ \rangle = \#T - \#B$ (where $\#T$ and $\#B$ are the numbers of elements in T respectively B). ⌟

We will denote extremal values of the scaffold norm, achieved on the bottom resp. top sections of a bundle p, by $\mathsf{scaff}_\pm(p) := \langle \mathsf{K}_p^\pm \rangle$. These values bound an interval of natural numbers, which described exactly the image of the scaffold norm.

Lemma 2.2.27 (Scaffold order of sections)**.** *Given a 1-truss bundle p, the scaffold norm $\langle - \rangle : \Gamma_p \to \mathbb{N}$ on sections is a bijection with its image; this image consist of all integers between $\mathsf{scaff}_-(p)$ and $\mathsf{scaff}_+(p)$. As a result, Γ_p obtains a total order, called the 'scaffold order of sections'.*

Proof of Lemma 2.2.27. In Construction 2.2.26 we showed that the scaffold norm has unique bottom and top sections $\mathsf{K}_p^\pm$. We now construct for each $K \neq \mathsf{K}_p^+$ a successor section $\mathsf{s}(K)$ with scaffold norm $\langle \mathsf{s}(K) \rangle = \langle K \rangle + 1$, and, conversely, for each section $K \neq \mathsf{K}_p^-$ a predecessor section $\mathsf{s}(K)$ with scaffold norm $\langle \mathsf{p}(K) \rangle = \langle K \rangle - 1$. Showing that successor and predecessor section constructions are mutually inverse will prove the

---

[13]In other words, $\mathsf{num} : (T, \preceq) \to (\mathbb{N}, \leq)$ restricts on fibers $p^{-1}(i)$, $i \in [m]$, to the unique isomorphisms $\mathsf{num} : (p^{-1}(i), \preceq) \cong [n] \hookrightarrow (\mathbb{N}, \leq)$ of frame ordered fibers with standard simplices.

claim. We can further make the following assumption (which simplifies the treatment of the 'boundary cases' in Remark 2.2.12). Recall, by Construction 2.2.15 we have an isomorphism $\Gamma_p \cong \Gamma_{\Sigma p}$, and by Observation 2.2.25 this isomorphism commutes with the scaffold norm. After potentially replacing p with its suspension Σp in the lemma, we may therefore assume that all sections K in $p : T \to [m]$ have transition index j in the range $0 < j < m + 1$ (in other words, p does not contain purely regular or purely singular sections).

Let K be a section of p with jump morphism $K(j-1) \to K(j)$ in T. Assume that $K \neq \mathsf{K}_p^+$, i.e. K is not the top section. We claim that then there is *either* an arrow $K(j-1) + 1 \to K(j)$ *or* an arrow $K(j-1) \to K(j) + 1$ in T (but there can never be both, due to bimonotonicity of 1-truss bordisms). Note, by assumption on K, there is at least on index $l \in [m]$ such that $K(l)$ has a successor $a \equiv K(l) + 1$ in the fiber over l. We verify the claim in the case where $l < j$ (the case $l \geq j$ follows by the same argument after dualizing p). Note if $l < j$ then $K(l)$ is regular. Thus the successor a is singular, and is mapped by the singular function of the 1-truss bordism $p^{-1}(l \to j-1)$ to a singular element b in the fiber over $(j-1)$. Since $K(l) \prec a$, bimonotonicity implies that $K(j-1) \prec b$. In particular, the regular element $K(j-1)$ has a singular successor $K(j-1) + 1$ in the fiber over $j-1$ in this case. Now, again by bimonotonicity, the 1-truss bordism $p^{-1}(j-1 \to j)$ must relate the singular element $K(j-1) + 1$ either to $K(j)$, or otherwise to some singular b with $K(j) \prec b$. Using the description of singular determined 1-truss bordisms (see Lemma 2.1.51), this implies there is an arrow $K(j-1) \to K(j) + 1$, which verifies the claim.

We now construct the successor section $\mathsf{s}(K)$ of K (still assuming $K \neq \mathsf{K}_p^+$) by distinguishing the two cases established above, namely, whether $K(j-1) + 1 \to K(j)$ or $K(j-1) \to K(j) + 1$. Both case are illustrated in Fig. 2.19. In each case we highlight singular elements in red, regular element in blue, and the jump morphism $K(j-1) \to K(j)$ of K is marked by green dot. The purple dot marks the jump morphism of the successor section of K. In general, the successor section can now be constructed as follows.
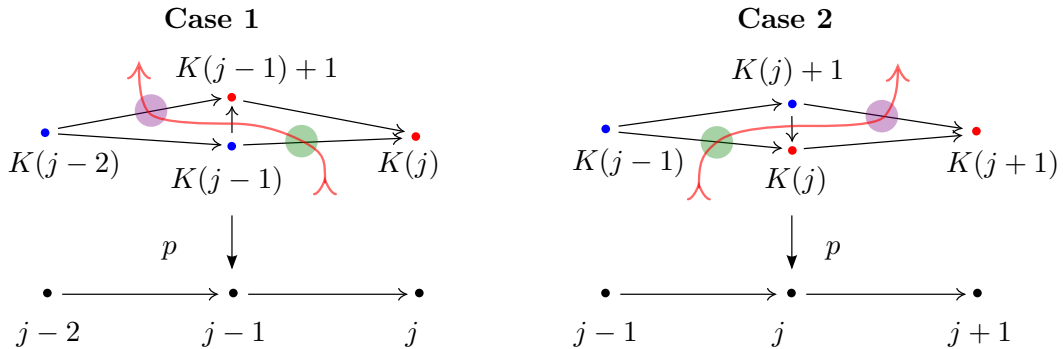
Figure 2.19: The construction of successor sections.

Case 1 If there is an arrow $K(j-1) + 1 \to K(j)$, we construct the successor section $\mathsf{s}(K)$ by setting $\mathsf{s}(K)(j-1) = K(j-1) + 1$ and $\mathsf{s}(K)(i) = K(i)$ if $i \neq j-1$. By assumption on p we must have $j > 1$ (otherwise, $\mathsf{s}(K)$ would be a purely singular section). Profunctoriality of the 1-truss bordism $p^{-1}(j-2 \to j-1)$ then implies there is an arrow $K(j-2) \to K(j-1) + 1$ in T, showing that $\mathsf{s}(K)$ is a valid section.

Case 2 If there is an arrow $K(j-1) \to K(j) + 1$ in T, we construct the successor $\mathsf{s}(K)$ by setting $\mathsf{s}(K)(j) = K(j) + 1$ and $\mathsf{s}(K)(i) = K(i)$ if $i \neq j$. By assumption on p we must have $j < m$ (otherwise, $\mathsf{s}(K)$ would be a purely regular section). Profunctoriality of the 1-truss bordism $p^{-1}(j \to j+1)$ then implies there is an arrow $K(j) + 1 \to K(j+1)$ in T, showing that $\mathsf{s}(K)$ is a valid section.

This completes the construction of successors. The construction of predecessor sections $\mathsf{p}(K)$ of K (for $K \neq \mathsf{K}_p^-$ not equal to the bottom section) reduces the construction of successors after passing to the opposite frame order (i.e. flipping the frame order on each fiber of p, which amounts to reading the total posets in Fig. 2.19 upside down). One verifies that $\mathsf{p}(\mathsf{s}(K)) = K$ and similarly $\mathsf{s}(\mathsf{p}(K)) = K$, which completes the proof. □

2.2.2.2 The case of spacers Similar to the case of sections, we will now show that the set of spacers Ψ_p in a 1-truss bundle $p : T \to [m]$ over the m-simplex carries a natural total order, called the 'scaffold order of spacers'.

Example 2.2.28 (Scaffold order on spacers). Recall from Example 2.2.23 that the scaffold norm of sections corresponds to a directed path that passes through all jump morphism of the suspended bundle once while intersecting exactly one fiber morphism in between any two consecutive jump morphisms—the order in which this path travels through fiber morphisms is exactly the scaffold order on spacers! We illustrate this in Fig. 2.20 for a truss bundle over the 2-simplex, highlighting fiber morphism by colored dots. ⌐
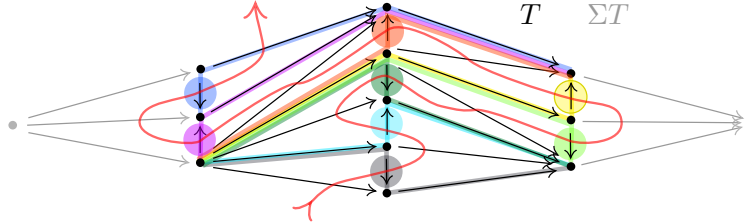
Figure 2.20: The scaffold order on spacers in a 1-truss bundle over the 2-simplex indicated by a directed path through fiber morphisms.

Fixing a spacer and its corresponding fiber morphism in Fig. 2.20, then the 'consecutive jump morphism' which the path intersects immediately before the given fiber morphisms correspond to so-called 'lower and upper boundary sections' of the spacer. We formally define these as follows.

Construction 2.2.29 (Upper and lower boundaries of spacer). For a spacer $L : [m+1] \to T$ of a 1-truss bundle $p : T \to [m]$ we construct sections $\partial_- L, \partial_+ L : [m] \to T$ called the **lower boundary** respectively **upper boundary** of L. Let j denote the transition index of L, that is, $L(j) \to L(j+1)$ is the fiber morphisms of L. If $L(j) \prec L(j+1)$, then we construct $\partial_- L$ as the $(j+1)$th face Ld_{j+1} of L, and $\partial_+ L$ as the jth face Ld_j of L. Otherwise, if $L(j+1) \prec L(j)$, then we construct $\partial_- L$ as the jth face Ld_j of L, and $\partial_+ L$ as the $(j+1)$th face Ld_{j+1}. ⌐

Example 2.2.30 (Upper and lower boundaries). We depict two spacers L and L' in a 1-truss bundle together with the lower and upper boundary sections $\partial_\pm L$ respectively $\partial_\pm L'$ in Fig. 2.21. ⌐
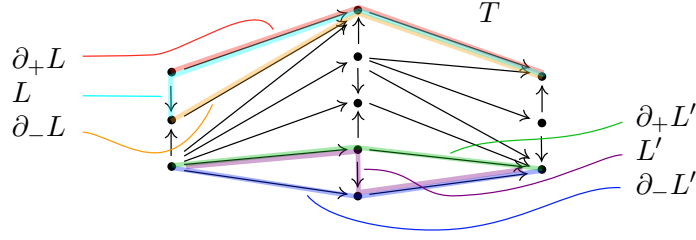
Figure 2.21: Upper and lower boundaries of spacers.

Remark 2.2.31 (Uppers boundaries succeed lower boundaries). Note that the preceding construction ensures that we have $\langle \partial_+ L \rangle = \langle \partial_- L \rangle + 1$, that is, $\partial_+ L$ is the successor of $\partial_- L$ in the totally ordered set $(\Gamma_p, \preceq)$ of sections in p. ⌐

In analogy to the definition of scaffold norms for sections (see Definition 2.2.24), we introduce the following scaffold norm of spacers.

Definition 2.2.32 (Scaffold norm of spacers). Consider a 1-truss bundle $p : T \to [m]$ and its set of spacers Ψ_p (see Notation 2.2.22). The **scaffold norm** $\langle - \rangle$ is the function

$$\langle - \rangle : \Psi_p \to \frac{1}{2}\mathbb{N}$$
$$L \mapsto \frac{\langle \partial_- L \rangle + \langle \partial_+ L \rangle}{2}$$

where $\langle \partial_\pm L \rangle$ denotes the scaffold norm of the sections $\partial_\pm L$. ⌐

In analogy to the construction of the scaffold order of sections in Lemma 2.2.27, we now construct the scaffold order of spacers; as before, the order is induced by exhibiting Ψ_p as a subset of a totally ordered set (in this case, the half integers $\frac{1}{2}\mathbb{N}$) via the scaffold norm function.

Lemma 2.2.33 (Scaffold order for spacers). *Let $p : T \to [m]$ be a 1-truss bundle over the m-simplex $[m]$. The scaffold norm $\langle - \rangle : \Psi_p \to \frac{1}{2}\mathbb{N}$ on spacers is a bijection with its image; this image consist of all half integers between $\mathsf{scaff}_-(p) + \frac{1}{2}$ and $\mathsf{scaff}_+(p) - \frac{1}{2}$. As a result, Ψ_p obtains a total order, called the 'scaffold order of spacers'.*

Proof of Lemma 2.2.33. Note that each spacer L in p is uniquely determined by its boundary sections $\partial_\pm L$. By the previous Lemma 2.2.27 it follows that $\langle - \rangle : \Psi_p \to \frac{1}{2}\mathbb{N}$ is injective. To see that it is also surjective, recall from the proof of Lemma 2.2.27 the construction of successors $\mathsf{s}(K)$ of sections K in p that are not maximal in $(\Gamma_p, \preceq)$. Inspecting the two cases for the construction of $\mathsf{s}(K)$, note that in case 1, the fiber morphism $\mathsf{s}(K)(j-1) \to K(j-1)$ determines a spacer L (see Construction 2.2.20) with $\partial_- L = K$ and $\partial_+ L = \mathsf{s}(K)$. Similarly, in case 2, the fiber morphism $K(j) \to \mathsf{s}(K)(j)$ determines a spacer L with $\partial_- L = K$ and $\partial_+ L = \mathsf{s}(K)$. This shows any two consecutive sections are the boundary sections of a spacer, which thus proves surjectivity of the scaffold norm of spacers as claimed. □

Finally, as a first application of the results here, we will prove several observation that will be useful for the later classification of framed regular cell complexes.

Terminology 2.2.34 (Fiber categories in 1-truss bundles). Let $p : T \to B$ be a 1-truss bundle and consider a non-degenerate simplex $z : [m] \to B$. The 'fiber category' $\Phi_1 \, p_z$ in p (over z) is the free category whose objects are sections K in Γ_{z^*p} (that is, sections of the pullback bundle z^*p) together with a generating morphism $L : \partial_- L \to \partial_+ L$ for each spacer L in Ψ_{z^*p}. ⌐

Definition 2.2.35 (Transition functors of fiber categories). Let $p : T \to B$ be a 1-truss bundle and consider non-degenerate simplices $z : [m] \to B$ and $y : [l] \to B$ such that z is a face of y, that is, z factors through y by an injective map $[m] \hookrightarrow [l]$. We can restrict sections and spacers in the bundle y^*p along the bundle inclusion $z^*p \hookrightarrow y^*p$ (note sections restrict to sections, while spacers restrict to sections or spacers). This induces a functor $-|_{z \subset y} : \Phi_1 \, p_y \to \Phi_1 \, p_z$ called the **fiber transition** functor from y to z in p. ⌐

Observation 2.2.36 ('Flatness' of 1-truss bundles). For all 1-truss bundles $p : T \to B$ the following holds.

1. All fiber categories in p are total orders.
2. All fiber transitions functors in p are endpoint preserving (and thus surjective).

Indeed, the first statement follows from Lemma 2.2.27 and Lemma 2.2.33 together with the observation that $\langle \partial_\pm L \rangle = \langle L \rangle \pm \frac{1}{2}$. The second statement follows by construction of top and bottom sections (see Construction 2.2.26) as the reader can check. ⌐

2.3 n-Trusses, n-truss bordisms, n-truss bundles, and truss blocks

We finally turn to the notion of n-trusses, which are 'n-dimensional' generalizations of 1-trusses. Just as 1-trusses geometrically translate to (framed) stratified intervals, n-trusses geometrically translate to certain (framed) stratified subspaces of $\mathbb{R}^n$. Before diving into the theory of n-trusses, we will define n-trusses and illustrate this geometric translation with examples. In fact, defining n-trusses is rather straight-forward: n-trusses are towers of 1-truss bundles over the trivial poset $[0]$ as follows.

Definition 2.3.1 (n-Trusses). An n**-truss** T is a tower of 1-truss bundles

$$T_n \xrightarrow{p_n} T_{n-1} \xrightarrow{p_{n-1}} \dots \xrightarrow{p_2} T_1 \xrightarrow{p_1} T_0 = [0]$$

where the total poset of each p_i is the base poset of p_{i+1}. We call T_n (with face order) the **total poset** of T. ⌐

Terminology 2.3.2 (Closed and open n-trusses). We call an n-truss T **closed** respectively **open** if all its 1-truss bundles p_i are closed respectively open. ⌐

Let us illustrate n-trusses by examples. In each case we also depict the 'corresponding stratifications' in $\mathbb{R}^n$: for now, this correspondence may be understood to identify entrance path posets of stratifications with face orders of the corresponding trusses. In Chapter 4 we will make the correspondence fully precise, by also accounting for framing structures on both stratifications and trusses.

Example 2.3.3 (A closed 2-truss)**.** In Fig. 2.22 we illustrate a closed 2-truss T. The bundle $p_1 : T_1 \to T_0$ has base post $T_0 = [0]$ and therefore its total poset is simply a 1-truss T_1. In contrast, the total poset T_2 of the bundle $p_2 : T_2 \to T_1$ looks more complicated—nonetheless it is '2-dimensional' in character as can be seen. Accordingly, the stratification that corresponds to T can be embedded in $\mathbb{R}^2$ as shown on the right in Fig. 2.22—in fact, this stratification can also be understood as flat 2-framed regular cell complex (see Definition 1.3.53). Note that the entrance path poset (see Definition 1.3.3) of this framed regular cell complex can be canonically identified with T_2. ⌐
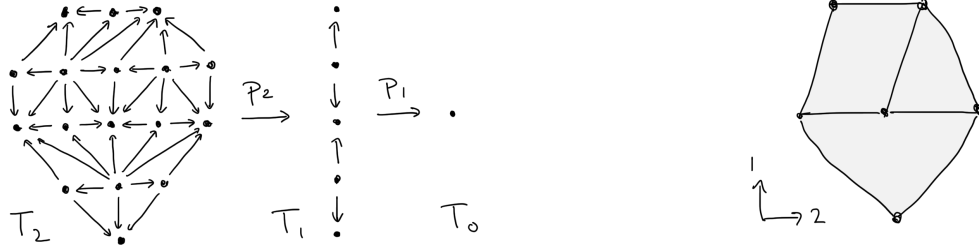
Figure 2.22: A closed 2-truss together with its corresponding stratification.

The previous example is an instance of a more general correspondence of *closed n-trusses* and flat framed regular cell complexes: a proof of this correspondence will be the subject of Chapter 3. The next example illustrates the case of an open 3-truss.

Example 2.3.4 (An open 3-truss)**.** Consider the open 3-truss T shown in Fig. 2.23: note that, in order to simplify our notation, we depicted only 'generating' arrows in T_2 and T_3 (with all other arrows being composites of the depicted ones). To the right of T we illustrate its corresponding stratification. Note that this now yields a stratification of the *open* 3-cube embedded in $\mathbb{R}^3$. Note further, that objects in the total poset T_3 are again in correspondence with strata in the stratification realizing T, that is, the face order of T_3 can be identified with the entrance path poset of the stratification (see Definition B.1.6). ⌐
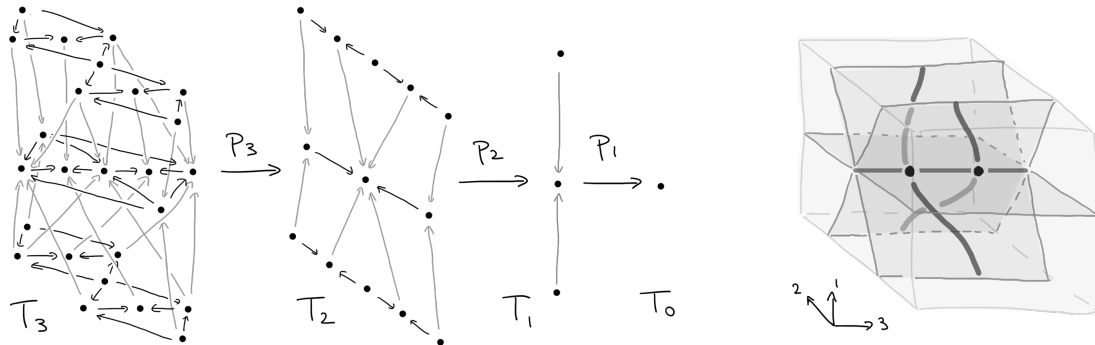
Figure 2.23: An open 3-truss together with its corresponding stratification.

Preview 2.3.5 (Classifying the braid)**.** The truss in the preceding example later plays a particularly interesting role: its corresponding stratification *canonically* refines the

braid[14]. This will formalized and generalized in Chapter 5 to a correspondence of so-called 'stratified trusses' and 'flat framed stratifications'. We revisit the completed combinatorial description of the braid in Example 5.3.22. ⌐

While our Definition 2.3.1 of n-trusses is short and simple, their combinatorial theory turns out to be rich. Moreover, properly dealing with notions such as 'n-truss bordism' and 'n-truss bundles' requires understanding a yet different aspect of truss theory, namely, of so-called 'labelings' of bundles which we shall first introduce in the case of 1-truss bundles in Section 2.3.1. Labeled truss bundles then allow to categorically organize n-trusses in several interesting ways, as we will discuss in Section 2.3.2 (they also provide the correct structure for capturing our later definition of 'stratified trusses'). In the final Section 2.3.3, we introduce the notion of 'block' and 'brace' trusses: these are the 'building blocks' of n-trusses analogous to cell complexes being 'built of cells'. More precisely, we will see that closed (and dually, open) trusses are gluings of blocks (resp. of braces). Importantly, considering (regular) presheaves on the category of such blocks will enable our classification of framed regular cell complexes in Chapter 3

2.3.1   Labeled 1-trusses, bordisms, and bundles Labeled 1-trusses are 1-trusses equipped with a 'labeling' in a category C as follows.

Definition 2.3.6 (Labeled 1-trusses). Let C be a 'labeling' category. A **C-labeled 1-truss** T is a pair consisting of an 'underlying' truss $\underline{T}$ together with a 'labeling' functor $\mathrm{lbl}_T : (\underline{T}, \trianglelefteq) \to C$. ⌐

Remark 2.3.7 (Labeled 1-trusses as spans). Given a C-labeled 1-truss $T \equiv (\underline{T}, \mathrm{lbl}_T)$ we may equivalently express its data as a span

$$C \xleftarrow{\mathrm{lbl}_T} \mathrm{Tot}(\underline{T}) \xrightarrow{\pi_{\underline{T}}} [0]$$

where the right leg $\pi_{\underline{T}}$ is a 1-truss bundle with underlying 1-truss $\underline{T} : [0] \to \mathsf{TBord}^1$ (see Construction 2.1.70), and the left leg is the labeling functor lbl_T up to the canonical identification of 1-trusses $(\mathrm{Tot}(\underline{T}), \trianglelefteq, \dim, \preceq) \cong (\underline{T}, \trianglelefteq, \dim, \preceq)$. ⌐

Many notions from the theory of 1-trusses as developed in Section 2.1 naturally generalize to the labeled case. In particular, we now discuss a notion of 'labeled 1-truss bordism', after which we define 'labeled 1-truss bundles'.

2.3.1.1   The definition of labeled 1-truss bordisms We define labeled 1-truss bordism. In analogy to the definition of labeled 1-trusses, we want to equip 1-truss bordisms with functorial 'labeling data' in a given category C. To make this precise, we will use that 1-truss bordisms R correspond to 1-truss bundles $\pi_R : \mathrm{Tot}(R) \to [1]$ over the 1-simplex $[1]$ via the classification-totalization correspondence for 1-truss bundles.

Definition 2.3.8 (Labeled 1-truss bordisms). For a category C, a **C-labeled 1-truss bordism** R is a pair consisting of an 'underlying' 1-truss bordism $\underline{R}$ together with a 'labeling' functor $\mathrm{lbl}_R : (\mathrm{Tot}(\underline{R}), \trianglelefteq) \to C$ from the total poset $\mathrm{Tot}(\underline{R})$ of $\underline{R}$ into C. ⌐

---

[14]Here, the 'braid' is the stratification of the 3-cube containing two 'crossing' wires.

In other words, a C-labeled 1-truss bordism $R \equiv (\underline{R}, \mathrm{lbl}_R)$ is a span of the form

$$\mathsf{C} \xleftarrow{\mathrm{lbl}_R} \mathrm{Tot}(\underline{R}) \xrightarrow{\pi_R} [1]$$

where π_R is the total bundle of $\underline{R} : [1] \to \mathsf{TBord}^1$.

Terminology 2.3.9 (Domain and codomain of labeled 1-truss bordisms). Given a C-labeled 1-truss bordism $R \equiv (\underline{R}, \mathrm{lbl}_R)$, the 'domain' $\mathrm{dom}(R)$ (resp. 'codomain' $\mathrm{cod}(R)$) of R is the C-labeled 1-truss $(\mathrm{Tot}(\underline{R})|_i, \mathrm{lbl}_R : \mathrm{Tot}(\underline{R})|_i \to \mathsf{C})$ obtained by restricting the totalization $\mathrm{Tot}\underline{R}$ and the labeling lbl_T to the fiber over the element $i = 0$ (resp. over $i = 1$) in the base $[1]$. We write this as $R : \mathrm{dom}(R) \twoheadrightarrow \mathrm{cod}(R)$. ⌐

The interesting question concerning the definition of labeled 1-truss bordisms is whether labeled 1-truss bordisms have well-defined compositions.

Definition 2.3.10 (Composition of labeled 1-truss bordisms). Given two C-labeled truss bordisms R_{01} and R_{12} which are composable (that is, $\mathrm{cod}(R_{01}) = \mathrm{dom}(R_{12})$), a **composition candidate** R_{02} of R_{12} and R_{01} is a C-labeled truss bordism for which there exists 'composition witness' $W \equiv (\underline{W} : [2] \to \mathsf{TBord}^1, \mathrm{lbl}_W : \mathrm{Tot}(\underline{W}) \to \mathsf{C})$ which restricts on each arrow $(i \to j) : [1] \hookrightarrow [2]$ to R_{ij}; that is, R_{ij} equals the C-labeled 1-truss bordism $(\mathrm{Tot}(\underline{W})|_{i \to j}, \mathrm{lbl}_W : \mathrm{Tot}(\underline{W})|_{i \to j} \to \mathsf{C})$. ⌐

As we now show composition candidates and composition witnesses always uniquely exist. The proof will use 'truss induction' as discussed in Section 2.2.

Lemma 2.3.11 (Existence and uniqueness of composition witnesses). *Composition candidates and witnesses of composable C-labeled 1-truss bordisms exist uniquely.*

Proof. Consider C-labeled 1-truss bordisms R_{01} and R_{12}, composable at a C-labeled 1-truss $S = \mathrm{cod}(R_{01}) = \mathrm{dom}(R_{12})$. We construct their unique composition witness $W = (\underline{W}, \mathrm{lbl}_W)$ of (their composition candidate is determined by restricting W over $(0 \to 2)$). First observe, the functor $\underline{W} : [2] \to \mathsf{TBord}^1$ must be defined by setting $\underline{W}(0 \to 1) = \underline{R_{01}}$ and $\underline{W}(1 \to 2) = \underline{R_{12}}$. The task is to see that the labeling functor $\mathrm{lbl}_W : \mathrm{Tot}\underline{W} \to \mathsf{C}$ exists uniquely.

Since $\underline{W}$ must restrict to R_{01} over $(0 \to 1)$, and to R_{12} over $(1 \to 2)$, it remains to define lbl_W on arrows f in $\mathrm{Tot}\underline{W}$ lying over $0 \to 2$. Since 1-truss bordisms compose as their underlying relations, any arrow f lying over $0 \to 2$ in $[2]$ can be written as a composite $f = gh$ for some arrow g over $(0 \to 1)$ and some arrow h over $(1 \to 2)$. Any candidate labeling $\mathrm{lbl}_W : \mathrm{Tot}\underline{W} \to \mathsf{C}$ will have to satisfy that

$$\mathrm{lbl}_W f = \mathrm{lbl}_{R_{12}} h \circ \mathrm{lbl}_{R_{01}} g$$

(thus the labeling lbl_W must be unique if it exists). Provided that this composition is independent of our choice of g and h, the value $\mathrm{lbl}_W f$ is well-defined, thus completing the definition of lbl_W.

Abbreviate the composite morphism $\mathrm{lbl}_{R_{12}} g \circ \mathrm{lbl}_{R_{01}} h$ in C by $\mathsf{lbl}(g, h)$. Given another pair of arrows g' and h' decomposing f, we need to verify that $\mathsf{lbl}(g, h) = \mathsf{lbl}(g', h')$. To prove this, we use truss induction (recall, in particular, the scaffold order $\preceq$ on sections, see Lemma 2.2.27). The arrows g and h (resp. g' and h') are the spine vectors of a section

simplex K (resp. K') in the bundle π_W. Assume that $K \preceq K'$ (otherwise switch K and K' in the following). Let $K = K_0, K_1, ..., K_k = K'$ be a chain of successor sections starting at K and ending in K'. Arguing inductively, we can assume $k = 1$, that is, $K' = K_1 = \mathsf{s}(K)$. By Lemma 2.2.33 there is a spacer L with lower boundary $\partial_- L = K$ and upper boundary $\partial_+ L = K'$. Note that the spine $L(2 \to 3) \circ L(1 \to 2) \circ L(0 \to 1)$ of L composes both to the spine $K(1 \to 2) \circ K(0 \to 1) = h \circ g$ of K and to the spine $K'(1 \to 2) \circ K'(0 \to 1) = h' \circ g'$ of K'. Functoriality of the labelings $\mathrm{lbl}_{R_{12}}$ of and $\mathrm{lbl}_{R_{12}}$ now implies

$$\mathsf{lbl}(g, h) = \mathrm{lbl}_{R_{12}} L(2 \to 3) \circ \mathrm{lbl}_S L(1 \to 2) \circ \mathrm{lbl}_{R_{01}} L(0 \to 1) = \mathsf{lbl}(g', h')$$

(where lbl_S is the restriction of both $\mathrm{lbl}_{R_{12}}$ and $\mathrm{lbl}_{R_{01}}$ to the fiber over $1 \in [2]$) as required. $\square$

The preceding result shows that labeled 1-truss bordisms have a well-defined notion of composition, allowing us to introduce the following category.

Definition 2.3.12 (Labeled 1-truss bordisms)**.** Given a category C, the **category of C-labeled 1-truss bordisms** $\mathsf{TBord}^1_{//\mathsf{C}}$ is the category whose objects are C-labeled 1-trusses (see Definition 2.3.6); whose morphisms are C-labeled 1-truss bordisms (see Definition 2.3.8); and whose composition is as given in Definition 2.3.10. $\quad\quad \lrcorner$

Noteworthily, the definition of the category of labeled 1-truss bordisms generalizes by allowing labelings to take values in '∞-categories' as well, which we record in the following remark (without using it later on).

Remark 2.3.13 (1-Truss bordism labeled in an ∞-category)**.** Let $\mathcal{C}$ be a quasicategory (i.e. a simplicial set in which inner horn fillers exist, see [BV06]). We define the quasicategory $\mathcal{T}Bord^1_{//\mathcal{C}}$ of '$\mathcal{C}$-labeled 1-truss bordisms' to be the simplicial set whose k-simplices are given by tuples consisting of a 'k-simplex of bordisms' $S : [k] \to \mathsf{TBord}^1$ together with a '$\mathcal{C}$-labeling' of its total space now given by an ∞-functor $\mathrm{lbl}_S : \mathrm{Tot}(S) \to \mathcal{C}$. Note that, for a simplicial map $f : [l] \to [k]$, we obtain an l-simplex in $\mathcal{T}Bord^1_{//\mathcal{C}}$ as the tuple consisting $S \circ f : [l] \to \mathsf{TBord}^1$ and the labeling $\mathrm{lbl}_S \circ \mathrm{Tot} f : \mathrm{Tot}(S \circ f) \to \mathcal{C}$. To see that the resulting simplicial set $\mathcal{T}Bord^1_{//\mathcal{C}}$ is indeed a quasicategory, i.e. as inner horn fillers, one uses truss induction for bundles over general k-simplices (conceptually following the proof of our earlier Lemma 2.3.11, which used truss induction only over the 2-simplex). $\quad \lrcorner$

Remark 2.3.14 (Unlabeled 1-truss bordisms are trivially labeled)**.** Unlabeled 1-truss bordisms can be thought of as having 'trivial' labelings, i.e. labelings in the terminal category $*$: indeed, the functor $\mathsf{TBord}^1_{//*} \to \mathsf{TBord}^1$ mapping $*$-labeled 1-trusses T (resp. bordisms R) to their underlying 1-truss $\underline{T}$ (resp. their underlying 1-truss bordism $\underline{R}$) is an isomorphism of categories. $\quad\quad \lrcorner$

The construction of $\mathsf{TBord}^1_{//\mathsf{C}}$ is, in fact, functorial in the category C, yielding the 'labeled 1-truss bordism' functor defined below. Its behavior on functors of categories can be described as follows.

Definition 2.3.15 (Relabeling along a functor)**.** Consider a functor $F : \mathsf{C} \to \mathsf{D}$ between categories C and D. The F**-relabeling functor** between categories of labeled 1-truss bordisms

$$\mathsf{TBord}^1_{//F} : \mathsf{TBord}^1_{//\mathsf{C}} \to \mathsf{TBord}^1_{//\mathsf{D}}$$

is defined to map a C-labeled 1-truss T to the D-labeled 1-truss with underlying truss $\underline{T}$ and labeling $F \circ \mathrm{lbl}_T$, and a C-labeled 1-truss bordism R to the D-labeled 1-truss bordism with underlying 1-truss bordism $\underline{R}$ and labeling $F \circ \mathrm{lbl}_R$. ⌐

Terminology 2.3.16 (The label forgetting functor). Let C be a category. Relabeling by the terminal functor $\mathsf{C} \to *$ yields the functor

$$\mathsf{TBord}^1_{/\!/\mathsf{C}\to *} : \mathsf{TBord}^1_{/\!/\mathsf{C}} \to \mathsf{TBord}^1_{/\!/*} \cong \mathsf{TBord}^1$$

which we call the 'forgetful functor' and usually abbreviate by an underline $(\underline{-})$. ⌐

Definition 2.3.17 (The labeled 1-truss bordism functor)**.** The **labeled 1-truss bordism functor** is the endofunctor
$$\mathsf{TBord}^1_{/\!/-} : \mathsf{Cat} \to \mathsf{Cat}$$

that takes a category C to the category $\mathsf{TBord}^1_{/\!/\mathsf{C}}$ of C-labeled 1-truss bordisms and a functor $\mathsf{F} : \mathsf{C} \to \mathsf{D}$ to the F-relabeling $\mathsf{TBord}^1_{/\!/\mathsf{F}} : \mathsf{TBord}^1_{/\!/\mathsf{C}} \to \mathsf{TBord}^1_{/\!/\mathsf{D}}$. ⌐

We end our discussion of labeled 1-truss bordisms by highlighting an alternative definition of the notion in abstract categorical terms. The reader without an inclination for abstract categorical structures may readily skip ahead to the next section.

Terminology 2.3.18 (Vertical comma categories). For a normal pseudofunctor $H : \mathsf{D} \to \mathsf{Prof}$ from a category D into the bicategory of profunctors, and a category C, define the 'vertical comma bicategory' $H_{/\!/\mathsf{C}}$ as follows: objects are tuples $(d \in \mathsf{D}, \mathsf{F} : H(d) \to \mathsf{C})$; 1-morphisms $(d, \mathsf{F}) \to (d', \mathsf{F}')$ are tuples $(f : d \to d', \alpha : H(f) \Rightarrow \mathrm{Hom}_\mathsf{C}(\mathsf{F}-, \mathsf{F}'-))$; composition uses pseudofunctoriality of H and the bicategorical structure of Prof. ⌐

To apply the notion of vertical comma categories to the case of 1-truss bordism, we next define a functor $\iota : \mathsf{TBord}^1 \to \mathsf{Prof}$. Recall the fully faithful inclusion $\mathsf{Bool} \hookrightarrow \mathsf{Set}$, mapping 0 to the empty set $\emptyset$, and 1 to the singleton set $\{*\}$.

Terminology 2.3.19 (Inclusion of truss bordisms into ordinary profunctors). We define the 'bordisms-as-profunctors' pseudofunctor

$$\iota : \mathsf{TBord}^1 \to \mathsf{Prof}$$

from category 1-truss bordisms to the bicategory of profunctors Prof, by mapping 1-trusses T to the category $(T, \trianglelefteq)$ and 1-truss bordisms R to the post-composition of their underlying Boolean profunctor with the inclusion $\mathsf{Bool} \hookrightarrow \mathsf{Set}$. ⌐

The fact that ι is a well-defined pseudofunctor is addressed in the following remark; we highlight that this claim relies on the rather special properties of 1-truss bordisms and a similar definition fails for general Boolean profunctors.

Remark 2.3.20 (The specialness of 1-truss bordisms). Ordinary profunctors compose by coends (see [Lor21, §5]). Importantly, given general Boolean profunctors $R : X \nrightarrow Y$ and $S : Y \nrightarrow Z$ between preorders it need *not* be the case that $\iota(S \circ R) \cong (\iota S) \circ (\iota R)$; that is, the composition of Boolean profunctors need not coincide with their composition as ordinary profunctors. However, in the case of 1-truss bordisms R and S, the isomorphism

$\iota(S \circ R) \cong (\iota S) \circ (\iota R)$ always uniquely exists: this follows by explicitly evaluating the colimit defining the profunctor composite $(\iota S) \circ (\iota R)$ and following the arguments in the proof of Lemma 2.3.11. ⌐

Combining the above notions, we now reach the following alternative description of labeled 1-truss bordisms.

Observation 2.3.21 (Labeled 1-truss bordisms). The category of C-labeled 1-truss bordisms is equivalent to the vertical comma category $\iota_{//\mathsf{C}}$ as defined above. We omit a verification of this observation. ⌐

2.3.1.2   The definition of labeled 1-truss bundles We next generalize 1-truss bundles to the labeled case; similar to the previous definitions of labeled 1-trusses and their bordism, this now endows the total poset of a 1-truss bundle with a labeling functor as follows.

Definition 2.3.22 (Labeled 1-truss bundles). Let C be a 'labeling' category and B a 'base' poset. A **C-labeled 1-truss bundle** p over B is a pair consisting of an 'underlying' 1-truss bundle $p : T \to B$ over B together with a 'labeling' functor $\mathrm{lbl}_p : (T, \trianglelefteq) \to \mathsf{C}$. ⌐

In other words, a C-labeled 1-truss bundle $p \equiv (\underline{p}, \mathrm{lbl}_p)$ is a span of the form

$$\mathsf{C} \xleftarrow{\ \mathrm{lbl}_p\ } T \xrightarrow{\ p\ } B \quad .$$

Labeled 1-truss bundles support the following notion of maps, which mirrors the definition of maps of their unlabeled counterpart (see Definition 2.1.15).

Definition 2.3.23 (Maps of labeled 1-truss bundles). For categories C and D, let p be a C-labeled 1-truss bundle, and q an D-labeled 1-truss bundle. A **map of C-labeled 1-truss bundles** $F : p \to q$ is a pair consisting of an 'underlying' 1-truss bundle map $\underline{F} : \underline{p} \to \underline{q}$, as well as a 'label category' functor $\mathrm{lbl}_F : \mathsf{C} \to \mathsf{D}$ such that $\mathrm{lbl}_F \circ \mathrm{lbl}_p = \mathrm{lbl}_q \circ \underline{F}$. ⌐

Equivalently, one may think of labeled 1-truss bundle maps $F : p \to q$ as maps of spans $p \to q$, i.e. commuting diagrams

$$
\begin{array}{ccccc}
\mathsf{C} & \xleftarrow{\ \mathrm{lbl}_p\ } & T & \xrightarrow{\ \underline{p}\ } & B \\
{\scriptstyle \mathrm{lbl}_F}\downarrow & & \downarrow{\scriptstyle \underline{F}} & & \downarrow \\
\mathsf{D} & \xleftarrow[\ \mathrm{lbl}_q\ ]{} & S & \xrightarrow[\ \underline{q}\ ]{} & C
\end{array}
$$

in which $\underline{F}$ is a 1-truss bundle map $\underline{p} \to \underline{q}$.

Terminology 2.3.24 (Singular, regular, and balanced labeled bundle maps). A labeled 1-truss bundle map F is said to be 'singular', 'regular' or 'balanced' if its underlying 1-truss bundle map $\underline{F}$ is ⌐

Maps of labeled 1-truss bundles compose in the obvious way, yielding the following category.

Notation 2.3.25 (The category of labeled 1-truss bundle). The category of labeled 1-truss bundles and their maps will be denoted by $\mathsf{LblTrsBun}_1$. ⌐

Remark 2.3.26 (All 1-truss bundles are trivially labeled). Every 1-truss bundle $p : T \to B$ admits a trivial labeling $T \to *$ in the terminal category $*$. A map between trivially labeled 1-truss bundles is simply a map between their underlying 1-truss bundles. This provides a fully faithful inclusion

$$\mathsf{TrsBun}_1 \hookrightarrow \mathsf{LblTrsBun}_1$$

and in this sense the notion of 1-truss bundles is properly generalized by the notion of labeled 1-truss bundles. ⌐

Terminology 2.3.27 (Label and base preserving maps). A labeled 1-truss bundle map (F, G, H) is called 'label preserving' if $H = \mathrm{id}_\mathsf{C}$ is the identity on the labeling category C, and 'base preserving' if $F = \mathrm{id}_B$ is the identity on the base poset B. ⌐

Definition 2.3.28 (Restrictions of labeled 1-truss bundles). Given a C-labeled 1-truss bundle $p = (\underline{p} : T \to B, \mathrm{lbl}_p : T \to \mathsf{C})$ and a subposet $C \hookrightarrow B$, then the **restriction** $p|_C$ is the C-labeled 1-truss bundle $(\underline{p}\big|_C : T|_C \to C, \mathrm{lbl}_p : T|_C \to \mathsf{C})$. ⌐

Note for any non-identity morphisms $f : [1] \hookrightarrow B$, we may think of the restriction $p|_f$ as a C-labeled 1-truss bordism. In this sense, analogous to the unlabeled case, labeled 1-truss bordisms describe 'fiber transitions' in labeled 1-truss bundles.

2.3.1.3   Classification and totalization for labeled 1-truss bundles We construct a classification-totalization equivalence between C-labeled 1-truss bundles and their concordances and functors into the category of C-labeled 1-truss bordisms (analogous to the classification-totalization equivalence for unlabeled 1-truss bundles, see Construction 2.1.74). We start with the construction of 'classifying functors'.

Construction 2.3.29 (Classifying functors of labeled 1-truss bundles). Consider a C-labeled 1-truss bundle p over a poset B. Its **classifying functor** χ_p is the functor $B \to \mathsf{TBord}^1_{/\!/\mathsf{C}}$ that maps an object $x : [0] \hookrightarrow B$ to the C-labeled 1-truss $p|_x$ and a non-identity arrow $f : [1] \hookrightarrow B$ to the C-labeled 1-truss bordism $p|_f$. The fact that this constructs a functor can be checked using the uniqueness of composition witnesses (see Lemma 2.3.11). ⌐

Conversely, one constructs 'total bundles' as follows.

Construction 2.3.30 (Totalizing bundles of labeled 1-truss bundles). For a poset B and a category C, consider a functor $\mathsf{F} : B \to \mathsf{TBord}^1_{/\!/\mathsf{C}}$. We define its **total bundle** $\pi_\mathsf{F} \equiv (\pi_\mathsf{F}, \mathrm{lbl}_\mathsf{F})$ to be the C-labeled 1-truss bundle
- whose underlying bundle π_F is the total 1-truss bundle $\pi_{\underline{\mathsf{F}}}$ (of the composition of F with the label forgetting functor $(\underline{-}) : \mathsf{TBord}^1_{/\!/\mathsf{C}} \to \mathsf{TBord}^{\underline{1}}$),
- and whose labeling functor $\mathrm{lbl}_\mathsf{F} : \mathrm{Tot}(\underline{\mathsf{F}}) \to \mathsf{C}$ is determined on fibers over objects $x \in B$ as the labeling functor $\mathrm{lbl}_{Fx} : \mathrm{Tot}(\underline{F}x) \to \mathsf{C}$ (of the labeled 1-truss Fx) and on fibers over non-identity arrows $f : [1] \to B$ as the labeling functor $\mathrm{lbl}_{Ff} : \mathrm{Tot}(\underline{F}f) \to \mathsf{C}$ (of the labeled 1-truss bordism Ff).

The fact that lbl_F is indeed a functor can be seen to follow again using the construction of composition witness of labeled 1-truss bordisms (see Lemma 2.3.11). ⌐

In order to promote the preceding constructions to functors we introduce a notion labeled 1-truss bundle concordances as follows.

Definition 2.3.31 (Concordances of labeled 1-truss bordisms)**.** For a poset B and a 'labeling' category C, a **C-labeled 1-truss bundle concordance** $u : p \Rightarrow q$ between C-labeled 1-truss bundles p and q over B is a C-labeled 1-truss bundle over $B \times [1]$ such that $u|_{B \times \{0\}} = p$ and $u|_{B \times \{1\}} = q$. ⌐⌐

Remark 2.3.32 (Invertible concordances need not be unique)*.* In contrast to Remark 2.1.72, invertible labeled bundle concordances between fixed labeled 1-truss bundles need not be unique if they exist (since the labeling category C may have non-trivial automorphisms). ⌐⌐

Note that, given a C-labeled 1-truss bundle concordance $u : p \Rightarrow q$, its classifying functor χ_u is equivalently a natural transformation $\chi_p \Rightarrow \chi_q : B \to \mathsf{TBord}^1_{/\!/\mathsf{C}}$; we often refer to this natural transformation as the 'classifying natural transformation' of u.

Definition 2.3.33 (Categories of 1-truss bundle concordances)**.** For a poset B and a category C, define the **category of concordances of C-labeled 1-truss bundles over B**, denoted by $\mathsf{TrsConc}_1(B, \mathsf{C})$, to have C-labeled 1-truss bundles over B as objects, and C-labeled 1-truss bundle concordances as morphisms. Composition of two such concordances $u : p \Rightarrow q$ and $v : q \Rightarrow r$ is determined by the condition that $\chi_{v \circ u} = \chi_v \circ \chi_u$.[15] ⌐⌐

Classification and totalization then organize into equivalences of categories as follows.

Construction 2.3.34 (Classification and totalization functors)**.** Given a poset, we define the **classification functor**

$$\chi_- : \mathsf{TrsConc}_1(B, \mathsf{C}) \to \mathrm{Fun}(B, \mathsf{TBord}^1_{/\!/\mathsf{C}})$$

to map labeled 1-truss bundles $p : T \to B$ to their classifying functor χ_p, and labeled concordances $u : p \Rightarrow q$ to their classifying natural transformations $\chi_u : \chi_p \Rightarrow \chi_q$. Conversely, we define the **totalization functor**

$$\pi_- : \mathrm{Fun}(B, \mathsf{TBord}^1_{/\!/\mathsf{C}}) \to \mathsf{TrsConc}_1(B, \mathsf{C})$$

to take functors $\mathsf{F} : B \to \mathsf{TBord}^1_{/\!/\mathsf{C}}$ to their total 1-truss bundles π_F, and similarly natural transformations $\alpha : B \times [1] \to \mathsf{TBord}^1$ to their total 1-truss bundles π_α. ⌐⌐

Observation 2.3.35 (Classification and totalization are inverse)*.* Classification of labeled 1-truss bundles and totalization of functors into labeled 1-truss bordisms provide an equivalence of categories. ⌐⌐

---

[15]Explicitly, working skeletally with underlying 1-truss bundles (that is, identifying unlabeled 1-truss bundles up to invertible concordances, see Remark 2.1.72), we may define $v \circ u = \pi_{\chi_v \circ \chi_u}$.

2.3.1.4 Pullback, dualization, and suspension of labeled 1-truss bundles We introduce pullbacks, duals, and suspensions of labeled 1-truss bundles. The first, in analogy to the unlabeled case (see Definition 2.1.78), can be defined as follows.

Definition 2.3.36 (Pullbacks of labeled 1-truss bundles). Given a C-labeled 1-truss bundle p over a poset B and a poset map $F : C \to B$, the **pullback F^*p of p along F** is the C-labeled 1-truss bundle with classifying functor $\chi_{F^*p} = \chi_p \circ F$. ⌐

In other words, $F^*p \equiv (\underline{F^*p}, \mathrm{lbl}_{F^*p})$ can be thought of as the span obtained as the upper row in the following diagram

$$
\begin{array}{c}
\mathsf{C} \xleftarrow{\quad \mathrm{lbl}_p \circ \mathrm{Tot}\, F \quad} F^*T \xrightarrow{\quad \underline{F^*p} \quad} C \\
\end{array}
$$

where $\mathrm{Tot}\, F$ is the canonical pullback map of 1-truss bundles $\mathrm{Tot}\, F : \underline{F^*p} \to \underline{p}$.

Remark 2.3.37 (Restrictions). Pullbacks of labeled 1-truss bundles p over B along subposets $F : C \hookrightarrow B$ recover restrictions, that is, $F^*p = p|_C$. ⌐

Nextwe consider dualization of labeled 1-truss bundles.

Definition 2.3.38 (Duals of labeled 1-truss bundles). Given a C-labeled 1-truss bundle $p \equiv (\underline{p}, \mathrm{lbl}_p)$, its **dual bundle $p^\dagger$** is the C^{op}-labeled 1-truss bundle whose underlying bundle $\underline{p^\dagger}$ is the dual $(\underline{p})^\dagger$ of the underlying bundle $\underline{p}$ (see Definition 2.1.82) and whose labeling $\overline{\mathrm{lbl}}_{p^\dagger}$ is the opposite labeling $(\mathrm{lbl}_p)^{\mathrm{op}}$. ⌐

Definition 2.3.39 (Duals of labeled dual 1-truss bundle maps). Given a labeled 1-truss bundle map $F : p \to q$ one defines the **dual bundle map $F^\dagger : p^\dagger \to q^\dagger$** to have underlying 1-truss bundle map is $\underline{F^\dagger} = \underline{F}^{\mathrm{op}}$ (i.e. $\underline{F}^\dagger$) and labeling functors $\mathrm{lbl}_{F^\dagger} = (\mathrm{lbl}_F)^{\mathrm{op}}$. ⌐

Definition 2.3.40 (Duals of labeled 1-truss bundle concordances). Given a labeled 1-truss bundle concordance $u : p \Rightarrow q$ one constructs the **dual bundle concordance $u^\dagger : q^\dagger \Rightarrow p^\dagger$** by setting $u^\dagger$ to be the dual bundle of u. ⌐

The preceding definitions now yield the following functors.

Observation 2.3.41 (Dualization functors on 1-truss bundles). The preceding definitions construct a dualization functor of labeled 1-truss bundles as follows:

$$\dagger : \mathsf{LblTrsBun}_1 \cong \mathsf{LblTrsBun}_1.$$

Similarly, for a category C and a poset B, we obtain a dualization functor of C-labeled 1-truss bundle concordances as follows:

$$\dagger : \mathsf{TrsConc}_1(B, \mathsf{C}) \cong \mathsf{TrsConc}_1(B, \mathsf{C}^{\mathrm{op}})^{\mathrm{op}}.$$

If we set $B = *$, then this further specializes to an isomorphism of C-labeled 1-truss bordisms:

$$\dagger : \mathsf{TBord}^1_{/\!/\mathsf{C}} \cong (\mathsf{TBord}^1_{/\!/\mathsf{C}^{\mathrm{op}}})^{\mathrm{op}}. \qquad ⌐$$

Remark 2.3.42 (Dualization via classifying functors). Using the classification and totalization equivalence for labeled 1-truss bundles, given a C-labeled 1-truss bundle p, its dual $p^\dagger$ may equivalently be defined to have classifying functor $\chi_{p^\dagger}$ equal to the composite of $(\chi_p)^{\mathrm{op}}$ with the isomorphism $(\mathsf{TBord}^1_{/\!/\mathsf{C}})^{\mathrm{op}} \cong \mathsf{TBord}^1_{/\!/\mathsf{C}^{\mathrm{op}}}$. ⌐

Remark 2.3.43 (Suspensions in the labeled case). Recall the definition of suspensions of 1-truss bundles from Construction 2.1.88. This again has a labeled analog, subject to the additional assumption that C has both initial and terminal objects, which can be used as labels for the newly adjoined initial and terminal objects in the total poset of the suspension—we leave details to the reader. ⌐

2.3.2 n-Truss bordisms and n-truss bundles We now discuss the combinatorial theory of n-trusses, as well as their bordisms and their bundles. Recall from the introduction to this section, which defined n-trusses as tower of 1-truss bundles ending in the 0-simplex $[0]$. We saw the definition of n-trusses already in Definition 2.3.1; for a more uniform description of truss theory, it will be useful to generalize this to a notion of 'labeled n-trusses' as follows.

Definition 2.3.44 (Labeled n-trusses). Given a category C, an C-**labeled** n-**truss** T is a pair consisting of an 'underlying' n-truss $\underline{T}$ together with a 'labeling' functor $\mathrm{lbl}_T : T_n \to \mathsf{C}$ (where T_n is the total poset of $\underline{T}$). ⌐

Unpacking the underlying n-truss $\underline{T}$ as a tower of 1-truss bundles $p_i : T_i \to T_{i-1}$ we may equivalently consider an n-truss to be a diagram of the form

$$\mathsf{C} \xleftarrow{\mathrm{lbl}_T} T_n \xrightarrow{p_n} T_{n-1} \xrightarrow{p_{n-1}} ... \xrightarrow{p_2} T_1 \xrightarrow{p_1} T_0 = [0]$$

where p_i are 1-truss bundles whose base poset is the total poset of p_{i-1}, and where $\mathrm{lbl}_T : T_n \to \mathsf{C}$ is a functor from the total poset T_n of p_n to the category C.

The goal of this section will be to study notions of (labeled) n-truss bordisms, n-truss bundles as well as their maps and concordances, which generalize previously introduced notions to 'higher dimensions'. This will involve an interesting inductive iteration of ideas from previous sections.

Remark 2.3.45 (On towers and sequences). We use the terms 'towers of maps' and 'sequences of maps' largely synonymously, but prefer 'tower' when maps have 'bundle character'. ⌐

2.3.2.1   The definition n-truss bordisms We introduce n-truss bordisms, which, in analogy to the case of 1-trusses of 1-truss bordisms, will describe 'fiber transitions' in bundles whose fibers are n-trusses.

Definition 2.3.46 (n-Truss bordisms). An n-**truss bordism** R is a tower of 1-truss bundles

$$R_n \xrightarrow{p_n} R_{n-1} \xrightarrow{p_{n-1}} ... \xrightarrow{p_2} R_1 \xrightarrow{p_1} R_0 = [1] \quad .$$

We refer to R_n as the **total poset** of R. ⌐

Similarly, we obtain the following labeled analog of the notion.

Definition 2.3.47 (Labeled n-truss bordism)**.** For a category C, an C**-labeled** n**-truss bordism** R is a pair consisting of an 'underlying' n-truss bordism $\underline{R}$ and a 'labeling' functor $\mathrm{lbl}_R : R_n \to \mathsf{C}$ (where R_n is the total poset of R). ⌐⌐

Unpacking the tower $\underline{R}$, we may equivalently think of R as a diagram of the form

$$\mathsf{C} \xleftarrow{\mathrm{lbl}_R} R_n \xrightarrow{p_n} R_{n-1} \xrightarrow{p_{n-1}} ... \xrightarrow{p_2} T_1 \xrightarrow{p_1} R_0 = [1]$$

where p_i are 1-truss bundles, and $\mathrm{lbl}_R : (R_n, \trianglelefteq) \to \mathsf{C}$ is a functor. Clearly, Definition 2.3.47 specializes to Definition 2.3.46 if we label in the terminal category $\mathsf{C} = *$, and we shall therefore work mainly in the more general case of labeled n-truss bordisms. Our main goal will be understanding how labeled n-truss bordisms organize (as morphisms) into a category.

Terminology 2.3.48 (Domain and codomain of a labeled n-truss bordism)*.* Given a C-labeled n-truss bordism $R \equiv (\underline{R}, \mathrm{lbl}_R)$, the 'domain' $\mathrm{dom}(R)$ is the C-labeled n-truss T whose underlying n-truss $\underline{T}$ is obtained by (iteratively) restricting the tower of bundles $\underline{R}$ to $0 \in [1]$, and whose labeling lbl_T is the restriction of lbl_R to total poset T_n of $\underline{T}$; in other words, T is the determined by the upper row in the following diagram of 1-truss bundle pullbacks (all of which are restrictions, see Remark 2.1.80)

$$
\begin{array}{ccccccccc}
\mathrm{lbl}_T & T_n & \longrightarrow & T_{n-1} & \longrightarrow & \cdots & \longrightarrow & T_1 & \longrightarrow & [0] \\
\mathsf{C} \overset{\nwarrow}{\underset{\swarrow}{}} & \downarrow & \lrcorner & \downarrow & \lrcorner & \cdots & & \downarrow & \lrcorner & \downarrow 0 \\
\mathrm{lbl}_R & R_n & \underset{p_n}{\longrightarrow} & R_{n-1} & \underset{p_{n-1}}{\longrightarrow} & \cdots & \underset{p_2}{\longrightarrow} & R_1 & \underset{p_1}{\longrightarrow} & [1]
\end{array}
$$

Similarly, one defines the 'codomain' $\mathrm{cod}(R)$ by restricting R to the object $1 \in [1]$. ⌐⌐

The following notion of composition immediately generalizes Definition 2.3.10.

Definition 2.3.49 (Composition of labeled n-truss bordisms)**.** Given two C-labeled truss bordisms R_{01} and R_{12} which are composable (that is, $\mathrm{cod}(R_{01}) = \mathrm{dom}(R_{12})$), a **composition candidate** R_{02} of R_{12} and R_{01} is a C-labeled truss bordism for which there exists a diagram W of the form

$$\mathsf{C} \xleftarrow{\mathrm{lbl}_W} W_n \xrightarrow{p_n} W_{n-1} \xrightarrow{p_{n-1}} ... \xrightarrow{p_2} W_1 \xrightarrow{p_1} W_0 = [2]$$

where p_i are 1-truss bundles and $\mathrm{lbl}_W : R_n \to \mathsf{C}$ is a functor, such that lbl_W restricts on each non-identity arrow $(i \to j)$ in $[2]$ to the C-labeled n-truss bordism R_{ij}. In this case, we call the above diagram a **composition witness** of R_{01} and R_{12}. ⌐⌐

Lemma 2.3.50 (Existence and uniqueness of n-truss bordism composition witnesses)**.** *Composition candidates and witnesses of labeled n-truss bordisms exist uniquely.*

A direct proof of this statement can be given, but since the statement will be a corollary of subsequent constructions, we will defer a proof until Observation 2.3.79. The preceding result allows us to define the following category.

Definition 2.3.51 (The category of C-labeled n-truss bordisms)**.** Given a category C, the **category** $n\mathsf{TBord}_{/\!/\mathsf{C}}$ **of C-labeled n-truss bordisms** is the category whose objects are C-labeled n-trusses and whose morphisms are C-labeled n-truss bordisms endowed with a notion of composition induced by the existence and uniqueness of composition candidates. ⌟

Importantly, the category of C-labeled 1-truss bordisms admits an equivalent formulation, which highlights its 'fundamentally iterative' nature. We introduce this formulation as follows.

Definition 2.3.52 (The labeled n-truss bordism functor)**.** The n-fold composite $\mathsf{TBord}^1_{/\!/-} \circ \mathsf{TBord}^1_{/\!/-} \circ ... \circ \mathsf{TBord}^1_{/\!/-} : \mathsf{Cat} \to \mathsf{Cat}$ of the labeled 1-truss bordism functor $\mathsf{TBord}^1_{/\!/-} : \mathsf{Cat} \to \mathsf{Cat}$ is called the **labeled n-truss bordism functor**, and denoted by $\mathsf{TBord}^n_{/\!/-}$. If $n = 0$, we take this definition to mean $\mathsf{TBord}^0_{/\!/-} = \mathrm{id}$. ⌟

Applying the labeled n-truss bordism functor to a specific category $\mathsf{C} \in \mathsf{Cat}$, we obtain the following.

Definition 2.3.53 (The 'iterative' category of C-labeled n-truss bordisms)**.** Given a category C, the **'iterative' category of C-labeled n-truss bordisms** is the category $\mathsf{TBord}^n_{/\!/\mathsf{C}}$. ⌟

Remark 2.3.54 (Unwinding the iteration)*.* Recall that the functor $\mathsf{TBord}^1_{/\!/-} : \mathsf{Cat} \to \mathsf{Cat}$, takes a category and maps it to the category of 1-truss bordisms labeled in that category. On the one hand, the 'iterative' category of C-labeled n-truss bordisms can be obtained as the category

$$\mathsf{TBord}^n_{/\!/\mathsf{C}} = \mathsf{TBord}^1_{/\!/\mathsf{TBord}^{n-1}_{/\!/\mathsf{C}}}$$

From this perspective, we can understand $\mathsf{TBord}^n_{/\!/\mathsf{C}}$ as the category of 1-truss bordisms *labeled* in the category of C-labeled $(n-1)$-truss bordisms. On the other hand, we may also write

$$\mathsf{TBord}^n_{/\!/\mathsf{C}} = \mathsf{TBord}^{n-1}_{/\!/\mathsf{TBord}^1_{/\!/\mathsf{C}}}$$

This means we can also understand $\mathsf{TBord}^n_{/\!/\mathsf{C}}$ as the category of $(n-1)$-truss bordisms *labeled* in the category of C-labeled 1-truss bordisms. Moreover, setting the labeling category $\mathsf{C} = *$ to be trivial, these observations specialize the case of unlabeled n-truss bordisms: they may either be understood as 1-truss bordisms labeled in $(n-1)$-truss bordisms, or as $(n-1)$-truss bordisms labeled in 1-truss bordisms. ⌟

The fact that Definition 2.3.51 and Definition 2.3.53 define the same category requires proof.

Proposition 2.3.55 (Equivalence of viewpoints on labeled n-truss bordism)**.** *The categories $n\mathsf{TBord}_{/\!/\mathsf{C}}$ and $\mathsf{TBord}^n_{/\!/\mathsf{C}}$ are canonically equivalent.*

Again, we defer a proof since the statement will be a corollary of subsequent constructions, and will be revisited in Observation 2.3.80.

2.3.2.2   The definition of n-truss bundles We discuss n-truss bundles which generalize 1-truss bundles. We start, as before, with the unlabeled case.

Definition 2.3.56 (n-Truss bundles). An **n-truss bundle** p over a poset B is a tower of 1-truss bundles

$$T_n \xrightarrow{p_n} T_{n-1} \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_2} T_1 \xrightarrow{p_1} T_0 = B \quad .$$

We call T_n the **total poset** of p. ⌐

Definition 2.3.57 (Labeled n-truss bundles). Given a category C, a **C-labeled n-truss bundle** p over a poset B is a pair consisting of an 'underlying' n-truss bundles $\underline{p}$ together with a 'labeling' functor $\mathrm{lbl}_p : T_n \to \mathsf{C}$ (where T_n is the total poset of p_n). ⌐

A C-labeled n-truss bundle $p = (\underline{p}, \mathrm{lbl}_p)$ can equivalently be understood as a diagram of the form

$$\mathsf{C} \xleftarrow{\mathrm{lbl}_p} T_n \xrightarrow{p_n} T_{n-1} \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_2} T_1 \xrightarrow{p_1} T_0 = B$$

where p_i are the 1-truss bundles in $\underline{p}$ and $\mathrm{lbl}_p : (T_n, \trianglelefteq) \to \mathsf{C}$ is a functor.

Remark 2.3.58 (n-Trusses and n-truss bordisms as bundles). Note that both C-labeled n-truss as well as n-truss bordisms are special instance of the preceding definition, which can be recover for base posets $B = [0]$ resp. $B = [1]$. ⌐

Terminology 2.3.59 (Open and closed bundles). If, for all i, ith bundles p_i of an n-truss bundle p are open (respectively closed), then we will call p itself 'open' (respectively 'closed'). ⌐

Definition 2.3.60 (Labeled n-truss bundle maps). Given labeled n-truss bundles p and q, a **labeled n-truss bundle map** $F : p \to q$ consists of a commuting diagram

$$
\begin{array}{ccccccccc}
\mathsf{C} & \xleftarrow{\mathrm{lbl}_p} & T_n & \xrightarrow{p_n} & T_{n-1} & \xrightarrow{p_{n-1}} & \cdots & \xrightarrow{p_2} & T_1 & \xrightarrow{p_1} & T_0 \\
\scriptstyle{\mathrm{lbl}_F}\downarrow & & \scriptstyle{F_n}\downarrow & & \scriptstyle{F_{n-1}}\downarrow & & \cdots & & \downarrow\scriptstyle{F_1} & & \downarrow\scriptstyle{F_0} \\
\mathsf{D} & \xleftarrow{\mathrm{lbl}_q} & S_n & \xrightarrow{q_n} & S_{n-1} & \xrightarrow{q_{n-1}} & \cdots & \xrightarrow{q_2} & S_1 & \xrightarrow{q_1} & S_0
\end{array}
$$

where, for $i > 0$, each $F_i : p_i \to q_i$ is a 1-truss bundle map (see Definition 2.1.64) and $\mathrm{lbl}_F : \mathsf{C} \to \mathsf{D}$ is a functor, called the 'label category' functor of F. ⌐

Terminology 2.3.61 (Label and base preserving). An n-truss bundle map F is 'base preserving' if $F_0 = \mathrm{id}$, and 'label preserving' if its label category functor $\mathrm{lbl}_F = \mathrm{id}$ is the identity. ⌐

Terminology 2.3.62 (Singular, regular, and balanced n-truss bundles maps). If for each i, the ith bundle map of an n-truss bundle map singular resp. regular resp. balanced (in the sense of Terminology 2.1.65) then we call the bundle map itself 'singular', resp. 'regular', resp. balanced. ⌐

The preceding definition of maps of labeled n-truss bundle specializes both to the case 'unlabeled' n-truss bundles as well as the case of 'n-trusses'.

Remark 2.3.63 (The unlabeled case). For unlabeled n-truss bundles p, q (which can be considered as labeled n-truss bundles labeled in the terminal category $*$) the labeling functor $\mathrm{lbl}_F : * \to *$ of any labeled n-truss bundle map $F : p \to q$ must be trivial; in this case we call F simply an 'n-truss bundle map'. Moreover, every labeled n-truss bundle map $F : p \to q$ induces an n-truss bundle map $\underline{F} : \underline{p} \to \underline{q}$ of underlying n-truss bundles, by setting $\underline{F}_i = F_i$. We call $\underline{F}$ the 'underlying' n-truss bundle map of F. ⌐

Remark 2.3.64 (The case of n-trusses). For labeled n-trusses T and S (which can be regarded as labeled n-truss bundles over the trivial base poset $[0]$) we usually refer to labeled n-truss bundle maps $F : T \to S$ as 'labeled n-truss maps'. ⌐

Categories of n-trusses as well as n-truss bundles and their maps can now be introduced as follows.

Notation 2.3.65 (n-Truss and n-truss bundle categories). We denote by $\mathsf{LblTrsBun}_n$ the category of labeled n-truss bundles, TrsBun_n the category of (unlabeled) n-truss bundles, $\mathsf{Trs}_n(B)$ for the category of n-truss bundles over a fixed base poset B (and base preserving maps), and Trs_n for the category of n-trusses. As before, decorations $\mathring{\mathsf{T}}$ and $\bar{\mathsf{T}}$ indicate that we are working in the open-regular resp. closed-singular case. ⌐

Definition 2.3.66 (Upper truncation). For $0 \leq k \leq n$, we define the **upper k-truncation functor**

$$(-)_{\geq k} : \mathsf{LblTrsBun}_n \to \mathsf{LblTrsBun}_{n-k}$$

which maps a labeled n-truss bundle $p = (\underline{p}, \mathrm{lbl}_p)$ where $\underline{p} = (p_n, p_{n-1}, ..., p_1)$ to the labeled k-truss bundles $p_{\geq k} = (\underline{p}_{\geq k}, \mathrm{lbl}_p)$ where $\underline{p}_{\geq k} = (p_n, p_{n-1}, ..., p_{k+1})$. ⌐

Definition 2.3.67 (Lower truncation). For $k \leq n$, we define the **lower k-truncation functor**

$$(-)_{\leq k} : \mathsf{LblTrsBun}_n \to \mathsf{TrsBun}_k$$

which maps a labeled n-truss bundle $p = (\underline{p}, \mathrm{lbl}_p)$ where $\underline{p} = (p_n, p_{n-1}, ..., p_1)$ to the unlabeled k-truss bundle $p_{\leq k}$ consisting of 1-truss bundles $(p_k, p_{k-1}, ..., p_1)$. ⌐

Definition 2.3.68 (Labeled n-truss bundle restrictions). Given a C-labeled n-truss bundle p over a poset B with underlying bundle $\underline{p} = (p_n, p_{n-1}, ..., p_1)$, as well as a subposet $C \hookrightarrow B$, we define the **restriction** $p|_C$ of p to C to be the C-labeled n-truss bundle over C obtained as the upper row in the following diagram

$$
\begin{array}{ccccccccc}
& T_n|_C & \xrightarrow{p_n|_C} & T_{n-1}|_C & \xrightarrow{p_{n-1}|_C} & \cdots & \xrightarrow{p_2|_C} & T_1|_C & \xrightarrow{p_1|_C} & [0] \\
\mathsf{C} {}^{\mathrm{lbl}_{p|_C}}\!\!\nwarrow\!\!\swarrow_{\mathrm{lbl}_p} & \downarrow & \llcorner & \downarrow & \llcorner & \cdots & & \downarrow & \llcorner & \downarrow \\
& T_n & \xrightarrow{p_n} & T_{n-1} & \xrightarrow{p_{n-1}} & \cdots & \xrightarrow{p_2} & T_1 & \xrightarrow{p_1} & B
\end{array}
$$

(note all pullbacks of 1-truss bundles in the diagram are in fact restrictions). ⌐

We briefly remark on how to construct a class of 'generating arrows' in the total poset of n-truss bundles, yielding a minimal set of arrows whose transitive closure recovers all non-identity arrows of the total poset. Note that, this class is sometimes referred to as the 'covering relation' of a poset and can be defined for any poset; however, n-truss bundles admit an explicit construction of this class as follows.

Construction 2.3.69 (Generating arrows). Given an n-truss bundle $p = (T_n \xrightarrow{p_n} T_{n-1} \xrightarrow{p_{n-1}} ... \xrightarrow{p_1} T_0 = B)$ over a base poset B, we inductively construct subsets of non-identity arrows $\mathrm{cov}(T_i) \subset \mathrm{mor}(T_i, \trianglelefteq)$, called the 'generating arrows' of T_i.

- If $i = 0$, define $\mathrm{cov}(B)$ to be the minimal subset of $\mathrm{mor}(B)$ that generates all non-identity arrows in B, i.e. its covering relation.
- If $i > 0$, then $f \in \mathrm{cov}(T_i)$ if and only if one of the following holds.
    - either $p_i f = \mathrm{id}$ (that is, f lies in a fiber of p_i),
    - or, $p_i f \in \mathrm{cov}(T_{i-1})$ and $f \in \mathsf{reg}(T_i) \cup \mathsf{sing}(T_i)$ (that is, f lies over a generating arrow of T_{i-1} and passes either between two regular or two singular objects, see Terminology 2.1.65).

The central observation about the generating arrow set $\mathrm{cov}(T_i)$ is that each non-identity arrow in $(T_i, \trianglelefteq)$ can be written as a composite of generating arrows (in other words, it is the covering relation of T_i). The observation will be particularly useful when illustrating n-trusses (and their bundles) in that we will often only depict their generating arrows. ⌐⌐

Example 2.3.70 (Generating arrows in n-truss bundles). In Fig. 2.24 we illustrate a closed 2-truss T on the left, and depict generating arrows of T_2 on the right. ⌐⌐
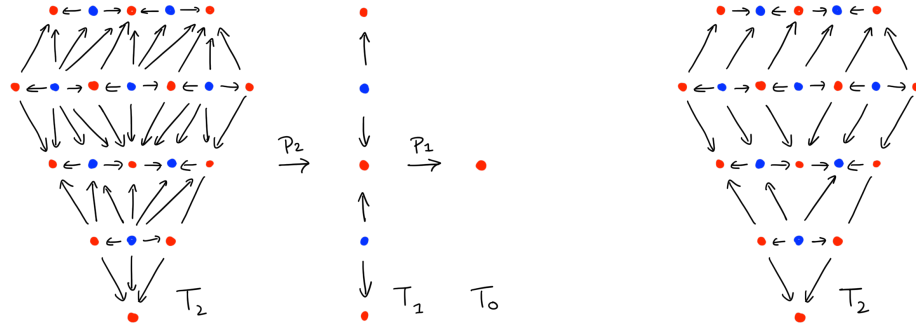
Figure 2.24: A 2-truss bundle and its generating arrows.

2.3.2.3 Classification and totalization for labeled n-truss bundles We discuss classification and totalization for labeled n-truss bundles, analogous to the case of labeled 1-truss bundles (see Construction 2.3.34). Crucially, this now uses the iterative nature of the category $\mathsf{TBord}^n_{/\!/\mathsf{C}}$ of C-labeled n-truss bordisms.

Construction 2.3.71 (Classification of labeled n-truss bundles). Let $p = (\underline{p}, \mathrm{lbl}_p)$ be an C-labeled n-truss bundle over a poset B with $\underline{p} = (p_n, p_{n-1}, ..., p_1)$. We construct the **classifying functor** $\chi_p : B \to \mathsf{TBord}^n_{/\!/\mathsf{C}}$ of p inductively as follows.

- Define a functor $\chi_p^n := \mathrm{lbl}_p : T_n \to \mathsf{C}$.
- Inductively in descending i, define a functor $\chi_p^{i-1} : T_{i-1} \to \mathsf{TBord}^{n-i}_{/\!/\mathsf{C}}$ to be the classifying map of the $\mathsf{TBord}^{n-i+1}_{/\!/\mathsf{C}}$-labeled 1-truss bundles (p_i, χ_p^i).

Then set $\chi_p = \chi_p^0$. ⌐⌐

Example 2.3.72 (The labeled tower of a truss bundle). Here is an illustration of the form of a labeled tower of a C-labeled 3-truss-bundle p with underlying 3-truss bundle

$T_3 \xrightarrow{p_3} T_2 \xrightarrow{p_2} T_1 \xrightarrow{p_1} T_0$ and with label lbl_p, together with its classifying maps χ_p^i.

$$
\begin{array}{ccc}
\mathsf{C} & \xleftarrow{\;\chi_T^3 \,=\, \mathrm{lbl}_p\;} & T_3 \\
& & \downarrow {\scriptstyle p_3} \\
\mathsf{TBord}^1_{/\!/\mathsf{C}} & \xleftarrow{\;\chi_p^2\;} & T_2 \\
& & \downarrow {\scriptstyle p_2} \\
\mathsf{TBord}^1_{/\!/\mathsf{TBord}^1_{/\!/\mathsf{C}}} = \mathsf{TBord}^2_{/\!/\mathsf{C}} & \xleftarrow{\;\chi_p^1\;} & T_1 \\
& & \downarrow {\scriptstyle p_1} \\
\mathsf{TBord}^1_{/\!/\mathsf{TBord}^2_{/\!/\mathsf{C}}} = \mathsf{TBord}^3_{/\!/\mathsf{C}} & \xleftarrow{\;\chi_p^0\;} & T_0
\end{array}
$$

Each of the following subsets of the above diagrams determines the entire truss bundle p: the functor χ_p^0, or the bundle p_1 together with the functor χ_p^1, or the bundles p_1 and p_2 together with the functor χ_p^2, or the bundles p_1, p_2 and p_3 together with the functor $\chi_p^3 := \mathrm{lbl}_p$. ⌐

Conversely, we may construct 'total bundles' as follows. This simply inverts each inductive step in the preceding construction separately.

Construction 2.3.73 (Totalization of labeled n-truss bundles)**.** Given a functor $\mathsf{F} : B \to \mathsf{TBord}^n_{/\!/\mathsf{C}}$ we construct its **total C-labeled n-truss bundle** $\pi_\mathsf{F} = (\underline{\pi}_\mathsf{F}, \mathsf{lbl}_\mathsf{F})$ inductively as follows.

- Define $\mathsf{lbl}_\mathsf{F}^0 = \mathsf{F}$.
- Inductively in ascending i, define $(\pi_\mathsf{F}^i, \mathsf{lbl}_\mathsf{F}^i)$ to be the $\mathsf{TBord}^{n-i}_{/\!/\mathsf{C}}$-labeled total 1-truss bundle constructed from $\mathsf{lbl}_\mathsf{F}^{i-1} : \mathrm{Tot}^{i-1}\mathsf{F} \to \mathsf{TBord}^1_{/\!/\mathsf{TBord}^{n-i}_{/\!/\mathsf{C}}}$.

Then set $\underline{\pi}_\mathsf{F} = (\pi_\mathsf{F}^n, \pi_\mathsf{F}^{n-1}, ..., \pi_\mathsf{F}^1)$ and $\mathsf{lbl}_\mathsf{F} = \mathsf{lbl}_\mathsf{F}^n$. ⌐

To promote the preceding constructions to functors we once more introduce the notions of bundle concordances as follows.

Definition 2.3.74 (Concordances of labeled n-truss bundles)**.** For a poset B and a 'labeling' category C, a **C-labeled n-truss bundle concordance** $u : p \Rightarrow q$ between C-labeled n-truss bundles p and q over B is a C-labeled n-truss bundle over $B \times [1]$ such that $u|_{B \times \{0\}} = p$ and $u|_{B \times \{1\}} = q$. ⌐

Remark 2.3.75 (Uniqueness of n-truss bundle concordances)**.** Fixing unlabeled n-truss bundles, any concordance between them must be unique if it exists (this follows by iteratively applying Remark 2.1.72). In analogy to Remark 2.3.32, this need not hold in the labeled case; but there is no harm in identifying labeled n-truss bundles up to concordance of their underlying (unlabeled) n-truss bundles. ⌐

Note that, given a C-labeled n-truss bundle concordance $u : p \Rightarrow q$, its classifying functor χ_u is equivalently a natural transformation $\chi_p \Rightarrow \chi_q : B \to \mathsf{TBord}^n_{/\!/\mathsf{C}}$; we often refer to this natural transformation as the 'classifying natural transformation' of u.

Definition 2.3.76 (Categories of n-truss bundle concordances). For a poset B and a category C, define the **category of concordances of C-labeled n-truss bundles over B**, denoted by $\mathsf{TrsConc}_1(B, \mathsf{C})$, to have C-labeled n-truss bundles over B as objects, and C-labeled n-truss bundle concordances as morphisms. Composition of two such concordances $u : p \Rightarrow q$ and $v : q \Rightarrow r$ is determined by the condition that $\chi_{v \circ u} = \chi_v \circ \chi_u$.[16]   ⌐

Classification and totalization then organize into equivalences of categories as follows.

Construction 2.3.77 (Classification and totalization functors). Given a poset, we define the **classification functor**

$$\chi_- : \mathsf{TrsConc}_n(B, \mathsf{C}) \to \mathrm{Fun}(B, \mathsf{TBord}^n_{/\!/\mathsf{C}})$$

to map labeled n-truss bundles p to their classifying functor χ_p, and labeled concordances $u : p \Rightarrow q$ to their classifying natural transformations $\chi_u : \chi_p \Rightarrow \chi_q$. Conversely, we define the **totalization functor**

$$\pi_- : \mathrm{Fun}(B, \mathsf{TBord}^1_{/\!/\mathsf{C}}) \to \mathsf{TrsConc}_1(B, \mathsf{C})$$

to take functors $\mathsf{F} : B \to \mathsf{TBord}^1_{/\!/\mathsf{C}}$ to their total n-truss bundles π_F, and similarly natural transformations $\alpha : B \times [1] \to \mathsf{TBord}^1$ to their total n-truss bundles π_α.   ⌐

Observation 2.3.78 (Classification and totalization are inverse). Classification of labeled n-truss bundles and totalization of functors into labeled n-truss bordisms provide an equivalence of categories.   ⌐

The correspondence between n-truss bundles and their classifying functors into the category $\mathsf{TBord}^n_{/\!/\mathsf{C}}$ enables a simple proof of our earlier claim that the 'non-iterative' and 'iterative' definitions of labeled n-truss bordisms coincide. We record this by the following two observations.

Observation 2.3.79 (Proof of Lemma 2.3.50). In Lemma 2.3.50 we claimed that the composition candidates and witnesses of composable C-labeled n-truss bordisms R_{01} and R_{12} always uniquely exist. Indeed, the composition witness (and thereby the candidate) can be easily constructed using totalization and classification as follows. First, composability implies $\chi_{\mathrm{dom}(R_{12})} = \chi_{\mathrm{cod}(R_{01})}$, and thus $\chi_{R_{01}} : [1] \to \mathsf{TBord}^n_{/\!/\mathsf{C}}$ and $\chi_{R_{12}} : [1] \to \mathsf{TBord}^n_{/\!/\mathsf{C}}$ are composable morphisms in $\mathsf{TBord}^n_{/\!/\mathsf{C}}$. Define a functor $W : [2] \to \mathsf{TBord}^n_{/\!/\mathsf{C}}$ by setting $W(0 \to 1) = \chi_{R_{01}}$ and $W(1 \to 2) = \chi_{R_{12}}$. The required composition witness is (up to unique invertible concordance of the underlying n-truss bundle) the totalization π_W of W.   ⌐

Observation 2.3.80 (Proof of Proposition 2.3.55). The equivalences between the categories $n\mathsf{TBord}_{/\!/\mathsf{C}}$, as defined in Definition 2.3.51, and $\mathsf{TBord}^n_{/\!/\mathsf{C}}$, as defined in Definition 2.3.53, can be constructed using totalization and classification as follows. A C-labeled n-truss T (resp. n-truss bordism R) in $n\mathsf{TBord}_{/\!/\mathsf{C}}$ can be classified by functors $\chi_T : [0] \to \mathsf{TBord}^n_{/\!/\mathsf{C}}$ (resp. $\chi_T : [1] \to \mathsf{TBord}^n_{/\!/\mathsf{C}}$) thus yielding objects (resp. morphisms) in $\mathsf{TBord}^n_{/\!/\mathsf{C}}$. Conversely, objects (resp. morphisms) in $\mathsf{TBord}^n_{/\!/\mathsf{C}}$ correspond to functors $T : [0] \to \mathsf{TBord}^n_{/\!/\mathsf{C}}$ (resp. functors $R : [1] \to \mathsf{TBord}^n_{/\!/\mathsf{C}}$) which we can totalize to C-labeled n-trusses π_T (resp. n-truss bordisms π_R) thus yielding objects (resp. morphisms) in $n\mathsf{TBord}_{/\!/\mathsf{C}}$.   ⌐

---

[16]Explicitly, working skeletally with underlying n-truss bundles (that is, identifying unlabeled n-truss bundles up to invertible concordances, see Remark 2.3.75), we may define $v \circ u = \pi_{\chi_v \circ \chi_u}$.

Observation 2.3.81 (Classifying unlabeled n-truss bundles). We write $\mathsf{TBord}^n \equiv \mathsf{TBord}^n_{/\!/*}$, and call this the 'category of n-truss bordisms'. Note that, since every (unlabeled) n-truss bundle is trivially labeled in the terminal category $*$, Construction 2.3.77 exhibits TBord^n as the classifying category of n-truss bundles. ⌐⌐

Construction 2.3.82 (The label forgetting functor for n-trusses). Applying the functor $\mathsf{TBord}^n_{/\!/-}$ to the terminal functor $\mathsf{C} \to *$ yields the 'label forgetting functor'

$$(\underline{-}) : \mathsf{TBord}^n_{/\!/\mathsf{C}} \to \mathsf{TBord}^n.$$ ⌐⌐

Observation 2.3.83 (The terminal and initial n-trusses). The terminal object of TBord^n is n-truss $\bar{\mathbb{T}}^n_0 = (p_n, p_{n-1}, ..., p_{\underline{1}})$ in which each p_i is the bundle $\bar{\mathbb{T}}_0 \to ast$ over the point with the trivial closed 1-truss $\bar{\mathbb{T}}_0$ as its fiber (see Observation 2.3.83). Similarly, the initial object in TBord^n is n-truss $\mathring{\mathbb{T}}^n_0 = (p_n, p_{n-1}, ..., p_1)$ in which each p_i is the bundle $\mathring{\mathbb{T}}_0 \to ast$ over the point with the trivial closed 1-truss $\mathring{\mathbb{T}}_0$ as its fiber. ⌐⌐

Finally, let us remark about the following immediate generalization of the classification/totalizations construction given here.

Remark 2.3.84 (Classifying n-truss bundles over categories). Fully analogous to Remark 2.3.84, given a category B, note that functors $\mathsf{B} \to \mathsf{TBord}^n$ up to natural isomorphism classify n-truss bundles over B up to bundle concordance isomorphism (where n-truss bundles over B are defined as towers of 1-truss bundles in the sense of Remark 2.1.63 ending in B). Yet more generally, we can consider functors $\mathcal{B} \to \mathsf{TBord}^n$ from an ∞-category $\mathcal{B}$ into TBord^n to describe n-truss bundles over $\mathcal{B}$; however, note that any such functor is equivalently a functor of the corresponding homotopy category $\mathrm{ho}(\mathsf{B}) \to \mathsf{TBord}^n$. ⌐⌐

2.3.2.4   Pullback, dualization, and suspension of labeled n-truss bundles Our usual constructions of pullbacks, duals, and suspensions immediately carry over from 1-trusses bundles to n-trusses bundles.

Construction 2.3.85 (Pullbacks of labeled n-truss bundles). Consider a C-labeled n-truss bundle $p = (\underline{p}, \mathrm{lbl}_p)$ over a poset B with underlying bundle $\underline{p} = (p_n, p_{n-1}, ..., p_1)$, as well as a poset map $F : C \to B$. We define the **pullback** $F^*p \equiv (F^*\underline{p}, \mathrm{lbl}_{F^*p})$ **of** p **along** F to be the C-labeled n-truss bundle with underlying bundle $F^*\underline{p} = (F^*\underline{p}_n, F^*\underline{p}_{n-1}, ..., F^*\underline{p}_1)$ constructed and labeling lbl_{F^*p} as follows.
  - Define $\mathrm{Tot}^0 F := F$.
  - For ascending i, define $\mathrm{Tot}^i F : F^*p_i \to p_i$ to be the 1-truss bundle pullback map of p_i along the poset map $\mathrm{Tot}^{i-1}F : F^*T_{i-1} \to T_{i-1}$ (where T_{i-1} resp. F^*T_{i-1} is the total poset of p_{i-1} resp. of F^*p_{i-1}).
Finally, define the labeling lbl_{F^*p} as the composite $\mathrm{lbl}_p \circ \mathrm{Tot}^n F$. ⌐⌐

In other words, the pullback F^*p of a labeled n-truss bundle p along a base poset map F can be obtained by the upper row in the following diagram

Observe that the poset maps $\mathrm{Tot}^i F$ (together with the labeling category functor $\mathrm{id} : \mathsf{C} \to \mathsf{C}$) assemble into a C-labeled n-truss bundle map $F^*p \to p$ which we call the 'pullback bundle map'. As before, in the special case where $F : C \hookrightarrow B$ is a subposet of B, this recovers our earlier definition of restrictions, i.e. $F^*p = F|_C$.

Next we define duals of C-labeled n-truss bundles.

Definition 2.3.86 (Duals of labeled n-truss bundles)**.** Given a C-labeled n-truss bundle $p \equiv (\underline{p}, \mathrm{lbl}_p)$ with underlying n-truss bundle $\underline{p} = (p_n, p_{n-1}, ..., p_1)$, its **dual bundle** $p^\dagger$ is the C^{op}-labeled n-truss bundle whose underlying n-truss bundle is $\underline{p}^\dagger = (p_n^\dagger, p_{n-1}^\dagger, ..., p_1^\dagger)$ (where $p_i^\dagger$ is the dual of p_i, see Definition 2.1.82), and whose labeling $\mathrm{lbl}_{p^\dagger}$ is the opposite labeling $(\mathrm{lbl}_p)^{\mathrm{op}}$. ⌐

Definition 2.3.87 (Duals of labeled n-truss bundle maps)**.** Given a labeled n-truss bundle map $F : p \to q$ one defines the **dual bundle map** $F^\dagger : p^\dagger \to q^\dagger$ to have underlying n-truss bundle map is $\underline{F}^\dagger = \underline{F}^{\mathrm{op}}$ (i.e. $\underline{F}^\dagger$) and labeling functors $\mathrm{lbl}_{F^\dagger} = (\mathrm{lbl}_F)^{\mathrm{op}}$. ⌐

Definition 2.3.88 (Duals of labeled n-truss bundle concordances)**.** Given a labeled n-truss bundle concordance $u : p \Rightarrow q$ one constructs the **dual bundle concordance** $u^\dagger : q^\dagger \Rightarrow p^\dagger$ simply as the dual bundle of u. ⌐

The preceding definitions now yield the following functors, which are (covariant or contravariant) involutive isomorphisms of categories.

Observation 2.3.89 (Dualization functors on labeled n-truss bundles). The preceding definitions construct a dualization functor of labeled n-truss bundles:

$$\dagger : \mathsf{LblTrsBun}_n \cong \mathsf{LblTrsBun}_n.$$

Similarly, for a fixed category C and a poset B, we obtain a dualization functor of labeled n-truss bundle concordances:

$$\dagger : \mathsf{TrsConc}_n(B, \mathsf{C}) \cong \mathsf{TrsConc}_n(B, \mathsf{C}^{\mathrm{op}})^{\mathrm{op}}.$$

If we set $B = *$, then this further specializes to an isomorphism of labeled n-truss bordisms:

$$\dagger : \mathsf{TBord}^n_{/\!/\mathsf{C}} \cong (\mathsf{TBord}^n_{/\!/\mathsf{C}^{\mathrm{op}}})^{\mathrm{op}}. \qquad ⌐$$

Remark 2.3.90 (Duality of closed and open trusses, and singular and regular maps). Dualization maps closed n-trusses to open n-truss bundles, and singular n-truss bundle maps to regular n-truss bundle maps; in particular, dualization of n-truss bundles specializes to an isomorphism

$$\dagger : \bar{\mathsf{Trs}}_n \rightleftarrows \mathring{\mathsf{Trs}}_n : \dagger$$

between closed n-trusses with singular maps and open n-truss with regular maps. ⌐

Finally, we construct suspension of (unlabeled) n-truss bundles.

Construction 2.3.91 (Suspension of unlabeled n-truss bundles)**.** For an (unlabeled) n-truss bundle $p = (p_n, p_{n-1}, ..., p_1)$ its **suspension** Σp is the n-truss bundle with 1-truss bundle maps $(\Sigma p_n, \Sigma p_{n-1}, ..., \Sigma p_1)$ (where Σp_i is the suspension bundle of the 1-truss bundle p_i, see Construction 2.1.88). ⌐

In analogy to Remark 2.3.43, the construction may further be generalized to the labeled case as long as the labeling category C has initial and terminal objects—we leave details to the reader.

2.3.2.5 Face, embedding, coarsening, & degeneracy truss maps and their rigidity Trusses are 'rigid' when considered with special classes of n-truss maps, that is, no two maps in that class admit a non-trivial natural transformation between them. As we will later see, the types of maps in question combinatorially mirror the properties of certain stratified topological maps, and we introduce them here with accordingly mirrored terminology.

Terminology 2.3.92 (Subtrusses, faces, and embeddings of 1-trusses). A map of 1-trusses $F : T \to S$ that is injective on objects is called an 'injection'. An injection F may further have the following properties.

1. *Balanced*: If F is balanced then F is a 'subtruss' inclusion map.
2. *Closed singular*: If T and S are closed and F is singular then we call F a 'closed face' (or simply, a 'face').
3. *Open regular*: If T and S are open and F is regular then we call F an 'open embedding' (or simply, a 'embedding'). ⌐

Observation 2.3.93 (Characterizing faces and embeddings). Note that closed faces are exactly subtrusses of closed 1-trusses, and open embeddings are exactly subtrusses of open 1-trusses. ⌐

Conversely, we introduce the following terminology for surjective 1-truss maps.

Terminology 2.3.94 (Degeneracies and coarsenings of 1-trusses). A map of 1-trusses $F : T \to S$ that is surjective on objects is called a 'surjection'. A surjection F may further have the following properties.

1. *Singular*: If T and S have the same endpoint type[17] and F is singular then F is called a 'degeneracy'.
2. *Regular*: If T and S have the same endpoint type and F is regular, then F is called a 'coarsening'.
3. *Closed singular*: If T and S are closed and F is singular then we call F a 'closed degeneracy'.
4. *Open regular*: If T and S are open and F is regular then we call F an 'open coarsenings'.

Note that if a surjective 1-truss map F is balanced then it must be an isomorphism. ⌐

Observation 2.3.95 (Characterizing closed degeneracies and open coarsenings). Note that closed degeneracies are exactly degeneracies of closed 1-trusses, and open coarsenings are exactly coarsenings of open 1-trusses. ⌐

Example 2.3.96 (Faces, embeddings, degeneracies, and coarsenings of 1-trusses)**.** In Fig. 2.25 we depict examples of 'subtrusses', ' faces', 'embeddings', as well as '(closed) degeneracies' and '(open) coarsenings' of 1-trusses. ⌐

---

[17]Recall that this means $\dim(\mathsf{end}_{\pm}T) = \dim(\mathsf{end}_{\pm}S)$, see Terminology 2.1.22. Note also that any surjection must be endpoint preserving, and thus the condition implies that F preserves dimensions of endpoints.
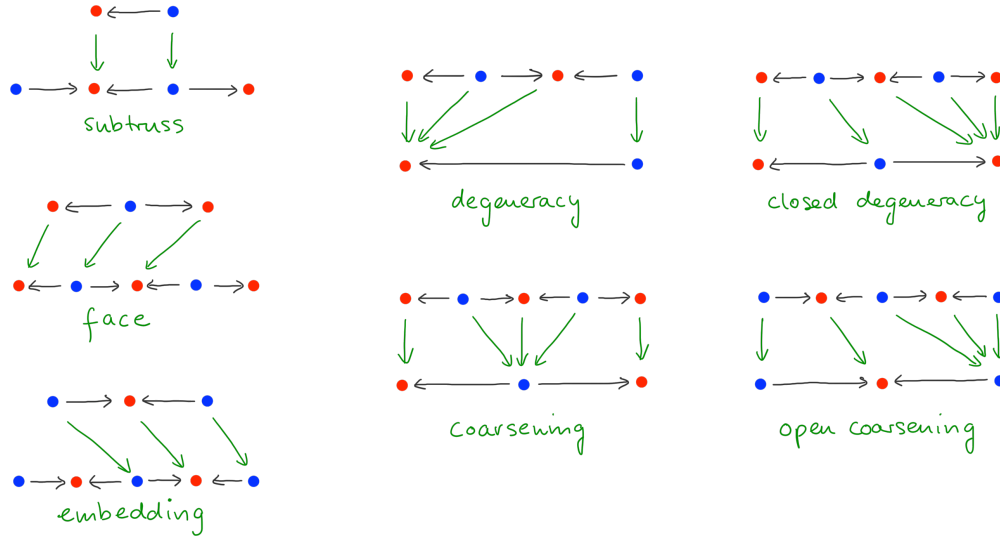
Figure 2.25: Faces, embeddings, degeneracies, and coarsenings.

We next generalize the preceding terminology to the case of (labeled) n-trusses as well as n-truss bundles.

Terminology 2.3.97 (Faces, embeddings, degeneracies, and coarsenings of n-trusses). Given n-trusses $T = (p_n, p_{n-1}, ..., p_1)$ and $S = (q_n, q_{n-1}, ..., q_1)$, a truss map $F : T \to S$ is said to be a 'injection' if each 1-truss bundle map $F_i : p_i \to q_i$ is fiberwise a 1-truss injection. Similarly, one says that F is a 'face', or an 'embedding', or a 'subtruss', or a 'surjection', or a '(closed) degeneracy', or a '(open) coarsening' if each F_i is fiberwise so.

The terminology also applies to n-truss bundle maps $F : T \to S$ (we require F to be base preserving), and further, to maps $F : T \to S$ of labeled n-trusses or labeled n-truss bundles (we require F to be base preserving and label preserving). ⌐

Notation 2.3.98 (Categories of truss coarsenings and truss degeneracies). Denote by $\mathsf{Trs}_n^{\mathrm{crs}}$ the category of n-trusses and their coarsenings, and by $\mathsf{Trs}_n^{\mathrm{deg}}$ the category of n-trusses and their degeneracies. ⌐

Remark 2.3.99 (Coarsenings vs refinements). Given a coarsening of n-trusses $F : T \to S$ we also call F a 'refinement' of S by T. That is, we use the term 'coarsening' and 'refinement' synonymously but describing dual processes: a coarsening 'coarsens' the domain, while a refinement, in opposite direction, 'refines' the codomain. ⌐

The role of dual categories $\bar{\mathsf{Trs}}_n$ of 'closed trusses with singular maps' and $\mathring{\mathsf{Trs}}_n$ of 'open trusses with regular maps' is special; these categories will be shown to combinatorially model framed regular cells (resp. its dual). In many ways, their properties reflect that of other 'categories of shapes', such as the category of simplices Δ. Our terminology here is meant to highlight this parallel: for instance, just as any morphisms in Δ factors into a degeneracy and face map (yielding its '(epi,mono)-factorization'), any singular n-truss map of closed trusses factors into degeneracy and a face map. Importantly, this existence of (epi,mono)-factorizations is special to the categories $\bar{\mathsf{Trs}}_n$ and $\mathring{\mathsf{Trs}}_n$, and fails for general truss bundle maps as illustrated in the next example.

110

Example 2.3.100 (Failure of (epi,mono)-factorization in general)**.** Fig. 2.26 shows a map $F : T \to S$ of 2-trusses: components $F_i : T_i \to S_i$ are highlighted in bold, and the mapping of the top components F_2 is determined by coloring its images corresponding preimages in the same color (note, as usual, singular elements are shown as red dots, while regular elements are shown as blue dots). The map F cannot admit an (epi,mono) factorization since the image $\mathrm{im}(F_2) \subset S_2$ does not describe a subtruss of S. ⌐⌐
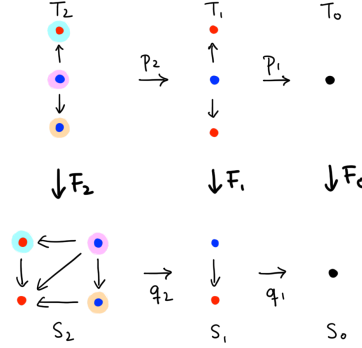
Figure 2.26: Failure of (epi,mono)-factorization of a 2-truss map.

In contrast, singular maps of closed n-trusses, and, dually, regular maps of open n-trusses, do have (epi,mono)-factorizations. We record the existence of (epi,mono)-factorization for the categories $\bar{\mathsf{Trs}}_n$ and $\mathring{\mathsf{Trs}}_n$ as follows.

Lemma 2.3.101 ((Epi,mono)-factorization)**.** *Any singular truss map F of closed n-trusses factors uniquely into a degeneracy F^{E} followed by a face F^{M}. Thus, after dualizing face posets, any regular truss map F of open n-trusses factors uniquely into a coarsening F^{E} followed by an embedding F^{M}.*

Proof. In both cases the factorization $F = F^{\mathsf{M}} F^{\mathsf{E}}$ is determined by factoring ith poset maps $F_i = F_i^{\mathsf{M}} F_i^{\mathsf{E}}$ using the (epi,mono)-factorization in Pos. We omit the verification that this determines n-truss bundle maps F^{E} and F^{M}. □

Example 2.3.102 ((Epi,mono)-factorization of closed singular truss maps)**.** In Fig. 2.27 we depict a singular map $F : T \to S$ of closed 2-trusses, together with its (epi,mono)-factorization $F = F^{\mathsf{M}} \circ F^{\mathsf{E}}$. As before, we indicate the mappings of the top components by coloring their images corresponding preimages in the same color. ⌐⌐

Centrally, we now observe the following 'rigidity of homs' in the categories $\bar{\mathsf{Trs}}_n$ and $\mathring{\mathsf{Trs}}_n$, meaning that 'hom posets' (consisting of maps and their natural transformations) are discrete. The result further applies to general 'coarsening' and 'degeneracies' as we record. We start with the case of 1-truss maps.

Lemma 2.3.103 (Rigidity of natural transformations for 1-trusses)**.** *Consider 1-trusses T and S, and 1-truss maps $E, F : T \to S$. Assume one of the following holds.*

1. *T and S are open and E and F are either embeddings, coarsenings, or (more generally) regular truss maps,*

2. *T and S are closed and E and F are either faces, degeneracies, or (more generally) singular truss maps,*
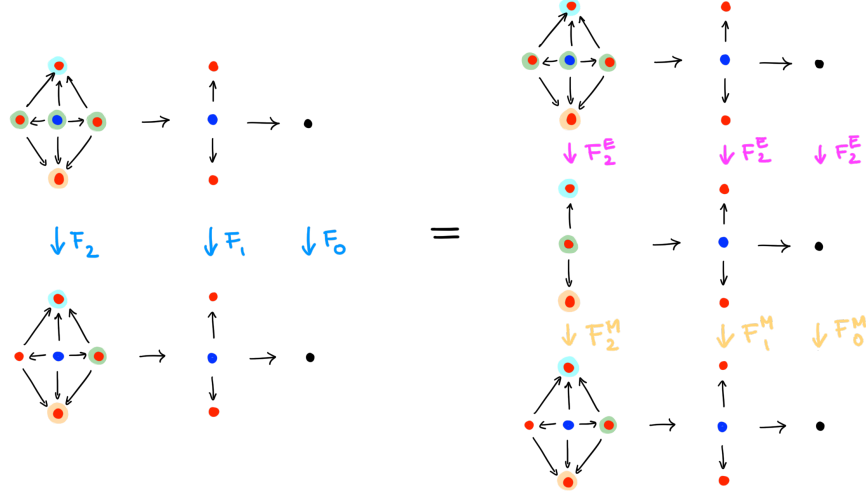
Figure 2.27: (Epi,mono)-factorization of a closed singular 2-truss map.

   3. E and F are coarsenings,
   4. E and F are degeneracies.
Then any natural transformation $\nu : E \Rightarrow F : (T, \unlhd) \to (S, \unlhd)$ must be the identity.

Proof. We argue in the first case of open 1-trusses and regular maps (the second case follows by duality, and the other cases follow similar arguments). Both embeddings E and F map the regular values a in T to regular values $E(a)$ resp. $F(a)$ in S; because there are no non-identity arrows between regular values in S, we must have $E(a) = F(a)$. Because T is assumed to be open, every singular value b in T has two adjacent regular values $b \pm 1$; since the maps E and F are functorial and monotone, both $E(b)$ and $F(b)$ must be the unique element $E(b) = F(b)$ in S such that $E(b-1) \unlhd E(b) \unrhd E(b+1)$. The functors E and F are thus identical and the natural transformation ν is necessarily trivial.   □

Lemma 2.3.104 (Rigidity of natural transformations for n-trusses)**.** *Consider n-trusses T and S, and n-truss maps $E, F : T \to S$. Assume one of the following holds*
   1. T and S are open and E and F are either embeddings, coarsenings, or (more generally) regular truss maps,
   2. T and S are closed and E and F are either faces, degeneracies, or (more generally) singular truss maps,
   3. E and F are coarsenings,
   4. E and F are degeneracies.
Then any natural transformation $\nu : E_n \Rightarrow F_n$ of poset maps $E_n, F_n : (T_n, \unlhd) \to (S_n, \unlhd)$ must be the identity. The statement further generalizes to base preserving n-truss bundle maps $E, F : p \to q$ over a base poset B.

Proof. We argue in the case of open n-trusses T, S and regular maps E, F (the second case follows by duality, and the other cases follow similar arguments). Write $T = (p_n, p_{n-1}, ..., p_1)$ and $S = (q_n, q_{n-1}, ..., q_1)$. Arguing inductively, we assume the statement holds for $(n-1)$-trusses (the bases case of 1-trusses was shown in the previous lemma). Postcomposing ν with the bundle map q_n yields a natural transformation $q_n \circ \nu : q_n \circ E_n \Rightarrow$

$q_n \circ F_n$, which equivalently is a natural transformation $E_{n-1} \circ p_n \Rightarrow F_{n-1} \circ p_n$. We must have $q_n \circ \nu = \nu_{n-1} \circ p_n$ for some natural transformation $\nu_{n-1} : E_{n-1} \Rightarrow F_{n-1}$.[18] Inductively, we deduce that $\nu_{n-1} = \mathrm{id}$. Applying the rigidity of 1-trusses (see Lemma 2.3.103) to the transformation ν restricted to the fibers of p_n and q_n shows that ν is itself trivial as claimed.

The statement generalizes to base preserving n-truss bundle maps $E, F : p \to q$ over a base poset B by applying the above argument fiberwise to each n-truss fiber over objects in B. □

2.3.3 Truss blocks and truss block sets

A truss block (or simply a 'block') is a closed truss with an initial element; the name reflects the idea that all closed trusses can be 'built from blocks', as we will make precise in this section. Just as closed trusses dualize to open trusses, blocks dualize to open trusses with terminal elements, which we will refer to as 'braces'.

2.3.3.1 The definition of truss blocks

Recall that the depth of an object in a poset is the maximal length of chains starting in that object (for instance, a maximal object is of depth 0).

Definition 2.3.105 (Truss blocks)**.** An n-**truss** k-**block** T is a closed n-truss T whose total poset $(T_n, \trianglelefteq)$ has an initial object, and that object is of depth k. ⌐

When referring to n-truss k-blocks, we often keep either or both of the dimensions n and k implicit, or, yet more simply, just refer to them as 'blocks'.

Remark 2.3.106 (Blocks truncate)**.** Note that, given an n-truss block $T = (p_n, p_{n-1}, ..., p_1)$, then each truncation $T_{\leq i} = (p_i, p_{i-1}, ..., p_1)$ is again a i-truss block; indeed, the minimal element in the total poset of $T_{\leq i}$ is simply the image of the initial object in the total poset of T under the map $p_{>i} = p_{i+1} \circ p_{i+2} \circ ... \circ p_n$. ⌐

Notation 2.3.107 (Initial elements of blocks)**.** We often denote the initial element in the total posets of a block by $\bot$. ⌐

Example 2.3.108 (A 2-truss 2-block)**.** In Fig. 2.28 we illustrate a 2-truss 2-block on the left, together with its 'geometric realization' on its right. In contrast to our earlier example of a general closed 2-truss in Fig. 2.22, note that the realization now consists of a single 2-cell. ⌐
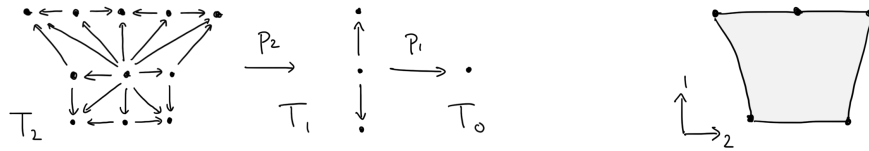
Figure 2.28: A 2-truss 2-block together with its geometric realization as a single framed 2-cell.

---

[18]In general, given poset maps $E, F : B \to C$ and $G : A \to B$, any natural transformation $\nu : E \circ G \Rightarrow F \circ G$ will be of the form $\mu \circ G$ for some $\Rightarrow$.

Notation 2.3.109 (Category of blocks). The category Blk_n of n-truss blocks and singular maps is defined to be the full subcategory of the category $\bar{\mathsf{Trs}}_n$ of closed trusses and singular maps whose objects are n-truss blocks. ⌐

Recall the notion of faces of closed trusses from Terminology 2.3.97. We construct faces in trusses that are blocks by taking closures of their elements, as follows.

Construction 2.3.110 (Face blocks in closed trusses). Let $T = (p_n, p_{n-1}, ..., p_1)$ be a closed n-truss, and consider an element $x \in T_n$ in the total poset T_n of T. We construct a subtruss inclusion $T^{\unrhd x} \hookrightarrow T$ such that $T^{\unrhd} = (p_n^{\unrhd x}, p_{n-1}^{\unrhd x}, ..., p_1^{\unrhd x})$ is a block, called the 'face block of x'. For each $i \leq n$, denote by $x_i = p_{>i} x$ the image of x under $p_{>i} = p_{i+1} \circ ... \circ p_n : T_n \to T_i$. Define $(T_i^{\unrhd x}, \unlhd)$ to be the subposet of T_i given by the upper closure of x in $(T_i, \unlhd)$. Define $p_i^{\unrhd x} : T_i^{\unrhd x} \to T_{i-1}^{\unrhd x}$ to be the restriction of $p_i : T_i \to T_{i-1}$ to these subposets. One verifies that we may uniquely endow p_i with 1-truss bundle structure such that the subposet inclusions $T_i^{\unrhd x} \hookrightarrow T_i$ induce 1-truss bundle maps $p_i^{\unrhd x} \hookrightarrow p_i$. The subposet inclusions $T_i^{\unrhd x} \hookrightarrow T_i$ also define the components of the claimed subtruss inclusion $T^{\unrhd x} \hookrightarrow T$. Note further that $T^{\unrhd x}$ is a k-block where k is the depth of x in T_n. ⌐

Example 2.3.111 (Blocks in closed trusses). In Fig. 2.29 we depict a 2-truss $T_2 \to T_1 \to T_0 = [0]$, and highlight two elements $x, y \in T_2$. For both elements we then construct their face block as shown. Identifying the total poset T_2 with the entrance path poset of the cell complex on the right, note that face blocks (obtained by taking 'upper closures' in T_2) correspond closed cells in the cell complex (obtained by taking 'topological closure). ⌐
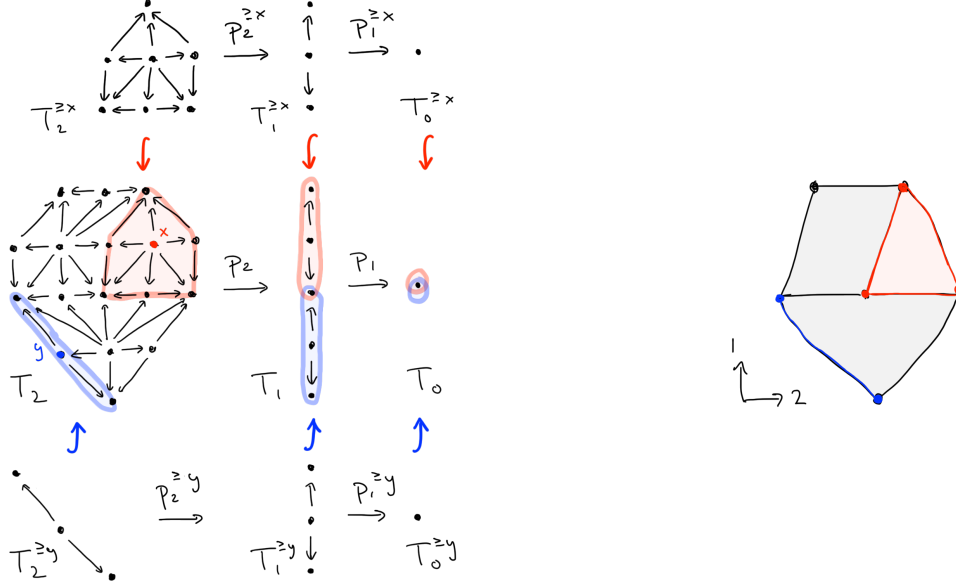
Figure 2.29: The face block of two elements x (red) and y (blue) in a 2-truss T.

In fact, all subtrusses that are blocks are obtained by taking closures as in the previous construction.

Remark 2.3.112 (All subtruss blocks are face blocks). Given an n-truss T, then subtrusses of T which are n-truss blocks are in 1-to-1 correspondence with objects of T_n: indeed, every such subtruss block of T determines an element of T_n as the image of its initial element, and conversely objects in T_n determined face blocks by taking closures as explained in Construction 2.3.110.   ⌐⌐

2.3.3.2 The definition of block sets In analogy with simplicial sets we now introduce the following notion of 'n-truss block sets'.

Definition 2.3.113 (Block sets). A n-**truss block set**, is a Set-valued presheaf on the category Blk_n of n-truss blocks.   ⌐⌐

We usually abbreviate 'n-truss block set' simply to 'block set', leaving the dimension n implicit.

Terminology 2.3.114 (Category of block sets). The 'category of block sets and their maps' is the category of Set-valued presheaves on the category Blk_n of n-truss blocks.   ⌐⌐

Terminology 2.3.115 (Faces, degeneracies, and non-degenerate blocks). Given a block set $X \in \mathsf{BlkSet}_n$ and $B \in \mathsf{Blk}_n$, we call elements of $X(B)$ 'blocks' of X of 'shape B'. Given a face map $F : C \to B$ in Blk_n such that $(X(F))(b) = c$ for $b \in X(B)$, then we call $c \in X(C)$ a 'face' (or 'F-face') of b. Given a degeneracy map $F : C \to B$ in Blk_n such that $(X(F))(b) = c$ for $b \in X(B)$, then we call $c \in X(C)$ a 'degeneracy (or 'F-degeneracy') of b. If $c \in X(C)$ is not an F-degeneracy for any non-identity F, then we call c a 'non-degenerate' block of X.   ⌐⌐

Notation 2.3.116 (Representable block sets). Given $B \in \mathsf{Blk}_n$ we often keep the Yoneda embedding notationally implicit, and write the representable presheaf $\mathsf{Blk}_n(-, B)$ simply as B.   ⌐⌐

Construction 2.3.117 (Block nerve of trusses). Given a closed n-truss T be can construct its 'block nerve' to be the block set $\bar{\mathsf{Trs}}_n(-, T)$ mapping a block B to the hom set $\bar{\mathsf{Trs}}_n(B, T)$. This construction is functorial and gives to the 'block nerve functor' $\mathrm{N}_{\mathsf{Blk}} : \bar{\mathsf{Trs}}_n \to \mathsf{BlkSet}_n$.   ⌐⌐

Remark 2.3.118 (Building trusses from their blocks). We can now make precise our previous statement that each closed n-truss T can be 'built from their blocks'. Given a closed n-truss T, denote by Blk_n/T the comma category of the inclusion $\mathsf{Blk}_n \hookrightarrow \bar{\mathsf{Trs}}_n$ over T (that is, objects of Blk_n/T are singular maps $B \to T$ from blocks into T, and morphisms are commuting block maps $B \hookrightarrow B'$). One verifies that T is given by the colimit

$$T = \mathrm{colim}(\mathsf{Blk}_n/T \to \bar{\mathsf{Trs}}_n)$$

of the forgetful functor $\mathsf{Blk}_n/T \to \bar{\mathsf{Trs}}_n$, which maps $(B \to T)$ to B.   ⌐⌐

The reader familiar with the general notion of nerves may also have observed that there are equivalent ways to state this remark: namely, it is equivalent to the observation that the functor $\mathsf{Blk}_n \hookrightarrow \bar{\mathsf{Trs}}_n$ is dense; it is also equivalent to the observation that the functor $\mathrm{N}_{\mathsf{Blk}}$ is fully faithful.
    We next introduce a notion of 'regular' block sets.

Definition 2.3.119 (Regular block set)**.** A block set $X \in \mathsf{BlkSet}_n$ is called **regular** if for each non-degenerate block $b \in X(B)$ the block set map $b : B \to X$ is a monomorphism. ⌟

Proposition 2.3.120 (Block nerves are regular block sets)**.** *Given a closed n-truss T, its block nerve $\mathrm{N}_{\mathsf{Blk}} T$ is regular.*

Proof. Non-degenerate blocks $b : B \to \mathrm{N}_{\mathsf{Blk}} T$ are of the form $\mathrm{N}_{\mathsf{Blk}}(F : B \hookrightarrow T)$ where $F : B \hookrightarrow T$ is a face block of T (see Remark 2.3.112). One verifies that $\mathrm{N}_{\mathsf{Blk}}(F : B \hookrightarrow T)$ is a monomorphism of block sets as required. □

Notation 2.3.121 (Category of regular block sets)*.* The full subcategory of the category of block set BlkSet_n containing only regular block sets will be denoted by $\mathsf{RegBlkSet}_n$. ⌟

2.3.3.3 Truss braces and truss brace sets Finally, let us briefly record the duals of truss blocks, which we will call 'truss braces'. Recall that open and closed n-trusses are related by duality functors $\dagger : \bar{\mathsf{Trs}}_n \cong \mathring{\mathsf{Trs}}_n : \dagger$ which dualize face orders on each truss. The preceding discussion of truss blocks and blocks sets may be completely dualized in this way. We mirror the most central definitions of the discussion for convenience. The 'height' of an object x in a poset measures the length k of maximal chains $x_{-k} \to x_{-k+1} \to ... \to x_0$ in the poset ending at that object, $x = x_0$.

Definition 2.3.122 (Truss braces)**.** An n**-truss** k**-brace** T is an open n-truss whose total poset $(T_n, \trianglelefteq)$ has a terminal object, and that object is of height k. ⌟

More concisely, we often refer to 'n-truss k-braces' simply as 'braces'. Dually to 'face blocks' we then find a notion of 'embedding braces' as follows.

Construction 2.3.123 (Embedding braces in open trusses)**.** Let T be an open n-truss. For any x in its total poset T_n, we construct the subtruss $T^{\trianglelefteq x} \hookrightarrow T$, called the **embedding brace of** x: it is the unique open subtruss of T such that $T^{\trianglelefteq x}$ is a brace whose terminal element maps to x. ⌟

Notation 2.3.124 (Category of braces)*.* The category Brc_n of braces and regular maps is defined to be the full subcategory of the category $\mathring{\mathsf{Trs}}_n$ of open trusses and regular maps whose objects are braces. ⌟

Definition 2.3.125 (Brace sets)**.** The **category of brace sets** BrcSet_n is the category of presheaves on the category Brc_n of braces and regular maps. Objects of BrcSet_n will be called **brace sets** and morphisms **brace set maps**. ⌟

Note that the dualization isomorphism $\dagger : \bar{\mathsf{Trs}}_n \cong \mathring{\mathsf{Trs}}_n$, induces an isomorphism of presheaf categories $\mathsf{Brc}_n \cong \mathsf{Blk}_n$ (by precomposition with $\dagger$).

Constructibility of framed combinatorial structures

The central theorems of this chapter will construct equivalences between the framed combinatorial structures introduced in Chapter 1 and the iterated constructible combinatorial structures discussed in Chapter 2.

3.1 Overview of equivalences

The most elementary such equivalence states that n-framed regular cells are classified by n-truss blocks as follows.

Theorem 3.1.1 (Truss blocks classify framed regular cells). *Framed regular cells are classified by truss blocks; that is, there is a canonical equivalence of categories* $\mathfrak{A}$

$$\mathsf{FrCell}_n \underset{\mathsf{C}^*\nabla}{\overset{\mathsf{T}_*\!\smallint}{\rightleftarrows}} \mathsf{Blk}_n$$

The equivalence is illustrated in Fig. 3.1: on the left, we depict a 3-framed regular cell (note this cell also appeared in Fig. 1.33); and on the right, we depict a 3-truss block T (note that we only depict generating arrows in the posets T_i, see Construction 2.3.69). The 3-framed regular cell will be mapped to the 3-truss by the 'truss integration' functor $\mathsf{T}_*\!\smallint$, and the inverse mapping will be described by the 'cell gradient' functor $\mathsf{C}^*\nabla$, as given in the previous theorem.

The correspondence of framed regular cells and truss blocks can be generalized to a correspondence of *flat* framed regular cell complexes and general trusses as follows.

Theorem 3.1.2 (Trusses classify flat framed regular cell complexes). *Flat framed regular cell complexes are classified by closed trusses and their singular maps; that is, there is a canonical equivalence of categories*

$$\mathsf{FlatFrCellCplx}_n \underset{\mathsf{C}^*\nabla}{\overset{\mathsf{T}_*\!\smallint}{\rightleftarrows}} \bar{\mathsf{Trs}}_n$$
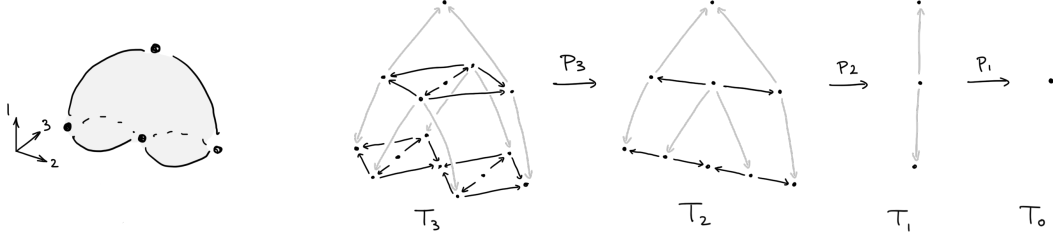
Figure 3.1: Framed regular cells correspond to truss blocks.

In Fig. 3.2 we illustrate an instance of this equivalence, depicting a flat 3-framed regular cell complex on the left (note that this complex also appeared in Fig. 1.41). On the right, we depict its corresponding closed 3-truss.
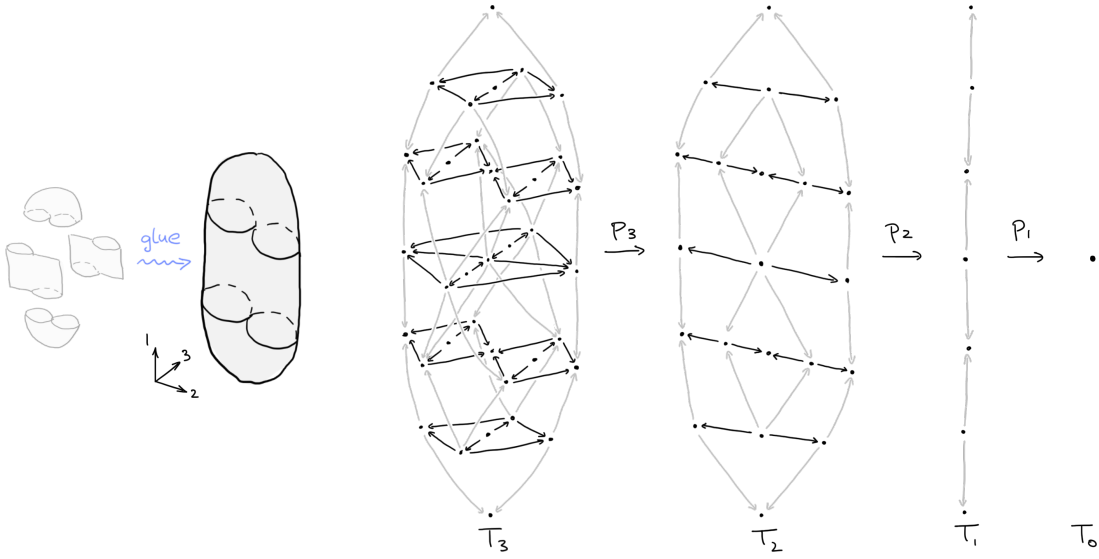
Figure 3.2: Flat framed regular cell complexes correspond to closed trusses.

Yet more generally, the preceding two equivalences are restrictions of the following equivalence.

Theorem 3.1.3 (Truss block sets classify framed regular cell complexes)**.** *Framed regular cell complexes are classified by regular truss block sets; that is, there is a canonical equivalence of categories*

$$\mathsf{FrCellCplx}_n \xrightleftharpoons[\mathsf{C}^*\nabla]{\mathsf{T}_*\int} \mathsf{RegBlkSet}_n$$

In Fig. 3.3 we illustrate an instance of this equivalence. On the left, we depict a 2-framed regular cell complex, consisting of two 0-cells, connected by two 1-cells, between which we suspend two 2-cells (note that this complex also appeared in Fig. 1.37). The regular truss block set on the right is indicated by its six non-degenerate truss blocks, each marked by its own box, which correspond to the cells on the left, and with face relations as indicated by colored inclusions.
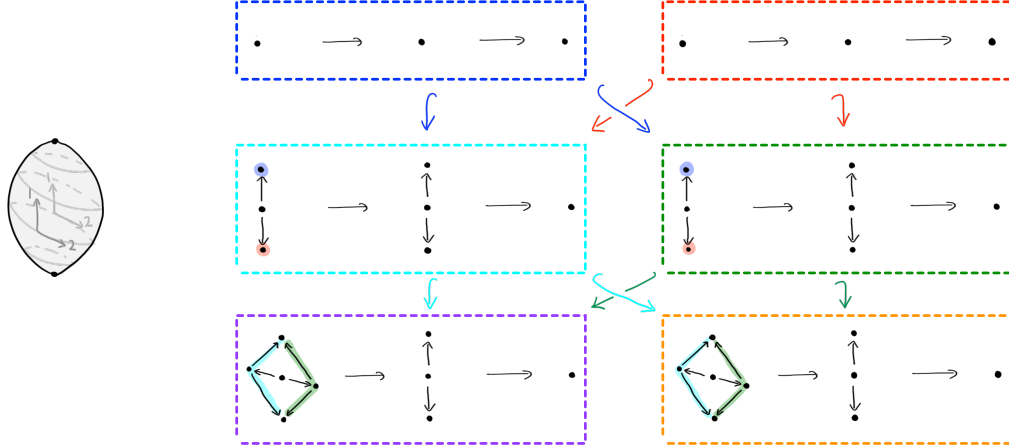
Figure 3.3: Framed regular cell complexes correspond to regular truss block sets.

In each of the above three equivalences, we call the functor $\mathsf{T}_*\!\int$ the 'truss integration', and the functor $\mathsf{C}^*\nabla$ the 'cell gradient'. In order to define these functors, we will introduce an intermediate notion between framed regular cells and truss blocks: this is the notion of *proframed* regular cells. Proframed regular cells are towers of cellular projections, and these can be easily translated into truss blocks. Framed regular cells and proframed regular cells are related, conceptually, by the idea of 'gradient vector fields' and 'vector-field-integrating functions' which can be concisely expressed in framed combinatorial terms. Note that, while proframed regular cells and truss blocks will each be comprised of very similar data (namely, towers of projections), the precise construction of their equivalence will be technical at times (for instance, due to the verification of cellularity). For those interested in the technical details, we now outline the structure of this chapter as follows.

1. In Section 3.2 we will introduce proframed combinatorial structures, and our discussion will roughly follow the outline of Chapter 1. Namely, we will start by defining proframes on individual simplices, and generalize this to proframes of simplicial complexes. A particularly important case will be that of *flat* proframed simplicial complexes (in which proframings are globally trivializable in an appropriate sense). This will allow us to define proframed regular cells, and, similarly, (flat) proframed regular cell complexes.

2. In Section 3.3 we establish the equivalence of framed and proframed regular cells via 'gradient' and 'integration' functors, yielding an equivalence $\mathsf{FrCell}_n \cong \mathsf{ProFrCell}_n$. There is a similar equivalence between flat framed and flat proframed regular cell complexes. These equivalences allow us to work with proframed cells in place of framed cells. Since proframed cells (like trusses) are defined by towers of maps, this equivalence will facilitate the construction of the truss corresponding to a framed cell.

3. In Section 3.4 we then construct the equivalence of proframed regular cells and truss blocks (as remarked above, the crucial step in this verification will be checking the cellularity of face posets of trusses, for which we employ a result relating 'shellability' and 'cellularity' of posets). Combining our results, we will proof the three theorems stated above.

3.2 Proframed combinatorial structures

3.2.1 Proframed simplices We start by recasting the definition of frames of simplices in the form of so called 'proframes'; these are towers of simplicial projections with 1-dimensional affine kernels. The usefulness of proframes stems from the 'non-local' perspective they provide: while we think of frames as choices of 'infinitesimal' vectors in tangent spaces, we can think of proframes as choices of 'extensional' projections defined on the entire simplex (this perspective also underlies our later discussion of gradients and integrals, which formally relates proframes back to frames). Just as frames on simplices have a classical analog in terms of linear frames of vector spaces, the notion of proframes also finds an analog in classical linear algebraic terms yielding a notion of 'linear proframes' (discussed in more detail in Appendix A.1); the prototypical model of a linear proframe is the standard euclidean proframe of $\mathbb{R}^n$ (from which all other proframes can then be obtained via trivializations $V \xrightarrow{\sim} \mathbb{R}^n$).

Terminology 3.2.1 (The standard euclidean proframe). The 'standard euclidean proframe' of $\mathbb{R}^n$ is the sequence of projections

$$\mathbb{R}^n \xrightarrow{\pi_n} \mathbb{R}^{n-1} \xrightarrow{\pi_{n-1}} \mathbb{R}^{n-2} \xrightarrow{\pi_{n-2}} \cdots \xrightarrow{\pi_2} \mathbb{R}^1 \xrightarrow{\pi_1} \mathbb{R}^0$$

where $\pi_i : \mathbb{R}^i \to \mathbb{R}^{i-1}$ forgets the last component of vectors in $\mathbb{R}^i$. ⎿

3.2.1.1 The definition of proframes As in the case of frames, we first introduce a combinatorial analog of ('standard' and 'general') proframes, in which the role of a codimension-1 projection is played by codimension-1 degeneracies.

Definition 3.2.2 (Proframe on a standard simplex)**.** A **proframe** $\mathcal{P}$ **of the standard** m**-simplex** $[m]$ is a sequence $(p_m, p_{m-1}, \ldots, p_1)$ of surjective simplicial maps of the form

$$[m] \xrightarrow{p_m} [m-1] \xrightarrow{p_{m-1}} [m-2] \to \cdots \to [1] \xrightarrow{p_1} [0].$$ ⎿

Definition 3.2.3 (Proframe on a simplex)**.** A **proframe** of an m-simplex S is an isomorphism $S \cong [m]$ together with a proframe $\mathcal{P}$ on $[m]$. ⎿

We usually denote proframes on S by tuples $(S \cong [m], \mathcal{P})$, or keep the isomorphism implicit and simply say $\mathcal{P}$ is a proframe on S.

Example 3.2.4 (Proframes on simplices)**.** In Fig. 3.4 we illustrate four proframed m-simplices $(S \cong [m], \mathcal{P})$. Each degeneracy is indicated by highlighting its affine kernel. ⎿

Recall that a linear map from a framed simplex $([m], \mathcal{F})$ to euclidean space is a framed realization when it takes each frame vector v_i into the ith positive component ϵ_i^+ of $\mathbb{R}^m$. There is an analogous notion of realization for proframes, as mappings into the standard proframed euclidean space, as follows.

Definition 3.2.5 (Proframed realization of a proframed simplex)**.** A **proframed realization** of a proframed simplex $(S \cong [m], \mathcal{P} = (p_m, p_{m-1}, \ldots, p_1))$ is a sequence of linear
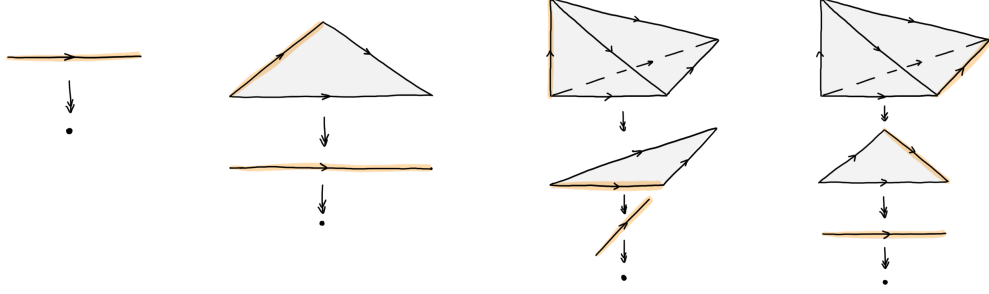
Figure 3.4: Proframes on simplices.

embeddings $r_i^{\mathcal{P}} : \Delta^i \hookrightarrow \mathbb{R}^i$, giving a commutative diagram,

$$
\begin{array}{ccccccccc}
|S| \cong \Delta^m & \xrightarrow{p_m} & \Delta^{m-1} & \xrightarrow{p_{m-1}} & \ldots & \xrightarrow{p_2} & \Delta^1 & \xrightarrow{p_1} & \Delta^0 \\
\downarrow{r_m^{\mathcal{P}}} & & \downarrow{r_{m-1}^{\mathcal{P}}} & & & & \downarrow{r_1^{\mathcal{P}}} & & \downarrow{r_0^{\mathcal{P}}} \\
\mathbb{R}^m & \xrightarrow{\pi_m} & \mathbb{R}^{m-1} & \xrightarrow{\pi_{m-1}} & \ldots & \xrightarrow{\pi_2} & \mathbb{R}^1 & \xrightarrow{\pi_1} & \mathbb{R}^0
\end{array}
$$

such that, for all i, $r_i^{\mathcal{P}}$ embeds the unique kernel vectors $v = \ker^{\mathrm{aff}}(p_i)$ orientation-preservingly into the $\mathbb{R}$-fiber of π_i over the point $r_{i-1}^{\mathcal{P}} \circ p_i(v)$. ⌐

Example 3.2.6 (Proframed realization of a proframed simplex)**.** In Fig. 3.5 we illustrate a proframed realization of a proframed 2-simplex. ⌐
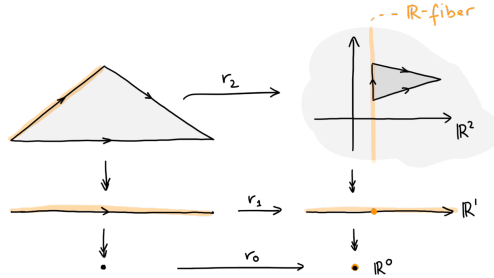
Figure 3.5: Proframed realization of a proframed simplex.

Remark 3.2.7 (Relations of proframed and framed realizations)**.** In Section 3.2.1.4 we will show that proframes $\mathcal{P}$ and frames $\mathcal{F}$ on the simplex $[m]$ are in correspondence (i.e. they define the same structure on $[m]$); for corresponding proframes $\mathcal{P}$ and frames $\mathcal{F}$ on $[m]$, a proframed realization $\{r_i^{\mathcal{P}}\}$ of $\mathcal{P}$ determines is determined by a framed realization $r_{\mathcal{F}}$ of $\mathcal{F}$ by setting $r_m^{\mathcal{P}} = r_{\mathcal{F}}$. ⌐

A partial frame of a simplex was a frame of a quotient of the simplex by a simplicial subspace (the 'unframed subspace'); similarly, a partial proframe of a simplex will be a proframe of a 'quotient' of the simplex, as follows.

Definition 3.2.8 (Partial proframe on a simplex)**.** A k-**partial proframe** on an m-simplex S is a degeneracy $S \twoheadrightarrow [k]$ together with a proframe $\mathcal{P} = (p_k, p_{k-1}, \ldots, p_1)$ of $[k]$. ⌐

We usually denote k-partial proframe $\mathcal{P}$ on a simplex S by tuples $(S \twoheadrightarrow [k], \mathcal{P})$. Analogous to the case of partial frames, we refer to the affine kernel $U = \ker^{\mathrm{aff}}(S \twoheadrightarrow [k])$ as the 'unframed subspace' of $(S \twoheadrightarrow [k], \mathcal{P})$. Note that in an m-partial proframe of an m-simplex $(S \twoheadrightarrow [m], \mathcal{P})$ the degeneracy $S \twoheadrightarrow [m]$ must be an isomorphism, and thus m-partial proframes of m-simplices are proframes on m-simplices.

Example 3.2.9 (Partial proframes on simplices)**.** In Fig. 3.6 we depict a few partially proframed simplices. As before, each degeneracy is given by highlighting its affine kernel. ⌐
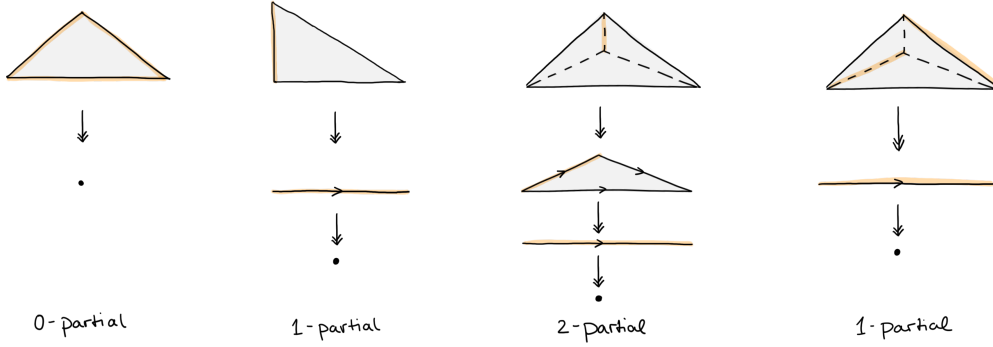
Figure 3.6: Partial proframes on simplices.

The notion of proframed realization adapts to the partial case as follows: a 'proframed realization' of a k-partially proframed m-simplex $(S \twoheadrightarrow [k], \mathcal{P})$ is simply a proframed realization $\{r_i^{\mathcal{P}}\}$ of $([k], \mathcal{P})$ which now further determines a composite $r^{\mathcal{P}} = r_k^{\mathcal{P}} \circ |S \twoheadrightarrow [k]|$ mapping $|S|$ to $\mathbb{R}^k$. Remark 3.2.7 will also apply with evident adjustments: a proframed realization $\{r_i^{\mathcal{P}}\}$ of a k-partially proframed simplex $(S \twoheadrightarrow [k], \mathcal{P})$ determines and is determined by a framed realization $r_{\mathcal{F}}$ of the corresponding k-partially framed simplex $(S \twoheadrightarrow [m], \mathcal{F})$ by setting $r^{\mathcal{P}} = r_{\mathcal{F}}$.

3.2.1.2 Embedded and partial proframes We introduce 'n-embedded proframes' as a combinatorial analog of our earlier notion of 'linear n-embedded proframes'. As in the linear case, while a proframed m-simplex is a sequence of codimension-1 degeneracies, an n-embedded proframed m-simplex will be a sequence of n degeneracies each either of codimension-1 or codimension-0. Again, we introduce both 'standard' and 'general' notions of embedded proframes.

Definition 3.2.10 (Embedded proframe on a standard simplex)**.** An n**-embedded proframe** $\mathcal{P}$ **of the standard** m**-simplex** $[m]$ is a sequence $(p_n, p_{n-1}, \ldots, p_1)$ of surjective simplicial maps

$$[m] = [m_n] \xrightarrow{p_n} [m_{n-1}] \xrightarrow{p_{n-1}} [m_{n-2}] \xrightarrow{p_{n-2}} \cdots \xrightarrow{p_2} [m_1] \xrightarrow{p_1} [m_0] = [0]$$

where for all i, either $m_{i-1} = m_i$ or $m_{i-1} = m_i - 1$. ⌐

Definition 3.2.11 (Embedded proframe on simplex)**.** An n**-embedded proframe** of an m-simplex S is an isomorphism $S \cong [m]$ together with an n-embedded proframe $\mathcal{P}$ on $[m]$. ⌐

122

We usually denote n-embedded proframed simplices by tuples $(S \cong [m], \mathcal{P})$ (where $\mathcal{P}$ is an n-embedded proframe of $[m]$). Note that an m-embedded proframe on the m-simplex is the same as a proframe on the m-simplex as previously defined.

Example 3.2.12 (Embedded proframes)**.** In Fig. 3.7 we illustrate a few n-embedded proframed m-simplices $(S \cong [m], \mathcal{P})$. As before, each degeneracy is given by highlighting its affine kernel. ⌐

2-embedded   3-embedded   3-embedded   4-embedded
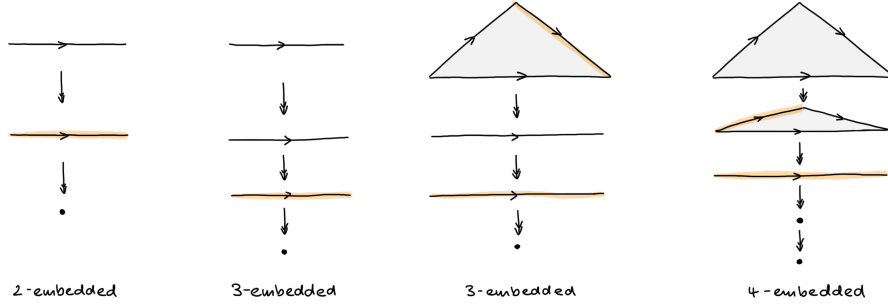
Figure 3.7: Embedded proframes on simplices.

As a proframe of an m-simplex has a notion of proframed realizations given by embeddings into the standard proframe of euclidean m-space, similarly an n-embedded proframe on an m-simplex has proframed realizations given by embeddings into the standard proframe of euclidean n-space, as follows.

Definition 3.2.13 (Proframed realization of an embedded proframed simplex)**.** A **proframed realization** of an n-embedded proframed m-simplex $(S \cong [m], \mathcal{P})$ with $\mathcal{P} = (p_n, p_{n-1}, \ldots, p_1)$ is a sequence of linear embeddings $r_i^{\mathcal{P}} : \Delta^{m_i} \hookrightarrow \mathbb{R}^i$, giving a commutative diagram,

$$\begin{array}{ccccccccc} |S| \cong \Delta^{m_n} & \xrightarrow{p_n} & \Delta^{m_{n-1}} & \xrightarrow{p_{n-1}} & \ldots & \xrightarrow{p_2} & \Delta^{m_1} & \xrightarrow{p_1} & \Delta^{m_0} \\ \downarrow{\scriptstyle r_n^{\mathcal{P}}} & & \downarrow{\scriptstyle r_{n-1}^{\mathcal{P}}} & & & & \downarrow{\scriptstyle r_1^{\mathcal{P}}} & & \downarrow{\scriptstyle r_0^{\mathcal{P}}} \\ \mathbb{R}^n & \xrightarrow{\pi_n} & \mathbb{R}^{n-1} & \xrightarrow{\pi_{n-1}} & \ldots & \xrightarrow{\pi_2} & \mathbb{R}^1 & \xrightarrow{\pi_1} & \mathbb{R}^0 \end{array}$$

such that $r_i^{\mathcal{P}}$ orientation-preservingly embeds kernel vectors $v = \ker^{\mathrm{aff}}(p_i)$ (for all i for which v exists) into the $\mathbb{R}$-fiber of π_i over the point $r_{i-1}^{\mathcal{P}} \circ p_i(v)$. ⌐

Example 3.2.14 (Proframed realization of an embedded proframed simplex)**.** In Fig. 3.8 we illustrate a proframed realization of a 3-embedded proframed 2-simplex. ⌐

The relation of proframed and framed realizations as described in Remark 3.2.7 generalizes to the embedded case: for corresponding n-embedded proframes $\mathcal{P}$ and n-embedded frames $\mathcal{F}$ on $[m]$, a proframed realization $\{r_i^{\mathcal{P}}\}$ of $\mathcal{P}$ determines is determined by a framed realization $r_{\mathcal{F}}$ of $\mathcal{F}$ by setting $r_n^{\mathcal{P}} = r_{\mathcal{F}}$. We will revisit this observation in more formal terms in Remark 3.2.40.

―――――――――

Recall, a k-partial proframe of an m-simplex is a degeneracy to the k-simplex followed by a proframe of that simplex. Similarly, an n-embedded k-partial proframe of an m-simplex is a degeneracy to the k-simplex followed by an n-embedded proframe of that simplex.
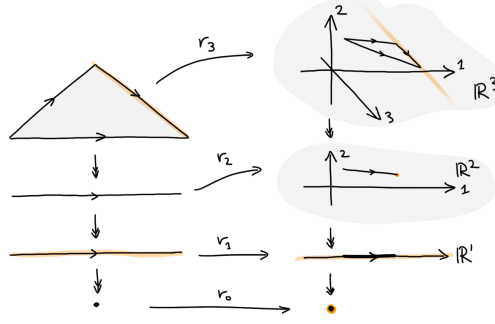
Figure 3.8: Proframed realization of an embedded proframed simplex.

Definition 3.2.15 (Embedded partial proframe on a simplex). An n-**embedded** k-**partial proframe** on an m-simplex S is a degeneracy $S \twoheadrightarrow [k]$ together with an n-embedded proframe $\mathcal{P} = (p_n, p_{n-1}, \ldots, p_1)$ of $[k]$. ⌟

We usually denote n-embedded k-partial proframe $\mathcal{P}$ on a simplex S by tuples $(S \twoheadrightarrow [k], \mathcal{P})$, and refer to the affine kernel $U = \ker^{\mathrm{aff}}(S \twoheadrightarrow [k])$ as the 'unframed subspace' of $(S \twoheadrightarrow [k], \mathcal{P})$. Of course, an n-embedded m-proframe of an m-simplex is the same structure as an n-embedded proframe of an m-simplex as previously defined

Example 3.2.16 (Embedded partial proframes). In Fig. 3.9 we illustrate three embedded partial proframes. As before, we depict degeneracies by highlighting their affine kernels. ⌟
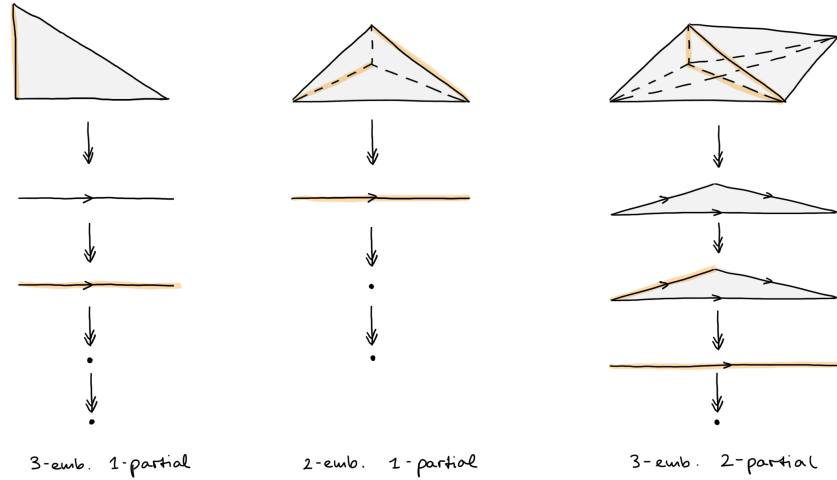
Figure 3.9: Embedded partial proframes on simplices.

The notion of proframed realization of embedded proframes adapts to the partial case as follows: a proframed realization of an n-embedded k-partially proframed m-simplex $(S \twoheadrightarrow [k], \mathcal{P})$ is simply a proframed realization $\{r_i^{\mathcal{P}}\}$ of the n-embedded proframe $([k], \mathcal{P})$ which further determines a composite $r^{\mathcal{P}} = r_k^{\mathcal{P}} \circ |S \twoheadrightarrow [k]|$ mapping $|S|$ to $\mathbb{R}^n$. Remark 3.2.7 will apply with evident adjustments: a proframed realization $\{r_i^{\mathcal{P}}\}$ of a k-partially proframed simplex $(S \twoheadrightarrow [k], \mathcal{P})$ determines and is determined by a framed

realization $r_{\mathcal{F}}$ of the corresponding k-partially framed simplex $(S \twoheadrightarrow [m], \mathcal{F})$ by setting $r^{\mathcal{P}} = r_{\mathcal{F}}$.

3.2.1.3 **Proframed maps** Recall that a framed map is a map of simplices that, for each vector of the source, either preserves the frame label of the vector or degenerates the vector. Proframed maps similarly are maps, now of simplicial sequences, that for each vector of the source, either preserve the proframe of the vector or degenerate that vector, as follows.

Definition 3.2.17 (Proframed maps)**.** Given n-embedded proframed simplices $(S \cong [l], \mathcal{P} = (p_n, \ldots, p_1))$ and $(T \cong [m], \mathcal{Q} = (q_n, \ldots, q_1))$, a **proframed map** $F : (S \cong [l], \mathcal{P}) \to (T \cong [m], \mathcal{Q})$ is a map of sequences

$$
\begin{array}{ccccccccc}
[l] = [l_n] & \xrightarrow{p_n} & [l_{n-1}] & \xrightarrow{p_{n-1}} & \cdots & \xrightarrow{p_2} & [l_1] & \xrightarrow{p_1} & [l_0] = [0] \\
\downarrow{\scriptstyle F_n} & & \downarrow{\scriptstyle F_{n-1}} & & \cdots & & \downarrow{\scriptstyle F_1} & & \downarrow{\scriptstyle F_0} \\
[m] = [m_n] & \xrightarrow[q_n]{} & [m_{n-1}] & \xrightarrow[q_{n-1}]{} & \cdots & \xrightarrow[q_2]{} & [m_1] & \xrightarrow[q_1]{} & [m_0] = [0]
\end{array}
$$

such that for every vector $v : [1] \to [l]$, either its proframe is preserved, i.e. $\mathcal{P}|_v = \mathcal{Q}|_{F_n \circ v}$, or the vector is degenerated, i.e. $F_n \circ v : [1] \to [m]$ is constant. ⌐

Terminology 3.2.18 (Proframed face and proframed degeneracy maps)*.* An injective proframed map is called a 'proframed face map', and a surjective proframed map is called a 'proframed degeneracy map'. ⌐

Observation 3.2.19 (Inclusions of proframe restrictions are proframed faces)*.* Given a proframed simplex $([m], \mathcal{Q})$ and a face $f : [l] \hookrightarrow [m]$ of the simplex, the inclusion of the restriction $([l], \mathcal{Q}|_f) \hookrightarrow ([m], \mathcal{Q})$ is a proframed face, and every proframed face is of this form. ⌐

Observation 3.2.20 ((Epi,mono)-factorization of proframed maps)*.* Every proframed map factors as a proframed degeneracy followed by a proframed face. ⌐

Example 3.2.21 (Proframed maps)**.** In Fig. 3.10 we illustrate proframed and non-proframed maps of proframed simplices. Degeneracies in proframes are indicated as usual by highlighting their affine kernels. Top-level map (which, in a maps of sequences of surjective maps, determine all other maps) coincide with the maps in Fig. 1.15. The first map of sequences is proframed and injective, thus is a proframed face. The second map is not a map of sequences (the upper square does not commute as indicated). The third map of sequences is proframed and surjective, thus is a proframed degeneracy. The last map, despite being a map of sequences, that is forming a commutative diagram as depicted, is not a proframed map: the map does not preserve the proframe of the diagonal vector of the 2-simplex of the source, i.e. the map restricted to that vector is not an isomorphism onto its image.v ⌐

Notation 3.2.22 (Category of proframed simplices)*.* Denote by $\mathsf{ProFrSimp}_n$ the category of n-embedded proframed simplices and their proframed maps. ⌐

The definition of proframed maps generalizes naturally from the embedded to the embedded partial case.
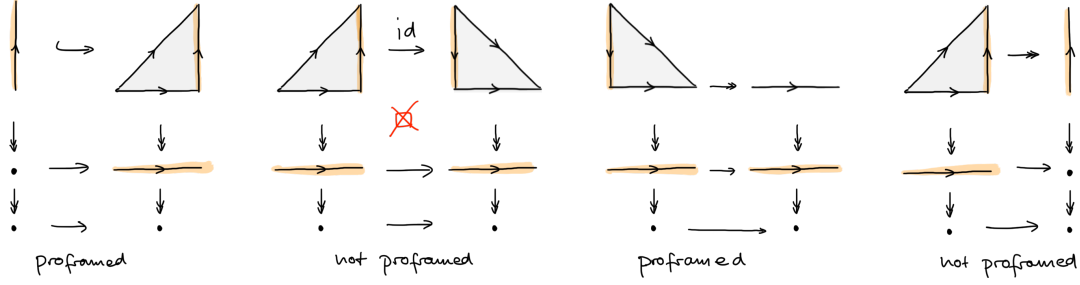
Figure 3.10: Proframed and non-proframed maps.

Definition 3.2.23 (Proframed maps of partially proframed simplices)**.** Given an n-embedded partially proframed simplices $(S \twoheadrightarrow [k], \mathcal{P})$ and $(T \twoheadrightarrow [j], \mathcal{Q})$ a **proframed map** $F : (S \twoheadrightarrow [k], \mathcal{P}) \to (T \twoheadrightarrow [j], \mathcal{Q})$ is a simplicial map $F : S \to T$ that descends to a proframed map $(F_n, F_{n-1}, ..., F_0) : ([k], \mathcal{P}) \to ([j], \mathcal{Q})$ whose top component $F_n : [k] \to [j]$ commutes with F and the degeneracies $S \twoheadrightarrow [k]$ and $T \twoheadrightarrow [j]$. ⌟

Note that any proframed map $F : (S \twoheadrightarrow [k], \mathcal{P}) \to (T \twoheadrightarrow [j], \mathcal{Q})$ must descend to a unique proframed map $([k], \mathcal{P}) \to ([j], \mathcal{Q})$ whose components will always be denoted by $(F_n, F_{n-1}, ..., F_0)$.

Notation 3.2.24 (Category of partially proframed simplices)*.* Denote by $\mathsf{PartProFrSimp}_n$ the category of n-embedded partially proframed simplices and their proframed maps. ⌟

Of course, we may regard proframed maps of proframed simplices as proframed maps of partially proframed simplices; thus $\mathsf{ProFrSimp}_n$ is a full and faithful subcategory of $\mathsf{PartProFrSimp}_n$.

Remark 3.2.25 (Subproframed maps)*.* Recall from Remark 1.1.67 and Definition 1.1.68 the notion of subframed map of framed simplices, in which framed vectors are allowed to specialize to have frame labels in lower strata of the standard stratification. The corresponding notion of 'subproframed map' of (partially) proframed simplices is remarkably simple: a subproframed map $F : (S \twoheadrightarrow [k], \mathcal{P} = (p_n, ..., p_1)) \to (T \twoheadrightarrow [j], \mathcal{Q} = (q_n, ..., q_1))$ is a collection of maps $(F, F_n, ..., F_0)$ of unordered simplices $F : S \to T$ and $F_i : [k_i]^{\mathsf{un}} \to [j_i]^{\mathsf{un}}$ forming a commutative diagram (i.e. $F_n \circ (S \twoheadrightarrow [k]) = (T \twoheadrightarrow [j]) \circ F$ and $F_{i-1} \circ p_i^{\mathsf{un}} = q_i^{\mathsf{un}} \circ F_{i+1}$). This does not require any further conditions on conditions on vectors: instead, the structure of the sequence itself controls the specialization of frame vectors without explicit mention of an order on frame labels or the standard stratification of frame vectors. ⌟

3.2.1.4 Gradient frames and integral proframes We conceive of frames as an infinitesimal notion, concerning 'tangential' vectors, and of proframes as extensional notion, concerning quotients of space. (This contrast becomes visible, for instance, in the classical case of frames and proframes on affine spaces, see Remark A.2.5 resp. Remark A.2.7.) We will thus refer to the passage from frames to proframes as 'integration', and to a converse operation as taking a 'gradient'.

We start by introducing the gradient frame of a proframe, and conversely the integral proframe of a frame.

126

Notation 3.2.26 (Composite degeneracies in proframes). For an n-embedded proframe $\mathcal{P} = (p_n, \ldots, p_1)$ of the m-simplex $[m]$, we abbreviate the composite $p_i p_{i+1} \cdots p_n : [m] \to [k_{i-1}]$ by $p_{\geq i}$ and similarly the composite $p_{i+1} \cdots p_n : [m] \to [k_i]$ by $p_{>i}$. ⏝

Terminology 3.2.27 (Gradient frame of a proframed standard simplex). Given an n-embedded proframed m-simplex $([m], \mathcal{P} = (p_n, \ldots, p_1))$, its 'gradient' $\nabla \mathcal{P}$ is the n-embedded framed m-simplex $([m], \nabla \mathcal{P} : \mathsf{spine}[m] \hookrightarrow \underline{n})$ with frame label i on the spine vector $\ker^{\mathrm{aff}}(p_{\geq i}) \setminus \ker^{\mathrm{aff}}(p_{>i})$ (when that complement is nonempty). ⏝

In other words, if the spine vector $v \in \mathsf{spine}[m]$ projects to a spine vector $p_{>i}v \in \mathsf{spine}[k_i]$ and the degeneracy $p_i : [k_i] \to [k_{i-1}]$ collapses that vector $p_{>i}v$, then the spine vector $v \in \mathsf{spine}[m]$ is given the frame label i.

Terminology 3.2.28 (Integral proframe of a framed standard simplex). Given an n-embedded framed m-simplex $([m], \mathcal{F} : \mathsf{spine}[m] \hookrightarrow \underline{n})$, an 'integral' $\int \mathcal{F}$ is an n-embedded proframed m-simplex $([m], \int \mathcal{F})$ whose gradient is $([m], \mathcal{F})$. ⏝

Construction 3.2.29 (Integral proframes of framed standard simplices exist). Given an n-embedded framed m-simplex $([m], \mathcal{F} : \mathsf{spine}[m] \hookrightarrow \underline{n})$, inductively set $p_i : [k_i] \to [k_{i-1}]$ to be the unique simplicial map collapsing the spine vector $p_{>i}(\mathcal{F}^{-1}(i))$ i.e. the spine vector that has frame label i; if there is no spine vector with frame label i, then p_i is taken to be the identity. The sequence of degeneracies $\mathcal{P} = (p_n, p_{n-1}, \ldots, p_1)$ is an n-embedded k-partial proframe of the m-simplex, and has the frame $\mathcal{F}$ as its gradient. ⏝

Observation 3.2.30 (Gradient and integral are inverse). By definition, or equivalently by the construction of the integral in the previous observation, the integral is right-inverse to the gradient; similarly, taking the integral of the gradient of a proframe evidently reconstructs that proframe, and so the integral is also left-inverse to the gradient. That is, for any n-embedded proframe $\mathcal{P}$ and any n-embedded frame $\mathcal{F}$, we have

$$\nabla \int \mathcal{F} = \mathcal{F} \quad \text{and} \quad \int \nabla \mathcal{P} = \mathcal{P}. \qquad ⏝$$

The gradients and integrals now carry over to the case of general simplices (and the case of partial frames and proframes) as follows.

Definition 3.2.31 (Gradient of a proframed simplex). Given an n-embedded partially proframed simplex $(S \twoheadrightarrow [k], \mathcal{P})$ its **gradient** is the n-embedded partially framed simplex $(S \twoheadrightarrow [k], \nabla \mathcal{P})$. ⏝

Definition 3.2.32 (Integral of a framed simplex). Given an n-embedded partially framed simplex $(S \twoheadrightarrow [k], \mathcal{F})$ its **integral** is the n-embedded partially proframed simplex $(S \twoheadrightarrow [k], \int \mathcal{P})$. ⏝

Example 3.2.33 (Gradient frame and integral proframe for a simplex). In Fig. 3.11 we illustrate a 4-embedded 3-proframed 4-simplex and its corresponding gradient 3-embedded 2-framed 4-simplex; equivalently, that framed simplex integrates to that proframed simplex. As further examples, note that the four embedded framed simplices in Fig. 1.7 are, in order, the gradients of the four embedded proframed simplices in Fig. 3.7. Similarly, the three embedded partially framed simplices in Fig. 1.9 are, in order, the gradients of the three embedded proframed simplices in Fig. 3.9. ⏝
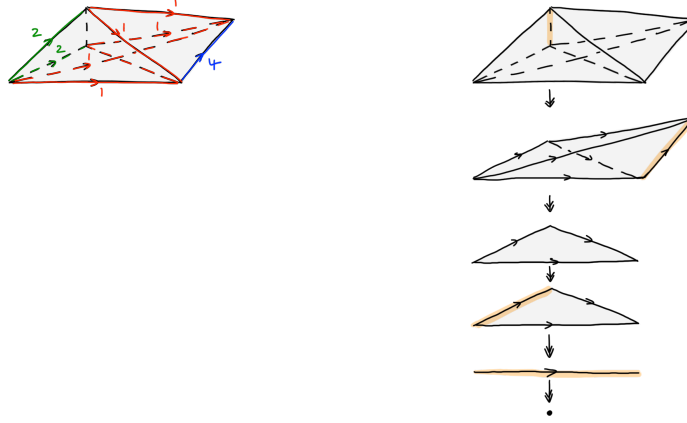
Figure 3.11: The gradient frame and the integral proframe of a simplex.

The gradient–integral relation between frames and proframes applies also to maps.

Observation 3.2.34 (Gradients and integrals respect restriction). Given an n-embedded partially proframed simplex $(S \twoheadrightarrow [k], \mathcal{P})$ and an n-embedded partially framed simplex $(S \twoheadrightarrow [k], \mathcal{F})$, and a face $f : [j] \to [m]$, the gradient of the restriction to the face is the restriction of the gradient, and similarly for the integrals:

$$\nabla(\mathcal{P}|_f) = (\nabla \mathcal{P})|_f \quad \text{and} \quad \textstyle\int(\mathcal{F}|_f) = (\textstyle\int \mathcal{F})|_f. \qquad \quad \lrcorner$$

Terminology 3.2.35 (Gradient framed map of a proframed map). Given an n-embedded partially proframed simplex $(S \twoheadrightarrow [j], \mathcal{P})$, an n-embedded partially proframed simplex $(T \twoheadrightarrow [k], \mathcal{Q})$, and a proframed map $F : (S \twoheadrightarrow [j], \mathcal{P}) \to (T \twoheadrightarrow [k], \mathcal{Q})$, the 'gradient' ∇F is simply the framed map $(S \twoheadrightarrow [j], \nabla \mathcal{P}) \to (T \twoheadrightarrow [k], \nabla \mathcal{Q})$ determined by the simplicial map $F : S \to T$. $\qquad \lrcorner$

Terminology 3.2.36 (Integral proframed map of a framed map). Given an n-embedded partially framed simplex $(S \twoheadrightarrow [j], \mathcal{F})$, an n-embedded partially framed simplex $(T \twoheadrightarrow [k], \mathcal{G})$, and a framed map $F : (S \twoheadrightarrow [j], \mathcal{F}) \to (T \twoheadrightarrow [k], \mathcal{G})$, an 'integral' $\int F$ is a proframed map $(S \twoheadrightarrow [j], \int \mathcal{F}) \to (T \twoheadrightarrow [k], \int \mathcal{G})$ whose gradient is the framed map F. $\qquad \lrcorner$

Example 3.2.37 (Gradient framed maps and integral proframed maps)**.** The two framed maps in Example 1.1.62 are the gradients of the two proframed maps in Example 3.2.21, or equivalently the latter maps are the integrals of the former maps. $\qquad \lrcorner$

As there exists a unique integral proframe of any framed simplex, similarly there exists a unique integral proframed map of any framed map. Altogether, we have an equivalence of categories, as follows.

Proposition 3.2.38 (Correspondence of partial frames and partial proframes)**.** *Gradient and integration are inverse isomorphic functors between the category of n-embedded partially proframed simplices with proframed maps and the category of n-embedded partially framed simplices with framed maps:*

$$\nabla : \mathsf{PartProFrSimp}_n \cong \mathsf{PartFrSimp}_n : \textstyle\int . \qquad \qquad \square$$

In practice we will be focused on a special case of this correspondence, namely between n-embedded frames and n-embedded proframes.

Corollary 3.2.39 (Correspondence of frames and proframes)**.** *Gradient and integration are inverse isomorphic functors between the categories of n-embedded framed respectively proframed simplices and framed respectively proframed maps:*

$$\nabla : \mathsf{ProFrSimp}_n \cong \mathsf{FrSimp}_n : \smallint .$$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Remark 3.2.40 (Correspondence of framed and proframed realizations). Given an n-embedded proframed simplex $(S \cong [m], \mathcal{P})$ with gradient framed simplex $(S \cong [m], \mathcal{F} = \nabla \mathcal{P})$, then any proframed realization $\{r_i^{\mathcal{P}} : \Delta^{m_i} \hookrightarrow \mathbb{R}^i\}$ determines is and determined by a framed realization $r_{\mathcal{F}} : |S| \cong \Delta^{m_n} \hookrightarrow \mathbb{R}^n$ by setting $r_n^{\mathcal{P}} = r_{\mathcal{F}}$. ⌐

3.2.2   Proframed simplicial complexes Generalizing the discussion in the previous section, we will now describe how simplicial complexes may be endowed with proframings.

3.2.2.1   The definition of proframings We next introduce proframings on simplicial complexes. The definition takes a 'global route to framings': a proframing endows a simplicial complex with a sequence of simplicial surjections defined on the entire complex (we also refer to these surjections as 'simplicial projections'). In particular, the definition of n-proframings differs from that of n-framings in that n-proframings do not merely require compatible choices of proframes on each simplex (however, such 'local' proframes can be derived from the definition). The 'global nature' of proframings will make them a useful tool for describing global properties of framings; most importantly, we will see that flat framings can be completely understood in terms of certain proframings.

Definition 3.2.41 (Proframings of simplicial complexes)**.** An n-**proframing** of a simplicial complex K is a ordering of K together with a sequence $\mathcal{P}$ of ordered simplicial surjections

$$K = K_n \xrightarrow{p_n} K_{n-1} \xrightarrow{p_{n-1}} ... \xrightarrow{p_2} [K_1] \xrightarrow{p_1} K_0 = [0]$$

such that on each simplex $x : [m] \hookrightarrow K$ the restricted sequence $\mathcal{P}|_x$ is an n-embedded proframe of $[m]$. ⌐

We will refer to the pair $(K, \mathcal{P})$ of a simplicial complex with an n-proframing $\mathcal{P} = (p_n, p_{n-1}, ..., p_1)$ on it, as an 'n-proframed simplicial complex'.

Example 3.2.42 (Proframings of simplicial complexes)**.** In Fig. 3.12 we depict three sequences of simplicial surjections. Each projection p_i is suggested as a geometric projection, but we also highlight affine kernels of p_i on each simplex (in blue). While the first three sequences define proframings, the last sequence fails to do so as it fails to be an 2-embedded proframe when restricted to its central 2-simplex. ⌐

As in the case of framings, we can further generalize the definition of proframings to allow for partial proframes.
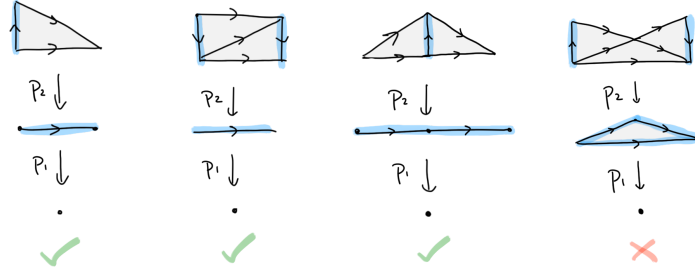
Figure 3.12: Two 2-proframings on a simplicial complex with two 2-simplices.

Definition 3.2.43 (Partial proframing). An **partial n-proframing** of a simplicial complex K is a sequence of surjective simplicial maps (all but the first of which are *ordered* simplicial maps)

$$K \twoheadrightarrow K_n \xrightarrow{p_n} K_{n-1} \xrightarrow{p_{n-1}} ... \xrightarrow{p_2} K_1 \xrightarrow{p_1} K_0 = [0]$$

such that, on each m-simplex $x : S \hookrightarrow K$ of K, the sequence restricts to an n-embedded partial proframe of S. ⌐

Example 3.2.44 (A familiar partial proframing). The sequence of ordered simplicial maps depicted earlier in Fig. 1.17 is, in fact, a partial 1-proframing. ⌐

While many subsequent definitions also apply (with evident adjustments) to partial proframings, we will henceforth be interested only in the non-partial case.

3.2.2.2 Maps, gradients and integrals We next define maps of proframed simplicial complexes. This generalizes our earlier definition of proframed maps of embedded proframed simplices (see Definition 3.2.17).

Definition 3.2.45 (Maps of proframings). Given n-proframed simplicial complexes $(K, \mathcal{P})$ and $(L, \mathcal{Q})$, a **proframed simplicial map** $F : (K, \mathcal{P}) \to (L, \mathcal{Q})$ is a map of sequences $(F_n, F_{n-1}, ..., F_1, F_0) : \mathcal{P} \to \mathcal{Q}$ (consisting of cellular maps F_i) which, on a simplex $x : [k] \hookrightarrow K$ with image $y = \mathrm{im}(F_n \circ x) : [l] \hookrightarrow L$, restricts to a proframed map $F : \mathcal{P}|_x \to \mathcal{Q}|_y$ of n-embedded proframed simplices. ⌐

Notation 3.2.46 (Category of proframings). The category of n-proframed simplicial complexes and their proframed maps will be denoted by $\mathsf{ProFrSimpCplx}_n$. ⌐

Notation 3.2.47 (Truncations of proframings). Given an n-proframing $\mathcal{P} = (K_n \xrightarrow{p_n} K_{n-1} \xrightarrow{p_{n-1}} ... \xrightarrow{p_1} K_0)$ of K, its i-**truncation** $\mathcal{P}_{\leq i}$ (for $i \leq n$) is the i-proframing $(K_i \xrightarrow{p_i} K_{i-1} \xrightarrow{p_{i-1}} ... \xrightarrow{p_1} K_0)$ of the simplicial complex K_i^{un}. Similarly truncating maps, one obtains functors $(-)_{\leq i} : \mathsf{ProFrSimpCplx}_n \to \mathsf{ProFrSimpCplx}_i$. ⌐

Definition 3.2.48 (Restrictions of proframings). Given an n-proframing $\mathcal{P}$ of K and a subcomplex $L \hookrightarrow K$, the **restriction** $\mathcal{P}|_L$ **of** $\mathcal{P}$ to L is the n-proframing of L obtained by restricting the sequence $\mathcal{P}$ to L. ⌐

130

From each proframed simplicial complex we can extract a framed simplicial complexes by the following 'gradient' operation, which generalizes gradients of framed simplices as described in Terminology 3.2.27.

Definition 3.2.49 (Gradients of proframed simplicial complexes)**.** Given an n-proframing $\mathcal{P}$ of a simplicial complex K, the **gradient framing** $\nabla \mathcal{P}$ is the n-framing of K with the same ordering as $\mathcal{P}$, and with an n-embedded frame $\nabla \mathcal{P}_x$ on each simplex $x : [m] \hookrightarrow K$ given by the gradient frame $\nabla \mathcal{P}|_x$. ⌐

The fact that choices of frames in the gradient framing $\nabla \mathcal{P}$ are compatible with face restrictions follows from the compatibility of gradient frames with face restrictions (see Observation 3.2.34).

Definition 3.2.50 (Gradients of proframed maps)**.** Given a proframed map $F = (F_n, F_{n-1}, ..., F_1, F_0) : (K, \mathcal{P}) \to (L, \mathcal{Q})$ of n-proframed simplicial complexes, the **gradient framed map** $\nabla F : (K, \nabla \mathcal{P}) \to (K, \nabla \mathcal{Q})$ is the framed map determined by the simplicial map $F_n : K \to L$. ⌐

Terminology 3.2.51 (The gradient framing functor)*.* The construction of gradients on proframings and their maps yield the 'gradient framing' functor

$$\nabla : \mathsf{ProFrSimpCplx}_n \to \mathsf{FrSimpCplx}_n.$$ ⌐

Example 3.2.52 (Proframings and their gradients)**.** We depict three 2-proframings in Fig. 3.13 together with their gradients framings; note that the gradient framings recover our examples from Example 1.2.16. ⌐
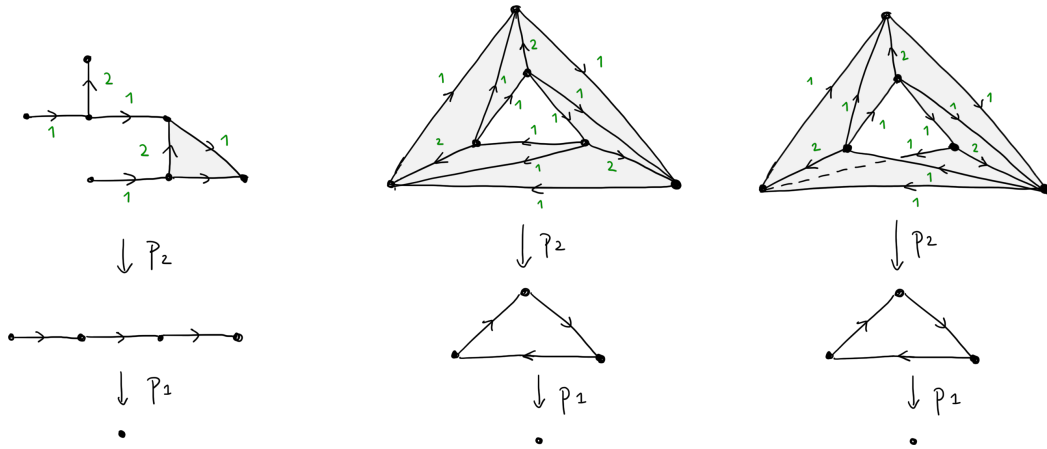
Figure 3.13: Proframed simplicial complexes with their gradient framings.

The 'converse' to taking gradients is taking integrals. However, not all framings are 'integrable'—and if they are, they need not have unique integral proframings. Only in special cases will there be a correspondence between framings and proframings via gradients and integrals (noteworthily, in the case of 'flat framings' which we will discuss shortly).

Definition 3.2.53 (Integrating proframings). Given an n-framed simplicial complex $(K, \mathcal{F})$, an **integral proframing** $(K, \mathcal{P})$ of $(K_n, \mathcal{F})$ is an n-proframing whose gradient n-framing $\nabla \mathcal{P}$ recovers $\mathcal{F}$. ⌐

Example 3.2.54 (A framing with two proframings). In Fig. 3.14 we depict a 2-framed simplicial complex on the left; to its right we depict two different integral 2-proframings $\mathcal{P} = (p_2, p_1)$ and $\mathcal{Q} = (q_2, q_1)$. ⌐
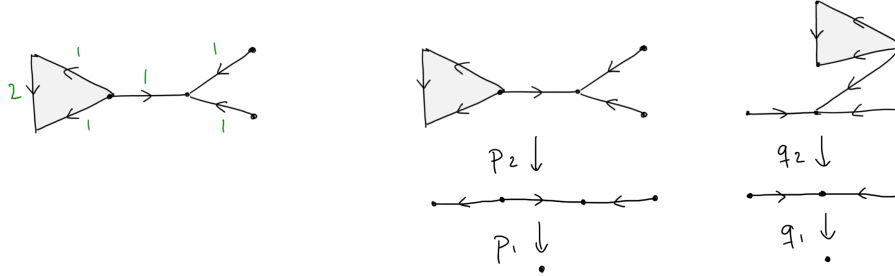
Figure 3.14: A 2-framing with two different integral proframings.

Example 3.2.55 (A non-integrable framing). In Fig. 3.15 we depict two 2-framings of the boundary $\partial[2]^{\mathsf{un}}$ of the unordered 2-simplex $[2]^{\mathsf{un}}$: neither framing admits an integral proframing. ⌐

Figure 3.15: Two framings without integral proframings.

We remark that the failure of integrability in the previous example may be explained by the following more general observation (which we state without proof).

Remark 3.2.56 (Integrating simplex boundary framings). An n-framing $\mathcal{F}$ of the unordered simplex boundary $\partial[m]^{\mathsf{un}}$ is integrable if and only if it is the restriction of some n-framing $\bar{\mathcal{F}}$ of the unordered m-simplex $[m]^{\mathsf{un}}$; that is, $\mathcal{F} = \bar{\mathcal{F}}\big|_{\partial[m]^{\mathsf{un}}}$. ⌐

3.2.2.3 Flat proframings In this section we discuss an alternative characterization of flatness in the context of proframings. The two definitions of flatness will turn out to coincide in a precise sense.

Terminology 3.2.57 (Section and spacers simplices). Let $(K, \mathcal{P})$ be an n-proframed simplicial complex, and pick a simplex $x : [m] \hookrightarrow K_i$ in some K_i, $1 \leq i \leq n$. Assume the restriction $\mathcal{P}_{\leq i}\big|_x$ of the truncation $\mathcal{P}_{\leq i}$ is an i-proframe of $[m]$ of the form $(p_i, ..., p_1)$. We call x a 'section simplex' of $\mathcal{P}$ if $p_i = \mathrm{id}$, and a 'spacer simplex' if $p_i \neq \mathrm{id}$. ⌐

Terminology 3.2.58 (Upper and lower sections of spacers). Consider an n-proframed simplicial complex $(K, \mathcal{P})$ and a spacer simplex $x : [m] \hookrightarrow K_i$. The top projection p_i of the restricted truncated proframe $\mathcal{P}_{\leq i}\big|_x = (p_i, ..., p_1)$ equals a jth degeneracy map $s_j : [m] \to [m-1]$ $(1 \leq j \leq m)$. Precomposing with simplicial face maps $d_k : [m-1] \hookrightarrow [m]$, we obtain the $(m-1)$-simplices $\partial_- x := x \circ d_{j+1} : [m-1] \hookrightarrow K_i$, called the 'lower section of x in $\mathcal{P}$', and $\partial_+ x := x \circ d_j : [m-1] \hookrightarrow K_i$, called the 'upper section of x in $\mathcal{P}$'. ⌐

Observe that, given a spacer simplex x in a proframed simplicial complex $(K, \mathcal{P})$, its upper and lower sections $\partial_{\pm}x$ are indeed section simplices in $(K, \mathcal{P})$.

Example 3.2.59 (Sections and spacers). In Fig. 3.16 we illustrate 2-proframed simplicial complex $K = K_2 \to K_1 \to K_0$. We highlight two spacer simplices in K_2 (in blue), and their respective upper and lower sections (in red). ⌟
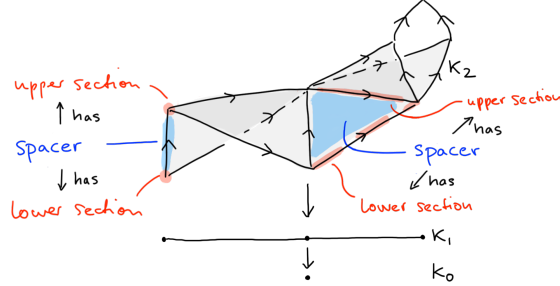
Figure 3.16: Sections and spacers in a proframed simplicial complex.

Terminology 3.2.60 (Fiber complexes and categories). Consider an n-proframed simplicial complex $(K, \mathcal{P})$, with $\mathcal{P} = (p_n, ..., p_1)$, and a k-simplex $z : [k] \hookrightarrow K_{i-1}$.
1. The 'fiber' K_z over z is the subset of non-degenerate simplices in K_i that are mapped to z by p_i.
2. The 'fiber complex' $\Phi_1^\Delta K_z$ of the fiber over z is the ordered simplicial complex whose vertices are sections $x \in K_z$ over z and whose 1-simplices $y : x_- \to x_+$ are spacers $y \in K_z$ with $\partial_{\pm}y = x_{\pm}$.
3. The 'fiber category' $\Phi_1 K_z$ is the free category on the complex $\Phi_1^\Delta K_z$ (with vertices as objects, and edges as generating morphisms). ⌟

Construction 3.2.61 (Transition functors of fiber categories). For an n-proframed simplicial complex $(K, \mathcal{P})$, consider simplices $z : [k] \hookrightarrow K_{i-1}$ and $w : [l] \hookrightarrow K_{i-1}$ such that w is a face of z (that is, w factors through z as a face $[l] \hookrightarrow [k]$). Note that each simplex $x \in K_z$ restricts over w to a simplex $x|_{w \subset z} \in K_w$: this restriction takes sections to sections, while spacers restrict either to spacers or to sections. The restriction thus induces the 'transition functor' $-|_{w \subset z} : \Phi_1 K_z \to \Phi_1 K_w$. ⌟

Example 3.2.62 (Fiber categories and transition functors). In Fig. 3.17, for the proframing given in the previous Fig. 3.16, we colored each simplex in K_1, and depicted its fiber category in the corresponding color (with a point for each object, and an arrow for each morphisms). We also indicated the transition functors between them (by mapping arrows '$\mapsto$'). ⌟

Fiber transition functors relate fiber categories across fibers. In order to define flat proframings, we will be interested in the case where fibers are 'linear', and transitions functors glue fibers onto fibers, as follows.

Definition 3.2.63 (Flat proframings). An n-proframed simplicial complex $(K, \mathcal{P})$ is said to be **flat** (or to have 'flat proframing') if the following holds.
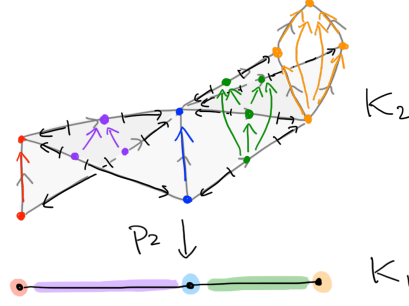
Figure 3.17: Fiber categories and transition functors.

1. *Fibers are linear*: For any simplex $z : [k] \hookrightarrow K_i$, the fiber category $\Phi_1 K_z$ is a total order.
2. *Fiber transition is endpoint-preserving*: For simplices $w \subset z$ in K_i the transition functor $-|_{w \subset z}$ is endpoint-preserving (meaning it preserves least and greatest elements as a map of total orders). ⌐

Terminology 3.2.64 (Linear complexes). The 'finite linear simplicial complex with j edges' $\mathsf{lin}[j]$ is the ordered simplicial complex with $j + 1$ vertices $\{0, 1, ..., j\}$ and j directed 1-simplices $i \to i + 1$. ⌐

Observation 3.2.65 (Fiber complexes in flat proframings are linear). The first condition in Definition 3.2.63 is equivalent to the condition that all fiber complexes $\Phi_1^\Delta K_z$ are linear. ⌐

Notation 3.2.66 (Categories of flat proframings). Denote the full subcategory of $\mathsf{ProFrSimpCplx}_n$ consisting of flat proframings by $\mathsf{FlatProFrSimpCplx}_n$. ⌐

Example 3.2.67 (Flat proframings)**.** In Fig. 3.18 we depict six 2-proframings (in each case highlight in blue the kernels of projection). The first example is not flat since it has non-linear fiber categories; the third example is not flat since its transition functors are not endpoint-preserving; the fifth example is not flat since, again, it has non-linear fiber categories (this time already at level 1 of the sequence). ⌐
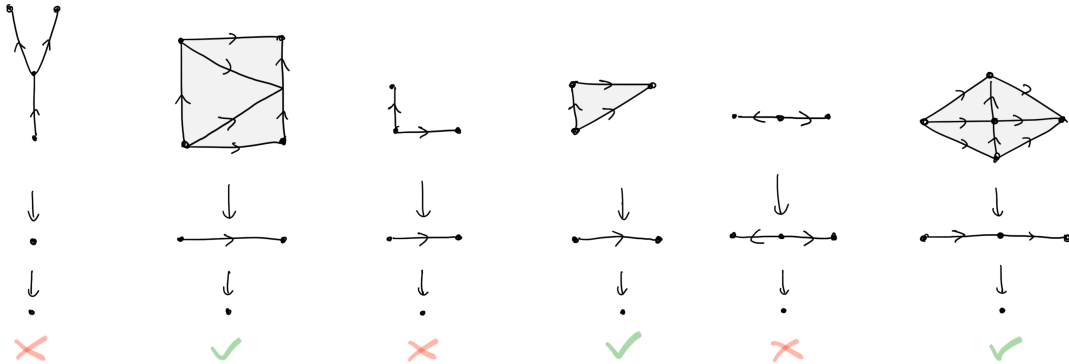
Figure 3.18: Flat and non-flat proframings of simplicial complexes.

Finally, just as flat n-framed simplicial complexes have framed realizations, flat proframed simplicial complexes 'proframed realizations'. These now embed in the standard euclidean n-proframe $\Pi = (\pi_n, \pi_{n-1}, ..., \pi_1)$ (see Terminology 3.2.1).

Remark 3.2.68 (Flat proframings admit proframed realizations). Any flat n-proframed simplicial complex $(K, \mathcal{P})$ admits a 'proframed realization', by which we mean an embedding of sequences $r = (r_n, r_{n-1}, ..., r_1, r_0) : |\mathcal{P}| \hookrightarrow \Pi$, with components $r_i : |K_i| \hookrightarrow \mathbb{R}^i$, which restricts on each simplex $x : [m] \hookrightarrow K$ to a proframed realization in the sense of Definition 3.2.13. As in the case of framed realization, note that any proframed realization of $(K, \mathcal{P})$ determines the proframing $\mathcal{P}$ of K uniquely. A proframed realization of a 3-proframed simplicial complex is illustrated in Fig. 3.19. ⌐
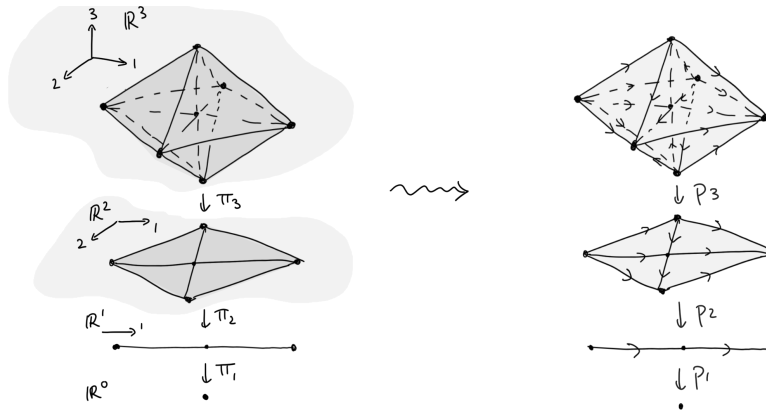
Figure 3.19: Flat proframings are determined by their proframed realizations.

Flatness fully restores the correspondence of framings and proframings. We record this in several statements below, that will be proven later in this chapter.

Proposition 3.2.69 (Gradients of flat proframings are flat). *Given a flat n-proframed simplicial complex $(K, \mathcal{P})$ its gradient n-framing $\nabla \mathcal{P}$ is a flat n-framing.*

Proposition 3.2.70 (Flat framings have flat integral proframings). *Every flat n-framing $(K, \mathcal{F})$ has a essentially unique integral flat n-proframing $\int \mathcal{F}$.*

Theorem 3.2.71 (Correspondence of flat proframings and flat framings). *The gradient framing functor restricts to an equivalence of categories*

$$\nabla : \mathsf{FlatProFrSimpCplx}_n \simeq \mathsf{FlatFrSimpCplx}_n \quad .$$

The inverse to ∇ will be called 'integration', and denoted by $\int$.

The theorem parallels (and, in fact, generalizes) the correspondence of framed simplices and proframed simplices that we've seen in Corollary 3.2.39. The proofs of Proposition 3.2.69, Proposition 3.2.70 and Theorem 3.2.71 will be deferred until Section 3.3.1.

3.2.3   Proframed regular cell complexes Finally, we also extend our discussion of proframing to the case of regular cells and their complexes.

3.2.3.1 The definition of proframed cells Just as simplicial complexes may be endowed both with framings or proframings, we now introduce proframings of regular cells. Recall that proframed simplicial complexes were defined as certain sequences of surjective ordered simplicial maps (see Definition 3.2.41). The role of surjective simplicial maps will now be played by surjective cellular maps. The following terminology will further be useful.

Terminology 3.2.72 (Ordering a sequence of simplicial maps). Recall from Terminology 1.2.9, that an 'ordering' of a simplicial map $F : K \to L$ is an ordered simplicial map which, after unordering, recovers F. Similarly, given a sequence of simplicial maps, an 'ordering' of it is a sequence of ordered simplicial maps which, after unordering, recovers the given sequence. ⌐

Definition 3.2.73 (Proframed regular cells). An n-**proframing** $\mathcal{P}$ **of a regular cell** X is a sequence of surjective cellular maps $X = X_n \xrightarrow{p_n} X_{n-1} \xrightarrow{p_{n-1}} \ldots \xrightarrow{p_1} X_0$, whose sequence of underlying simplicial maps is endowed with an ordering, such that the resulting sequence of ordered simplicial maps (denoted again by $\mathcal{P}$) is an n-proframing of the simplicial complex X, and with the condition that, for each $x \in X$, the restriction $\mathcal{P}|_{X^{\geq x}}$ of $\mathcal{P}$ to the subcomplex $X^{\geq x}$ is flat. ⌐

We will refer to the pair $(X, \mathcal{P})$, of a regular cell X together with an n-proframing $\mathcal{P}$ on it, as an 'n-proframed regular cell'. Note that '$\mathcal{P}$' refers, similarly to previous notation, to multiple structures simultaneously.

Remark 3.2.74 (Notation for data of proframes). Abusing notation, we may refer to $\mathcal{P}$ as a 'sequence of cellular maps', a 'sequence of simplicial maps', or a 'sequence *ordered* simplicial maps' all of which refer to different parts of the data of the proframed cell $(X, \mathcal{P})$ (see also Notation 1.3.35). ⌐

Notation 3.2.75 (Restricting framing to subcells). We abbreviate the cell restrictions $\mathcal{P}|_{X^{\geq x}}$ by $\mathcal{P}|_x$. ⌐

Example 3.2.76 (2-proframed regular cells). We illustrate four 2-proframed regular cells $(X, \mathcal{P})$ in Fig. 3.20. In each case we depict the sequence of surjective cellular maps $X_2 \to X_1 \to X_0$ embedded in the standard proframe $\mathbb{R}^2 \to \mathbb{R}^1 \to \mathbb{R}^0$. In analogy to our examples of framed regular cells, each cell embedding $X_i \hookrightarrow \mathbb{R}^i$ represents an embedding $|X_i| \hookrightarrow \mathbb{R}^i$ of the realized underlying simplicial complex X_i, and requiring the latter embeddings to be the components of a proframed realization (see Remark 3.2.68) fully determines the proframing $\mathcal{P}$. ⌐
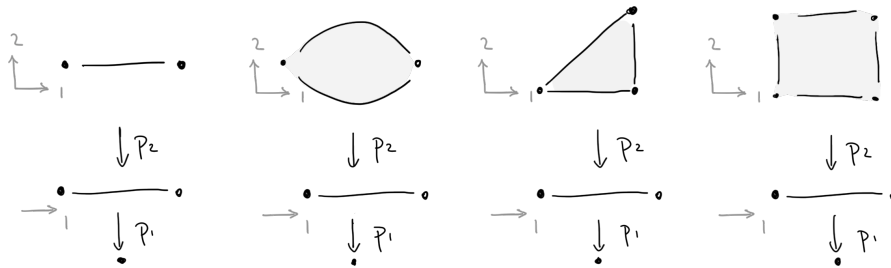
Figure 3.20: 2-Proframed regular cells.

136

Example 3.2.77 (3-proframed regular 3-cells)**.** Recall the 'simplest' 3-framed 3-cell from Fig. 1.32. The analogous 3-proframe is depicted in Fig. 3.21 (the 'analogy' will be made precise in Theorem 3.2.91). We also illustrate four (less simple) 3-proframed regular 3-cells in Fig. 3.22. ⌐
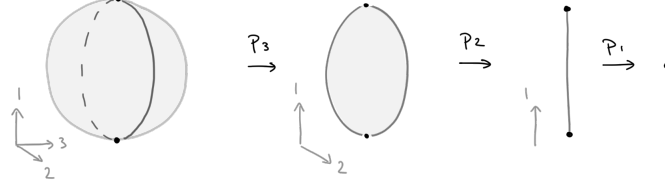
Figure 3.21: The simplest 3-proframed regular 3-cell.
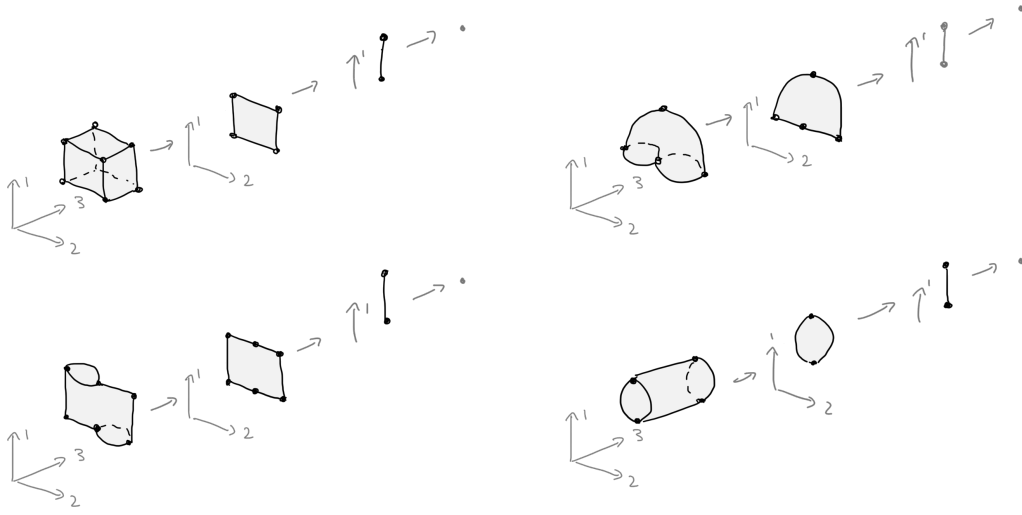
Figure 3.22: 3-proframed regular 3-cells.

3.2.3.2   Proframed cellular maps We define maps of proframed regular cells. Like in the case of framed maps of framed regular cells, we are confronted with the difficulty of regular cells not have 'spine vectors'; as before, we will remedy this using the observation that framed regular cells have 'final frame vectors'. The definition of proframed maps then takes the following form, which is parallel to the earlier definition of proframed maps of framed simplices (see Definition 3.2.17).

Definition 3.2.78 (Proframed maps)**.** Given n-proframed regular cells $(X, \mathcal{P} = (p_n, ..., p_1))$ and $(Y, \mathcal{Q} = (q_n, ..., q_1))$, a **proframed cellular map** $F : (X, \mathcal{P}) \to (Y, \mathcal{Q})$ (or simply, a 'proframed map') is a map of cellular poset sequences

$$\begin{array}{ccccccccc}
X = X_n & \xrightarrow{p_n} & X_{n-1} & \xrightarrow{p_{n-1}} & & \xrightarrow{p_2} & X_1 & \xrightarrow{p_1} & X_0 = [0] \\
\downarrow{\scriptstyle F_n} & & \downarrow{\scriptstyle F_{n-1}} & & \cdots & & \downarrow{\scriptstyle F_1} & & \downarrow{\scriptstyle F_0} \\
Y = Y_n & \xrightarrow{q_n} & Y_{n-1} & \xrightarrow{q_{n-1}} & \cdots & \xrightarrow{q_2} & Y_1 & \xrightarrow{q_1} & Y_0 = [0]
\end{array}$$

where each F_i is a cellular map, and such that for every $x \in X$, either the proframing of the final frame vectors $\mathsf{vec}\, x$ is preserved, i.e. F restricts to a proframed simplicial

isomorphism $F : (\mathsf{vec}\, x, \mathcal{P}|_{\mathsf{vec}\, x}) \cong (F_n(\mathsf{vec}\, x), \mathcal{Q}|_{F_n(\mathsf{vec}\, x)})$, or the final frame vectors are degenerated, i.e. $F_n(\mathsf{vec}\, x) \subset Y_n$ is a point. $\quad\quad$⌐

We will forego giving examples of proframed cellular maps here (earlier examples of framed maps of framed regular cells may be adapted appropriately).

Notation 3.2.79 (The category of proframed cells). The category of n-proframed regular cells and their proframed cellular maps will be denoted by $\mathsf{ProFrCell}_n$. $\quad\quad$⌐

3.2.3.3 Proframed regular cell complexes complexes We next discuss proframings of regular cell complexes. The definition is an almost verbatim generalization of the case of proframed regular cells. Recall the notion of 'orderings' of sequences of simplicial maps (see Terminology 3.2.72).

Definition 3.2.80 (Proframed regular cells). **An n-proframing $\mathcal{P}$ of a regular cell complex** X is a sequence of surjective cellular maps $X = X_n \xrightarrow{p_n} X_{n-1} \xrightarrow{p_{n-1}} ... \xrightarrow{p_1} X_0$, whose sequence of underlying simplicial maps is further endowed with an ordering, such that the resulting sequence of ordered simplicial maps (denoted again by $\mathcal{P}$) is an n-proframing of the simplicial complex X, and with the condition that, for each $x \in X$, the restriction $\mathcal{P}|_{X^{\geq x}}$ of $\mathcal{P}$ to the subcomplex $X^{\geq x}$ is flat. $\quad\quad$⌐

We will refer to the pair $(X, \mathcal{P})$, of a regular cell X together with an n-proframing $\mathcal{P}$ on it, as an 'n-proframed regular cell'. As before, we usually abbreviate the cell restrictions $\mathcal{P}|_{X^{\geq x}}$ by $\mathcal{P}|_x$ (see Notation 3.2.75).

Terminology 3.2.81 (Flat proframed regular cell complexes). An n-proframed regular cell complex $(X, \mathcal{P})$ is called 'flat' if the proframing $\mathcal{P}$ is flat. $\quad\quad$⌐

Example 3.2.82 (Non-flat 2-proframed regular cell complexes). In Fig. 3.23 we depict two 2-proframed regular cell complexes. (Our notation is an immediate generalization of the case of proframed regular cells.) Neither proframing is flat. In both case, we also indicate cell-wise 'gradient' framings of the cell complexes; these will be further discussed in the next section. $\quad\quad$⌐
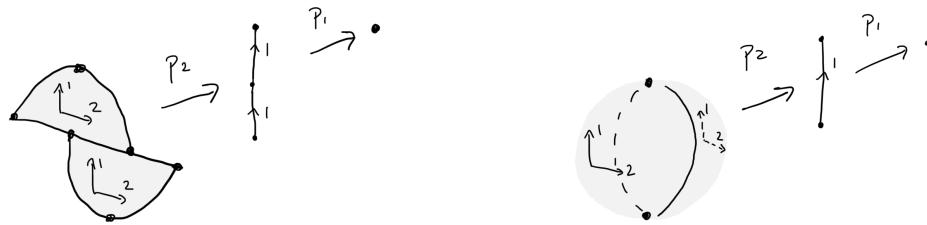
Figure 3.23: Two 2-proframed regular cell complexes together with their gradient framings.

Example 3.2.83 (Flat 3-proframed regular cell complexes). In Fig. 3.24 we depict two 3-proframed regular cell complexes. Both of proframings are flat (and they are depicted as proframed realized in the standard euclidean proframe). $\quad\quad$⌐

We next define framed maps of framed regular cell complexes; the definition is directly parallel to the definition of proframed simplicial maps (see Definition 3.2.45).
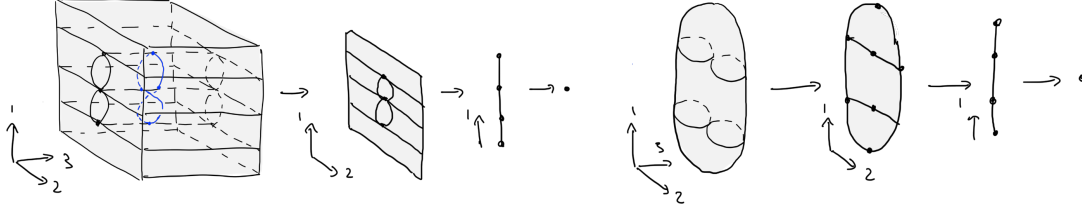
Figure 3.24: Proframing of the Hopf circle cellulation and a proframing of its dual.

Definition 3.2.84 (Proframed maps of proframed regular cell complexes). Consider n-proframed regular cell complexes $(X, \mathcal{P})$ and $(Y, \mathcal{Q})$. A **proframed cellular map** $F : (X, \mathcal{P}) \to (Y, \mathcal{Q})$ is a map of sequences $F = (F_n, F_{n-1}, ..., F_1, F_0) : \mathcal{P} \to \mathcal{Q}$ (consisting of cellular maps F_i) which, for all $x \in X$, restricts to a proframed cellular map $F : (X^{\geq x}, \mathcal{P}|_x) \to (Y^{\geq Fx}, \mathcal{Q}|_{Fx})$. ⌐

We often refer to 'proframed cellular maps' simply as 'proframed maps'.

Notation 3.2.85 (The category of proframed regular cell complexes). The category of n-proframed regular cell complexes and their proframed cellular maps will be denoted by $\mathsf{ProFrCellCplx}_n$. The full subcategory consisting of flat n-proframings will be denoted by $\mathsf{FlatProFrCellCplx}_n$. ⌐

3.2.3.4 Gradient cellular framings and integral cellular proframings Parallel to the definition of gradient framings of proframed simplicial complexes (see Definition 3.2.49), proframed regular cell complexes too have gradient framings as follows.

Definition 3.2.86 (Gradient framings of proframed regular cell complexes). Given an n-proframed regular cell complex $(X, \mathcal{P})$, its **gradient framing** is the n-framed regular cell complex $(X, \nabla \mathcal{P})$ whose framing is given by the gradient framing $\nabla \mathcal{P}$ of the simplicial complex P. Further, given a proframed cellular map $F = (F_n, F_{n-1}, ..., F_1, F_0) : (X, \mathcal{P}) \to (Y, \mathcal{Q})$ its **gradient map** is the framed cellular map $(X, \nabla \mathcal{P}) \to (Y, \nabla \mathcal{Q})$ determined by the cellular map $F_n : X \to Y$. ⌐

Terminology 3.2.87 (Gradient framing functor). Gradients of proframings of regular cell complexes and their proframed maps assemble into a 'gradient framing' functor as follows:

$$\nabla : \mathsf{ProFrCellCplx}_n \to \mathsf{FrCellCplx}_n.$$ ⌐

Example 3.2.88 (Gradients of proframed regular cells). Passing to the gradient 2-framings of the 2-proframed cell depicted in Fig. 3.20 recovers the upper four examples of framings given in Fig. 1.30. ⌐

Example 3.2.89 (Gradients of proframed regular cell complexes). In our earlier Fig. 3.23 we depicted two 2-proframed regular cell complexes; their respective gradient framings recover the 2-framed regular cell complexes from our earlier Fig. 1.37. Similarly, in Fig. 3.24 we depicted two 2-proframed regular cell complexes whose respective gradient framings recover the 2-framed regular cell complexes from our earlier Fig. 1.40 and Fig. 1.41. ⌐

Definition 3.2.90 (Integral proframings). Given an n-framed regular cell complex $(X, \mathcal{F})$, an n-proframed regular cell complex $(X, \mathcal{P})$ is an **integral n-proframing** for $(X, \mathcal{F})$ if $(X, \mathcal{F})$ is the gradient framing of $(X, \mathcal{P})$. ⌐

139

The previous definitions leave open the question of whether each n-framed regular cell complexes has an integral n-proframing: unlike in the case of simplices, this is now even non-trivial in the case of framed cells, since the definition of proframed regular cells requires a 'sequence of surjective cellular maps' as part of its data. Nonetheless, we have the following result.

Theorem 3.2.91 (Equivalence of framed and proframed regular cells)**.** *The gradient framing functor is an equivalence of categories of proframed regular cells and framed regular cells:*

$$\nabla : \mathsf{ProFrCell}_n \cong \mathsf{FrCell}_n.$$

The inverse to ∇ will be called integration*, and denoted by $\int$.*

The proof will be given in Section 3.3.2. The theorem may, in fact, be phrased in yet more general form: namely, parallel to the case of simplicial complexes, the notion of framings and proframings on regular cell complexes become equivalent when flatness is imposed.

Theorem 3.2.92 (Equivalence of flat framed and flat proframed regular cell complexes)**.** *The gradient framing functor is an isomorphism of categories of flat proframed regular cell complexes and flat framed regular cell complexes:*

$$\nabla : \mathsf{FlatProFrCellCplx}_n \cong \mathsf{FlatFrCellCplx}_n$$

The inverse to ∇ will be called integration*, and denoted by $\int$.*

The proof can be found in Section 3.3.2.

Both theorems, together with our earlier results and constructions, then organize into the diagram in Fig. 3.25, which summarizes the categories defined in this chapter. (Note that except for the isomorphism $\mathsf{ProFrSimp}_n \cong \mathsf{FrSimp}_n$, which was explained in Corollary 3.2.39, all isomorphisms in Fig. 3.25 are yet to be constructed; we will construct the remaining isomorphisms in Section 3.3).

3.3   Equivalences of framings and proframings

Our goal in this section will be the construction of equivalences between framings and proframings in three instances: (1) flat framings and flat proframings of simplicial complexes, (2) framings and proframings of regular cells, and (3) flat framings and flat proframings of regular cell complexes.

3.3.1   Equivalence of flat framed and flat proframed simplicial complexes We will prove that the gradient framing functor

$$\nabla : \mathsf{FlatProFrSimpCplx}_n \to \mathsf{FlatFrSimpCplx}_n$$

(see Terminology 3.2.51) provides an equivalence of flat proframed simplicial complexes with flat framed simplicial complexes.

ProFrSimp$_n$ $\xrightarrow[\text{(see 3.2.39)}]{\nabla}$ FrSimp$_n$ $\;\;\;\overset{\int}{\longleftarrow}$

Entr$^{\text{fr}}$

ProFrCell$_n$ $\xrightarrow[\int]{\text{(see 3.2.91)}}\overset{\nabla}{\longleftarrow}$ FrCell$_n$ $\xleftarrow{\text{Entr}^{\text{fr}}}$ FlatProFrSimpCplx$_n$ $\xrightarrow[\int]{\text{(see 3.2.71)}}\overset{\nabla}{\longleftarrow}$ FlatFrSimpCplx$_n$

Entr$^{\text{fr}}$ · · · Entr$^{\text{fr}}$

FlatProFrCellCplx$_n$ $\xrightarrow[\int]{\text{(see 3.2.92)}}\overset{\nabla}{\longleftarrow}$ FlatFrCellCplx$_n$ · · · ProFrSimpCplx$_n$ $\xrightarrow{\nabla}$ FrSimpCplx$_n$ $\xrightarrow{\text{Unframe}}$ SimpCplx

Entr$^{\text{fr}}$ · · · Entr$^{\text{fr}}$

ProFrCellCplx$_n$ $\xrightarrow{\nabla}$ FrCellCplx$_n$ $\xrightarrow{\text{Unframe}}$ CellCplx
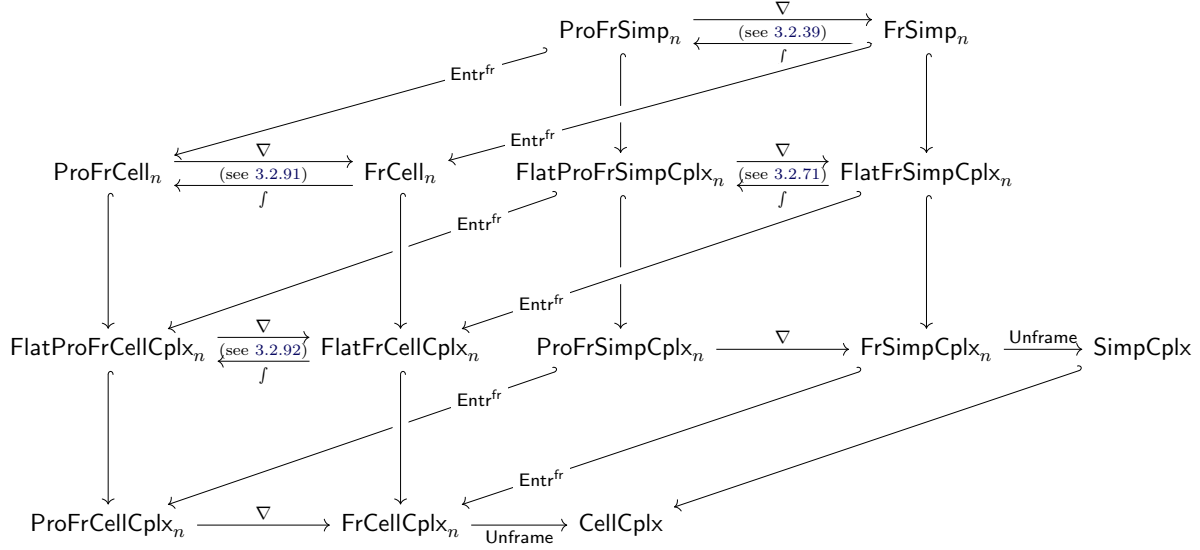
Figure 3.25: Categories of framings and proframings of simplicial complexes and cell complexes.

3.3.1.1 Gradients of flat proframings are flat We start with a proof of Proposition 3.2.69—recall, this stated that the gradient framing of a flat proframing is flat itself.

Proof of Proposition 3.2.69. Consider a flat n-proframing $\mathcal{P} = (p_n, ..., p_1)$ on a simplicial complex K. Since flatness of proframings truncates, we can argue inductively and assume that the gradient framing $\nabla \mathcal{P}_{<n}$ of the truncation $\mathcal{P}_{<n}$ of $\mathcal{P}$ is flat: that is, it admits a framed bounded realization $e_{n-1} : |K_{n-1}| \hookrightarrow \mathbb{R}^{n-1}$. We now construct a geometric realization $|p_n| : |K_n| \to |K_{n-1}|$ of the map p_n, and a framed bounded realization $e_n : |K_n| \hookrightarrow \mathbb{R}^n$ such that the following commutes

$$
\begin{array}{ccc}
|K_n| & \overset{e_n}{\hookrightarrow} & \mathbb{R}^n \\
\scriptstyle{|p_n|}\downarrow & & \downarrow\scriptstyle{\pi_n} \\
|K_{n-1}| & \overset{e_{n-1}}{\hookrightarrow} & \mathbb{R}^{n-1}
\end{array}
$$

We first define $e_n : |K_n| \hookrightarrow \mathbb{R}^n$ on vertices $x \in |K_n|$ as follows. Assume x lies over a base vertex $y = p_n x \in |K_{n-1}|$. Define $e_n(x) = (e_{n-1}(y), e_y(x))$ where $e_x : |K_y| \hookrightarrow \mathbb{R}$ is some choice of framed realization of the linear fiber complex K_y into $\mathbb{R}$ (see Observation 3.2.65). Now extend the embedding e_n linearly to all other simplices in $|K_n|$. We then define $|p_n|$ to be the composite $e_{n-1}^{-1} \circ \pi_n \circ e_n$. Using the definition of flat proframings, one check that e_n defines a framed realization whose image is framed bounded (as required in the definition of flat framings), and that $|p_n|$ realizes p_n as claimed. $\qquad\square$

3.3.1.2 Existence of integral proframings for simplicial complexes We next prove Proposition 3.2.70—recall this stated that any flat framing has an essentially unique integral flat proframing. We will prove this statement in two steps, first observing existence and then essential uniqueness.

Lemma 3.3.1 (Existence of integral proframings of flat framings). *Every flat n-framed simplicial complex $(K, \mathcal{F})$ has an integral flat n-proframing $\mathcal{P}$.*

Proof. Our strategy is to construct $\mathcal{P}$ inductively in n. For this, we construct a flat $(n-1)$-framing of K_{n-1}, denoted by cF_{n-1}, and uniquely determined by the following condition: $(K_{n-1}, \mathcal{F}_{n-1})$ admits a framed bounded realization $e_{n-1} : |K_{n-1}| \hookrightarrow \mathbb{R}^{n-1}$ (see Terminology 1.2.29) such that the composite $e_{n-1}^{-1} \circ \pi_n \circ e_n$ geometrically realizes a simplicial map $p_n : K_n \to K_{n-1}$ (where K_n is the ordering of K determined by its n-framing). The construction of $(K_{n-1}, \mathcal{F}_{n-1})$ relies on the framed boundedness of the embedding e_n and we outline its steps as follows. By definition of flatness, $e_n |K_n|$ is a framed bounded subspace of $\mathbb{R}^n$ of the form $\bigcap_{i \leq n}(\mathbb{R}^n_{\geq \gamma_i^-} \cap \mathbb{R}^n_{\leq \gamma_i^+})$. Therefore, $\pi_n : \mathbb{R}^n \to \mathbb{R}^{n-1}$ projects $e_n |K|$ to the bounded subspace $\bigcap_{i \leq n-1}(\mathbb{R}^{n-1}_{\geq \gamma_i^-} \cap \mathbb{R}^{n-1}_{\leq \gamma_i^+})$ of $\mathbb{R}^{n-1}$ (which, in particular, is again framed bounded). The definition of framed realizations implies $\pi_n e_n$ maps simplices of $|K|$ linearly to simplices in $\pi_n e_n |K|$. Framed boundedness of e_n implies that the collection of such image simplices in fact triangulates $\pi_n e_n |K|$ (see Fig. 3.26). Moreover, one checks this triangulation is the embedding $e_{n-1} : |K_{n-1}| \hookrightarrow \mathbb{R}^{n-1}$ of a unique ordered simplicial complex K_{n-1} subject to the condition that $e_{n-1}^{-1} \pi_n e_n : |K| \to |K_{n-1}|$ realizes an ordered simplicial map $p_n : K_n \to K_{n-1}$. Finally, endow K_{n-1} with the unique framing $\mathcal{F}_{n-1}$ making e_{n-1} a framed realization. This completes the construction $(K_{n-1}, \mathcal{F}_{n-1})$.

Now, arguing inductively, define $\mathcal{P}_{<n}$ to be a flat $(n-1)$-proframing obtained as the integral proframing of $\mathcal{F}_{n-1}$. Then define the n-proframing $\mathcal{P}$ of K by extending the tower $\mathcal{P}_{<n}$ by the map p_n. Once more using the definition of framed realizations and boundedness, one verifies that the proframed simplicial complexes $(K, \mathcal{P})$ is indeed flat proframed. $\square$

Terminology 3.3.2 (Projected framings). The $(n-1)$-framing $\mathcal{F}_{n-1}$ of $(K_{n-1}$ constructed in the preceding proof will be referred to as the 'projected framing' of the flat n-framing $\mathcal{F}$ of K, and the map $p_n : K_n \to K_{n-1}$ as its 'projection map'. $\quad\lrcorner$

Example 3.3.3 (Projected framings). In Fig. 3.26, we illustrate the projected framing $\mathcal{F}_2$ for a flat 3-framing $\mathcal{F}$ of a simplicial complex K. $\quad\lrcorner$
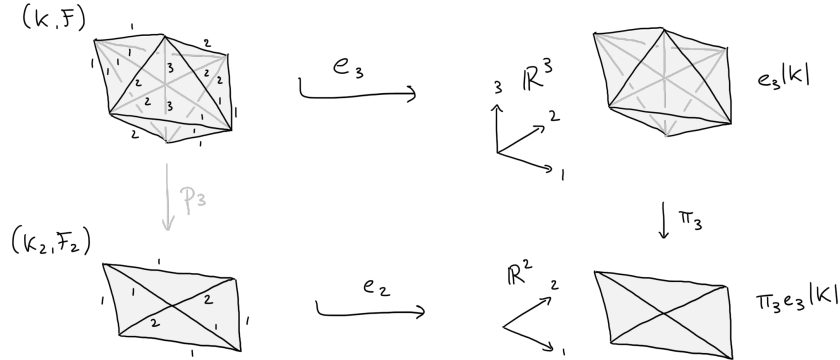

Figure 3.26: The projected framing of a flat framing.

Lemma 3.3.4 (Essential uniqueness of integral proframings of flat framings)**.** *Every flat n-framed simplicial complex $(K, \mathcal{F})$ has, up to unique isomorphism, a unique integral flat n-proframing.*

The following terminology will be useful.

Terminology 3.3.5 (Simplex directions in proframings). Given an n-proframed simplicial complex $(K, \mathcal{F})$, we say 1-simplex $x : [1] \to K$ is 'i-directed' if $p_{>i}(x)$ is a 1-simplex that is degenerated by p_i (where $p_{>i}$ abbreviates $p_{i+1} \circ p_{i+2} \circ ... \circ p_n$ as before). $\quad\quad\quad\llcorner\!\lrcorner$

Proof of Lemma 3.3.4. Let $\mathcal{P} = (p_n, ..., p_1)$ and $\mathcal{Q} = (q_n, ..., q_1)$ be flat n-proframings of simplicial complexes K resp. L whose gradient n-framings $(K, \nabla \mathcal{P})$, $(L, \nabla \mathcal{Q})$ are isomorphic. We will prove that $\mathcal{P}$ and $\mathcal{Q}$ are isomorphic proframings related by a unique isomorphisms $\mathcal{P} \cong \mathcal{Q}$ whose nth component $K_n \cong L_n$ must equal the framed isomorphism $(K, \nabla \mathcal{P}) \cong (L, \nabla \mathcal{Q})$. Our strategy is to argue by induction, this time, 'from the bottom up'. We will first consider fibers of the map $p_{>1} : K_n \to K_1$. Note, since $\mathcal{P}$ is flat, K_1 is a finite linear simplicial complex of the form $\mathsf{lin}[j]$ for some $j \in \mathbb{N}$ (see Terminology 3.2.64). Define 'fiber' subcomplexes $K(i) := p_{>1}^{-1}(i)$ of K for each vertex i in $\mathsf{lin}[j] \cong K_1$. Note we can restrict the tower of projections $\mathcal{P}$ to $K(i) \hookrightarrow K$, and drop the trivial last projection of the resulting restricted tower, to obtain an $(n-1)$-proframing $\mathcal{P}(i)$ on $K(i)$. Since $\mathcal{P}$ is flat it follows by definition of flatness that $\mathcal{P}(i)$ is flat as well. Note that no 1-simplex in $K(i) \hookrightarrow K$ can be a 1-directed 1-simplex in K (see Terminology 3.3.5). Conversely, any simplex from a vertex in $K(i)$ to a vertex $K(i')$, $i \neq i'$, must be 1-directed in p and we must have $i' = i + 1$. There must also be at least one 1-simplex between $K(i)$ and $K(i + 1)$ since $p_{>1}$ is assumed surjective on simplices (note, applying this observation inductively, one sees that $K_n(i)$ are in fact connected subcomplexes).

We exhibited $K(i)$ as connected components of 'K without its 1-directed 1-simplices'; and further, 1-directed 1-simplices of K fix an order $K(0)$, $K(1)$, ..., $K(j)$ of these components, as they only run between $K(i)$ and $K(i + 1)$. The assumed isomorphism $(K, \nabla \mathcal{P}) \cong (L, \nabla \mathcal{Q})$ must therefore restrict to isomorphisms $(K(i), \nabla \mathcal{P}(i)) \cong (L(i), \nabla \mathcal{Q}(i))$ (as $(n-1)$-framed simplicial complexes). Arguing inductively, deduce unique isomorphism $\mathcal{P}(i) \cong \mathcal{Q}(i)$ for all i such that nth components of these isomorphisms equal the isomorphisms $(K(i), \nabla \mathcal{P}(i)) \cong (L(i), \nabla \mathcal{Q}(i))$. One checks that this extends to the claimed unique isomorphism of n-proframings $(K, \mathcal{P}) \cong (L, \mathcal{Q})$ as claimed. $\quad\square$

Proof of Proposition 3.2.70. The proposition follows from Lemma 3.3.1 and Lemma 3.3.4.
$\quad\square$

As a corollary to the preceding observations, we can now record the following more explicit construction of the essentially unique integral proframings of flat framings.

Construction 3.3.6 (The integration of flat framings via iterated projection)**.** Combining the proofs of Lemma 3.3.1 and Lemma 3.3.4, it follows that the essentially unique integral proframing of a flat framed simplicial complex $(K, \mathcal{F})$ can be obtained by inductively constructing, for $i = n, n - 1, ...$, the projected framings $(K_i, \mathcal{F}_i)$ and their projections $p_i : K_i \to K_{i-1}$ (see Terminology 3.3.2) yielding a tower of simplicial maps $(p_n, ..., p_1)$. Moreover, inspecting the proof of Lemma 3.3.1, we see each p_i acts by quotienting vertices of K_i along 1-simplices x with framing $\mathcal{F}_i(x) = i$. We illustrate this in Fig. 3.27. $\quad\quad\llcorner\!\lrcorner$
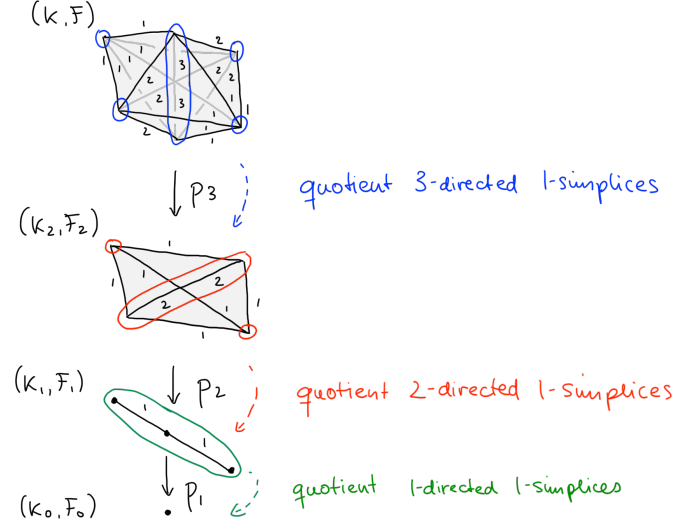
Figure 3.27: Constructing the essentially unique integration of flat framings as a tower of quotient maps.

Remark 3.3.7 (Equivalence of all framed realizations of flat framings). The proof of Lemma 3.3.1 shows that any framed bounded realization $e = e_n : |K| \hookrightarrow \mathbb{R}^n$ of a flat n-framed simplicial complex $(K, \mathcal{F})$ descends through a tower of simplicial maps $p_i : K_i \to K_{i-1}$ to framed bounded realizations $e_i : |K_i| \hookrightarrow \mathbb{R}^i$ such that $\pi_i e_i = e_{i-1} |p_i|$. The proof of Lemma 3.3.4 then implies that all such framed bounded realizations are 'equivalent' in that, given another framed bounded realization $e' = e'_n : |K| \hookrightarrow \mathbb{R}^n$, we have unique isomorphisms $\phi_i : e_i |K_i| \cong e'_i |K_i|$ (which are linear on each simplex) such that $\pi_i \phi_i = \phi_{i-1} \pi_i$. Namely, using notation from the proof of Lemma 3.3.4, the isomorphisms ϕ_i may be constructed by first inductively defining isomorphisms $\phi_i(j) : e_i |K_i(j)| \to e_i |K_i(j)|$ on the 'fibers of $p_{>1}$ over $j \in K_1$', and then extending these isomorphism of $p_{>1}$-fibers linearly to the remaining simplices of $(K_i, \mathcal{F}_i)$ (those containing 1-directed edges). In fact, a similar argument applies to *any* framed realization e' of $(K, \mathcal{F})$ (without requiring boundedness of e') which similarly shows equivalence of e and e' in the above sense. Note, in particular, that this entails that all framed realizations of flat framings are framed bounded. ⌐

Construction 3.3.8 (The integration functor for flat framings)**.** We construct the functor $\int : \mathsf{FrSimpCplx}_n \to \mathsf{ProFrSimpCplx}_n$, called the integration functor of flat framings. On objects, this functor maps flat n-framed simplicial complexes to their essentially unique integral flat n-proframed simplicial complexes, as constructed by the preceding two lemmas. On morphisms, construct the functor as follows. Let $F : (K, \mathcal{F}) \to (L, \mathcal{G})$ be a framed map of flat n-framed simplicial complexes. Denote the integral proframings of $(K, \mathcal{F})$ resp. of $(L, \mathcal{G})$ by $(K, \int \mathcal{F})$ resp. $(L, \int \mathcal{G})$. Note that, since F is a framed map, it must map k-directed 1-simplices in K either to 0-simplices or to k-directed 1-simplices in L. In particular, it preserves fibers of p_n and q_n which allows us to construct $F_{n-1} : K_{n-1} \to L_{n-1}$ such that $q_n F = F_{n-1} p_n$. Continuing inductively, this constructs a map of towers $\int F : (K, \int \mathcal{F}) \to (L, \int \mathcal{G})$. Comparing definitions, one checks that $\int F$ is proframed since F is framed. ⌐

Finally, we prove Theorem 3.2.71, which stated that the gradient framing functor yields an equivalence of categories $\nabla : \mathsf{ProFrSimpCplx}_n \to \mathsf{FrSimpCplx}_n$.

Proof of Theorem 3.2.71. The inverse of the 'gradient' functor ∇ is the 'integration' functor $\int$ defined in Construction 3.3.8. $\qquad\square$

3.3.2 Equivalence of flat framed and flat proframed cell complexes

We next show that the gradient framing functor, that takes proframed regular cells to framed regular cells, has an inverse. The inverse, as before, will be called 'integration'. Recall, an n-framed regular k-cell $(X, \mathcal{F})$ consists of a combinatorial regular k-cell X together with a flat n-framing $\mathcal{F}$ of the simplicial complex X. Throughout this section, let $(X, \mathcal{F})$ denote an n-framed regular cell with $\bot$ being the initial element of X, and denote by $\mathcal{P} = (X = X_n \xrightarrow{p_n} X_{n-1} \xrightarrow{p_{n-1}} ... \xrightarrow{p_1} X_0)$ the unique integral flat n-proframing of the *simplicial complex* X that integrates $\mathcal{F}$ (as constructed in the previous section). Our goal will be to endow the simplicial complexes X_i with cellular poset structures (i.e. to define cellular posets whose underlying simplicial complexes recover X_i), such that $\mathcal{P}$ becomes an n-proframing of the *regular cell* $X = X_n$. This will then provide the unique integral n-proframed regular cell for the n-framed regular cell $(X, \mathcal{F})$.

3.3.2.1 Upper and lower section cells

We now discuss a distinction of framed cells into 'section' and 'spacer' cells, such that spacer cells 'fill the space' between their 'upper' and 'lower' section cells (the idea is analogous to the discussion of section and spacer simplices in the context of simplices in proframings, see Terminology 3.2.57).

Proposition 3.3.9 (Section and spacer cells)**.** *For any n-framed regular cell $(X, \mathcal{F})$, one of the following holds true.*
1. *Either, the fiber $X_{p_n(\bot)}$ over $p_n(\bot)$ is trivial (i.e. contains only $\bot \in X$) and the ordered simplicial map $p_n : X_n \to X_{n-1}$ is an isomorphism.*
2. *Or, the fiber $X_{p_n(\bot)}$ is isomorphic to the 3-element linear complex $(\bot - 1) \to \bot \to (\bot + 1)$ with $\bot$ as its middle element.*

Terminology 3.3.10 (Section and spacer cells)*.* If a framed regular cell $(X, \mathcal{F})$ satisfies the first case of Proposition 3.3.9 then we call it a 'section cell'. In the second case, we call it a 'spacer cell', and call the elements $(\bot \pm 1)$ the (upper resp. lower) 'central fiber bounds' of the cell. $\qquad\lrcorner$

Example 3.3.11 (Section and spacer cells)**.** We illustrate both cases of Proposition 3.3.9 in Fig. 3.28, highlighting the poset structure X of the framed cell $(X, \mathcal{F})$ in blue and central fiber elements in red. $\qquad\lrcorner$

Proof of Proposition 3.3.9. We first note that the fiber $X_{p_n(\bot)}$ is a non-empty linear order with *at most* three elements. Indeed, by definition of flat proframings, the ordered simplicial complex $X_{p_n(\bot)}$ is a linear complex of the form $\mathsf{lin}[j]$ for some $j \in \mathbb{N}$, and, since $\bot$ is the initial object in the poset X, this implies that $X_{p_n(\bot)}$ can have at most two 1-simplices which must contain $\bot$ as a vertex (and thus at most three objects). Accordingly, we distinguish the following three cases.

First, assume the fiber $X_{p_n(\bot)}$ has a single element $\bot$. We claim that $p_n : X_n \to X_{n-1}$ does not have spacer simplices (which, by surjectivity of p_n and flatness of $\mathcal{P}$, equivalently
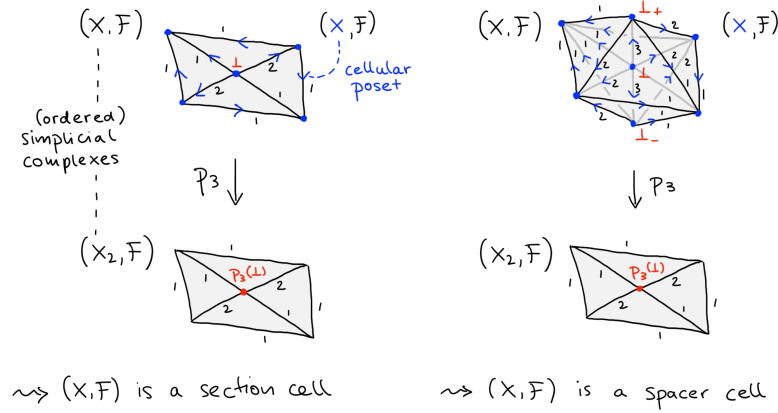
Figure 3.28: Framed regular cells are either section or spacer cells.

says that p_n is an isomorphism). Arguing by contradiction, assume there is a spacer simplex x of X_n, and pick such x of maximal dimension. Since $\bot$ is initial in X, x must contain $\bot$ in its vertices. Note x cannot be the face of any other simplex (since this would have to be again a spacer simplex). Consequently, $\dim(x) = k$ equals dimension of the cell X. Denote by $z = p_n x$ the $(k-1)$-simplex in X_{n-1} that x projects to. Recall, the fiber category $\Phi_1 X_z$ over z is the category consisting of section over z with spacers in between section acting as generating morphisms (see Terminology 3.2.60). Let y be the initial section in the fiber category $\Phi_1 X_z$ (the terminal section works just as well). Note $\dim(y) = k - 1$. Note $\bot$ is a vertex of y. We show that y is the $(k-1)$-face of exactly one k-simplex in X. By our choice of y, it is the face of exactly one spacer (note each spacer has exactly two faces that are sections, its upper and lower section). Assume y were the face of a section y'. Since X is a k-cell, we must have $\dim(y') = k$ and $\Phi_1 X_{z'} = [0]$ where $z' = p_n y'$. But since z has non-trivial fiber category, this contradicts the transition functor $\Phi_1 X_{z'} \to \Phi_1 X_z$ being endpoint-preserving. We deduce that y is indeed the $(k-1)$-face of exactly one k-simplex in X. This, however, is impossible since $\bot$ is a vertex in y: indeed, it would imply that the point $\bot$ in $|X|$ has no neighborhood homeomorphic to $\mathbb{R}^k$, which contradicts X being a regular k-cell. The contradiction proves that p_n cannot have spacers, and thus p_n is an isomorphism.

Next, assume that the fiber $X_{p_n(\bot)}$ has two elements. We show this is impossible. Note $X_{p_n(\bot)}$ is of the form $\bot \to (\bot + 1)$ or $(\bot - 1) \to \bot$. Argue in the former case (the latter case is similar). Since p_n has at least one spacer simplex (namely in the fiber over $p_n(\bot)$), we can pick a spacer simplex x of X_n of maximal dimension, and, as before, we must have $\dim(x) = k$. By the same arguments as in the previous paragraph, construct an initial section y over $p_n x$ and show that $|\bot \in X|$ does not have euclidean neighborhood, thus contradicting that X is a regular k-cell.

This leaves only the case that $\Phi_1 X_{p_n(\bot)}$ has three elements, in which case it must be of the form $(\bot - 1) \to \bot \to (\bot + 1)$. $\qquad\square$

Terminology 3.3.12 (Upper and lower sections). Define ordered simplicial maps $\gamma^- : X_{n-1} \to X_n$ resp. $\gamma^+ : X_{n-1} \to X_n$, called the 'lower section' resp. 'upper section' of

$(X, \mathcal{F})$, mapping a j-simplex z of X_{n-1} to the j-simplex of X_n that is initial resp. terminal in the fiber category $\Phi_1 X_z$ over z. ⌐

Note that $\gamma^- = \gamma^+$ if and only if $(X, \mathcal{F})$ is a section cell. Indeed, if $(X, \mathcal{F})$ is spacer cell, then $(\perp \pm 1) \in \mathrm{im}(\gamma^\pm)$. In fact, images of upper and lower sections are exactly the cells determined by $(\perp \pm 1)$, as the following result shows.

Lemma 3.3.13 (Section images are cells). *If $(X, \mathcal{F})$ is a spacer cell, with lower and upper sections $\gamma^\pm$, then $X^{\geq (\perp \pm 1)}$ are cells of dimension $(k-1)$, whose simplices are exactly those in the image of $\gamma^\pm$.*

Proof. We argue in the case of the lower section γ^- (the case of the upper section is similar). First observe that any simplex in X containing $(\perp -1)$ but not $\perp$ is a section simplex for p_n (indeed, otherwise we could pick a spacer simplex containing $(\perp -1)$ but not $\perp$, and, by flatness of $\mathcal{P}$ and since $(X, \mathcal{F})$ is a spacer k-cell, we could choose such spacer of dimension k, which is impossible).

We show that $X^{\geq (\perp -1)}$ is a $(k-1)$-cell in ∂X. Pick any $x \in \partial X$ such that $X^{\geq x}$ is a $(k-1)$-cell and such that $(\perp -1) \in X^{\geq x}$. Since we assumed $(\perp -1) \in X^{\geq x}$, the framed cell $(X^{\geq x}, \mathcal{F}|_x)$ must be a section cell (indeed, $X^{\geq x}$ will contain a $(k-1)$-simplex containing $(\perp -1)$ which, as we've just observed, must be a section simplex). In fact, each $(k-1)$-simplex in $X^{\geq x}$ must either contain $(\perp -1)$ or $(\perp +1)$: this follows, since taking the cone of the section $(k-1)$-simplices in $X^{\geq x}$ with cone point $\perp$ must yield spacer k-simplices. Observe that $X^{\geq x}$ cannot, however, contain both $(\perp -1)$ and $(\perp +1)$ (for instance, this would contradict the framed boundedness of all framed realizations of the flat framing $\mathcal{F}|_x$, see Remark 3.3.7). It follows that all $(k-1)$-simplices of the $(k-1)$-cell $X^{\geq x}$ must contain the vertex $(\perp -1)$. But this is only possible if $x = (\perp -1)$.

Finally, we check $\mathrm{im}(\gamma^-)$ contains the same simplices as $X^{\geq (\perp -1)}$. Certainly simplices of $X^{\geq (\perp -1)}$ are lowest sections and must lie in the image of $\mathrm{im}(\gamma^-)$. Conversely, since every simplex X lies in the star of $\perp$ and thus every simplex in X_{n-1} lies in the star of $p_n(\perp)$, which implies $\mathrm{im}(\gamma^-)$ must be contained in $X^{\geq (\perp -1)}$ as required. □

Given a spacer cell $(X, \mathcal{F})$, we often refer to the subposets $X^{\geq (\perp \pm 1)}$ of X, determined by the images of the sections $\gamma^\pm$, as the 'upper' resp. 'lower section cells' in $(X, \mathcal{F})$.

3.3.2.2 Projecting cells We explain that spacer cells 'project down' (resp. 'up') onto their lower (resp. upper) section cells. This will provide the cellular projection maps necessary for the comparison of framed regular cells and proframed regular cells. The following notation will be useful. A framed regular cell $(X, \mathcal{F})$ restricts, by definition, to a framed cell $(X^{\geq x}, \mathcal{F}|_x)$ on the subcell $X^{\geq x} \hookrightarrow X$, for any $x \in X$. We denote by $\mathcal{P}^x = \int \mathcal{F}|_x$ the integral proframing of the restricted n-framing $\mathcal{F}|_x$. Write $\mathcal{P}^x = (p_n^x, p_{n-1}^x, ..., p_1^x)$ with $p_i^x : X_i^x \to X_{i-1}^x$. Importantly, while $X_n^x = X_n^{\geq x}$ is a subcomplex of X_n, the same need not a priori hold for X_i^x and X_i with $i < n$—the fact that it does hold, follows from the next result.

Lemma 3.3.14 (Projections restrict to cells). *Given a framed regular cell $(X, \mathcal{F})$ with subcell $(X^{\geq x}, \mathcal{F}|_x)$, to top simplicial projection p_n^x of the integral proframing of $(X^{\geq x}, \mathcal{F}|_x)$ is obtained by restricting the top simplicial projection p_n of the integral proframing of $(X, \mathcal{F})$ along $X_n^x \hookrightarrow X_n$.*

Proof. We can restrict any framed realization of $(X, \mathcal{F})$ to $X^{\geq x} \hookrightarrow X$ to obtain a framed realization of $(X^{\geq x}, \mathcal{F}|_x)$ which, by Remark 3.3.7, is necessarily bounded. Tracing through the construction of integral proframings via framed bounded realizations in Lemma 3.3.1 then proves the claim. $\qquad\square$

Lemma 3.3.15 (Projected cells). *For a framed cell $(X, \mathcal{F})$, with top simplicial projection $p_n : X = X_n \to X_{n-1}$ in its integral proframing, we can chose a unique poset structure on X_{n-1} making p_n a cellular map of posets.*

Proof. If $(X, \mathcal{F})$ is a section cell, then p_n is a simplicial isomorphism and thus we must have $p_n : X_n \cong X_{n-1}$ as cellular posets. If $(X, \mathcal{F})$ is a spacer cell, we define the poset structure on X_{n-1} by setting $X_{n-1} \cong X^{\geq(\perp-1)}$, where we identify the corresponding simplicial complexes using the lower section map $\gamma^- : X_{n-1} \hookrightarrow X_n$. To see that $p_n : X = X_n \to X_{n-1} \cong X^{\geq(\perp-1)}$ is a cellular poset map note that p_n maps each cell $X^{\geq x} \hookrightarrow X$ exactly to the cell $X^{\geq \gamma^- \circ p_n(x)} \hookrightarrow X^{\geq(\perp-1)} \cong X_{n-1}$ (which can be seen by combining Lemma 3.3.14 and our earlier construction of section cells). $\qquad\square$

Remark 3.3.16 (Isomorphism of upper and lower cells). If $(X, \mathcal{F})$ is a spacer cell, then the cellular poset map $p_n : X_n \to X_{n-1}$ must restrict on lower resp. upper section cells $X^{\geq(\perp\pm1)} \hookrightarrow X$ to cellular poset isomorphisms $p_n : X^{\geq(\perp\pm1)} \cong X_{n-1}$. In particular, $X^{\geq(\perp-1)}$ and $X^{\geq(\perp+1)}$ are canonically cellular isomorphic. $\qquad\lrcorner$

Observation 3.3.17 (Framed cell structures project). Having constructed a cellular poset structure on the simplicial complex X_{n-1} (which is a regular cell, since $X_{n-1} \cong X^{\geq(\perp+1)}$ is a regular cell) we may now further endow the poset X_{n-1} with a framing. Recall our construction of the projected framing $\mathcal{F}_{n-1}$ of $\mathcal{F}$ (see Terminology 3.3.2), which equips the simplicial complex X_{n-1} with an $(n-1)$-framing. Note that this $(n-1)$-framed simplicial complex coincides (up to increasing the embedding dimension of frame labels by postcomposing with $\underline{n-1} \hookrightarrow \underline{n}$) with the n-framed simplicial complex $(X^{\geq(\perp\pm1)}, \mathcal{F}|_{\perp\pm1})$. Therefore, considering $X_{n-1} \cong X^{\geq(\perp+1)}$ as cellular posets, since $\mathcal{F}|_{\perp\pm1}$ defines an n-framing of the latter cell, it follows $\mathcal{F}_{n-1}$ defines an $(n-1)$-framing of the former cell. $\qquad\lrcorner$

Terminology 3.3.18 (Projected cells). We call the $(n-1)$-framed regular cell $(X_{n-1}, \mathcal{F}_{n-1})$ the 'projected cell' of the n-framed cell $(X, \mathcal{F})$ and the cellular map $p_n : X_n \to X_{n-1}$ its 'cell projection'. $\qquad\lrcorner$

Note that if $(X, \mathcal{F})$ is a section cell, then $X_{n-1} \cong X$ canonically. The next example illustrates the projected cell of a spacer cell.

Example 3.3.19 (Projected cells). In Fig. 3.29 we illustrate the projected cell of the 3-framed regular cell $(X, \mathcal{F})$ given in Fig. 3.28, highlighting the 'projected' poset structure on X_2 constructed by Lemma 3.3.15 in green. $\qquad\lrcorner$

3.3.2.3   Existence of integral proframings for framed cells Finally, we construct and verify uniqueness of integral proframings of framed cells.

Lemma 3.3.20 (Existence of integral n-proframings of framed cells). *Any n-framed regular cell $(X, \mathcal{F})$ has an integral n-proframing $(X, \mathcal{P})$.*
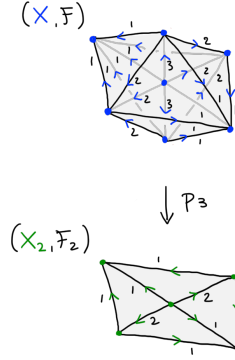
Figure 3.29: The projected 2-framed cell of a 3-framed regular cell.

Proof. Recall, the definition of n-proframed cells first requires the construction of a tower $\mathcal{P}$ of cellular maps $X = X_n \xrightarrow{p_n} X_{n-1} \to ... \xrightarrow{p_1} X_0$. We inductively construct the cellular posets X_i resp. the surjective cellular maps p_i as 'projected cells' and 'cell projections' in the sense of Terminology 3.3.18. Further, by the above constructions, the integral proframing $\int \mathcal{F}$ of the simplicial framing $\mathcal{F}$ endows the tower $\mathcal{P}$ with the structure of a simplicial proframing as required in the definition of proframed cell. Finally, the fact that $\mathcal{F}$ is flat on each cell implies that $\mathcal{P}$ is flat on each cell, which completes the definition of the integral proframing $\mathcal{P}$ of the cell X. $\qquad\square$

Lemma 3.3.21 (Essential uniqueness of integral n-proframings of framed cells)**.** *Integrating n-proframings of framed cells are essentially unique.*

Proof. Let $(X, \mathcal{P})$ and $(Y, \mathcal{Q})$ be n-proframed regular cells with isomorphic gradient framed cells. Then the unique isomorphism $\mathcal{P} \cong \mathcal{Q}$ can be constructed as in Lemma 3.3.4. The fact that this simplicial isomorphism of towers is also a cellular isomorphisms of towers can be seen inductively using the relation of projected cells with lower (or upper) section cells (see Observation 3.3.17). $\qquad\square$

Construction 3.3.22 (The integration functor of framed regular cells)**.** We construct the functor $\int : \mathsf{FrCell}_n \to \mathsf{ProFrCell}_n$, called the integration functor of framed regular cells. On objects, this maps n-framed regular cells $(X, \mathcal{F})$ to their essentially unique integral n-proframed regular cells, denoted by $(X, \int \mathcal{F})$, which was shown to exist above. On morphisms, it maps a framed cellular map $F : (X, \mathcal{F}) \to (Y, \mathcal{G})$ to the proframed cellular map $\int F : (X, \int \mathcal{F}) \to (Y, \int \mathcal{G})$ determined by setting the component $(\int F)_n$ to equal F—the fact that this descents to a tower of cellular poset maps can be seen inductively the relation of projected cells with lower (or upper) section cells (see Observation 3.3.17). Comparing definitions, one further checks that framedness of F implies proframedness of $\int F$ as required. $\qquad\lrcorner$

Finally, our results assemble into the following proof of Theorem 3.2.91, which stated that the gradient functor $\nabla : \mathsf{ProFrCell}_n \to \mathsf{FrCell}_n$ is an equivalence of categories.

Proof of Theorem 3.2.91. The inverse of the 'gradient' functor ∇ is given by the 'integration' functor $\int$ as defined in Construction 3.3.22. $\qquad\square$

As a consequence of the tools and observations described in this section, we may now revisit our earlier claim in Remark 1.3.48 that the 'final frame vector $\mathsf{vec}\perp$ is in fact a 'vector', or more precisely, the entrance path poset of a 1-simplex.

Corollary 3.3.23 (Characterizing final frame vectors)**.** *The final frame vectors $\mathsf{vec}\perp$ of any n-framed regular k-cell $(X, \mathcal{F})$, $k > 0$, span a subposet canonically isomorphic to the entrance path poset $\mathsf{Entr}[1] \cong (- \leftarrow 0 \rightarrow +)$ of a 1-simplex.*

Proof. The claim is trivial if $n = 1$. If $(X, \mathcal{F})$ is a spacer simplex then $\mathsf{vec}\perp \cong ((\perp - 1) \leftarrow \perp \rightarrow (\perp + 1))$. If $(X, \mathcal{F})$ is a section, then $\mathsf{vec}\perp \cong \mathsf{vec}\perp_{n-1}$ where $\perp_{n-1}$ is the initial object of the projected cell $(X_{n-1}, \mathcal{F}_{n-1})$ and thus the statement follows inductively. $\square$

3.3.2.4 Existence of integral proframings for flat framed cell complexes Building on the previous two sections, we now indicate the proof of Theorem 3.2.92: recall, this states that the gradient functor $\nabla : \mathsf{FlatProFrCellCplx}_n \rightarrow \mathsf{FlatFrCellCplx}_n$ is an equivalence of categories. Since the proof closely follows the corresponding arguments for simplicial complexes and for cells, we will only outline its main steps.

Proof of Theorem 3.2.92. As before, we want to construct an inverse $\int$ to ∇. Let $(X, \mathcal{F})$ be a flat n-framed regular cell. Set $\mathcal{P}$ to be the integral proframing of the framed simplicial complex $(X, \mathcal{F})$. Analogously to Terminology 3.3.12, define upper and lower sections $\gamma^\pm : X_{n-1} \rightarrow X_n$. In analogy to Lemma 3.3.13, one shows that images of lower (resp. upper) sections are in fact exactly simplices in an upper-closed subposets X_- (resp. X_+) of X. In analogy to Lemma 3.3.14 one finds that $p_n : X_n \rightarrow X_{n-1}$ induces a poset map $p_n : X \rightarrow X_{n-1}$, where we can identify $X_{n-1} \cong X_-$ via γ^- (or similarly $X_{n-1} \cong X_+$ via γ^+). As before, the projected framing $\mathcal{F}_{n-1}$ now yields another flat framed regular cell complex $(X_{n-1}, \mathcal{F}_{n-1})$. Arguing inductively, we can construct an integral proframing for $(X_{n-1}, \mathcal{F}_{n-1})$, and augmenting it with the cellular map p_n we obtain the integral proframing $(X, \mathcal{P})$ of $(X, \mathcal{F})$, as required. One checks essential uniqueness as before. The construction of $\int$ on morphisms proceeds similarly to Construction 3.3.22. $\square$

3.4 Equivalence of framed regular cell complexes & regular block sets

We now prove the equivalence of framed regular cells and truss blocks, and, more generally, of flat framed regular cell complexes and trusses, and *yet* more generally, of framed regular cell complexes and truss block sets. A first ingredient in these proofs is the equivalence of flat framed and flat proframed regular cell complexes, described in the previous section, which will allow us to think of flat framed regular cell complexes as 'towers' of surjective cellular maps. A second important ingredient concerns the structure of trusses: namely, in Section 3.4.1 we will show that face posets of closed trusses are in fact cellular posets (moreover, they are *PL* cellular posets). Equipped with both of these results, in Section 3.4.2, we first construct the correspondence of flat framed cell complexes and trusses, and then generalize this to a correspondence of framed cell complexes and truss block sets.

3.4.1 Piecewise linear cellularity of closed trusses In this section, we prove that face posets of trusses are PL cellular posets. Recall, this means that the strict upper closure of any element in a poset geometrically realizes to a PL sphere.

Lemma 3.4.1 (Cellularity of closed trusses). *Given a closed n-truss T, each face poset $(T_i, \unlhd)$ is a PL cellular poset.*

While there are several direct ways of proving the result, our approach will be based on the 'shellability' properties of trusses. Our proof will rely on the following result.

Proposition 3.4.2 (See [Bjö84, Prop. 4.5 ff.]). *The realization $|X|$ of a poset X is a regular cell complex PL homeomorphic to the PL m-sphere if X is pure of dimension m, shellable, and thin.* $\qquad\qquad\square$

Terminology 3.4.3 (Purity, shellability, and thinness). A simplicial complex is called 'pure of dimension m' if its facets (that is, non-degenerate simplices which are not the face of any other non-degenerate simplex) are all of the same dimension d. Similarly, a poset X is called pure of dimension m if its nerve NX is pure of dimension m.

A poset X is called 'shellable' if the simplicial complex NX is pure of dimension m and its facets admit an ordering $K_0, K_1, K_2, ..., K_j$, such that, for all $0 < k \le j$, the subcomplex $(\cup_{i<k} K_i) \cap K_k$ of X (obtained by intersecting of the union K_i, $i < k$, with K_k) is a pure simplicial complex of dimension $(m-1)$.

Finally, a poset X is called 'thin' if for every non-refinable length-2 chain $x < y < z$ in X there is exactly one $y' \ne y$ such that $x < y' < z$ (this is also sometimes called the 'diamond property'). $\qquad\qquad\lrcorner$

Proof of Lemma 3.4.1. Recall the following notation: given a poset $(X, \le)$ and an element $x \in X$, the 'upper closure' $X^{\ge x}$ (resp. 'strict upper closure' $X^{>x}$) is the subposet of X consisting of all elements greater than x (resp. those that are strictly greater than x). Given an n-truss T and an element $x \in T_i$ in a face poset $(T_i, \unlhd)$ of T, for the proof of Lemma 3.4.1 we need to show that the strict upper closure $T_i^{\rhd x}$ realizes to a PL m-sphere. Inductively, we may assume this holds for $i < n$ and only consider strict upper closures $T_n^{\rhd x}$ in the top level poset. Moreover, by passing to the face block of $T_n^{\unrhd x}$ of x, we may assume T is in fact an n-truss block, with initial element $\bot \equiv x$. What is left to show is that the truss block's 'boundary' $\partial T_n \equiv T_n^{\rhd \bot}$ realizes to a PL m-sphere. By Proposition 3.4.2 this is equivalent to showing that ∂T_n is pure, thin and shellable.

Denote by $\bot_{n-1} = p_n(\bot)$ the projection of $\bot$ to T_{n-1}. Denote by $T_{<n}$ the $(n-1)$-truncation of T. Since T is an n-truss k-block, note that $T_{<n}$ is an $(n-1)$-truss l-block (note $l \in \{k-1, k\}$). Arguing by induction, we assume the statement of the lemma to hold for $T_{<n}$; that is, we assume that the strict upper closure ∂T_{n-1} of $\bot_{n-1}$ is thin, shellable and pure of dimension $l-1$. There are now two basic cases. Either, $\bot$ is singular in the fiber over $\bot_{n-1}$ or it is regular. In the first case, the 1-truss bundle $p_n : T_n \to T_{n-1}$ is an isomorphism of face posets (and $k = l$) so all claimed properties of ∂T_n follow from that of ∂T_{n-1}. Thus, it remains to prove the lemma in the second case, that is, for $\bot$ being regular in the fiber over $\bot_{n-1}$. Note that initiality of $\bot$ implies the fiber $p_n^{-1}(\bot_{n-1})$ must be of the form $(\bot - 1) \rhd \bot \lhd (\bot + 1)$. We separately prove purity, shellability and thinness separately in this case.

Purity. We first show ∂T_n is pure. Observe that facets of T_n must be mapped to facets of T_{n-1} by p_n (this follows since p_n is surjective on simplices, and since simplex fiber transition maps are surjective too, see Observation 2.2.36). Each facet in T_{n-1} must contain the vertex $\perp_{n-1}$. Since the fiber over $\perp_{n-1}$ has spacers, so must the fiber of each facet in T_{n-1}. Consequently, facets in T_n must themselves be spacers. Since all facets in T_{n-1} have dimension $(k-1)$, it follows that facets of T_n have dimension k. Note that this further implies that all facets of ∂T_n are of dimension $k-1$.

Shellability. We next show ∂T_n is shellable. Observe first that facets in ∂T_n are either first or last sections lying over facets in T_{n-1}, or they are spacers lying over facets in ∂T_{n-1}. Inductively, construct a shelling $K_1, K_2, ..., K_{\#T_{n-1}}$ of facets of ∂T_{n-1} (where $\#T_{n-1}$ is the number of facets in ∂T_{n-1}). This also induces a shelling $K_\bullet = (K_1^\perp, K_2^\perp, ...K_{\#T_{n-1}}^\perp)$ of T_{n-1} (where $K_i^\perp$ is obtained from K_i by adjoining a new first vertex $\perp_{n-1}$). Now build a shelling $L_\bullet = (L_1, L_2, ...L_{\#T_n})$ of ∂T_n (where $\#T_n$ is the number of facets in ∂T_n) in the following three steps.

1. *Lower section shelling*: We define the first $\#T_{n-1}$ facets

$$L_1, L_2, ..., L_{\#T_{n-1}}$$

   in the sequence $L_\bullet$, by setting L_i to be the bottom section (see Construction 2.2.26) lying over $K_i^\perp$. Note that this satisfies $L_i(0) = (\perp - 1)$.

2. *Side shelling*: We next define the subsequence

$$L_{\#T_{n-1}+1}, L_{\#T_{n-1}+2}, ..., L_{\#T_m - \#T_{n-1}}$$

   of $L_\bullet$ to be the sequence $L_{(1,1)}, L_{(1,2)}, ...L_{(1,j_0)}, L_{(2,1)}, L_{(2,2)}, ..., L_{(2,j_1)}, ..., L_{(\#T_{m-1},1)},$ ..., $L_{(\#T_{m-1},j_1)}$, where $L_{(i,j)}$ is the jth spacer (in scaffold order) lying over K_i.

3. *Upper section shelling*: Finally, we define the last $\#T_{n-1}$ facets

$$L_{\#T_m - \#T_{n-1}+1}, L_{\#T_m - \#T_{n-1}+2}, ..., L_{\#T_{n-1}}$$

   in the sequence $L_\bullet$ by setting $L_{\#T_m - \#T_{n-1}+i}$ to be the top section lying over $K_i^\perp$. Note that this satisfies $L_{\#T_m - \#T_{n-1}+i}(0) = (\perp + 1)$.

This constructs a shelling of ∂T_n. We illustrate an instance of the construction in Fig. 3.30: for the 3-truss T shown on the left, a shelling $(K_1, K_2, ..., K_6)$ of ∂T_2 is depicted on the (lower) right. Above it, we depict the shelling $(L_1, L_2, ..., L_{16})$ of ∂T_3 as constructed by the above procedure.

Thinness. It remains to check that ∂T_n is thin. We show that T_n is thin which implies thinness of ∂T_n (since T_n adjoins an initial element to ∂T_n). Inductively assume T_{n-1} is thin. Take a 2-simplex $K : [2] \to T$ such that the chain $K = (x \to y \to z)$ is non-refinable. We distinguish two cases based on the dimension of the base projection $\mathrm{im}(p_n K) : [j] \hookrightarrow T_{n-1}$, with $j \in \{1, 2\}$.

First assume $j = 2$. Then the base projection $J := \mathrm{im}(p_n K)$ is a chain $(x_{n-1} \to y_{n-1} \to z_{n-1})$ in T_{n-1}. Note J must be non-refinable (otherwise, L would be refinable). Thinness of T_{n-1} implies there is exactly one other non-refinable chain $J' : x_{n-1} \to y'_{n-1} \to z_{n-1}$. Since the 1-truss bordism lying over the J compose to the same 1-truss bordism as the 1-truss bordism lying over J', there must be at least one chain K' from x to z lying over J'. Moreover, there cannot be a third chain K'' from x to z, since that would have to lie
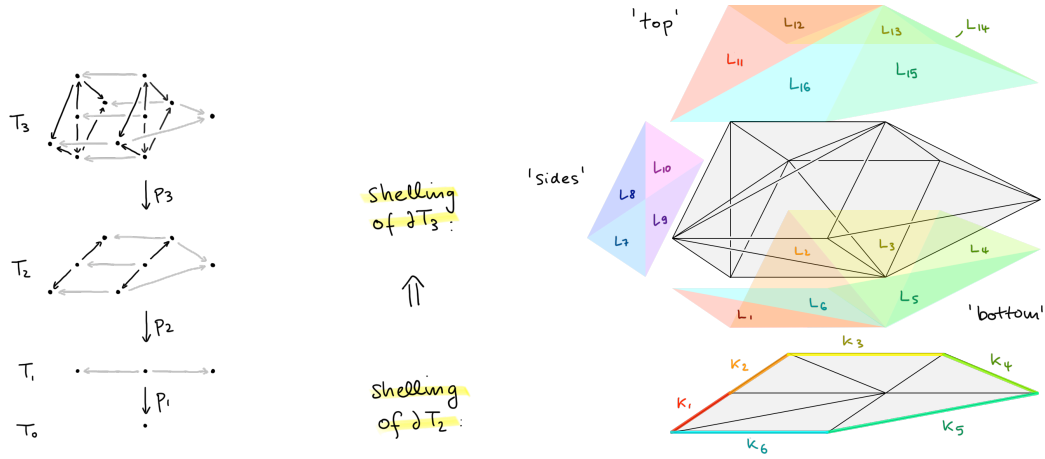
Figure 3.30: Inductive shelling of a face poset boundary ∂T_n, by sequentially shelling its bottom, sides, and top part.

over either J or J'; assume, w.l.o.g., that it lies over J and that $K \prec K''$ in the scaffold order of sections over J (see Section 2.2); all spacers over J between K and K'' must now have fiber morphisms in the fiber over y_{n-1}. Thus the 3-spacer containing the 2-section K as its lower section, has a spine that refines K; contradicting that K was non-refinable. It follows that a third chain K'' cannot exist.

Next assume $j = 1$. In this case, the base projection $J := \mathrm{im}(p_n K)$ is a 1-simplex $(x_{n-1} \to z_{n-1})$ in T_{n-1}. Thus, K must be a spacer over J. Arguing by truss induction on the 1-truss bundle p_n over J, we find that exactly two non-refinable chains from x to z must exist. Namely, either the lower or the upper section of K must have a jump morphism that lies over J (this follows from the arguments in the proof of Lemma 2.2.27, but it can be graphically understood when thinking of the section order as a directed path through jump morphisms, see Fig. 2.17). In the former case the two non-refinable chains are given by K's spine and its predecessor's spine, in the latter case, by K's spine and its successor's spine.

This completes the verification that ∂T_n is pure of dimension $k - 1$, shellable and thin. □

3.4.2 Correspondence of flat framed regular cell complexes and trusses

3.4.2.1 From flat framed regular cell complexes to trusses
We define the 'truss integration' functor from flat framed regular cell complexes to trusses. As a first step, we define a 'truss translation' functor

$$\mathsf{T} : \mathsf{FlatProFrCellCplx}_n \to \overline{\mathsf{Trs}}_n$$

of flat *pro*framed regular cell complexes.

We start by constructing T on objects.

Construction 3.4.4 (Truss translation of flat proframed regular cell complexes). Given a flat n-proframed regular cell complex $(X, \mathcal{P})$ with projections $\mathcal{P} = (p_n, ..., p_1)$, we construct

153

a closed n-truss $\mathsf{T}(X, \mathcal{P})$. Recall, that each projection $p_i : X_i \to X_{i-1}$ is both a cellular poset map as well as an ordered simplicial map (see Remark 3.2.74). Denote the tower of (to be constructed) 1-truss bundles $\mathsf{T}(X, \mathcal{P})$ by $(T_n \xrightarrow{p_n} T_{n-1} \xrightarrow{p_{n-1}} ... \xrightarrow{p_1} T_0)$. In order to construct the 1-truss bundles $p_i : T_i \to T_{i-1}$ we need to: (1) define poset maps $p_i : (T_i, \trianglelefteq) \to (T_{i-1}, \trianglelefteq)$ of face orders, (2) define a dimension functor $\dim : (T_i, \trianglelefteq^{\mathrm{op}}) \to [1]$, (3) define a frame order $(T_i, \preceq)$, (4) verify that fibers of p_i over objects are closed 1-trusses, and (4') verify that fibers over morphisms are 1-truss bordism.

1. We set $(T_i, \trianglelefteq) = X_i$ and define $p_i : (T_i, \trianglelefteq) \to (T_{i-1}, \trianglelefteq)$ to be $p_i : X_i \to X_{i-1}$.

2. We define $\dim : (T_i, \trianglelefteq^{\mathrm{op}}) \to [1]$ to map $x \in T_i$ to 0 if $X^{\geq x}$ is a section cell and to 1 if $X^{\geq x}$ is a spacer cell in X (see Terminology 3.3.10). Since section cell can only border on other section cells, this defines a poset map $(T_i, \trianglelefteq^{\mathrm{op}}) \to [1]$ as required.

3. We define two elements x, y in T_i to be related in the frame order $(T_i, \preceq)$ such that $x \prec y$ if and only if they lie in the same fiber of p_i and there is a sequence of 1-simplices $x \to \cdots \to y$ in the simplicial complex X_i.

4. We check that the structures $\trianglelefteq$, $\dim$, and $\preceq$ restrict on fibers $p_i^{-1}(z)$ over objects z in T_{i-1} to yield closed 1-trusses $T_z = (p_i^{-1}(z), \trianglelefteq, \dim, \preceq)$. Since $\mathcal{P}$ is flat, the fiber over z is a linear subcomplex $x_0 \to x_1 \to ... \to x_k$ of the simplicial complex X_i; this shows, $(p_i^{-1}(z), \preceq)$ is a total order. Moreover, if $X^{\geq x_i}$ is a spacer cell, then $X^{\geq x_{i\pm 1}}$ must be its upper and lower section cells; thus $(p_i^{-1}(z), \trianglelefteq)$ is a linear fence. Our definition of the functor $\dim$ then implies that T_z is a closed 1-truss as claimed (spacer cells have both upper and lower section cells, and thus cannot lie at fiber endpoints).

(4') It remains to verify that fibers over morphisms are 1-truss bordisms. Let $f : z \to w$ be an arrow in T_{i-1}. Denote by $R : T_z \nrightarrow T_w$ the relation $p_i^{-1}(f)$. Since $T_i = X_i$ is a poset, R is a boolean profunctor. Since the proframing $\mathcal{P}$ is flat, it follows that $R \subset (T_z, \preceq) \times (T_w, \preceq)$ is bimonotone. Since the transition functors are surjective, it follows that R fully matches objects. Finally, if $x \in \mathsf{sing}(T_z)$ there is a unique $y \in \mathsf{sing}(T_w)$ such that $R(x, y)$: indeed, p_n restricts on sections cells $X^{\geq x}$ to poset isomorphisms $p_x : X^{\geq x} \cong X^{\geq z}$, and thus $R(x, y)$ holds if and only if $X^{\geq y} = p_x^{-1}(X^{\geq w})$. The statement that R is a 1-truss bordism now follows from Corollary 2.1.53.

This completes the construction of the closed n-truss $\mathsf{T}(X, \mathcal{P})$. ⌐⌐

Next, we construct T on morphisms.

Construction 3.4.5 (Truss translation of proframed maps)**.** Given a proframed cellular map $F : (X, \mathcal{P}) \to (Y, \mathcal{Q})$ of flat proframed regular cell complexes we construct the singular truss map $\mathsf{T}F : \mathsf{T}(X, \mathcal{P}) \to \mathsf{T}(Y, \mathcal{Q})$. The ith component of $\mathsf{T}F$ is defined by the ith component $F_i : X_i \to Y_i$ of F. We need to check that F being a proframed cellular map translates to $\mathsf{T}F$ being a singular map, i.e. $\mathsf{T}F$ maps singular objects to singular objects. Indeed, singular objects $x \in T_i$ yield section cells $X_i^{\geq x}$ in X_i and the definition of proframed maps implies that F_i must map section cells (in $(X_i, \mathcal{P}_{\leq i})$) to section cells (in $(Y_i, \mathcal{Q}_{\leq i})$). It follows that $\mathsf{T}F_i$ preserves singular objects as required. ⌐⌐

Terminology 3.4.6 (Truss translation functors)**.** We define the 'truss integration' of flat framed regular cell complexes to be the functor

$$\mathsf{T}_* \textstyle\int : \mathsf{FlatFrCellCplx}_n \to \bar{\mathsf{Trs}}_n$$

obtained as the composite of the functors $\smallint : \mathsf{FlatFrCellCplx}_n \to \mathsf{FlatProFrCellCplx}_n$ and $\mathsf{T} : \mathsf{FlatProFrCellCplx}_n \to \bar{\bar{\mathsf{Trs}}}_n$.

We define the 'truss integration' of framed regular cells to be the functor

$$\mathsf{T}_*\smallint : \mathsf{FrCell}_n \to \mathsf{Blk}_n$$

obtained by restricting the functor $\mathsf{T}_*\smallint : \mathsf{FlatFrCellCplx}_n \to \bar{\bar{\mathsf{Trs}}}_n$ to the full subcategory $\mathsf{FrCell}_n \hookrightarrow \mathsf{FlatFrCellCplx}_n$. ⌙

3.4.2.2   From trusses to flat framed regular cell complexes Inverse to truss integration, we next construct the cell gradient of n-truss blocks and closed n-trusses. First we construct a 'proframed cell translation functor'

$$\mathsf{C} : \bar{\bar{\mathsf{Trs}}}_n \to \mathsf{FlatProFrCellCplx}_n.$$

We start by defining the functor C on objects.

Construction 3.4.7 (Proframed complex translation of closed trusses)**.** Given a closed n-truss $T = (T_n \xrightarrow{p_n} T_{n-1} \xrightarrow{p_{n-1}} ... \xrightarrow{p_1} T_0)$ we construct a flat n-proframed regular cell complexes $\mathsf{C}T$. We write this proframed complex as $(X, \mathcal{P})$ with X a cellular poset and $\mathcal{P} = (p_n, ..., p_1)$ a tower of (both cellular and ordered simplicial) projections. In order to define $(X, \mathcal{P})$, we need to: (1) define the sequence $\mathcal{P}$ of cellular maps $p_i : X_i \to X_{i-1}$ (with $X = X_n$), (2) define an ordering of the underlying simplicial sequence $\mathcal{P}$, making $\mathcal{P}$ a simplicial proframing, and (3) verify that $\mathcal{P}$ is both a flat proframing itself and flat on each cell of X.

1. We define the poset map $p_i : X_i \to X_{i-1}$ to be $p_i : (T_i, \trianglelefteq) \to (T_{i-1}, \trianglelefteq)$. The fact that each X_i is cellular was proven in Lemma 3.4.1. The fact that the bundle maps p_i are cellular maps follows since 1-truss bundle have lifts (see Observation 2.1.62).

2. We define the ordered simplicial maps $p_i : X_i \to X_{i-1}$. Inductively, we assume to have defined the ordered simplicial complex X_{i-1}. To obtain the ordered simplicial complex X_i, we need to consistently order vertices of all k-simplices $x : [k]^{\mathsf{un}} \hookrightarrow X_i$ in the unordered simplicial complex X_i. Take $k = 1$. Either the 1-simplex x lies over an object y of X_{i-1}, or it lies over a (ordered) 1-simplex $z : z(0) \to z(1)$ in X_{i-1}. In the first case, order $x = x(0) \to x(1)$ such that $x(0) \prec x(1)$ in the frame order $(T_i, \preceq)$. Otherwise, if x lies over a 1-simplex $y : y(0) \to y(1)$, then order $x = x(0) \to x(1)$ such that its first vertex $x(0)$ lies over $y(0)$ while its second vertex $x(1)$ lies over $y(1)$. This determines a vertex order on all 1-simplices, which extends an ordering of X_i. The construction also entails that the poset map $p_i : X_i \to X_{i-1}$ is an ordered simplicial map $p_i : X_i \to X_{i-1}$.

3. Finally, we verify that $\mathcal{P}$ is both flat itself and flat on each cell of X. Recall, flatness requires fiber categories to be linear and transition functors to be endpoint-preserving. But these are exactly the conditions verified via truss induction in Observation 2.2.36. Thus, $\mathcal{P}$ is flat. Similarly it follows that all cell restricted proframings $(X^{\geq x}, \mathcal{P}|_x)$ are flat (namely, by applying Observation 2.2.36 to the 1-truss bundles in the n-truss block $T^{\trianglerighteq x}$). ⌙

Next, we define the functor C on morphisms.

Construction 3.4.8 (Proframed cell translation of singular maps). Given a singular truss map $F : T \to S$ we construct a proframed cellular map $\mathsf{C}\, F : \mathsf{C}\, T \to \mathsf{C}\, S$. Write the complex $\mathsf{C}\, T$ as $(X, \mathcal{P})$ (consisting of a cellular poset X and a proframing $\mathcal{P}$) and similarly write the complex $\mathsf{C}\, S$ as $(Y, \mathcal{Q})$. The ith component $(\mathsf{C}\, F)_i : X_i \to Y_i$ is defined by the ith component $F_i : T_i \to S_i$. We need to check that $\mathsf{C}\, F$ is cellular and proframed. Both can be seen by an inductive argument. For cellularity, inductively assume F_{i-1} is cellular. To see that F_i is cellular, we check that, for each $x \in X_i$, the cell $X_i^{\geq x}$ in X_i maps onto some cell $Y_i^{\geq y}$ in Y_i. If x is singular (i.e. $X_i^{\geq x}$ is a section cell) then this claim follows from our inductive assumption that F_{i-1} is cellular, and since F_i preserves singular objects. If x is regular (i.e. $X_i^{\geq x}$ is a spacer cell) the claim follows from the singular case since spacer cells are 'sandwiched' between their lower and upper section cells.

Next, to verify that $\mathsf{C}\, F$ is proframed, we may inductively assume that the truncation $\mathsf{C}\, F_{<n} : \mathsf{C}\, T_{<n} \to \mathsf{C}\, S_{<n}$ is proframed. Since F is a singular truss map it follows that $\mathsf{C}\, F_n$ maps section cells to sections cell, which (together with the inductive assumption) implies that $\mathsf{C}\, F$ preserves final frame vectors as required in the definition of cellular proframed maps. ⌐

Terminology 3.4.9 (Cell gradient). We define the 'cell gradient of closed trusses' to be the functor

$$\mathsf{C}^*\nabla : \bar{\mathsf{T}}\mathsf{rs}_n \to \mathsf{FlatFrCellCplx}_n$$

obtained by taking the composite of the functor $\mathsf{C} : \bar{\mathsf{T}}\mathsf{rs}_n \to \mathsf{FlatProFrCellCplx}_n$ and the functor $\nabla : \mathsf{FlatProFrCellCplx}_n \to \mathsf{FlatFrCellCplx}_n$.

This functor specializes as follows. We define the 'cell gradient of truss blocks' to be the functor

$$\mathsf{C}^*\nabla : \mathsf{Blk}_n \to \mathsf{FrCell}_n$$

obtained by restricting the functor $\mathsf{C}^*\nabla : \bar{\mathsf{T}}\mathsf{rs}_n \to \mathsf{FlatFrCellCplx}_n$ to the full subcategory $\mathsf{Blk}_n \hookrightarrow \bar{\mathsf{T}}\mathsf{rs}_n$. ⌐

As a last formality, we record that truss integration and cell gradient assemble into the following equivalences of categories.

Proof of Theorem 3.1.1 and Theorem 3.1.2. The functor $\mathsf{T}_*\!\int$ defined in Terminology 3.4.6 and the functor $\mathsf{C}^*\nabla$ defined in Terminology 3.4.9 are inverse; moreover, there is a unique choice for the natural isomorphisms $\mathrm{id} \cong \mathsf{T}_*\!\int \circ \mathsf{C}^*\nabla$ and $\mathrm{id} \cong \mathsf{C}^*\nabla \circ \mathsf{T}_*\!\int$. The equivalence further restricts to an equivalence of the subcategories $\mathsf{FrCell}_n \hookrightarrow \mathsf{FlatFrCellCplx}_n$ and $\mathsf{Blk}_n \hookrightarrow \bar{\mathsf{T}}\mathsf{rs}_n$. □

Proof of Theorem 3.1.3. We show that the category of regular truss block sets (i.e. of presheafs on truss blocks which are regular) is equivalent to the category of framed regular cell complexes. Having shown the equivalence of categories of truss blocks Blk_n and of framed regular cells FrCell_n, it will suffice to show that the category of framed regular cell complexes is equivalent to a category of 'regular' presheafs on framed regular cells, which can be seen as follows. Given a presheaf $W \in \mathrm{PSh}(\mathsf{FrCell}_n)$, a cell $y \in W(Y, \mathcal{G})$ is said to be non-degenerate if the map $y : (Y, \mathcal{G}) \to W$ does not factor through a framed cellular surjection $(Y, \mathcal{G}) \twoheadrightarrow (Y', \mathcal{G}')$ (except the identity). We say the presheaf W is

regular if each non-degenerate cell $y \in W(Y, \mathcal{G})$ includes back into W by an injective presheaf map $y : (Y, \mathcal{G}) \hookrightarrow W$. Every regular presheafs W defines a framed regular cell complex $(X_W, \mathcal{F}_W)$, whose cellular poset $(X_W, \leq)$ is the poset of all non-degenerate cells $y \in W(Y, \mathcal{G})$ such that $y \leq y'$ whenever y factors through y' by a framed cellular injection, and whose framing $\mathcal{W}$ restricts on each cell $X_W^{\geq y} \cong Y$ to the framing $\mathcal{G}$ of the framed regular cell $(Y, \mathcal{G})$. Conversely, every framed regular cell complexes $(X, \mathcal{F})$ defines a regular presheaf $W_{X,\mathcal{F}}$ such that $W_{X,\mathcal{F}}(Y, \mathcal{G})$ is the set of all framed cellular inclusion $(Y, \mathcal{G}) \hookrightarrow (X, \mathcal{F})$. The constructions are inverse up to canonical isomorphism. The correspondence canonically extends to a correspondence of maps between regular presheafs and framed cellular maps which provides the claimed equivalence of categories. $\qquad\square$

Corollary 3.4.10 (Framed regular cells are shellable PL cells). *Given an n-framed regular cell $(X, \mathcal{F})$ then cellular poset X is PL cellular, and both X and ∂X are shellable.*

Proof. By Lemma 3.4.1 we know that face posets of $\mathsf{T}_* \int (X, \mathcal{F})$ are PL cellular. Since the poset X is isomorphic to the top face poset of $\mathsf{T}_* \int (X, \mathcal{F})$ (see Construction 3.4.4) it follows that X itself is PL cellular. $\qquad\square$

Constructible framed topology: meshes

In this chapter we introduce the stratified topological notion of 'meshes', which are towers of constructible stratified bundles whose fibers are points or framed stratified 1-manifolds. We begin in Section 4.1 by describing these fibers, called '1-meshes', and their constructible bundles, before then generalizing these to notions of n-meshes and n-mesh bundles. In Section 4.2 we discuss how meshes are the 'topological counterpart' of trusses, by constructing the *entrance path n-trusses* of n-meshes and the *classifying n-meshes* of n-trusses, and showing that these constructions exhibit a 'weak equivalence' between meshes and trusses.

Convention 4.0.1 (∞-Categories modeled by Top-enriched categories). All '∞-categories' will be Top-enriched categories (see [Lur09a, §1]). ⌐

Convention 4.0.2 (Finiteness). We assume all stratification to be finite. ⌐

4.1 1-Meshes, 1-mesh bundles, and n-meshes

We introduce 1-meshes and n-meshes as certain structured stratifications (a recollection of basic ideas in the theory of stratified spaces can be found in Appendix B). 1-Meshes will be defined as a stratified 0- or 1-manifolds endowed with the structure of a flat framing, while n-meshes will be defined by iterating 'constructible stratified bundles' of 1-meshes. Here, the notion of 'stratified bundle' is a generalization of the ordinary notion of fiber bundles to the context of stratified topology (see Definition B.2.24). The term 'constructible' bundle refers, generally, to bundles which can be (re)constructed from functors on the 'fundamental category' of their base: while in the context of stratified topology this fundamental category is usually the entrance path ∞-category of the base stratification (see Appendix B.3) in case of meshes constructibility holds already at the level of entrance path posets (which will follow from the comparison of meshes and trusses in Section 4.2).

The section will be organized as follows. In Section 4.1.1 we will introduce meshes and their maps; our definitions will closely mirror the definition of 1-trusses and 1-truss maps in Chapter 2. We then discuss bundles of 1-meshes and introduce a condition for their constructibility in Section 4.1.2. Finally, in Section 4.1.3, we define n-meshes and briefly comment how this further gives rise to a notion of 'n-mesh bundles'.

4.1.1   1-Meshes

4.1.1.1   1-Meshes as 1-framed stratified manifolds We start with a notion of 1-framing of manifolds. Classically, a 'tangent framing' of a smooth manifold is a trivialization of its tangent bundle. In the absence of smooth structures, we can similarly require trivializations of tangent microbundles. However, independent of whether we work in the smooth or topological case, the following holds true: every (sufficiently nice) codimension-k submanifold of a framed manifold inherits a framing of its k-stabilized tangent bundle. Specifically, in the case $n = 1$, this motivates the following definitions.

Terminology 4.1.1 (Manifolds). In the following, 'manifold' will mean connected topological manifold with or without boundary. $\mathbb{R}$ will mean the 'standard euclidean 1-space' and S^1 the 'oriented standard circle'.   ⊔

Definition 4.1.2 (1-Framings on manifolds)**.** A **1-framing of a manifold** M is an embedding $\gamma : M \hookrightarrow S^1$.   ⊔

Remark 4.1.3 (Dimension of 1-framed manifolds). Given a 1-framed manifold (M, γ) if $\dim(M) = 0$ then M obtains a 'normal 1-frame' from the ambient S^1 and otherwise a 'tangent 1-frame'.   ⊔

We will be particular interested in the case of 'flat framings' where the ambient manifold is standard $\mathbb{R}^n$.

Definition 4.1.4 (Flat 1-framings on manifolds)**.** A **flat 1-framing of a manifold** M is a bounded embedding $\gamma : M \hookrightarrow \mathbb{R}$.   ⊔

Remark 4.1.5 (Flat 1-framings are framings). Any orientation preserving embedding $\mathbb{R} \hookrightarrow S^1$ can be used to translate flat framings into framings by post-composition.   ⊔

Remark 4.1.6 (The space of framings of a 1-manifold). Given a 1-manifold M, the subspace $\{\gamma \mid (M, \gamma)$ is a 1-framing$\}$ of the mapping space $\mathrm{Map}(M, S^1)$ is homotopy equivalent to $\mathbb{Z}_2$. That is, 'up to homotopy' there are two framings on a 1-manifold. The same observation applies to flat 1-framings.   ⊔

A mesh is a nicely stratified manifold with 1-framing: its strata are manifolds (without boundary).

Definition 4.1.7 (General 1-meshes)**.** A **1-mesh** (M, f, γ) is a manifold M with a finite stratification f, stratifying M by manifolds (without boundary), together with a 1-framing γ on M.
   1. If M has a single stratum, we call the 1-mesh **trivial**.
   2. If M is a contractible 1-manifold, we call the 1-mesh **linear**.
   3. If M is the circle, we call the 1-mesh **circular**.   ⊔

Example 4.1.8 (General meshes)**.** 1-Meshes of different types are shown in Fig. 4.1. In each case, we color 0-dimensional strata in red, and 1-dimensional strata in blue. For linear 1-meshes we depict the ambient euclidean space $\mathbb{R}$; more commonly we will simply be indicate by 'coordinate arrow' (indicated in green) which fixes the framing up to contractible choice. Similarly, we indicated the framing of circular meshes by a green arrow (describing the orientation of S^1). Note that, we can distinguish three types of
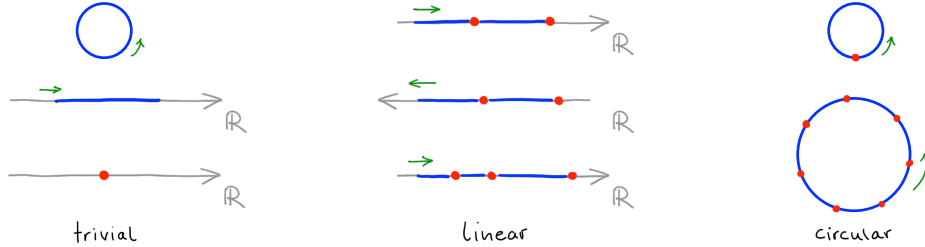

Figure 4.1: 1-Meshes of different types.

trivial 1-meshes: the 'trivial 0-dimensional mesh', the 'trivial linear 1-dimensional mesh' and the 'trivial circular 1-dimensional mesh'.[19]

While much of the theory of meshes can be developed in parallel for the 'linear' and 'circular' cases, our interest will (analogous to the case of 1-trusses) ultimately lie with the linear case. Going forward, we will therefore adopt the following convention.

Convention 4.1.9 (Linear 1-meshes by default)*.* Going forward, we will refer to 'linear 1-meshes' simply as '1-meshes', and always endow 1-meshes with flat framings (i.e. embeddings into $\mathbb{R}$).

Notation 4.1.10 (Flat 1-framing bounds)*.* For a 1-mesh (M, f, γ) with flat 1-framing $\gamma : M \hookrightarrow \mathbb{R}$ denote by $\gamma^-(M)$ and $\gamma^+(M)$ the lower resp. upper bound of the end $\gamma(M) \subset \mathbb{R}$.

Terminology 4.1.11 (Open and closed meshes)*.* A 1-mesh (M, f, γ) is called 'closed' resp. 'open' if the image $\gamma(M) \subset \mathbb{R}$ is closed respectively open in $\mathbb{R}$.

4.1.1.2   Maps of 1-meshes Let us first discuss framed maps of 1-framed manifolds (in the sense of Definition 4.1.2).

Terminology 4.1.12 (Framed maps of standard framed spaces)*.* A 'framed map' of the standard circle S^1 is an orientation preserving map $S^1 \to S^1$ (by which we mean map of the form $e^{ix} \mapsto e^{i\phi(x)}$ where $\phi : \mathbb{R} \to \mathbb{R}$ is monotone). A 'framed map' of standard $\mathbb{R}$ is an orientation preserving map $\mathbb{R} \to \mathbb{R}$ (by which we mean a monotone map $\mathbb{R} \to \mathbb{R}$). The notions similarly apply to connected subspaces of S^1 and $\mathbb{R}^1$.

Definition 4.1.13 (Framed maps of 1-framed manifolds)**.** For (flat) 1-framed manifolds (M, γ) and (N, ρ), a **framed map** $F : (M, \gamma) \to (N, \rho)$ is a continuous map $F : M \to N$ which induces a framed map $\tilde{F} : \gamma(M) \to \rho(N)$ between the images of their 1-framings, such that $\tilde{F} \circ \gamma = \rho \circ F$.

---

[19]Note, in contrast, we had described only two trivial 1-trusses in Fig. 2.2. This indicate that the trivial 1-truss with one element of dimension 1 should have two distinct combinatorial incarnations: the 'trivial linear' and the 'trivial circular' 1-truss. We will not bother further with rectifying the combinatorial situation, since we are ultimately interested only in the linear case.

A 1-mesh map preserves both framing and stratification structure as follows.

Definition 4.1.14 (Maps of 1-meshes). A **map of 1-meshes** $F : (M, f, \gamma) \to (N, g, \rho)$ is a continuous map $F : M \to N$ that is both a stratified map $F : (M, f) \to (N, g)$ as well as a framed map $F : (M, \gamma) \to (N, \rho)$. ⌐

Fully parallel to our earlier definitions of 'regular', 'singular', and 'balanced' maps of trusses (see Definition 2.1.16), we introduce the following terminology for mesh maps.

Terminology 4.1.15 (Singular, regular, and balanced maps). A map of 1-meshes $F : (M, f, \gamma) \to (N, g, \rho)$ is called 'singular' if it maps point strata to point strata, 'regular' if it maps interval strata into interval strata, and 'balanced' if it is both singular and regular. ⌐

In analogy to our earlier definitions of 'subtrusses', 'truss degeneracies' and 'truss coarsenings' (see Terminology 2.3.92 and Terminology 2.3.94), we further introduce the following properties of mesh maps.

Terminology 4.1.16 (Submeshes of 1-meshes). A map of 1-meshes $F : (M, f, \gamma) \to (N, g, \rho)$ is called a 'submesh' if $F : (M, f) \to (N, g)$ is a substratification such that γ is a restriction of ρ, that is, the induced map $\tilde{F} : \mathrm{im}(\gamma) \to \mathrm{im}(\rho)$ (see Definition 4.1.13) is a subspace inclusion (of subspaces in $\mathbb{R}$). ⌐

Note that any submesh inclusion is necessarily balanced.

Terminology 4.1.17 (Mesh degeneracies and coarsenings of 1-meshes). A map of 1-meshes $F : (M, f, \gamma) \to (N, g, \rho)$ is called a 'mesh degeneracy' if it is a surjective singular mesh map that maps interval strata either homeomorphically onto their image stratum, or 'degenerates' them into a point stratum. Dually, F is called a 'mesh coarsening' if it is a mesh map that descents to a coarsening of stratifications $F : (M, f) \to (N, g)$. (Note that, by definition of stratified coarsenings, see Definition B.2.4, mesh coarsenings are necessarily surjective regular mesh maps, and homeomorphisms on underlying spaces.) ⌐

Example 4.1.18 (Maps of 1-meshes). In Fig. 4.2 we depict singular, regular, balanced and 'mixed' 1-mesh maps (whose mappings we indicate by green arrows). Note that the first example is in fact a mesh degeneracy, the second a mesh coarsening, and the third a submesh. ⌐
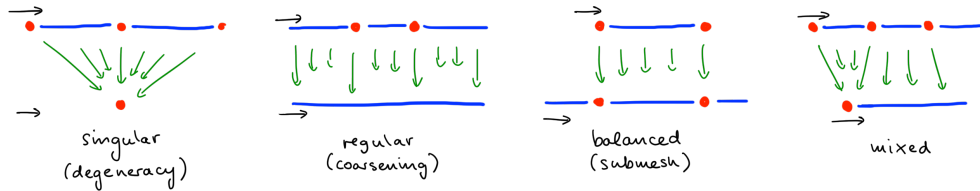

Figure 4.2: 1-Mesh maps.

4.1.2   1-Mesh bundles

4.1.2.1   The definition of 1-mesh bundles We introduce 1-mesh bundles. Recall the notion of stratified bundles (see Definition B.2.24): these are stratified maps $p : (E, f) \to (B, g)$ which, for each point x in stratum u in the base (B, g), locally trivialize over a neighborhood U of x in u to a bundle of the form $U \times (F, h) \to U$.

Notation 4.1.19 (Fiber dimensions). Given a stratified bundle $p : (E, f) \to (B, g)$ in which strata of stratified fibers are manifolds, then for t a stratum in (E, f) we write $\mathrm{fibdim}(t) = k$, if t restricts to a k-dimensional manifold in the fibers of p.[20]   ⌐

A 1-mesh bundle is a stratified bundles, whose fibers are (continuously) endowed with the structure of a 1-mesh, together with a constructibility condition that ensure 'entrance paths lift uniquely to fiber dimension-0 strata' in the following sense.

Definition 4.1.20 (1-Mesh bundles)**.** A **1-mesh bundle** (p, γ) is a stratified bundle $p : (M, f) \to (B, g)$ together with a 'fiber 1-framing' bundle embedding $\gamma : M \hookrightarrow B \times \mathbb{R}$ into the trivial bundle $\pi : B \times \mathbb{R} \to B$ such that:

   Fiberwise 1-mesh structure: f and γ restrict over $x \in B$ to a 1-mesh $(p^{-1}(x), f, \gamma)$, and the framing bounds $x \to \gamma^{\pm} p^{-1}(x)$ are continuous in x (see Notation 4.1.10),
   Constructibility: For any stratum t in (M, f) lying over a stratum $s = p(t)$ in (B, g) with $\mathrm{fibdim}(t) = 0$, given an entrance path $s \to r$ in $\mathsf{Entr}(g)$ there exists a unique lift $t \to u$ in $\mathsf{Entr}(f)$ (such that $\mathsf{Entr}(p)(t \to u) = s \to r$) and this lift satisfies $\mathrm{fibdim}(u) = 0$.   ⌐

Note that fiber dimension in 1-mesh bundle are either 0 or 1.

Terminology 4.1.21 (Regular and singular strata). Given a 1-mesh bundle $p : (M, f) \to (B, g)$ and a stratum s in f, we say t is 'singular' if $\mathrm{fibdim}(t) = 0$, or a 'regular' stratum if $\mathrm{fibdim}(t) = 1$.   ⌐

Notation 4.1.22 (Framing bounds). For a 1-mesh bundle $p : (M, f) \to (B, g)$ with fiber 1-framing γ, the maps $\gamma^{\pm} : B \to B \times \mathbb{R}$, mapping $b \to \gamma^{\pm} p^{-1}(b)$, will be called 'fiber 1-framing bounds'.   ⌐

Terminology 4.1.23 (Open and closed 1-mesh bundles). A 1-mesh bundle p is called 'open' resp. 'closed' if all its fibers are open resp. closed 1-meshes.   ⌐

Example 4.1.24 (A 1-mesh bundle)**.** In Fig. 4.3 we depict a 1-mesh bundle whose base stratification is the classifying stratification $(B, g) = \mathsf{CStr}\, P$ of the poset $P = (a \leftarrow b \leftarrow c \to d)$ (this poset also appeared as the base poset in Fig. 2.9). Note that fibers of the stratum $\mathsf{str}(c) \subset \mathsf{CStr}\, P$ are open 1-meshes, fibers of the stratum $\mathsf{str}(a)$ and $\mathsf{str}(d)$ are closed 1-meshes, while the fiber over $\mathsf{str}(b)$ is neither open nor closed. Note that the bundle map is depicted as a (restriction of the) projection $B \times \mathbb{R} \to B$ (with the orientation of $\mathbb{R}$ indicated by a green arrow) which provides the fiber 1-framing of the bundle.   ⌐

Example 4.1.25 (Constructible and non-constructible 1-mesh bundles)**.** In Fig. 4.4 we compare a 1-mesh bundle (on the left) a 'non-constructible' 1-mesh bundle (on the right), whose fibers are 1-meshes but which fails the constructibility condition. In each case, singular strata are colored in red and regular strata in blue. Note that the 1-mesh bundle on the right fails the constructibility condition on *both* its singular 1-strata: firstly, for

---

[20]Note, since manifold dimension is invariant under homeomorphism, $\mathrm{fibdim}(t)$ is well-defined.
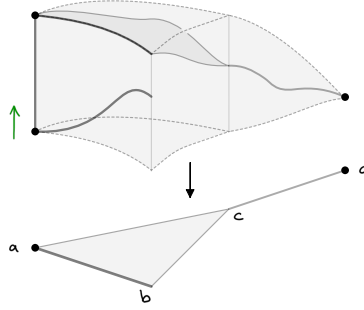
Figure 4.3: A 1-mesh bundle.

the upper singular 1-stratum the base entrance path lifts, but ends in a regular stratum; secondly, for the lower singular 1-stratum the base entrance path does not lift.[21]
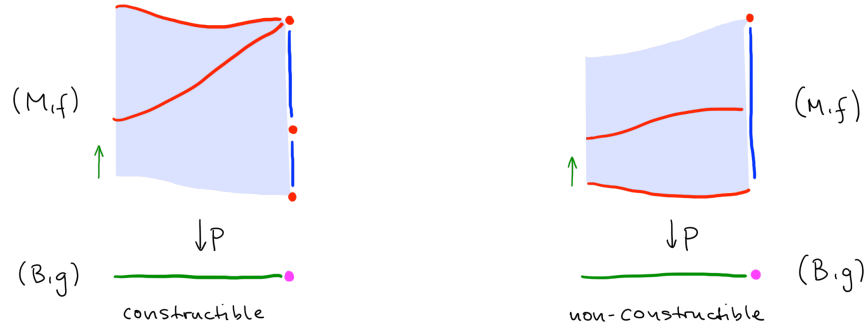
Figure 4.4: Constructible and non-constructible 1-mesh bundles.

Example 4.1.26 (A conical non-constructible bundle)**.** The constructibility condition is still non-trivial even in the 'better behaved' conical case: in Fig. 4.5 we depict a 1-mesh bundle which, despite its base and total stratification being conical, is non-constructible.
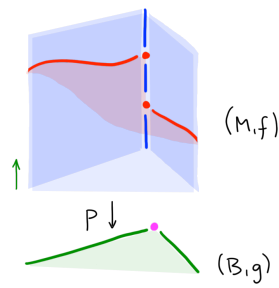
Figure 4.5: A conical but non-constructible 1-mesh bundle.

Remark 4.1.27 (Omitting orientations). When depicting bundle maps of 1-mesh bundles as projections $B \times \mathbb{R} \to B$ we will (analogously to our depictions of 1-truss bundles, see

---

[21]The two illustrated failures of the constructibility condition are somewhat different in nature, and weaker notion of 1-mesh bundle (for which, in particular, fibers may be empty) may, in fact, allow the second type of failure to happen.

Remark 2.1.58) sometimes omit indicating the standard orientation of $\mathbb{R}$, which leaves a $\mathbb{Z}_2$-ambiguity for such a choice.   ⌐

While stratified bundles generally only trivialize locally over base strata, 1-mesh bundles trivialize globally over base strata as the next observation records.

Notation 4.1.28 (Restrictions over base strata). Let $p : (M, f) \to (B, g)$ be a 1-mesh bundle. Over each stratum s in the base (B, g), p restricts to a 1-mesh bundle denoted by $p|_s : (p^{-1}(s), f) \to s$.   ⌐

Observation 4.1.29 (Trivialization over base strata). Since automorphism spaces of 1-meshes are contractible, the restricted 1-mesh bundle $p|_s$ is (non-uniquely) isomorphic to a trivial 1-mesh bundle which we will denote as follows

$$(p^{-1}(s), f) \xrightarrow{\;\cong\;} s \times \mathsf{fib}(s)$$

$$p|_s \searrow \qquad \swarrow$$
$$s$$

where $\mathsf{fib}(s)$ is a 1-mesh.   ⌐

Let us briefly discuss maps of 1-mesh bundles.

Definition 4.1.30 (Maps of 1-mesh bundles). Given 1-mesh bundles $p : (M, f) \to (B, g)$ and $p' : (M', f') \to (B', g')$, a **1-mesh bundle map** $F : p \to p'$ is a stratified map $F : (M, f) \to (M', f')$ that factors through p and p' by a map $(B, g) \to (B', g')$, such that F restricts on fibers to 1-mesh maps.   ⌐

Terminology 4.1.31 (Singular, regular and balanced 1-mesh bundles maps). We call a 1-mesh bundle map $F : p \to p'$ singular resp. regular resp. balanced if it is fiberwise singular resp. regular resp. balanced in the sense of Terminology 4.1.15.   ⌐

Remark 4.1.32 (Bundle maps commute with fiber framings). Every 1-mesh bundle map $F : p \to p'$ induces a commutative diagram of continuous maps (where γ resp. γ' are fiber 1-framings of p resp. of p')

$$\begin{array}{ccccc}
\mathrm{im}(\gamma) & \xleftarrow{\;\gamma\;} M \xrightarrow[p]{\pi} & B \\
\tilde{F} \downarrow & F \downarrow & \downarrow \\
\mathrm{im}(\gamma') & \xleftarrow{\;\gamma'\;} M' \xrightarrow[\pi]{p'} & B' \; .
\end{array}$$
   ⌐

We end with the following important remark about how our notion of constructible 1-mesh bundle could reasonably be called a '0-constructible' bundle, and how this can be generalized to 'higher constructibility' as well.

Remark 4.1.33 (On 0-constructibility and 1-constructibility). As we will see, the constructibility condition in Definition 4.1.20 precisely guarantees that isomorphism classes of 1-mesh bundles over (sufficiently nice) base (B, g) are classified by functors $\mathsf{Entr}(g) \to \mathsf{TBord}^1$ on the entrance path poset of the base (into the category of 1-truss bordisms): this will follow from Proposition 4.2.26 and Observation 4.2.68.

The entrance path poset $\mathsf{Entr}\,g$ of (B, g) is, however, merely the '0th level' of categorical data of the stratification g: formally, $\mathsf{Entr}\,g$ is the 0-truncation of the *entrance path* $(\infty, 1)$-*category* $\mathcal{E}ntr(g)$ of (B, g) (see section B.3). For 1-mesh bundles to 'see' higher structure of the base, we can amend the constructibility condition in our Definition 4.1.20 of 1-mesh bundles $p : (M, f) \to (B, g)$ as follows.

> *1-Constructibility*: For every stratum t in (M, f) with $\mathrm{fibdim}(s) = 0$, lying over $s = p(t)$ in (B, g), given an entrance path $\alpha : s \to r$ in $\mathcal{E}ntr(g)$, there exists a unique lift $\beta : t \to u$ in $\mathcal{E}ntr(f)$ (with $p \circ \beta = \alpha$) and this satisfies $\mathrm{fibdim}(u) = 0$.

Amending the definition in this way yields a notion of '1-constructible 1-mesh bundle' (in contrast, we may refer to bundles in the sense of Definition 4.1.20 as '0-constructible' 1-mesh bundles). This now satisfies the following analog of our earlier observation: bundle isomorphisms classes of 1-constructible 1-mesh bundles over (sufficiently nice) base (B, g) are classified by ∞-functors $\mathcal{E}ntr(g) \to \mathsf{TBord}^1$, and thus equivalently ordinary functors $\mathrm{ho}(\mathcal{E}ntr(g)) \to \mathsf{TBord}^1$ (see Remark 2.1.77). As noted in Remark 4.2.4, the proof of this observation is fully analogous to the proof in the 0-constructible case. ⌐

4.1.2.2 Pullbacks and compactifications of 1-mesh bundles

We discuss two important constructions on 1-mesh bundles: firstly, 1-mesh bundles can be fiberwise compactified and secondly, they can be pulled back along stratified maps in the base.

Construction 4.1.34 (Fiberwise compactifications of 1-mesh bundle)**.** Given a 1-mesh bundle $p : (M, f) \to (B, g)$ with fiber 1-framing $\gamma : M \hookrightarrow B \times \mathbb{R}$, we construct a 1-mesh bundle $\overline{p} : (\overline{M}, \overline{f}) \to (B, g)$ called the 'fiberwise compactification' of p. Define $\overline{M}$ to be the closure of $\gamma(M)$ in $B \times \mathbb{R}$, and obtain $\overline{p} : \overline{M} \to B$ by restricting $\pi : B \times \mathbb{R} \to B$ to $\overline{M}$. The stratification $(\overline{M}, \overline{f})$ is defined to have strata to be images $\gamma(r)$ of strata r in f, or images $\gamma^{\pm}(s)$ of strata s in g. ⌐

Example 4.1.35 (1-Mesh bundle compactification)**.** In Fig. 4.6 we depict the compactification of a 1-mesh bundle p. ⌐
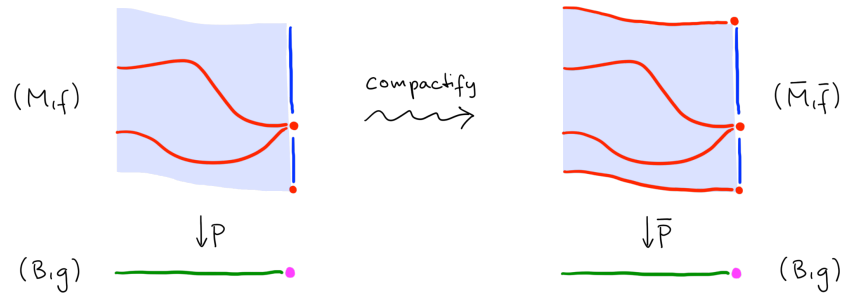

Figure 4.6: The compactification of a 1-mesh bundle.

Construction 4.1.36 (Pullbacks of 1-mesh bundles)**.** Given a 1-mesh bundle $p : (M, f) \to (B, g)$ and a stratified map $G : (B', g') \to (B, g)$ we construct a 1-mesh bundle $G^*p : (G^*M, G^*f) \to (B', g')$ called the 'pullback of p along G'. As a map of stratified spaces,

construct G^*p by a pullbacks (see Terminology B.2.26) as follows.

$$
\begin{array}{ccc}
(G^*M, G^*f) & \xrightarrow{\mathrm{Tot}G} & (M, f) \\
{\scriptstyle G^*p}\big\downarrow & \lrcorner & \big\downarrow {\scriptstyle p} \\
(B', g') & \xrightarrow{\ \ G\ \ } & (B, g)
\end{array}
$$

Note that G^*p is a stratified bundle since p is. Construct a fiber 1-framing $G^*\gamma : G^*M \hookrightarrow B' \times \mathbb{R}$ for G^*p by defining, for $x \in G^*M$, $G^*\gamma(x) \in B' \times \mathbb{R}$ to have first component $G^*p(x)$ and second component equal to the second component of $\gamma \circ \mathrm{Tot}G(x)$. ⌐

Observation 4.1.37 (Pullbacks preserve fiberwise compactifications). Consider a 1-mesh bundle $p : (M, f) \to (B, g)$ and a stratified map $G : (B', g') \to (B, g)$. Then the pullback $G^*\overline{p}$ of the fiberwise compactification of p is the fiberwise compactification $\overline{G^*p}$ of the pullback G^*p. ⌐

4.1.2.3 1-Mesh bundles over nice stratifications When working with 1-mesh bundles we usually assume the base stratification to be sufficiently nice. As we will discuss here, niceness of base often lifts to the total stratification of the bundle.

Our main interest will lie with so-called 'cellulable' stratifications, which are those stratifications that can be subdivided by cell complexes. Recall the notion of regular cell complexes from Definition 1.3.1 and that of constructible substratifications (see Definition B.2.9).

Terminology 4.1.38 (Regular cell stratifications, cf. Definition B.3.16). An '(open) regular cell' stratification is a stratification that is stratified homeomorphic to a constructible (open) substratification of some regular cell complex. ⌐

Terminology 4.1.39 (Cellulable stratification, cf. Definition B.3.14). A '(open) cellulable stratification' is a stratification that can be refined by a (open) regular cell stratification. ⌐

Importantly, if the base stratification of a 1-mesh bundles is cellulable, then so is its total stratification—we will rigorously prove this fact only in the slightly simpler case of *open* cellulable stratifications and closed/open 1-mesh bundles.

Proposition 4.1.40 (Cellulability lifts). *Let* $p : (M, f) \to (B, g)$ *be a closed (or open) 1-mesh bundle. If* (B, g) *is open cellulable then so is* (M, f).

The proof of this statement will use the following lemma.

Lemma 4.1.41 (Regular cellularity lifts). *Let* $p : (M, f) \to (B, g)$ *be a closed (or open) 1-mesh bundle. If* (B, g) *is open regular cell then so is* (M, f).

The proof of Lemma 4.1.41 will use the following general observation about cell of 'quotient-cylindrical shape'.

Lemma 4.1.42 (Cells bundled over cells). *Let* D^m *be the closed m-disk, S^{n-1} its boundary, and let* $p : X \to D^m$ *be a subbundle of the projection* $\pi : D^m \times \mathbb{R} \to D^m$ *whose fibers over* $x \in D^m$ *are subsets of* $\mathbb{R}$ *of the form* $[\gamma^-(x), \gamma^+(x)]$, *where* $\gamma^\pm : D^m \to D^m \times \mathbb{R}$ *are continuous sections. If* $\gamma^-(x) < \gamma^+(x)$ *everywhere, except possibly when* $x \in S^{n-1}$, *then* X *is a closed $(m+1)$-disk.*

Proof. Define the 'trivial fiber locus' of p to be the closed subset $\mathsf{triv}(p) = \{x \mid \gamma^-(x) = \gamma^+(x)\}$ of D^m. If $\mathsf{triv}(p) = S^{m-1}$, then X is the quotient of $D^m \times [-1, 1]$ by $S^{m-1} \times [-1, 1]$ which is an m-disk. If $\mathsf{triv}(p) \subsetneq S^{m-1}$ is a proper subset, the claim holds by the following standard argument. Define $V \subset S^{m-1} \times \mathbb{R}$ to be the subspace with points (x, y) where $x \in S^{m-1}$ and $y \in [-\mathrm{dist}(x, \mathsf{triv}(p)), +\mathrm{dist}(x, \mathsf{triv}(p))] \subset \mathbb{R}$ (where dist is some metric on S^{m-1}). Let W be the convex hull of V in $\mathbb{R}^{m+1}$. This projects to the convex hull of S^{m-1} in $\mathbb{R}^m$, which is D^m. By construction, the projection $q : W \to D^m$ is bundle isomorphic to $p : X \to D^m$, for instance, by a linearly identifying fibers $q^{-1}(z) \cong p^{-1}(z)$ for all $z \in D^m$. Since compact convex sets with non-empty interior are disks, we have $W \cong D^{m+1}$, and thus $X \cong D^{m+1}$ as claimed. $\qquad\square$

Proof of Lemma 4.1.41. We first argue in the case of *closed* 1-mesh bundles $p : (M, f) \to (B, g)$ with open regular cell base. By definition, (B, g) is a open constructible substratification of a regular cell complex (Y, e) (where e stratifies Y by its cells). By removing cells from Y that are not in the closure of B, we may further assume $\overline{B} = Y$. Let $\mathsf{triv}(p) \subset B$ be the closed subset of B over which p has point fibers, and denote by $\mathsf{triv}(p)^*$ the union of $\mathsf{triv}(p)$ and $Y \setminus B$. We build a bundle $q : (X, d) \to (Y, e)$ with (X, d) being a regular cell complex containing (M, f) as an open constructible substratification. The underlying map $q : X \to Y$ is the subbundle of the projection $\pi : Y \times \mathbb{R} \to Y$ with fibers over x given by $[-\mathrm{dist}(x, \mathsf{triv}(p)^*), +\mathrm{dist}(x, \mathsf{triv}(p)^*)]$.[22] Denote by $q|_B$ the restriction of q to $B \subset Y$ in the base. Observe that we can identify bundles $q|_B \cong p$ by linearly identifying their fibers $q^{-1}(z) \cong p^{-1}(z)$, $z \in B$ (note $p^{-1}(z)$ inherits linear structure from the fiber 1-framing γ of p, which embeds $p^{-1}(z) \hookrightarrow \mathbb{R}$). In particular, this identifies $q^{-1}(B) \cong M$. Now, stratify X by defining strata s of (X, d) to be either strata s in (M, f) or of the form $r \times \{0\}$ where r is a stratum of $(Y \setminus B, e)$. Verify that, by constructibility of p, this makes q itself a closed 1-mesh bundle. Using Lemma 4.1.42 as well as the constructibility of q, one further verifies that (X, d) is a regular cell complex. The constructible substratification $(M, f) \hookrightarrow (X, d)$ then exhibits (M, f) as a open regular cell stratification. We illustrate the construction in Fig. 4.7.
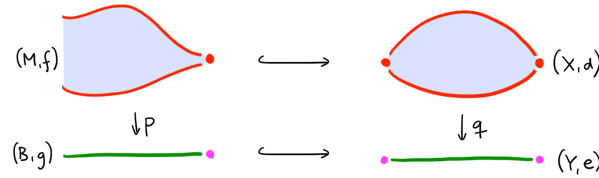
Figure 4.7: Closed 1-mesh bundles over open regular cell base have open regular cell total stratification.

Finally, to see the statement in the case of open 1-mesh bundles $p : (M, f) \to (B, g)$ over open regular cell stratifications (B, g), one first compactifies p to a closed 1-mesh bundle $\overline{p} : (\overline{M}, \overline{f}) \to (B, g)$ using Construction 4.1.34. The previous argument shows that $(\overline{M}, \overline{f})$ is open regular cell; since (M, f) is an open constructible substratification of $(\overline{M}, \overline{f})$ it follows that (M, f) is open regular cell as claimed. $\qquad\square$

We can prove that cellulability lifts.

---

[22]Note every regular cell complex Y is metrizable, for instance by using the 'barycentric metric' of its barycentric subdivision. The same need not hold for non-regular cell complexes!

Proof of Proposition 4.1.40. Since (B, g) is cellulable, we may refine it by a open regular cell stratification $G : (B, c) \to (B, g)$. Then the pullback bundle $G^*p : (M, d) \to (B, c)$ has open regular cell total stratifications (M, d) by Lemma 4.1.41. The coarsening $\mathrm{Tot}G : (M, d) \to (M, f)$ exhibits (M, f) to be open cellulable as claimed. $\qquad\square$

Remark 4.1.43 (General regular cell 1-mesh bundles). Working with 'cellulable' in place of 'open cellulable' stratifications (resp. 'regular cell' in place of 'open regular cell' stratifications) the preceding results still hold and, in fact, it apply to all 1-mesh bundles. The additional assumption of *openness* simplifies the proof, and will cover the cases of interest to us. $\qquad\rule{8mm}{0.3mm}$

While our main interest will lie with cellulable stratifications, for later use we also record following (easily verified) observations about further conditions that 'lift' in 1-mesh bundles.

Observation 4.1.44 (Finiteness and local finiteness lifts). Consider a 1-mesh bundle $p : (M, f) \to (B, g)$ in which (B, g) is finite (resp. locally finite). Then (M, f) is finite (resp. locally finite) as well. $\qquad\rule{8mm}{0.3mm}$

Observation 4.1.45 (Frontier-constructibility lifts). A 'frontier-constructible' stratification (B, g) is a stratification that satisfies $(\overline{s} \cap r \neq \emptyset) \Rightarrow (r \subset \overline{s})$ (equivalently, this asks for the characteristic map to be open, see Lemma B.2.10). Consider a 1-mesh bundle $p : (M, f) \to (B, g)$ in which (B, g) is finite (resp. locally finite). Then (M, f) is finite (resp. locally finite) as well. $\qquad\rule{8mm}{0.3mm}$

4.1.3 n-Meshes and their bundles

4.1.3.1 The definition of n-meshes We now generalize the definition of 1-meshes to dimension n.

Definition 4.1.46 (n-Meshes). An n-**mesh** M is a tower of 1-mesh bundles $(M_n, f_n) \xrightarrow{p_n} (M_{n-1}, f_{n-1}) \xrightarrow{p_{n-2}} ... \xrightarrow{p_2} (M_1, f_1) \xrightarrow{p_1} (M_0, f_0) = \mathrm{pt}$. $\qquad\rule{8mm}{0.3mm}$

Just as 1-meshes have flat 1-framings (embedding a 1-mesh in $\mathbb{R}$), n-meshes have 'flat n-framings' (embedding an n-mesh in $\mathbb{R}^n$). This can be constructed from the individual fiber 1-framings of the 1-mesh bundles in an n-mesh as follows. Recall the standard euclidean n-proframe $\Pi = (\pi_n, \pi_{n-1}, ..., \pi_1)$ consisting of projections $\pi_i : \mathbb{R}^i \to \mathbb{R}^{i-1}$ (see Terminology 3.2.1).

Construction 4.1.47 (Flat n-framings of n-meshes)**.** Given an n-mesh M consisting of 1-mesh bundles $p_i : (M_i, f_i) \to (M_{i-1}, f_{i-1})$, recall that each p_i comes equipped with a fiber 1-framing $M_i \hookrightarrow M_{i-1} \times \mathbb{R}$. This allows us to inductively construct a map of towers of spaces

$$
\begin{array}{ccccccccc}
M_n & \xrightarrow{p_n} & M_{n-1} & \xrightarrow{p_{n-1}} & \cdots & \xrightarrow{p_2} & M_1 & \xrightarrow{p_1} & M_0 = \mathrm{pt} \\
\downarrow{\scriptstyle\gamma_n} & & \downarrow{\scriptstyle\gamma_{n-1}} & & \cdots & & \downarrow{\scriptstyle\gamma_1} & & \downarrow \\
\mathbb{R}^n & \xrightarrow{\pi_n} & \mathbb{R}^{n-1} & \xrightarrow{\pi_{n-1}} & \cdots & \xrightarrow{\pi_2} & \mathbb{R}^1 & \xrightarrow{\pi_1} & \mathbb{R}^0
\end{array}
$$

where γ_i is obtained by postcomposing the fiber 1-framing $M_i \hookrightarrow M_{i-1} \times \mathbb{R}$ with the product map $\gamma_{i-1} \times \mathbb{R} : M_{i-1} \times \mathbb{R} \hookrightarrow \mathbb{R}^{i-1} \times \mathbb{R}$. We refer to the embedding of towers

$\gamma : M \hookrightarrow \Pi$ as the 'flat n-framing' (or simply 'n-framing') of the n-mesh M. Note that γ is fully determined by its top component $\gamma_n : M_n \hookrightarrow \mathbb{R}^n$ and, abusing terminology, we sometimes refer the embedding γ_n itself as the 'n-framing' of M. ⌟

Given an n-mesh M with n-framing γ, note that components of γ may be considered either as subspace embeddings $\gamma_i : M_i \hookrightarrow \mathbb{R}^i$ or as stratified maps $\gamma : (M_i, f_i) \to \mathbb{R}^i$ (which are then coarsenings onto their images in $\mathbb{R}^i$).

Example 4.1.48 (A mixed 2-mesh)**.** In Fig. 4.8 we depict a 2-mesh via its n-framing $\gamma : M \hookrightarrow \Pi$ into the standard euclidean tower Π (we indicate this by providing coordinate axis for the ambient euclidean space $\mathbb{R}^i$). Note that the 2-mesh is 'mixed' in that it is neither open nor closed. ⌟
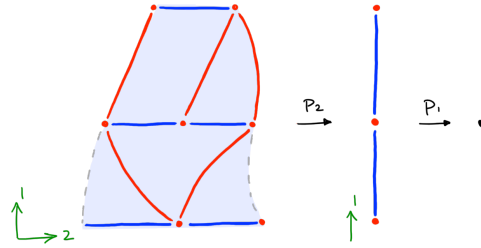
Figure 4.8: A mixed 2-mesh.

Example 4.1.49 (Closed and open n-Meshes)**.** In Fig. 4.9 we depict closed and open meshes in dimension 2 and 3. In each case we again depict the mesh via its n-framing $\gamma : M \hookrightarrow \Pi$. ⌟
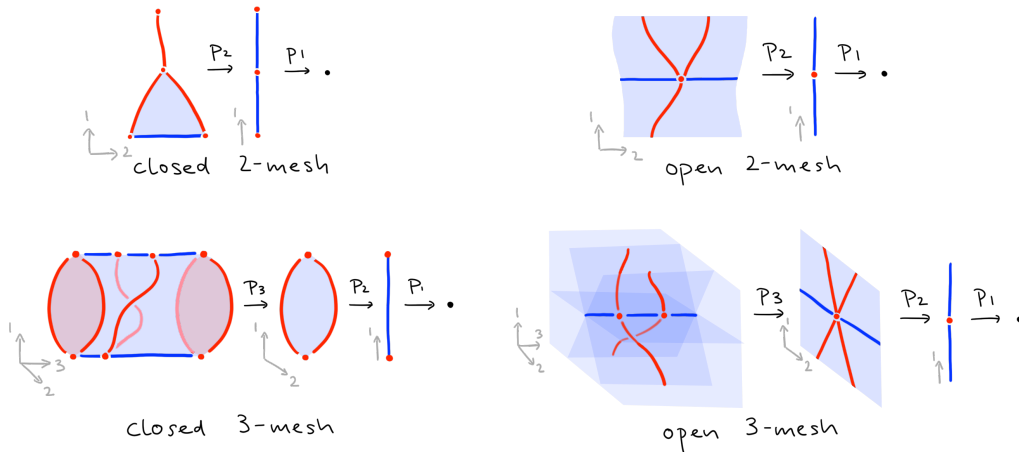
Figure 4.9: A closed and open meshes in dimension 2 and 3.

Terminology 4.1.50 (Closed or open n-meshes)**.** A mesh is called 'closed' (resp. 'open') if each 1-mesh bundle in its tower is closed (resp. open). ⌟

Observation 4.1.51 (Closed and open n-meshes are regular cell)**.** Our earlier Lemma 4.1.41, that 'open regular cellularity lifts in closed resp. open 1-mesh bundles', implies that for closed resp. open n-meshes M each (M_i, f_i) is an open regular cell stratification, and thus, in particular, conical. ⌟

Remark 4.1.52 (General n-meshes are also regular cell). The previous observation generalizes: general n-meshes are regular cell stratifications (while not necessarily open ones) and thus, in particular, conical. We omit a proof of this more general statement, since we will be mainly interested in the case of open and closed meshes. ⌙

4.1.3.2 n**-Mesh bundles** Yet more generally, we may consider n-meshes in 'bundles' parametrized by a base stratification.

Definition 4.1.53 (n-Mesh bundles)**.** Given a stratification (B, g), an n**-mesh bundle** p over (B, g) is a tower of 1-mesh bundles $(M_n, f_n) \xrightarrow{p_n} (M_{n-1}, f_{n-1}) \xrightarrow{p_{n-2}} \ldots \xrightarrow{p_2} (M_1, f_1) \xrightarrow{p_1} (M_0, f_0) = (B, g)$. ⌙

The construction of n-framings immediately generalizes to the bundle case.

Construction 4.1.54 (Fiber n-framings)**.** Given an n-mesh bundles p over a base stratification (B, g), replacing π_i by $B \times \pi_i$ in Construction 4.1.47, on constructs the 'fiber n-framing' γ of p as a map of towers

$$
\begin{array}{ccccccccc}
M_n & \xrightarrow{p_n} & M_{n-1} & \xrightarrow{p_{n-1}} & \cdots & \xrightarrow{p_2} & M_1 & \xrightarrow{p_1} & M_0 = B \\
\downarrow{\gamma_n} & & \downarrow{\gamma_{n-1}} & & \cdots & & \downarrow{\gamma_1} & & \downarrow{\mathrm{id}_B} \\
B \times \mathbb{R}^n & \xrightarrow{B \times \pi_n} & B \times \mathbb{R}^{n-1} & \xrightarrow{B \times \pi_{n-1}} \cdots & \xrightarrow{B \times \pi_2} & B \times \mathbb{R}^1 & \xrightarrow{B \times \pi_1} & B \times \mathbb{R}^0
\end{array}
$$

As before, we sometimes refer to the top map $\gamma_n : M_n \hookrightarrow B \times \mathbb{R}^n$ itself as the 'fiber n-framing' of p. ⌙

Following our earlier comparison of '0-constructibility' and '1-constructibility' for 1-mesh bundles (see Remark 4.1.33), we also record the following further variation of our notion of n-mesh bundle.

Remark 4.1.55 (1-constructible n-mesh bundles)*.* Given a stratification (B, g), an '1-constructible n-mesh bundle' p over (B, g) is a tower of 1-constructible 1-mesh bundles $(M_n, f_n) \xrightarrow{p_n} (M_{n-1}, f_{n-1}) \xrightarrow{p_{n-2}} \ldots \xrightarrow{p_2} (M_1, f_1) \xrightarrow{p_1} (M_0, f_0) = (B, g)$. Fully analogous to Remark 4.1.33, while we will show that ('0-constructible') n-mesh bundles over (B, g) can be classified by functors $\mathsf{Entr}\, g \to \mathsf{TBord}^n$, 1-constructible n-mesh bundles can be classified by ∞-functors $\mathcal{E}ntr(g) \to \mathsf{TBord}^n$ (which, equivalently, are ordinary functors $\mathrm{ho}(\mathcal{E}ntr(g)) \to \mathsf{TBord}^n$). ⌙

4.1.3.3 **Categories of n-meshes and n-mesh bundles** We define maps and categories of n-meshes and their bundles.

Definition 4.1.56 (Maps of n-mesh bundles)**.** Consider n-mesh bundles p and q over (B, g) resp. (C, h). An n**-mesh bundle map** $F : p \to q$ is a map of towers

$$
\begin{array}{ccccccccc}
(M_n, f_n) & \xrightarrow{p_n} & (M_{n-1}, f_{n-1}) & \xrightarrow{p_{n-1}} & \cdots & \xrightarrow{p_2} & (M_1, f_1) & \xrightarrow{p_1} & (M_0, f_0) = (B, g) \\
\downarrow{F_n} & & \downarrow{F_{n-1}} & & \cdots & & \downarrow{F_1} & & \downarrow{F_0} \\
(N_n, g_n) & \xrightarrow{q_n} & (N_{n-1}, g_{n-1}) & \xrightarrow{q_{n-1}} & \cdots & \xrightarrow{q_2} & (N_1, g_1) & \xrightarrow{q_1} & (N_0, g_0) = (C, h)
\end{array}
$$

such that F_0 is a stratified map, and each F_i (for $i > 0$) is a 1-mesh bundle map $p_i \to q_i$. Setting $(B, g) = (C, h) = *$, the definition specializes to a notion of n**-mesh maps**. ⌙

Remark 4.1.57 (*n*-Mesh bundle maps commute with fiber *n*-framings). In analogy to Remark 4.1.32, given *n*-mesh bundles p and q withfiber *n*-framing γ resp. ρ, an *n*-mesh map $F : (M, f) \to (N, g)$ induces a map of towers $\tilde{F} : \mathrm{im}(\gamma) \to \mathrm{im}(\rho)$ (with components $\tilde{F}_i : \mathrm{im}(\gamma_i) \to \mathrm{im}(\rho_i)$) such that $\tilde{F} \circ \gamma = \rho \circ F$. ⌐

Terminology 4.1.58 (Singular, regular and balanced maps). We call an *n*-mesh bundle map F 'singular' resp. 'regular' resp. 'balanced' if its is fiberwise resp. regular resp. balanced, that is, for all $1 \leq i \leq n$, the bundle maps F_i restrict on 1-mesh fibers to singular resp. regular resp. balanced 1-mesh maps in the sense of Terminology 4.1.15. ⌐

Terminology 4.1.59 (Subbundles and submeshes). We call an *n*-mesh bundle map F a 'subbundle' if F_0 is a substratification and all F_i (for $i > 0$) are fiberwise submesh inclusions of 1-meshes in the sense of Terminology 4.1.16. In the case of *n*-mesh maps we speak of 'submeshes'. ⌐

Of particular interest to us will be mesh bundles that live over the same base stratification (B, g), with maps that act identically on the base.

Terminology 4.1.60 (Base preserving maps). Given *n*-mesh bundles p and q, a mesh bundle map 'over (B, g)' (or a 'base preserving' mesh bundle map) is a mesh bundle map $F : p \to q$ with $F_0 = \mathrm{id}_B$. ⌐

Notation 4.1.61 (Ordinary categories of *n*-mesh bundles). We denote by $\mathsf{MeshBun}_n$ the category of *n*-mesh bundles and their maps. Given a stratification (B, g), we denoted by $\mathsf{Mesh}_n(B, g)$ the category of *n*-mesh bundles and their maps over (B, g). We denote by Mesh_n the categories of *n*-meshes and their maps. In each case, we add a bar decoration $\bar{\mathsf{M}}$ to mean the subcategory of *closed* *n*-mesh bundles (resp. *n*-meshes) and *singular* maps, and a circle decoration $\mathring{\mathsf{M}}$ to mean the subcategories of *open* *n*-mesh bundles (resp. *n*-meshes) and *regular* maps. ⌐

The category of meshes is naturally Top-enriched, and thus an ∞-category (recall from Convention 4.0.1 that we model ∞-categories by topologically enriched categories in this chapter).

Notation 4.1.62 (∞-Categories of *n*-mesh bundles). The ∞-category $\mathcal{M}eshBun_n$ is obtained by topologizing the hom sets $\mathsf{MeshBun}_n(M, N)$ of $\mathsf{MeshBun}_n$ as subspaces of the product

$$\mathrm{Map}(M_n, N_n) \times \mathrm{Map}(M_{n-1}, N_{n-1}) \times ... \times \mathrm{Map}(M_0, N_0) \quad .$$

This further restricts to subcategories of mesh bundles over a fixed base (B, g) and to *n*-meshes, yielding the categories $\mathcal{M}esh_n(B, g)$ resp. $\mathcal{M}esh_n$. As before we add a bar decoration $\bar{\mathcal{M}}$ to mean the sub-Top-categories of *closed* *n*-mesh bundles (resp. *n*-meshes) and *singular* maps, and a circle decoration $\mathring{\mathcal{M}}$ to mean the sub-Top-categories of *open* *n*-mesh bundles (resp. *n*-meshes) and *regular* maps. ⌐

It will be useful to separately record the following classes of maps.

Terminology 4.1.63 (Mesh degeneracies and coarsenings). An *n*-mesh map F will be called a 'mesh degeneracy' resp. a 'mesh coarsenings' if F_i ($i > 0$) are fiberwise a 1-mesh degeneracy resp. a 1-mesh coarsening in the sense of Terminology 4.1.17.[23] ⌐

---

[23]Similar to Remark 2.3.99, we also refer to 'mesh coarsenings' as 'mesh refinements' (of their codomain).

Note, in particular, any mesh coarsening induces a tower of homeomorphisms on underlying spaces, and is thus a tower of coarsenings of stratifications in the usual sense (see Definition B.2.4).

Notation 4.1.64 (Categories of degeneracies/coarsenings). Passing to sub-Top-categories of $\mathcal{M}esh_n$, one obtains the ∞-categories $\mathcal{M}esh_n^{\mathrm{deg}}$ of n-meshes and degeneracies, and $\mathcal{M}esh_n^{\mathrm{crs}}$ of n-meshes and coarsenings. ⌐

4.1.3.4 Truncations and pullbacks Like n-truss bundles, n-mesh bundles 'truncate' and truncation is continuous.

Terminology 4.1.65 (Truncations). Given an n-mesh bundle $p = (p_n, p_{n-1}, ..., p_1)$ over (B, g) its i-**truncation** $p_{\leq i}$ is the i-mesh $(p_i, p_{i-1}, ..., p_1)$ over (B, g) obtained by truncating the tower p to its lower i degrees. ⌐

Remark 4.1.66 (Truncating is continuous). i-Truncation induces a Top-enriched functor $(-)_{\leq i} : \mathcal{M}esh_n(B, g) \to \mathcal{M}esh_i(B, g)$ of ∞-categories of meshes bundles over (B, g). ⌐

We may also generalize our construction of pullbacks from the case of 1-mesh bundles (see Construction 4.1.36) to dimension n as follows. Of particular interest will be the following 'inductive' usage of the construction.

Construction 4.1.67 (Inductive pullbacks of n-meshes). Consider an n-mesh bundle $p = ((M_n, f_n) \xrightarrow{p_n} ... \xrightarrow{p_1} (M_0, f_0))$ and, for $0 \leq i < n$, an i-mesh $q = ((N_{n-1}, g_{n-1}) \xrightarrow{q_n} ... \xrightarrow{q_1} (N_0, g_0))$ (if $i = 0$ this is simply a stratification (N_0, g_0)). Given an i-mesh bundle map $G : q \to p_{\leq i}$, we can use Construction 4.1.36 inductively, to define a 'pullback n-mesh bundle' $\mathrm{Tot}G : G^*p \to p$ by the tower of maps

$$
\begin{array}{ccccccccc}
(G^*M_n, G^*f_n) & \xrightarrow{G^*p_n} & \cdots & \xrightarrow{G^*p_{i+2}} & (G^*M_{i+1}, G^*f_{i+1}) & \xrightarrow{G^*p_{i+1}} & (N_i, f_i) & \xrightarrow{q_i} \cdots \xrightarrow{q_1} & (N_0, f_0) \\
\mathrm{Tot}^{n-i}G \downarrow & \lrcorner & & \lrcorner & \mathrm{Tot}^1 G \downarrow & \lrcorner & \downarrow G_i & & \downarrow G_0 \\
(M_n, f_n) & \xrightarrow{p_n} & \cdots & \xrightarrow{p_{i+2}} & (M_{i+1}, f_{i+1}) & \xrightarrow{p_{i+1}} & (M_i, f_i) & \xrightarrow{p_i} \cdots \xrightarrow{p_1} & (M_0, f_0)
\end{array}
$$

where, for $j = 1, ..., n$, we define $\mathrm{Tot}^j G$ and G^*p_{i+j} by pulling back p_{i+j} along $\mathrm{Tot}^{j-1}G$ (with $\mathrm{Tot}^0 G = G_i$) using Construction 4.1.36. ⌐

Terminology 4.1.68 (Bundle restrictions). If $i = 0$ in the preceding construction, and if $G_0 : (C, h) \equiv (N_0, f_0) \hookrightarrow (M_0, f_0) \equiv (B, g)$ is a substratification, then we call $G^*p \to p$ the 'restriction of p to $C \hookrightarrow B$' in its base, and write this as $p|_C \hookrightarrow p$. ⌐

Note, restricting n-mesh bundles to a point in their base in this way yields the 'fiber n-meshes' at that point.

4.2 Weak equivalence of meshes and trusses

In this section we prove the equivalence of n-meshes and n-trusses and, more generally, of their respective bundles (for sufficiently nice base stratification). The equivalence will be witnessed by the 'entrance path truss' functor $\mathsf{ETrs} : \mathsf{Mesh}_n \to \mathsf{Trs}_n$ and, conversely, by the 'classifying mesh' functor $\mathsf{CMsh}- : \mathsf{Trs}_n \to \mathsf{Mesh}_n$. As the names suggest, the former functor is a variation of the entrance path poset functor, while the latter functor

is a variation of the classifying stratification functor. Namely, the entrance path truss functor will take an n-mesh, given by a tower M of 1-mesh bundles, to a tower T of 1-truss bundles defined by applying the entrance path poset functor to the tower M and endowing fibers in T with 1-truss structure obtained from the 1-mesh structure of fibers in M. Conversely and analogously, the classifying mesh functor will take an n-truss T to a tower M of 1-mesh bundles obtained by applying the classifying stratification functor to the tower T and endowing fibers in M with 1-mesh structure obtained from the 1-truss structure of fibers in T (note, however, we will see that additional care has to be taken if T is not closed). The central theorem of this chapter states the following.

Theorem 4.2.1 (Weak equivalence of meshes and trusses). *The entrance path truss and classifying mesh functors restrict to weak equivalences*

$$\bar{\mathcal{M}esh}_n \underset{\mathsf{CMsh}}{\overset{\mathsf{ETrs}}{\rightleftarrows}} \bar{\mathsf{Trs}}_n \qquad\qquad \overset{\circ}{\mathcal{M}esh}_n \underset{\mathsf{CMsh}}{\overset{\mathsf{ETrs}}{\rightleftarrows}} \overset{\circ}{\mathsf{Trs}}_n$$

between the ∞-categories of closed n-meshes and closed n-trusses, resp. between open n-meshes and open n-trusses.

The theorem will, in fact, naturally follow from the following more general statement about mesh and truss *bundles*.

Theorem 4.2.2 (Weak equivalence of mesh and truss bundles). *Let (B, g) be an open cellulable stratification. The entrance path truss and classifying mesh functors restrict to weak equivalences*

$$\bar{\mathcal{M}esh}_n(B, g) \underset{\mathsf{CMsh}}{\overset{\mathsf{ETrs}}{\rightleftarrows}} \bar{\mathsf{Trs}}_n(\mathsf{Entr}\, g) \qquad\qquad \overset{\circ}{\mathcal{M}esh}_n(B, g) \underset{\mathsf{CMsh}}{\overset{\mathsf{ETrs}}{\rightleftarrows}} \overset{\circ}{\mathsf{Trs}}_n(\mathsf{Entr}\, g)$$

between the ∞-categories of closed n-mesh bundles over (B, g) and closed n-truss bundles over $\mathsf{Entr}\, g$, resp. between open n-mesh bundles over (B, g) and open n-truss bundles over $\mathsf{Entr}\, g$.

Note that Theorem 4.2.2 reduces to Theorem 4.2.1 if we choose to work with a trivial base $(B, g) = *$. We mention three further variations of the theorem.

Remark 4.2.3 (General mesh and truss bundles). Note that both theorems are stated for open and closed meshes resp. trusses only, but we will, in fact, construct a pair of functors

$$\mathcal{M}esh_n(B, g) \underset{\mathsf{CMsh}}{\overset{\mathsf{ETrs}}{\rightleftarrows}} \mathcal{T}rs_n(\mathsf{Entr}\, g)$$

relating general n-mesh bundles and n-truss bundles over sufficiently nice base (B, g) (see Proposition 4.2.17 and Construction 4.2.67). As briefly explained in Remark 4.2.18, while one could show that these functors too form a weak equivalence, such a proof would lead outside the 'natural' realm of homotopy theory (i.e. of ∞-categories). ⌐

Remark 4.2.4 (1-Constructible bundles). A variation of Theorem 4.2.2 can be obtained by working with 1-constructible n-mesh bundles (see Remark 4.1.55) and n-truss bundles over categories (see Remark 2.3.84). In this case, ETrs and CMsh yield weak equivalences between $\bar{\mathcal{M}}esh_n^1(B, g)$ and $\bar{\mathsf{Trs}}_n(\mathcal{E}ntr(g))$, and similarly between $\mathring{\mathcal{M}}esh_n^1(B, g)$ and $\mathring{\mathsf{Trs}}_n(\mathcal{E}ntr(g))$. In fact, the proof of Theorem 4.2.2 given here will immediately generalize to this case.[24]   ⌐

Remark 4.2.5 (Non-finite base). Recall from Convention 4.0.2 that we assume (B, g) to be finite. Our proof of Theorem 4.2.2 can be adapted to the case of infinite base stratifications as well; however, a bit more care needs to be taken due to the usual intricacies of Top-enriched categories modelling ∞-categories.   ⌐

The proof of Theorem 4.2.1 and Theorem 4.2.2 will take up the rest of this section. After first defining the entrance path truss functor in Section 4.2.1, we will demonstrate its 'conservativity' in Section 4.2.2, as well as its 'weak faithfulness' in Section 4.2.3. We then define the classifying mesh functor in Section 4.2.4, first in the (simpler) case of closed trusses and then for general trusses. Finally, the proof of Theorem 4.2.1 will be assembled in Section 4.2.5. This last section will also discuss several immediate applications of Theorem 4.2.1, which we now briefly outline.

Most immediately, composing this equivalence of closed n-trusses and flat n-framed regular cell complexes established in Chapter 3 with the weak equivalence of Theorem 4.2.1, we obtain the following corollary.

Corollary 4.2.6 (Equivalence of closed meshes and flat framed cell complexes). *The equivalences of Theorem 4.2.1 and Theorem 3.1.2 compose to weak equivalences* $\mathsf{C}^*\nabla \circ \mathsf{ETrs} : \bar{\mathcal{M}}esh_n \simeq \mathsf{FlatFrCellCplx}_n : \mathsf{CMsh} \circ \mathsf{T}_*\!\!\int$.

Terminology 4.2.7 (Cell mesh and mesh complex). We will denote the composite $\mathsf{CMsh} \circ \mathsf{T}_*\!\!\int$ by CellMsh, called the 'cell mesh' functor, and the composite $\mathsf{C}^*\nabla \circ \mathsf{ETrs}$ by MshCplx, called the 'mesh complex' functor.   ⌐

Using the notion of mesh refinements, the translation of framed cells into meshes will allow us to define 'framed subdivisions' of framed cells. We will then deduce the following result, which shows that the notion of framed subdivision is 'combinatorializable'.

Corollary 4.2.8 (Classifying subdivision of flat framed cells). *Up to homotopy, framed subdivisions of any framed cell* $(X, \mathcal{F})$ *are in correspondence with truss refinements of the truss* $\mathsf{T}_*\!\!\int(X, \mathcal{F})$.

Finally, the equivalence of trusses and meshes in Theorem 4.2.1 will also allow us to 'transfer' the duality of closed and open n-trusses to a dualization equivalences between closed n-meshes and open n-meshes.

Corollary 4.2.9 (Dualization of meshes). *Dualization of trusses induces weak equivalences of* ∞-*categories* $\dagger : \bar{\mathcal{M}}esh_n \simeq \mathring{\mathcal{M}}esh_n : \dagger$.

---

[24]This uses the observation that cellulable stratifications (B, g) can be refined by regular cell stratifications (X, f), which are 0-truncated, and thus notions of 1-constructible and 0-constructible mesh bundles over (X, f) coincide, and $\mathcal{E}ntr(f) \simeq \mathsf{Entr}\, f$ canonically.

This last result is central. While the 'cellular' world of framed regular cell complexes provided us with (combinatorial) topological realizations of *closed* trusses, a similar topological realization of *open* trusses wasn't readily available. The stratified topological notion of meshes accommodates topological realizations of both closed and open trusses as well as of their duality. This observation underlies later, more powerful dualization operations of so-called 'flat framed stratifications', and provides a general bridge between the dual worlds of 'cellular' geometry and 'string' geometry (see also Section 5.4).

4.2.1 Entrance path trusses

We formally construct an entrance path truss functor

$$\mathsf{ETrs} : \mathcal{Mesh}_n(B, g) \to \mathcal{Trs}_n(\mathsf{Entr}\, g)$$

of ∞-categories, which takes mesh bundles to truss bundles (resp. meshes to trusses). Proceeding inductively, we start with the case of 1-mesh bundles.

Observation 4.2.10 (Restriction 1-mesh bundles to closures of strata). Given a 1-mesh bundle $p : (M, f) \to (B, g)$ with fiber 1-framing γ, and a stratum t in f lying over a stratum $s = p(t)$ in g, then p restricts on the closure $\bar{t}$ to a 1-mesh bundle $p|_{\bar{t}} : \bar{t} \to \bar{s}$ whose fiber 1-framing $\gamma_{\bar{t}}$ is obtained by restricting γ to $\bar{t}$. The resulting upper and lower bounds will be denoted by $\gamma_{\bar{t}}^{\pm} : \bar{s} \to \bar{s} \times \mathbb{R}$. Note that if t is singular then these bounds coincide, namely, $\gamma_{\bar{t}}^{\pm} \circ p|_{\bar{t}} = \gamma|_{\bar{t}}$. ⌐

Construction 4.2.11 (Entrance path 1-truss bundles). Given a 1-mesh bundle $p : (M, f) \to (B, g)$ we endow the entrance path poset map $\mathsf{Entr}(p) : \mathsf{Entr}(f) \to \mathsf{Entr}(g)$ with structure of a 1-truss bundle, yielding the 'entrance path 1-truss bundle' $\mathsf{ETrs}(p)$.

We first endow fibers of $\mathsf{Entr}(p)$ with 1-truss structure. Trivializing $p^{-1}(s) \cong s \times \mathsf{fib}(s)$ over a base stratum s, where $\mathsf{fib}(s)$ is a 1-mesh (see Observation 4.1.29), note that $\mathsf{Entr}(p)^{-1}(s) = \mathsf{Entr}(p^{-1}(s)) \cong \mathsf{Entr}(\mathsf{fib}(s))$ canonically. Endow $\mathsf{Entr}(p)^{-1}(s)$ with a frame order $\preceq$ ordering strata using the order determined by the flat 1-framing of the 1-mesh $\mathsf{Entr}(\mathsf{fib}(s))$, and with a dimension map $\dim : \mathsf{Entr}(p)^{-1}(s) \to [1]^{\mathrm{op}}$ that maps strata t in $p^{-1}(s)$ to the fiber dimension $\mathsf{fibdim}(t)$.

It remains to check, given an entrance path $s \to r$ in the base (B, g), the Boolean profunctor $R \equiv p^{-1}(s \to r) : \mathsf{Entr}(p)^{-1}(s) \nrightarrow \mathsf{Entr}(p)^{-1}(r)$ defines a 1-truss bordisms. Pick $t \in \mathsf{Entr}(p)^{-1}(s)$. If t is singular then by constructibility there is a unique $u \in \mathsf{Entr}(p)^{-1}(r)$ such that $R(t, u)$ holds and u too is singular. Otherwise, if t is regular then we can apply Observation 4.2.10 to see that $R(t, u)$ holds if and only if $\gamma_{\bar{t}}^-(r) \leq \gamma(u)$ and $\gamma(u) \leq \gamma_{\bar{t}}^+(r)$. Comparing the two cases to the construction of 'singular determined' 1-truss bordisms in Lemma 2.1.51 verifies that R is a 1-truss bordism as claimed. ⌐

Example 4.2.12 (Entrance path 1-truss bundles). The entrance path 1-truss bundle of the 1-mesh bundle depicted in Fig. 4.3 recovers the 1-truss bundle depicted in Fig. 2.9. ⌐

Definition 4.2.13 (Entrance path truss bundles). Given an n-mesh bundle p over $(B, g) \equiv (M_0, f_0)$, consisting of 1-mesh bundles $p_i : (M_i, f_i) \to (M_{i-1}, f_{i-1})$, its **entrance path truss** $\mathsf{ETrs}\, p$ is the n-truss bundle over $\mathsf{Entr}(g)$ defined by the tower of 1-truss bundles

$$\mathsf{Entr}(f_n) \xrightarrow{\mathsf{ETrs}(p_n)} \mathsf{Entr}(f_{n-1}) \xrightarrow{\mathsf{ETrs}(p_{n-1})} \cdots \xrightarrow{\mathsf{ETrs}(p_2)} \mathsf{Entr}(f_1) \xrightarrow{\mathsf{ETrs}(p_1)} \mathsf{Entr}(f_0)$$

where $\mathsf{ETrs}(p_i)$ is the entrance path 1-truss bundle of p_i (see Construction 4.2.11). ⌐

The construction of entrance path trusses further extends to mesh bundle maps as follows.

Definition 4.2.14 (Entrance path truss maps)**.** Given a mesh bundle map $F : p \to q$ be a map of n-mesh bundles p and q over (B, g) with components $F_i : (M_i, f_i) \to (N_i, g_i)$. The **entrance path truss map** $\mathsf{ETrs}\, F : \mathsf{ETrs}\, p \to \mathsf{ETrs}\, q$ is truss bundle map over $\mathsf{Entr}\, g$ with components $(\mathsf{ETrs}\, F)_i$ given by $\mathsf{Entr}(F_i) : \mathsf{Entr}(f_i) \to \mathsf{Entr}(g_i)$. ⌟

Notation 4.2.15 (The entrance path truss functor). The previous definitions yield the functor $\mathsf{ETrs} : \mathsf{Mesh}_n(B, g) \to \mathsf{Trs}_n(\mathsf{Entr}\, g)$ from mesh bundles over (B, g) to truss bundles over $\mathsf{Entr}\, g$. ⌟

As a first observation, we record that the functor preserves pullbacks.

Observation 4.2.16 (ETrs preserves pullbacks). Recall from Construction 4.1.67 the notion of pullbacks G^*p of n-mesh bundles p along base maps $G : (C, f) \to (B, g)$. The above construction satisfies $\mathsf{ETrs}(G^*p) = (\mathsf{Entr}\, G)^* \mathsf{ETrs}\, p$, where $(\mathsf{Entr}\, G)^* \mathsf{ETrs}\, p$ is the pullback of the n-truss bundle $\mathsf{ETrs}\, p$ along $\mathsf{Entr}\, G : \mathsf{Entr}\, f \to \mathsf{Entr}\, g$ in the sense of Construction 2.3.85. ⌟

Let us next discuss, how the entrance path truss functor is in fact an ∞-functor. To make this precise, we first need to endow the category of trusses (resp. truss bundles) with the structure of an ∞-category. A natural approach would be to note that the the the category $\mathsf{Trs}_n(\mathsf{Entr}\, g)$ is Pos-enriched: this endows the set of truss bundle maps with arrows given by (level-wise commuting) natural transformations between them. By passing to specialization topologies we obtain the $k\mathsf{Top}$-enriched category $\mathcal{Trs}_n$ (see Notation B.0.1 and Convention B.0.2). As a result, we can define the following $k\mathsf{Top}$-enriched functor.

Proposition 4.2.17 (ETrs as an ∞-functor)**.** *The entrance path truss functor induces an* ∞-*functor* $\mathsf{ETrs} : \mathcal{Mesh}_n(B, g) \to \mathcal{Trs}_n(\mathsf{Entr}\, g)$.

Proof. We need to check that the assignment $F \mapsto \mathsf{ETrs}\, F$ on hom spaces is continuous on each hom space. This follows from continuity of $\mathsf{Entr} : \mathcal{Strat} \to \mathcal{Pos}$ as established in Construction B.2.21 (note also Remark B.2.20 applies since all posets are finite). □

Remark 4.2.18 (Non-invertible 2-structure). The immediate 'issue' is that the category $\mathcal{Trs}_n$ (and similarly $\mathcal{Trs}_n(\mathsf{Entr}\, g)$) is $k\mathsf{Top}$-enriched and not Top-enriched: that is, hom spaces of $\mathcal{Trs}_n$ need not be (weak) Hausdorff. Moreover, these hom spaces do not 'honestly' represent the higher categorical structures of trusses: morally, natural transformations equip $\mathcal{Trs}_n$ with *non-invertible* 2-morphisms, which are inverted when passing to the specialization topology. This goes back to an earlier 'mistake': the category of stratifications should not be defined as an ∞-category, but as an $(\infty, 2)$-category.[25] Thus, while one can show that the functor ETrs in Proposition 4.2.17 is a weak equivalence as given (with a candidate weak inverse defined in Construction 4.2.67) dealing with non-Hausdorff spaces homotopically is cumbersome; a better approach in establishing the equivalence $\mathcal{Mesh}_n \simeq \mathcal{Trs}_n$ would be to construct both $\mathcal{Mesh}_n$ and $\mathcal{Trs}_n$ as $(\infty, 2)$-categories. ⌟

---

[25]Since stratifications have fundamental $(\infty, 1)$-categories, their category naturally has $(\infty, 2)$-categorical structure. This is similar to spaces having fundamental $(\infty, 0)$-categories, and thus their category having $(\infty, 1)$-categorical structure.

For us there will be no need to go down the route of $(\infty, 2)$-categories: this is because, in the cases of interest to us, non-invertible 2-structure disappears. Namely, in the closed-singular case $\bar{\mathsf{Trs}}_n(\mathsf{Entr}\, g)$ (resp. the open-regular case $\mathring{\mathsf{Trs}}_n(\mathsf{Entr}\, g)$) we may topologize hom sets in either category discretely and still obtain the following ∞-functor.

Proposition 4.2.19 (Continuity of ETrs on bundles)**.** *The entrance path truss functor restricts to ∞-functors* $\mathsf{ETrs} : \bar{\mathcal{M}esh}_n(B, g) \to \bar{\mathsf{Trs}}_n(\mathsf{Entr}\, g)$ *and* $\mathsf{ETrs} : \mathring{\mathcal{M}esh}_n(B, g) \to \mathring{\mathsf{Trs}}_n(\mathsf{Entr}\, g)$.

Proof. This follows from the fact that singular resp. regular maps of closed resp. open truss bundles over a fixed base poset do not admit non-trivial natural transformations, see Lemma 2.3.104. $\qquad\square$

Setting $(B, g) = *$ we in particular obtain.

Corollary 4.2.20 (Continuity of ETrs on meshes)**.** *The entrance path truss functor restricts to ∞-functors* $\mathsf{ETrs} : \bar{\mathcal{M}esh}_n \to \bar{\mathsf{Trs}}_n$ *and* $\mathsf{ETrs} : \mathring{\mathcal{M}esh}_n \to \mathring{\mathsf{Trs}}_n$.

For later use we will also need the following variation of the statement. Note that ETrs maps mesh degeneracies to truss degeneracies, and mesh coarsenings to truss coarsenings. We thus obtain ordinary functors $\mathsf{ETrs} : \mathsf{Mesh}_n^{\mathrm{deg}} \to \mathsf{Trs}_n^{\mathrm{deg}}$ and $\mathsf{ETrs} : \mathsf{Mesh}_n^{\mathrm{crs}} \to \mathsf{Trs}_n^{\mathrm{crs}}$ (see Notation 2.3.98). Now, again by rigidity of hom posets (this time in the case of truss degeneracies and coarsenings, see Lemma 2.3.104) we may observe the following.

Observation 4.2.21 (More continuity of ETrs)*.* The entrance path truss functor induces ∞-functors $\mathsf{ETrs} : \mathcal{M}esh_n^{\mathrm{deg}} \to \mathsf{Trs}_n^{\mathrm{deg}}$ and $\mathsf{ETrs} : \mathcal{M}esh_n^{\mathrm{crs}} \to \mathsf{Trs}_n^{\mathrm{crs}}$. $\qquad\lrcorner \qquad\qquad \lrcorner$

The restricted functors in Proposition 4.2.19 (and similarly in Corollary 4.2.20 and Observation 4.2.21) provide the first half of 'weak equivalences' of ∞-categories. In the next sections we will discuss the two main ingredients of proving this equivalence, which are 'conservativity' and 'weak faithfulness'.

4.2.2 Conservativity of the entrance path truss functor For n-meshes M, M', the functoriality of ETrs implies that if $M \cong M'$ then $\mathsf{ETrs}\, M = \mathsf{ETrs}\, M'$ (note we write equality in place of isomorphism since isomorphisms between closed resp. between open trusses are necessarily unique). In this section we will see that the converse holds as well; namely, the functor ETrs (restricted to closed resp. open n-meshes) is 'conservative'. More generally, we record this in the case of n-mesh bundles by the following result.

Proposition 4.2.22 (Conservativity)**.** *For an open cellulable stratification (B, g), the functor* $\mathsf{ETrs} : \bar{\mathsf{Mesh}}_n(B, g) \to \bar{\mathsf{Trs}}_n(\mathsf{Entr}\, g)$ *(resp.* $\mathsf{ETrs} : \mathring{\mathsf{Mesh}}_n(B, g) \to \mathring{\mathsf{Trs}}_n(\mathsf{Entr}\, g)$*) is conservative: that is, given closed (resp. open) n-mesh bundles p, p' over (B, g), if* $\mathsf{ETrs}\, p = \mathsf{ETrs}\, p'$ *then* $p \cong p'$.

The proof of Proposition 4.2.22 will take up the rest of this section. Furthermore, the properties of the construction of the isomorphism $p \cong p'$ (namely, its 'continuity in families') will be reused in our later proof of 'weak faithfulness' in the next section.

We also record the following further variation of the result.

Observation 4.2.23 (More conservativity). Note that any isomorphism of n-meshes is both a mesh degeneracy and a mesh coarsening. The preceding proof of conservativity will therefore immediately imply conservativity of the functors $\mathsf{ETrs} : \mathsf{Mesh}_n^{\deg} \to \mathsf{Trs}_n^{\deg}$ and $\mathsf{ETrs} : \mathsf{Mesh}_n^{\mathrm{crs}} \to \mathsf{Trs}_n^{\mathrm{crs}}$ when restricting these functors to open and closed n-meshes. (With enough care, the proof can be extended to include all other n-meshes as well, bur our main interest lies with the closed and open case.) ⌐

4.2.2.1   Reduction to 1-mesh bundles over open regular cell base As a first step we show that it suffices to prove the statement of Proposition 4.2.22 for bundles over regular cell base.

Remark 4.2.24 (Reduction to open regular cell base). Fix a refinement $G : (B, c) \to (B, g)$ of (B, g) by an open regular cell stratification. Using Construction 4.1.67 we may pullback both p and p' to n-mesh bundles G^*p resp. G^*p' over (B, c). The assumption $\mathsf{ETrs}\, p = \mathsf{ETrs}\, p'$ in Proposition 4.2.22 now implies that $\mathsf{ETrs}\, G^*p = \mathsf{ETrs}\, G^*p'$ (see Observation 4.2.16). Any n-mesh bundle isomorphism $G^*p \cong G^*p'$ that fixes the base (B, c) will induce an n-mesh bundle isomorphism $p \cong p'$. Thus it is sufficient to prove Proposition 4.2.22 in the case of open regular cell base. ⌐

Next, by an inductive argument, the statement of Proposition 4.2.22 may further be reduced to a statement about 1-mesh bundles, as the following remark records.

Remark 4.2.25 (Reduction to 1-mesh bundle isomorphism). Consider closed (respectively open) n-meshes p and p' given by towers of 1-mesh bundles $p_i : (M_i, f_i) \to (M_{i-1}, f_{i-1})$ resp. $p'_i : (M'_i, f'_i) \to (M'_{i-1}, f'_{i-1})$ with $(M_0, f_0) = (B, g) = (M'_0, f'_0)$. Assume $\mathsf{ETrs}\, p = \mathsf{ETrs}\, p'$ as in Proposition 4.2.22. This implies $(\mathsf{ETrs}\, p)_{<n} = (\mathsf{ETrs}\, p')_{<n}$, and thus we can inductively construct the isomorphism $G : p_{<n} \cong p'_{<n}$ of truncated mesh bundles. Define the n-mesh G^*p' to be the inductive pullback of p' along G (see Construction 4.1.67); write the top-level 1-mesh bundle of G^*p' as $\tilde{p}_n : (\tilde{M}_n, \tilde{f}_n) \to (M_{n-1}, f_{n-1})$ and denote the canonical map $(\tilde{M}_n, \tilde{f}_n) \to (M'_n, f'_n)$ by F, as shown in the diagram below.

$$
\begin{array}{ccc}
\tilde{f}_n & \xrightarrow{\ \ F\ \ } & f'_n \\
{\scriptstyle \tilde{p}_n}\downarrow & \lrcorner & \downarrow{\scriptstyle p'_n} \\
f_{n-1} & \xrightarrow{\ G_{n-1}\ } & f'_{n-1}
\end{array}
$$

Note, since G is assumed to be an isomorphism, F too is an isomorphism. To prove Proposition 4.2.22 it remains to construct a bundle isomorphism $\kappa_{\tilde{p}}^p$ as shown below.

$$
\begin{array}{ccc}
f_n & \xrightarrow[\ \sim\ ]{\ \ \kappa_{\tilde{p}_n}^{p_n}\ \ } & \tilde{f}_n \\
& {\scriptstyle p_n}\searrow \quad \swarrow {\scriptstyle \tilde{p}_n} & \\
& f_{n-1} &
\end{array}
$$

Moreoever, note that since open cellulability 'lifts' (see Proposition 4.1.40) and since (B, g) is assumed to be open cellulable, the stratification f_{n-1} itself is open cellulable. ⌐

The inductive step for Proposition 4.2.22 is therefore provided by the following result.

Proposition 4.2.26 (Isomorphisms between 1-mesh bundles). *For open cellulable (B, g) and 1-mesh bundles $p : (M, f) \to (B, g)$ and $\tilde{p} : (\tilde{M}, \tilde{f}) \to (B, g)$ such that $\mathsf{ETrs}(p) = \mathsf{ETrs}(\tilde{p})$ coincide, there is a 1-mesh bundle isomorphism $\kappa_{\tilde{p}}^{p} : p \cong \tilde{p}$ that fixes the base (B, g).*

The proof of this statement will take up the next few sections. Let us start by reducing the statement even further to the following case.

Remark 4.2.27 (Reduction closed bundles). Assume Proposition 4.2.26 holds for closed 1-mesh bundles over an open regular cell stratification (B, g). The the statement holds for general 1-mesh bundles p and $\tilde{p}$. Indeed, fiberwise compactifying both bundles we obtain closed 1-mesh bundles $\bar{p}$ and $\bar{\tilde{p}}$ (see Construction 4.1.34). This satisfies $\mathsf{ETrs}(\bar{p}) = \mathsf{ETrs}(\bar{\tilde{p}})$ and thus we find $\kappa : \bar{p} \cong \bar{\tilde{p}}$, which restricts to a bundle isomorphism $\kappa : p \cong \tilde{p}$ as required. ⌐

In light of the previous remark, we henceforth assume p, $\tilde{p}$ to be closed 1-mesh bundles, and (B, g) to be an open regular cell stratification. The construction of $\kappa_{\tilde{p}}^{p}$ will still need care, and as a motivation for our approach we will first discuss how *not* to construct $\kappa_{\tilde{p}}^{p}$. When defining the stratified isomorphism $\kappa_{\tilde{p}}^{p} : f \cong \tilde{f}$ fiberwise, note that when traveling along an entrance paths between two strata $r \to s$ in the base (B, g), new singular strata can appear in the 'special' fiber over s, which were not present in the 'generic' fiber over r. This implies that, for instance, we generally cannot define $\kappa_{\tilde{p}}^{p}$ by fiberwise mapping point strata to point strata and extending this mapping of points linearly to interval strata (note that fibers in 1-mesh bundle inherit linear structure from the standard linear structure of $\mathbb{R}$ via the fiber 1-framing embedding). We illustrate the failure of this approach in the following example.

Example 4.2.28 (Failure of continuity in fiberwise linear interpolation). To illustrate the failure of continuity that can happen in attempting to fiberwise linearly map between bundles, consider the bundles $p : (M, f) \to (B, g)$, $\tilde{p} : (\tilde{M}, \tilde{f}) \to (B, g)$ as shown in Fig. 4.10 whose entrance path 1-truss bundles coincide (which we depicted, via their fiber 1-framings, as embedded in $B \times \mathbb{R}$). We indicated an entrance path in (B, g), together with a generic fiber as well as a special fiber in both p and $\tilde{p}$. If we were to build a bundle isomorphism fiberwise, by first identifying point strata of fibers (as indicated by the mappings on the right in Fig. 4.10) and then linearly interpolating these mappings on interval strata, we would end up with a *discontinuous* bundle isomorphism between p and $\tilde{p}$. ⌐

Our strategy to ensure continuity in our construction of $\kappa_{\tilde{p}}^{p}$ will be to use 'affine combinations' of maps on generic and special fibers whenever we are getting 'critically close' to special fibers (this will be referred to as fibers in the 'critical region' of a stratum), and use simple linear interpolation otherwise.

4.2.2.2 Constructing critical regions We start with the construction of 'critical regions' in the base stratification (B, g), over which fiberwise maps will be defined as certain affine combinations of 'general' and 'special fiber maps' later on. Recall we assume the base stratification (B, g) to be an open regular cell stratification.
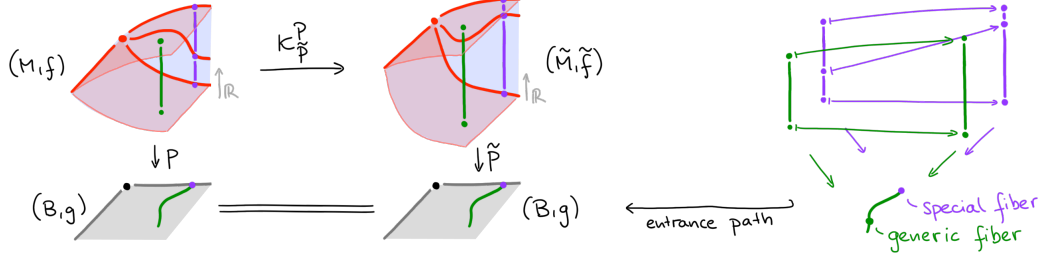
Figure 4.10: The isomorphism needed for the inductive step in the proof of conservativity of the entrance path truss functor generally cannot be constructed by simple fiberwise interpolation.

Terminology 4.2.29 (Interval contour). Let $p : (M, f) \to (B, g)$ be a closed 1-mesh bundle, and consider a regular stratum s of f lying over a stratum $r = p(s)$ of g. The 'interval contour under s', denoted by c_s, is the subset of $\partial r = \bar{r} \setminus r$, over which the restricted bundle $p|_{\bar{s}} : \bar{s} \to \bar{r}$ (see Observation 4.2.10) has interval fibers. ⌐

The complement of c_s in ∂r will also be called the 'point contour under s'. For the bundle p from Fig. 4.10, we highlight the point and interval contour of a chosen stratum s in Fig. 4.11 below
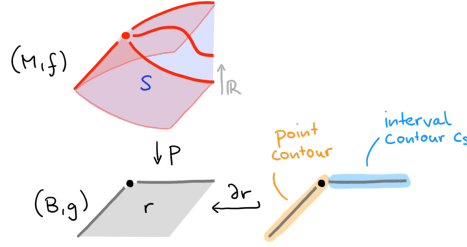

Figure 4.11: Point and interval contour.

Remark 4.2.30 (Contours only depend on truss structure). Importantly, if $\tilde{p} : (\tilde{M}, \tilde{f}) \to (B, g)$ is another closed 1-mesh bundle over (B, g) and $\mathsf{ETrs}\, p = \mathsf{ETrs}\, \tilde{p}$ (which in particular allows us to identify strata s of f with strata $\tilde{s}$ of $\tilde{f}$) then c_s and $c_{\tilde{s}}$ coincide as subspaces of B. ⌐

We will now thicken the interval contour by 'pushing it into' the interior of the base stratum r under s. This thickening will provide the 'critical region' for our later definition of $\kappa_{\tilde{p}}^p$. The construction relies on the base being *open* regular cell.

Construction 4.2.31 (Critical regions). For a closed 1-mesh bundle $p : (M, f) \to (B, g)$ with open regular cell base and a regular stratum s lying over r in f, we construct a 'critical region' $\mathsf{cr}_s \hookrightarrow r$ as follows. First, take Y to be a regular cell complex containing (B, g) as an open constructible substratification. Denote by $\bar{c}_s$ the closure of the interval contour in the boundary $S^{k-1} \cong \partial r$ of the open k-cell r in Y, and set $\partial c_s = \bar{c}_s \setminus c_s$. Now 'thicken' the interval contour's interior by defining $\overline{\mathsf{cr}}_s$ to be the quotient of $\bar{c}_s \times [0, 1]$ by $\partial c_s \times [0, 1]$. Define $\mathsf{cr}_s \hookrightarrow \overline{\mathsf{cr}}_s$ to be the subspace $c_s \times (0, 1)$. Form the gluing G_s by gluing $\bar{r} \cong D^n$ (where the closure is taken in Y) and cr_s along $\bar{c}_s \hookrightarrow \bar{r}$ resp. $\bar{c}_s \cong \bar{c}_s \times \{0\} \hookrightarrow \overline{\mathsf{cr}}_s$. A 'choice of critical region for s' is a choice of an isomorphism $\mathsf{R}_s : \mathsf{G}_s \cong \bar{r}$ which maps

180

$\bar{c}_s \cong \bar{c}_s \times \{1\} \hookrightarrow G_s$ identically to $\bar{c}_s \hookrightarrow \partial r$, and $\partial r \setminus c_s \hookrightarrow G_s$ identically to $\partial r \setminus c_s \hookrightarrow r$. Note such isomorphism can always be chosen since $c_s \hookrightarrow \partial r = S^{k-1}$ is an open subset. Finally, restricting $R_s : G_s \cong \bar{r}$ to cr_s yields the 'critical region' $cr_s \hookrightarrow G_s$ ⌟

For our previous example of p and choice of s in Fig. 4.11 we illustrate a choice of critical region R_s for s in Fig. 4.12.

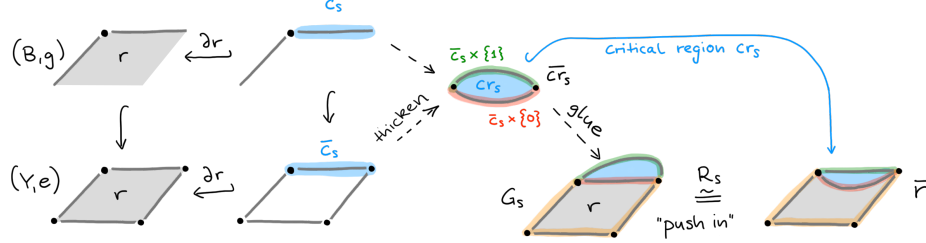Figure 4.12: Choosing a critical region in r for a regular stratum s.

Finally, note a valid choice of critical regions only depends on the entrance path truss structure of p, as recorded by the following remark.

Remark 4.2.32 (Critical regions only depend on 1-truss structure over the base stratification). Importantly, if $\tilde{p} : (\tilde{X}, \tilde{f}) \to (Y, g)$ is another closed 1-mesh bundle and $\mathsf{ETrs}(p) = \mathsf{ETrs}(\tilde{p})$ then any choice of critical region for p is also choice of critical regions for $\tilde{p}$. ⌟

4.2.2.3 Constructing the bundle isomorphism Having a notion of critical regions at hand, we can proceed to construct the bundle isomorphism $\kappa_{\tilde{p}}^p$, which will yield proofs of both Proposition 4.2.26 and Proposition 4.2.22.

Construction 4.2.33 (The bundle isomorphism κ). Consider closed 1-mesh bundles $p : (M, f) \to (B, g)$ and $\tilde{p} : (\tilde{M}, \tilde{f}) \to (B, g)$ with open regular cell base g and such that $\mathsf{ETrs}(p) = \mathsf{ETrs}(\tilde{p})$. Fix choices of critical regions R_s for each regular stratum in p (see Construction 4.2.31) which, by Remark 4.2.32 equally provides a choice of critical regions for $\tilde{p}$. We will define the bundle isomorphism $\kappa_{\tilde{p}}^p$ fiberwise by maps $\kappa_{\tilde{p}}^p(x, -) : p^{-1}(x) \to \tilde{p}^{-1}(x)$ over points $x \in r$ where r is a stratum in g.

We argue inductively in $\dim(r)$. If $\dim(r) = 0$ then $\kappa_{\tilde{p}}^p(x, -)$ is simply defined by mapping point strata of $p^{-1}(x)$ monotonicly to point strata of $\tilde{p}^{-1}(x)$ (note, since $\mathsf{ETrs}(p) = \mathsf{ETrs}(\tilde{p})$ both in particular have the same number of point strata) and extending this mapping linearly to interval strata. Next assume $\dim(r) > 0$. Again, we define $\kappa_{\tilde{p}}^p(x, -)$ is simply defined by mapping point strata of $p^{-1}(x)$ monotonicly to point strata of $\tilde{p}^{-1}(x)$ (note, since $\mathsf{ETrs}(p) = \mathsf{ETrs}(\tilde{p})$ both in particular have the same number of point strata). Note that interval strata s_x in $p^{-1}(x)$ are restrictions of regular strata s in (M, f), and correspond to interval strata $\tilde{s}_x$ in $p^{-1}(x)$ obtained by restricting the corresponding regular strata $\tilde{s}$ in $(\tilde{M}, \tilde{f})$. We define the restricted map $\kappa_{\tilde{p}}^p(x, -) : s_x \to \tilde{s}_x$. If $x \notin cr_s$ lies outside the choice of critical region for s, define $\kappa_{\tilde{p}}^p(x, -) : s_x \to \tilde{s}_x$ by linearly extending the mapping $\kappa_{\tilde{p}}^p(x, -)$ on the endpoints of the interval stratum s_x. Otherwise, if $x \in cr_s$, by Construction 4.2.31 we have $cr_s = c_s \times (0, 1)$ and can thus write

181

x as a pair $(u \in \mathsf{c}_s, t \in (0,1))$. This yields points $x_0 = \mathsf{R}_s(u,0) \in r$ and $x_1 = \mathsf{R}_s(u,1) \in \partial r$. Define $\kappa_{\tilde{p}}^p(x,-) : s_x \to \tilde{s}_x$ by the affine combination

$$\kappa_{\tilde{p}}^p(x,-) = t\kappa_{\tilde{p}}^p(x_1,-) + (1-t)\kappa_{\tilde{p}}^p(x_0,-)$$

where the first term on the right is defined since x_1 lies in a stratum of dimension lower than r, and the second term is defined since x_0 lies in r but outside the critical region. One checks that this continuously extends the domain of definition of $\kappa_{\tilde{p}}^p$ to all fibers over r, and thus completes our inductive construction of the bundle isomorphism $\kappa_{\tilde{p}}^p$. ⎯⎦

The construction of $\kappa_{\tilde{p}}^p$ now makes proofs of earlier statement immediate.

Proof of Proposition 4.2.26. The preceding construction defines $\kappa_{\tilde{p}}^p$ as required in the statement of Proposition 4.2.26 for closed 1-mesh bundles and open regular cell base. By Remark 4.2.24 (applied in the case $n = 1$) and Remark 4.2.27 this implies Proposition 4.2.26 in the general case. □

Proof of Proposition 4.2.22. By Remark 4.2.25, this follows from Proposition 4.2.26. □

4.2.2.4   Continuity of construction We mention two important properties of the construction of $\kappa_{\tilde{p}}^p$ in Construction 4.2.33. Firstly, the construction 'preserves identities' as recorded by the following remark.

Observation 4.2.34 (Constructing κ on identical bundles). Consider closed 1-mesh bundles $p : (M, f) \to (B, g)$ and $\tilde{p} : (\tilde{M}, \tilde{f}) \to (B, g)$ with open regular cell base g and such that $\mathsf{ETrs}(p) = \mathsf{ETrs}(\tilde{p})$. Identify M (resp. $\tilde{M}$) as a subspace of $B \times \mathbb{R}$ using its fiber 1-framing $\gamma : M \hookrightarrow B \times \mathbb{R}$ (resp. $\tilde{\gamma} : \tilde{M} \hookrightarrow B \times \mathbb{R}$). If (M, f) and $(\tilde{M}, \tilde{f})$ are identical as stratifications of subspaces of $B \times \mathbb{R}$ then the inductive construction of $\kappa_{\tilde{p}}^p$ (in Construction 4.2.33) outputs the bundle identity map $\mathrm{id} : p = \tilde{p}$. ⎯⎦

Secondly, the construction is 'continuous in families'. To make this precise, we first introduce a notion of families of 1-mesh bundles.

Definition 4.2.35 (Families of 1-mesh bundles)**.** For a connected topological space Z and an open regular cell stratification (B, g), a Z-**family** of closed 1-mesh bundles over g is a closed 1-mesh bundle $p : (M, f) \to Z \times (B, g)$. For $z \in Z$ denote by $p_z : (M_z, f_z) \to (B, g)$ the restriction of p to the subspace $B \cong \{z\} \times B \hookrightarrow Z \times B$, called the z-**slice** of p. ⎯⎦

Note, by Lemma 4.1.41, all slices p_z in a Z-family are open regular cell bundles.

Terminology 4.2.36 (ETrs-constant families). A Z-family of closed 1-mesh bundles $p : (M, f) \to Z \times (B, g)$ is call 'ETrs-constant' if the canonical inclusion $\mathsf{ETrs}(p_z) \hookrightarrow \mathsf{ETrs}(p)$ is a 1-truss bundle isomorphism for each $z \in Z$. ⎯⎦

Remark 4.2.37 (Constancy of entrance path truss bundle in families). Given a Z-family p for a path-connected space Z, then p is automatically ETrs-constant. ⎯⎦

The 'continuity in families' of our construction of κ now takes the following form.

Observation 4.2.38 (Constructing κ for families). For an open regular cell stratifications (B, g), consider ETrs-constant Z-families of closed 1-mesh bundles $p : (M, f) \to Z \times (B, g)$ and $\tilde{p} : (\tilde{M}, \tilde{f}) \to Z \times (B, g)$, such that $\mathsf{ETrs}(p) = \mathsf{ETrs}(\tilde{p})$. Pick any $z_0 \in Z$, and a chose critical regions for p_{z_0}. This choice equally provides critical regions to all bundles p_z and $\tilde{p}_z$ for all $z \in Z$, and we may thus construct the 1-mesh bundle isomorphisms $\kappa_{\tilde{p}_z}^{p_z} : p_z \cong \tilde{p}_z$ using Construction 4.2.33. The construction of these bundle isomorphisms is now 'continuous in $z \in Z$' in that we obtain a continuous bundle isomorphisms $\kappa_{\tilde{p}}^{p} : (M, f) \to (\tilde{M}, \tilde{f})$ whose restriction to $(M_z, f_z) \to (\tilde{M}_z, \tilde{f}_z)$ is defined to be $\kappa_{\tilde{p}_z}^{p_z}$. ⌙

Finally, let us mention a particular way of constructing ETrs-constant Z-families of 1-mesh bundles. Namely, ETrs-constant Z-families of 1-mesh bundles may be constructed by pullback along 'Entr-constant Z-families of stratified maps' as the following remark explains.

Remark 4.2.39 (Families of bundles from pullbacks along families of maps). Consider open regular cell stratifications (B, g) and $(B, \tilde{g})$ and let $F : Z \to \mathcal{S}trat(g, \tilde{g})$ be a continuous map from a space Z to the space of stratified maps between g and $\tilde{g}$, such that F is constant on entrance paths (that is, $\mathsf{Entr} \circ F : Z \to \mathcal{P}os(\mathsf{Entr}(g), \mathsf{Entr}(\tilde{g}))$ is constant). By Top-tensoredness of stratified spaces (see Observation B.2.23), we can consider F as a stratified map $F : Z \times g \to \tilde{g}$. Now, given a closed 1-mesh bundle $\tilde{p} : (\tilde{M}, \tilde{f}) \to (\tilde{B}, \tilde{g})$ we can construct a ETrs-constant Z-family of closed 1-mesh bundle as the pullback $F^*\tilde{p}$ of $\tilde{p}$ along F (see Construction 4.1.36). ⌙

4.2.3 Weak faithfulness of the entrance path truss functor

Our next goal will be to show that the entrance path truss functor ETrs is (in a weak sense) a 'faithful' functor of ∞-categories in the following sense.

Proposition 4.2.40 (Weak faithfulness of entrance path truss functor)**.** *Given two closed n-mesh bundles p and p' with open cellulable base (B, g), then hom space map*

$$\mathsf{ETrs} : \bar{\mathcal{M}}esh_n(B, g)(p, p') \to \bar{\mathsf{Trs}}_n(\mathsf{Entr}\, g)(\mathsf{ETrs}\, p, \mathsf{ETrs}\, p')$$

has empty or weakly contractible preimages. The same holds in the open-regular case.

The proof of Proposition 4.2.40 will take up the rest of this section. We will give the proof only in the case of closed n-meshes, noting that the case of open n-meshes is fully analogous (in fact, the proof for the closed case applies verbatim to the open case when replacing 'closed' by 'open' meshes resp. trusses, and 'singular' by 'regular' maps).

Remark 4.2.41 (Entrance path truss functor is 'weakly fully faithful'). Once we have constructed the weak inverse of ETrs it will follow that fibers of the hom space maps of ETrs are, in fact, never empty. ⌙

Remark 4.2.42 (More weak faithfulness). Similar to Observation 4.2.23, the proof of Proposition 4.2.40 may be adapted to show that the functors $\mathsf{ETrs} : \mathcal{M}esh_n^{\deg} \to \mathsf{Trs}_n^{\deg}$ and $\mathsf{ETrs} : \mathcal{M}esh_n^{\mathrm{crs}} \to \mathsf{Trs}_n^{\mathrm{crs}}$ are weakly faithful functors as well (the proof goes through without change in the closed resp. open case, and needs a little more care in the case of general meshes). ⌙

As a first step in the proof of Proposition 4.2.40, we observe that the statement only needs to be proven for open regular cell base (B, g).

Remark 4.2.43 (Reduction to open regular cell base). Fix a refinement $G : (B, c) \to (B, g)$ of (B, g) by an open regular cell stratification. Using Construction 4.1.67 we may pullback both p and p' to n-mesh bundles G^*p resp. G^*p' over (B, c). Let $F : \mathsf{ETrs}\, p \to \mathsf{ETrs}\, p'$ be a map of corresponding n-truss bundles. This pull backs to an n-truss bundle map $(\mathsf{Entr}\, G)^*F : (\mathsf{Entr}\, G)^* \mathsf{ETrs}\, p \to (\mathsf{Entr}\, G)^* \mathsf{ETrs}\, p'$ along $\mathsf{Entr}\, G : \mathsf{Entr}\, c \to \mathsf{Entr}\, g$ (see Construction 2.3.85). The the fiber of ETrs in $\bar{\mathcal{M}}esh_n(B, c)(G^*p, G^*p')$ over $(\mathsf{Entr}\, G)^*F$ is homeomorphic to the fiber of ETrs in $\bar{\mathcal{M}}esh_n(B, g)(p, p')$ over F. Thus it is sufficient to prove Proposition 4.2.40 in the case of open regular cell base. ⎯⎯⌋

Next, by an inductive argument, the statement of Proposition 4.2.40 may further be reduced to a statement about 1-mesh bundles. For $m \in \mathbb{N}$, denote by D^{m+1} the closed $(m+1)$-ball and by S^m its boundary. Recall, a topological space U is weakly contractible if for all maps $\zeta : S^m \to U$ there exists a map $\theta : D^{m+1} \to U$ such that $\theta|_{S^m} : S^m \to U$ equals ζ. We call θ a 'filler' for ζ.

Let p and p' be closed n-mesh bundles consisting of 1-mesh bundles $p_i : (M_i, f_i) \to (M_{i-1}, f_{i-1})$ resp. of $p'_i : (M'_i, f'_i) \to (M'_{i-1}, f'_{i-1})$ with $(M_0, f_0) = (B, g) = (M'_0, f'_0)$. Consider a map $\zeta : S^m \to \bar{\mathcal{M}}esh_n(B, g)(p, p')$ such that $\mathsf{ETrs} \circ \zeta$ is constant (in other words, ζ maps into a single fiber of ETrs). Note that, by rigidity of singular truss maps of closed trusses (see Lemma 2.3.104) this constancy condition is satisfied automatically except when $m = 0$. Recall that truncation of meshes is an ∞-functor (see Remark 4.1.66). Truncating ζ to degrees below n, we obtain the map $\zeta_{<n} : S^m \to \bar{\mathcal{M}}esh_{n-1}(M_{<n}, M'_{<n})$ which we denote by β. Arguing inductively, β has a filler which we denote by $\eta : D^{m+1} \to \bar{\mathcal{M}}esh_{n-1}(M_{<n}, M'_{<n})$. Using the Top-tensoredness of stratified spaces (see Observation B.2.23), we may consider ζ as a stratified map $S^m \times f_n \to f'_n$, β as a stratified map $S^m \times f_{n-1} \to f'_{n-1}$, and η as a stratified map $D^{m+1} \times f_{n-1} \to f'_{n-1}$. To show that ζ has a filler it will be sufficient to prove the following.

Proposition 4.2.44 (Lifting fillers in closed 1-mesh bundles). *Consider closed 1-mesh bundles $p : (M, f) \to (B, g)$ and $\tilde{p} : (\tilde{M}, \tilde{f}) \to (\tilde{B}, \tilde{g})$ with open regular cell base, and maps $\zeta : S^m \times f \to \tilde{f}$, $\beta : S^m \times g \to \tilde{g}$ such that, for each $e \in S^m$, $(\zeta(e, -), \beta(e, -)) : p \to \tilde{p}$ is a 1-mesh bundle map (if $m = 0$, further assume $\mathsf{ETrs}(\zeta(e, -), \beta(e, -))$ is independent of $e \in S^m$). Then any filler $\eta : D^{m+1} \times g \to \tilde{g}$ of β 'lifts' to a filler $\theta : D^{m+1} \times f \to \tilde{f}$ of ζ such that, for each $e \in D^{m+1}$, $(\theta(e, -), \eta(e, -)) : p \to \tilde{p}$ is a 1-mesh bundle map.*

It will be convenient to consider D^{m+1} as the quotient of $[0, 1] \times S^m$ by the subset $\{1\} \times S^m$. In particular, we will regard η as a map $\eta : [0, 1] \times S^m \times g \to \tilde{g}$ such that $\eta(1, -)$ is constant.

To construct the filler θ of ζ that lifts the filler η of β as claimed in Proposition 4.2.44, we will proceed in two steps. In the first step, by 'pulling back along η', we construct 'homotopy 1' $\theta_1 : [0, 1] \times S^m \times f \to \tilde{f}$ which homotopes $\theta_1(0, -) = \zeta$ into a map $\theta_1(1, -) : S^m \times f \to \tilde{f}$ that descends to a map of base stratifications $S^m \times g \to \tilde{g}$ that is constant in S^m. The second step, using 'fiberwise contractions', constructs 'homotopy 2' $\theta_2 : [0, 1] \times S^m \times f \to \tilde{f}$, which homotopes $\theta_2(0, -) = \theta_1(1, -)$ into a map $\theta_2(1, -) : S^m \times f \to \tilde{f}$ that is constant on fibers as well and thus fully constant in S^m. Concatenating the homotopies θ_1 and θ_2 will provide us with the required filler θ of ζ.

Proof of Proposition 4.2.44. Define a closed 1-mesh bundle $\widehat{p}$ by pulling back p along β and define a map $\widehat{\zeta}$ as the factorization of ζ through this pullback as shown below.

$$
\begin{array}{ccccc}
S^m \times f & \xrightarrow{\widehat{\zeta}} & \beta^*\tilde{f} & \xrightarrow{\quad} & \tilde{f} \\
{\scriptstyle S^m \times p}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{p}} & \lrcorner & \big\downarrow{\scriptstyle \tilde{p}} \\
S^m \times g & \xjustequal{\text{id}} & S^m \times g & \xrightarrow{\beta} & \tilde{g}
\end{array}
$$

with the arc ζ from $S^m \times f$ to $\tilde{f}$.

Note that $\widehat{p}$ is an ETrs-constant S^m-family of closed 1-mesh bundles over g (see Remark 4.2.39). We may trivially turn this into a ETrs-constant $([0,1] \times S^m)$-family by taking the product $[0,1] \times -$: the resulting closed 1-mesh bundle $[0,1] \times \widehat{p}$ is bundle isomorphic to the closed 1-mesh bundle $\breve{p}$ defined by the pullback on the right below.

$$
\begin{array}{ccccc}
[0,1] \times \beta^*\tilde{f} & \xrightarrow[\sim]{\kappa} & \eta^*\tilde{f} & \xrightarrow{\quad} & \tilde{f} \\
{\scriptstyle [0,1]\times\widehat{p}}\big\downarrow & & \big\downarrow{\scriptstyle \breve{p}} & \lrcorner & \big\downarrow{\scriptstyle \tilde{p}} \\
[0,1] \times S^m \times g & \xrightarrow{\text{id}} & [0,1] \times S^m \times g & \xrightarrow{\eta} & \tilde{g}
\end{array}
$$

Indeed, κ can be constructed using Observation 4.2.38 since g is assumed open regular cell. Our 'homotopy 1' map $\theta_1 : [0,1] \times S^m \times f \to \tilde{f}$ is now simply defined as the composite

$$
[0,1] \times S^m \times f \xrightarrow{[0,1]\times\widehat{\zeta}} [0,1] \times \beta^*\tilde{f} \xrightarrow{\kappa} \eta^*\tilde{f} \to \tilde{f} \quad .
$$

Since the κ construction preserves identities (see Observation 4.2.34) and since $\beta = \eta(0,-)$, we find that $\kappa(0,-)$ is the identity on $\beta^*\tilde{f}$. Thus, homotopy 1 satisfies $\theta_1(0,-) = \zeta$, and 'lifts ' η in the sense that

$$
\begin{array}{ccc}
[0,1] \times S^m \times f & \xrightarrow{\theta_1} & \tilde{f} \\
{\scriptstyle [0,1]\times S^m\times p}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{p}} \\
[0,1] \times S^m \times g & \xrightarrow{\eta} & \tilde{g}
\end{array}
$$

This completes the construction of the first part of the homotopy.

It remains to construct the 'homotopy 2' map $\theta_2 : [0,1] \times S^m \times f \to g$ such that $\theta_2(0,-) = \theta_1(1,-)$. We may construct the homotopy θ_2 'fiberwise', by convexly combining fiberwise maps $\theta_1(1,e,x) : (p^{-1}(y), f) \to (\tilde{p}^{-1}\eta(1,e,y), \tilde{f})$ on fiber over points $y \in B$ (note that $\eta(1,e,y)$ is in fact independent of $e \in S^m$). Namely, pick any $e_0 \in S^m$, and, for $t \in [0,1]$, $e \in S^m$ and $y \in B$, define the restriction of $\theta_2(t,e,-)$ to the fiber over y to be the map

$$
\begin{aligned}
\theta_2(t,e,-) : (p^{-1}(y), f) &\to (\tilde{p}^{-1}\eta(1,e,y), \tilde{f}) \\
x &\mapsto (1-t) \cdot \theta_1(1,e,x) + t \cdot \theta_1(1,e_0,x)
\end{aligned}
$$

Note that $\theta_2(t,e,-)$ is indeed a 1-mesh map for all t and e since ETrs $\theta_1(1,e,x) = $ ETrs $\theta_1(1,e_0,x)$ induce the same maps on 1-trusses (which in turn follows since, by assumption, ETrs$(\zeta(e,-), \beta(e,-))$ is independent of $e \in S^m$). Note also at $t = 1$, the map

185

$\theta_2(t, e, -)$ becomes independent of $e \in S^m$. We can chain the homotopies θ_1, θ_2 into a single homotopy

$$\theta := \theta_1 * \theta_2 : [0, 1] \times S^m \times f \to \tilde{f}$$

which defines the required filler θ of ζ lifting the filler η of β as required. $\square$

Proof of Proposition 4.2.40. By Remark 4.2.43, the statement of Proposition 4.2.40 only needs to be proven for open regular cell base. As we've seen, this inductively reduces to the statement of Proposition 4.2.44. $\square$

4.2.4 Classifying meshes Our goal in this section will be the construction of a weak inverse to the entrance path truss functor $\mathsf{ETrs} : \mathcal{M}esh_n(B, g) \to \mathcal{T}rs_n(\mathsf{Entr}\, g)$ constructed in the Section 4.2.1. We will construct this functor in the following instances. Firstly, in the unbundled case $(B, g) = *$, we will construct the 'classifying mesh functor'.

$$\mathsf{CMsh} : \mathsf{Trs}_n \to \mathcal{M}esh_n$$

Secondly, for fixed open cellulable base (B, g) we will construct the 'classifying mesh bundle functor'

$$\mathsf{CMsh} : \mathsf{Trs}_n(\mathsf{Entr}\, g) \to \mathcal{M}esh_n(B, g)$$

Finally, we will also discuss how the realization of truss coarsenings as mesh coarsening requires special attention.

We will start our construction with the case of closed trusses: this case is particularly easy since we will obtain the classifying mesh construction CMsh as a direct analog of the classifying stratification construction CStr. In contrast, in the case of general trusses (and their bundles), more care needs to be taken to ensure that dimensions of strata do not incorrectly 'degenerate'. To illustrate this difficulty in constructing classifying meshes of general trusses consider the following basic case. The classifying stratification of the poset underlying the closed 1-truss with one element is a point: this thus correctly produces the underlying space of the closed 1-mesh with one stratum. However, the classifying stratification of the poset underlying the *open* 1-truss with one element is again a point, and this does not equal the underlying space of the *open* 1-mesh with one stratum (which is an open interval). As we will see, the problem of constructing classifying meshes of general trusses may be reduced to the case of closed trusses, by first 'compactifying' trusses, and then applying the classifying mesh functor for closed trusses.

4.2.4.1 Classifying meshes of closed trusses We construct classifying meshes of closed n-trusses. We start by discussing the case of closed 1-truss bundles. Recall the construction of classifying stratifications of posets (Terminology 1.3.11 and Terminology 1.3.12).

Construction 4.2.45 (Classifying 1-mesh bundles of closed 1-truss bundles)**.** Given a closed 1-truss bundle $p : T \to X$, we endow the classifying stratified map $\mathsf{CStr}\, p : \mathsf{CStr}\, T \to \mathsf{CStr}\, X$ with the structure of a closed 1-mesh bundle, yielding the 'classifying 1-mesh bundle' $\mathsf{CMsh}\, p$ of p. Recall, the classifying stratification $\mathsf{CStr}\, T$ stratifies the geometric realization $|T|$. We need to define a fiber 1-framing $\gamma : |T| \hookrightarrow |X| \times \mathbb{R}$ for $\mathsf{CStr}\, p$. Order-preservingly identify objects in the (frame ordered) fiber $(p^{-1}(x), \preceq)$ over $x \in X$ with objects i in a total order $[m_x] = (0 \to ... \to m_x)$. Since objects in T

186

correspond to vertices in $|T|$, we then define γ to map vertices $i \in |p|^{-1}(x) \subset |T|$ to vertices $(x, i) \in |X| \times \mathbb{N} \hookrightarrow |X| \times \mathbb{R}$, and further extend this mapping on vertices linearly to simplices in $|T|$. Using truss induction (see Section 2.2) one verifies that γ is indeed an embedding such that the bounding maps $\gamma^{\pm} : |X| \to |X| \times \mathbb{R}$ are continuous. This endows $\mathsf{CStr}\, p$ with the structure of a closed 1-mesh bundle. ⌐

Example 4.2.46 (Classifying 1-mesh bundles of 1-truss bundles)**.** In Fig. 4.13 we depict a closed 1-truss bundle $p : T \to X$ on the left (note that we only depict generating arrows, see Construction 2.3.69), and on the right its classifying 1-mesh bundle $\mathsf{CMsh}\, p : \mathsf{CStr}\, T \to \mathsf{CStr}\, X$. ⌐
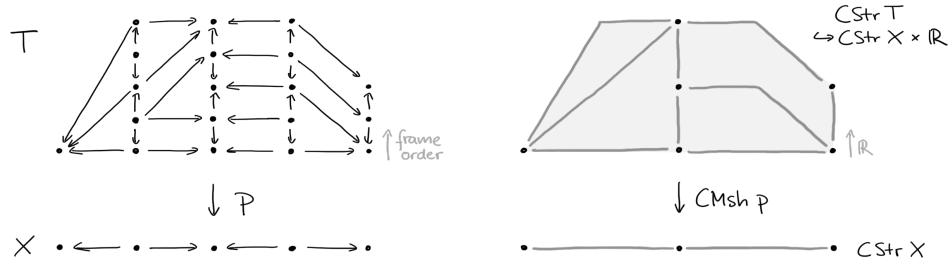

Figure 4.13: Classifying 1-mesh bundle of a 1-truss bundle.

Definition 4.2.47 (Classifying meshes of closed n-trusses)**.** Given a closed n-truss T, consisting of 1-truss bundle $p_i : T_i \to T_{i-1}$, its **classifying mesh** $\mathsf{CMsh}\, T$ is the closed n-mesh defined by the tower of 1-mesh bundles

$$\mathsf{CStr}\, T_n \xrightarrow{\mathsf{CStr}\, p_n} \mathsf{CStr}\, T_{n-1} \xrightarrow{\mathsf{CStr}\, p_{n-1}} \dots \mathsf{CStr}\, T_1 \xrightarrow{\mathsf{CStr}\, p_1} \mathsf{CStr}\, T_0$$

where $\mathsf{CMsh}(p_i)$ is the classifying 1-mesh bundle of p_i (see Construction 4.2.45). ⌐

The definition extends to maps of closed trusses as follows.

Definition 4.2.48 (Classifying mesh maps of closed n-truss maps)**.** Given a map of closed n-trusses $F : T \to S$ with components $F_i : T_i \to S_i$, the **classifying mesh map** $\mathsf{CMsh}\, F : \mathsf{CMsh}\, T \to \mathsf{CMsh}\, S$ is the n-mesh map with components given by classifying stratified maps $\mathsf{CStr}\, F_i : \mathsf{CStr}\, T_i \to \mathsf{CStr}\, S_i$. ⌐

Notation 4.2.49 (Classifying mesh functor of closed trusses)**.** The previous definitions assemble into the partial functor $\mathsf{CMsh} : \mathsf{Trs}_n \to \mathcal{M}esh_n$ on the full subcategory of closed n-trusses called the 'classifying mesh functor'. ⌐

Our earlier construction of entrance path trusses is right inverse to the classifying mesh functor as follows.

Observation 4.2.50 (Entrance path trusses of classifying meshes)**.** Unwinding definitions, one checks that the composite $\mathsf{ETrs} \circ \mathsf{CMsh}\, T$ is the identity on the category of closed n-trusses (up to a unique natural isomorphism). ⌐

4.2.4.2 Classifying meshes of general trusses We now turn to the construction of classifying meshes of general n-trusses. The construction needs care because, as discussed, the 'naive' geometric realization of posets (by passing to their nerve, and using the usual geometric realization of simplicial sets) does not dualize dimensions correctly.

The correct construction of classifying meshes will reduce the case general trusses to that of closed trusses by first 'compactifying' general trusses to obtain closed trusses. The basic idea for truss compactifications is a combinatorial analog to fiberwise compactifications of 1-mesh bundles that were defined in Construction 4.1.34. However, when employing fiberwise compactifications 'inductively' to a *tower* of bundles, we are left with a choice on how to extend bundles to compactifications of their base. This leads to several possible notions of compactifications. We mention the following universal choice, which we call 'cubical compactification' of a truss, that admits a useful explicit construction. Rather than working only with trusses, we will define cubical compactifications in the more general setting of truss bundles.

Terminology 4.2.51 (Dense subposet and truss subbundles). A subposet $P \hookrightarrow Q$ is called 'dense' if the upward closure of P is all of Q. A base preserving truss subbundle $F : T \hookrightarrow S$ is similarly called dense if all i-level maps F_i are dense (recall, 'base preserving' means $F_0 = \mathrm{id}$). ⌟

Definition 4.2.52 (Retractable compactifications)**.** Given an n-truss bundle p over a poset X, a **retractable compactification** of p is pair of regular n-truss bundle maps $F : p \rightleftarrows q : G$, where q is closed, F is a dense subtruss bundle map, and $G \circ F = \mathrm{id}_p$. ⌟

Definition 4.2.53 (Cubical compactification)**.** Given an n-truss bundle p over a poset X, its **cubical compactification** $\mathsf{ci} : p \rightleftarrows \overline{p} : \mathsf{cr}$ (sometime simply called 'compactification') is the universal retractable compactification, satisfying the following universal property: for any other retractable compactification $F : p \rightleftarrows q : G$ there is a unique n-truss bundle r over $X \times [1]$, with $r|_{X \times \{0\}} = \overline{p}$ and $r|_{X \times \{1\}} = q$ which on the images of ci in $\overline{p}$ and i in q restricts to the product bundle $p \times [1]$. ⌟

Example 4.2.54 (Cubical compactifications)**.** In Fig. 4.14 we indicate the inclusion $\mathsf{ci} : T \hookrightarrow \overline{T}$ of an open 2-truss T (in black) into its cubical compactification $\overline{T}$ (extending T by the red structure). ⌟
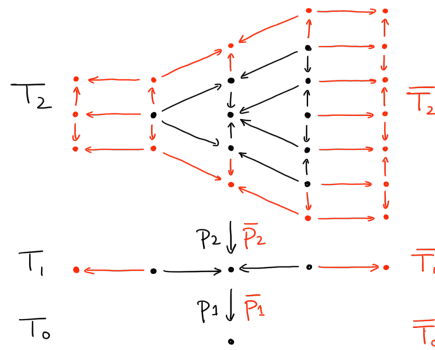

Figure 4.14: Cubical compactifications of two 2-trusses.

The fact that cubical compactifications always exist is established by the following sequence of constructions, which explicitly construct cubical compactification with increasing generality first for 1-trusses, then for 1-truss bundles, and finally for n-truss bundles.

Construction 4.2.55 (Cubical compactifications of 1-trusses)**.** Given a 1-truss T, its cubical compactification $\mathsf{ci} : T \rightleftarrows \overline{T} : \mathsf{cr}$ is the unique closed 1-truss containing T, which satisfies the conditions in Definition 4.2.53 (in the case $n = 1$ and $X = [0]$): explicitly, $\overline{T}$ is obtained from T by adjoining a new upper (resp. lower) singular endpoint if the upper (resp. lower) endpoint of T is regular. ⌐

Construction 4.2.56 (Cubical compactifications of 1-truss bundles)**.** Given a 1-truss bundle $p : T \to X$, its cubical compactification $\mathsf{ci} : p \rightleftarrows \overline{p} : \mathsf{cr}$ is the unique closed 1-truss bundle containing T and satisfying the conditions in Definition 4.2.53 (in the case $n = 1$): explicitly, $\overline{p} : \overline{T} \to X$ is obtained from p by compactifying each fiber of p using Construction 4.2.55, and then extending truss bordisms to compactified fibers in the unique endpoint preserving way. ⌐

Construction 4.2.57 (Cubical compactification for n-truss bundles)**.** Let p be an n-truss bundle over a base poset X with bundle maps $p_i : T_i \to T_{i-1}$. Inductively construct its $(n-1)$-truncated compactification $\mathsf{ci} : p_{<n} \rightleftarrows \overline{p}_{<n} : \mathsf{cr}$ (the case $n = 1$ is Construction 4.2.56).

First pull back $p_n : T_n \to T_{n-1}$ along $\mathsf{cr}_{n-1} : \overline{T}_{n-1} \to T_{n-1}$ to obtain a 1-truss bundle $\mathsf{cr}_{n-1}^* p_n$ over $\overline{T}_{n-1}$. Pulling back again, now by ci_{n-1}, recovers p_n, that is, $\mathsf{ci}_{n-1}^* \mathsf{cr}_{n-1}^* p_n = p_n$. We thus have two 1-truss bundle maps: the 1-truss subbundle $\mathrm{Tot}(\mathsf{ci}_{n-1}) : p_n \hookrightarrow \mathsf{cr}_{n-1}^* p_n$ and the 1-truss bundle map $\mathrm{Tot}(\mathsf{cr}_{n-1}) : \mathsf{cr}_{n-1}^* p_n \to p_n$. The two 1-truss bundle maps form an inclusion-retraction pair $p_n \rightleftarrows \mathsf{cr}_{n-1}^* p_n$. Composing this pair with the inclusion-retraction pair $\mathsf{cr}_{n-1}^* p_n \rightleftarrows \overline{\mathsf{cr}_{n-1}^* p_n}$ constructed using Construction 4.2.56, then defines the inclusion-retraction pair $\mathsf{ci}_n : p_n \rightleftarrows \overline{p}_n : \mathsf{cr}_n$ (where $\overline{p}_n := \overline{\mathsf{cr}_{n-1}^* p_n}$).

We define the cubical compactification $\overline{p}$ to be the truss bundle obtained by augmenting $\overline{p}_{<n}$ with the bundle map $\overline{p}_n$, and obtain an inclusion-retraction pair $\mathsf{ci} : p \rightleftarrows \overline{p} : \mathsf{cr}$ as required. ⌐

Remark 4.2.58 (Universal property of cubical stratifications)**.** The fact that the above construction satisfies the universal property stated in Definition 4.2.53 can be verified in a similar inductive fashion. We give a sketch. Given a cubical compactification $\mathsf{ci} : p \rightleftarrows \overline{p} : \mathsf{cr}$ as constructed above, consider a retractable compactification $i : p \rightleftarrows q : r$. Arguing inductively to have defined the $(n-1)$-truncation $r_{<n}$ of the 'factorizing' bundle r over $X \times [1]$. As an inductive hypothesis assume that $(0, x) \to (1, y)$ in the total poset of $r_{<n}$ then $\mathsf{cr}(x) \to r(y)$ in the total poset of $p_{<n}$. We now construct r from $r_{<n}$ by adding the 1-truss bundle r_n, defined by setting $r_n|_{(0,x)\to(1,y)} = q_n|_{r(y)\to y} \circ \overline{p}_n|_{\mathsf{cr}(x)\to r(y)}$. One verifies that this indeed defines a 1-truss bundle, which further satisfies the inductive hypothesis. ⌐

We now give a definition of classifying meshes of general trusses. Recall the notion of 'submeshes' of n-meshes from Terminology 4.1.59 (which are mesh inclusions the n-framing of the codomain restricts to the n-framing of the domain). We will also use the notion of 'constructible substratifications' (see Definition B.2.9): a substratification $F : (Y, g) \hookrightarrow (X, f)$ is constructible if $Y = f^{-1}(Q)$ for some subposet $Q \hookrightarrow \mathsf{Entr}(f)$ of the

entrance path poset of f. Constructible substratifications F are thus fully determined by their entrance path poset maps $\mathsf{Entr}\, F$.

Definition 4.2.59 (Classifying meshes of general n-trusses)**.** For an n-truss T, the **classifying n-mesh** $\mathsf{CMsh}\, T$ of T is the submesh $\mathsf{CMsh}\, \mathsf{ci} : \mathsf{CMsh}\, T \hookrightarrow \mathsf{CMsh}\, \overline{T}$ given by constructible substratifications $\mathsf{CMsh}\, \mathsf{ci}_i : \mathsf{CMsh}\, T_i \hookrightarrow \mathsf{CMsh}\, \overline{T}_i$ with entrance path poset map $\mathsf{Entr}\, \mathsf{CMsh}\, \mathsf{ci}_i = \mathsf{ci}_i : T_i \hookrightarrow \overline{T}_i$. ⌟

(The verification that $\mathsf{CMsh}\, T$ indeed defines an n-mesh proceeds inductively, using the inductive construction of cubical compactifications.)

Note that if T is closed, then the preceding definition of $\mathsf{CMsh}\, T$ specializes to our earlier Definition 4.2.47.

Example 4.2.60 (Classifying meshes of general n-trusses)**.** Recall the open 2-truss T from Fig. 4.14. In Fig. 4.15 we depict the closed classifying mesh $\mathsf{CMsh}\, \overline{T}$ of the compactification $\overline{T}$ of T, together with the resulting open classifying mesh $\mathsf{CMsh}\, T$ of T. The latter mesh includes into $\mathsf{CMsh}\, \overline{T}$ as a submesh $\mathsf{CMsh}\, \mathsf{ci} : \mathsf{CMsh}\, T \hookrightarrow \mathsf{CMsh}\, \overline{T}$ as indicated. ⌟
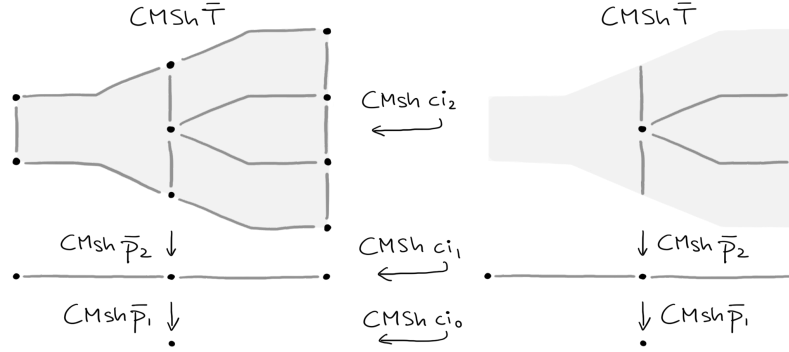

Figure 4.15: The classifying mesh of an open 2-truss constructed via the classifying mesh of its compactification.

Given an n-truss T, then classifying stratifications $\mathsf{CStr}\, T_k$ and classifying meshes $\mathsf{CMsh}\, T_k$ are in general distinct stratifications (the simplest example being obtained by taking T to be the open 1-truss with one element). However, the former stratification canonically includes into the latter stratification as follows.

Remark 4.2.61 (Classifying meshes vs. classifying stratifications)**.** For an n-truss T, the classifying stratified map $\mathsf{CStr}\, \mathsf{ci}_k : \mathsf{CStr}\, T_k \hookrightarrow \mathsf{CStr}\, \overline{T}_k = \mathsf{CMsh}\, \overline{T}_k$ is a substratification that factors through the constructible substratification $\mathsf{CMsh}\, T_k \hookrightarrow \mathsf{CMsh}\, \overline{T}_k$ by a substratification $\mathsf{CStr}\, T_k \hookrightarrow \mathsf{CMsh}\, T_k$. Conversely, there is a retraction

$$\mathsf{CMsh}\, T_k \twoheadrightarrow \mathsf{CStr}\, T_k = \left(\mathsf{CMsh}\, T_k \xrightarrow{\mathsf{CMsh}\, \mathsf{ci}} \mathsf{CMsh}\, \overline{T}_k = \mathsf{CStr}\, \overline{T}_k \xrightarrow{\mathsf{CStr}\, \mathsf{cr}_k} \mathsf{CStr}\, T_k \right).$$

Precomposed with the inclusion $\mathsf{CStr}\, T_k \hookrightarrow \mathsf{CMsh}\, T_k$ this is the identity on $\mathsf{CStr}\, T_k$. ⌟

It remains to define classifying mesh maps for general truss maps.

Definition 4.2.62 (Classifying mesh maps)**.** Given a map of n-trusses $F : T \to S$ with components $F_i : T_i \to S_i$, its **classifying mesh map** $\mathsf{CMsh}\, F : \mathsf{CMsh}\, T \to \mathsf{CMsh}\, S$ is the n-mesh map with components $\mathsf{CMsh}\, F_i$ defined by the composite

$$\mathsf{CMsh}\, T_k \twoheadrightarrow \mathsf{CStr}\, T_k \xrightarrow{\mathsf{CStr}\, F_k} \mathsf{CStr}\, S_k \hookrightarrow \mathsf{CMsh}\, S_k$$

where the first and last map are the canonical retraction and inclusion maps from Remark 4.2.61. One checks that the association $F \mapsto \mathsf{CMsh}\, F$ is functorial as required.
⌐

Note that $\mathsf{CMsh}\, F$ is a singular resp. regular n-mesh map whenever F is a singular resp. regular n-truss map.

Notation 4.2.63 (Classifying mesh functor)**.** Definition 4.2.59 and Definition 4.2.62 together yield the 'classifying mesh functor'

$$\mathsf{CMsh} : \mathsf{Trs}_n \to \mathcal{M}esh_n$$

from the category of trusses to the category of n-meshes. ⌐

Observation 4.2.64 (CMsh as a right inverse)**.** Unwinding definitions, one checks that $\mathsf{ETrs} \circ \mathsf{CMsh} \cong \mathrm{id}_{\mathsf{Trs}_n}$ by a unique natural isomorphism, and thus CMsh is 'right inverse' to ETrs. ⌐

4.2.4.3 Classifying mesh bundles over cellulable base We now turn to the case of truss bundles over open cellulable base.

Terminology 4.2.65 (Cellulation triples)**.** Given an open cellulable stratification (B, g), a 'cellulation triple' (g, c, X) is a refinement of (B, g) by a open regular cell stratification (B, c), together with a cellular poset X such that (B, c) is a open dense constructible substratification of the regular cell complex $\mathsf{CStr}\, X$ (that is, $(B, c) \hookrightarrow \mathsf{CStr}\, X$ is a constructible substratification and $B \hookrightarrow |X|$ is an open subset satisfying $\overline{B} = |X|$). ⌐

Our constructions of the classifying mesh bundle functor will depend on a choice of cellulation tuple for the base stratification. We first discussed the closed case.

Construction 4.2.66 (Closed classifying mesh bundles)**.** We construct the functor

$$\mathsf{CMsh} : \bar{\mathsf{T}}\mathsf{rs}_n(\mathsf{Entr}\, g) \to \bar{\mathcal{M}}esh_n(B, g)$$

for open cellulable (B, g) and a given cellulation triple (g, c, X).

We start with the case $n = 1$, defining CMsh on objects. Let $p : T \to B$ be a closed 1-truss bundle over $\mathsf{Entr}\, g$. Since the inclusion $B \hookrightarrow |X|$ is open, note that $\mathsf{Entr}\, c \hookrightarrow X$ is an open subposet. Define the 'cellulation' $q : Y \to X$ of p to be the closed 1-truss bundle over X, whose classifying functor $\chi_q : X \to \mathsf{TBord}^1$ restricts on $\mathsf{Entr}\, c$ to the composite $\chi_p \circ \mathsf{Entr}(c \to g)$, and otherwise maps to the terminal 1-truss in $\bar{\mathbb{T}}_0^n$ (see Observation 2.1.46). Using Construction 4.2.45, we construct the 1-mesh bundle $\mathsf{CMsh}\, q : \mathsf{CStr}\, Y \to \mathsf{CStr}\, X$. Restricting this along the inclusion $c \hookrightarrow \mathsf{CStr}\, X$ yields the 1-mesh bundle $\mathsf{CMsh}\, q|_c : (M, d) \to (B, c)$. The composite of $\mathsf{CMsh}\, q|_c$ and the coarsening $(B, c) \to (B, g)$ now factors uniquely through a coarsening $(M, d) \to (M, f)$ and a 1-mesh bundle $\mathsf{CMsh}\, p : (M, f) \to (B, g)$ (whose fiber 1-framing is inherited from that of $\mathsf{CMsh}\, q|_c$).

191

This constructs the 'classifying mesh bundle' $\mathsf{CMsh}\,p$ of p. Note, moreover, we obtain a canonical choice of cellulation triple (f, d, Y) for (M, f).

Next we consider the action of CMsh on 1-truss bundles maps for the following (slightly more general) case. Let (B', g') be another cellulable base with cellulation triple (g', c', X'). A 'cellulation triple map' $(E, G, \tilde{G})$ is a triple of poset maps fitting into the following diagram:

$$
\begin{array}{ccccc}
\mathsf{Entr}\,g & \xleftarrow{\mathsf{Entr}(g \to c)} & \mathsf{Entr}\,c & \longrightarrow & X \\
{\scriptstyle E}\downarrow & & \downarrow{\scriptstyle G} & & \downarrow{\scriptstyle \tilde{G}} \\
\mathsf{Entr}\,g' & \xleftarrow[\mathsf{Entr}(g' \to c')]{} & \mathsf{Entr}\,c' & \longrightarrow & X'
\end{array}
$$

Given any 1-truss bundle map $F : p \to p'$ over E, there now exist unique 1-truss bundle maps $H : \mathsf{Entr}(g \to c)^* p \to \mathsf{Entr}(g' \to c')^* p'$ and $\tilde{H} : q \to q'$ over G resp. $\tilde{G}$, such that the following commutes:

$$
\begin{array}{ccccc}
p & \longleftarrow & \mathsf{Entr}(g \to c)^* p & \longrightarrow & q \\
{\scriptstyle F}\downarrow & & \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle \tilde{H}} \\
p' & \longleftarrow & \mathsf{Entr}(g' \to c')^* p' & \longrightarrow & q'
\end{array}
$$

Construct $\mathsf{CMsh}\,p : (M, f) \to (B, g)$ and $\mathsf{CMsh}\,p' : (M', f') \to (B', g')$; denote canonical cellulation triples of (M, f) and (M', f') by (f, d, Y) resp. (f', d', Y'). Define the 1-mesh bundle $\mathsf{CMsh}\,\tilde{H} : \mathsf{CMsh}\,q \to \mathsf{CMsh}\,q'$ as the stratified map $\mathsf{CStr}\,\tilde{H} : \mathsf{CStr}\,Y \to \mathsf{CStr}\,Y'$. Restrict this map along $d \hookrightarrow \mathsf{CStr}\,Y$ resp. $d' \in \mathsf{CStr}\,Y'$ to obtain $\mathsf{CMsh}\,H : d \to d'$. The latter map further factors through the coarsenings $d \to f$ and $d' \to f'$. This yields the 'classifying 1-mesh map' $\mathsf{CMsh}\,F : \mathsf{CMsh}\,p \to \mathsf{CMsh}\,p'$. Note that $(F, H, \tilde{H})$ now provides a canonical cellulation triple map for (f, d, Y) and (f', d', Y').

Finally, we consider the case of general n. For an n-truss bundle $p = (p_n, p_{n-1}, ..., p_1)$ over $\mathsf{Entr}\,g$, using the classifying mesh bundle construction for 1-truss bundles, we construct construct the classifying n-mesh bundle $\mathsf{CMsh}\,p$ as the tower of bundles $\mathsf{CMsh}\,p_i$ (using the canonical cellulation triple output by the construction of $\mathsf{CMsh}\,p_{i-1}$ for the construction of $\mathsf{CMsh}\,p_i$). For an n-mesh bundle map $F : p \to p'$ over fixed base (B, g), we similarly construct $\mathsf{CMsh}\,F : \mathsf{CMsh}\,p \to \mathsf{CMsh}\,p'$ by inductively constructing $\mathsf{CMsh}\,F_i$ (using the canonical cellulation triple map from the construction of $\mathsf{CMsh}\,p_{i-1}$ for the construction of $\mathsf{CMsh}\,p_i$; the first such triple is $(\mathrm{id}_{\mathsf{Entr}\,g}, \mathrm{id}_{\mathsf{Entr}\,c}, \mathrm{id}_X)$. $\quad\quad\quad\sqcup\!\!\!\!\!\!\!\lrcorner$

Note that the previous construction also has a concise non-inductive description. For open cellulable base (B, g) and cellulation triple (g, c, X) as before, given a (closed) n-truss bundle p over $\mathsf{Entr}\,g$ define its 'cellulation' q to be the n-truss bundle with classifying map $\chi_q : X \to \mathsf{TBord}^n$, that restrict on $\mathsf{Entr}\,c$ to the composite $\mathsf{Entr}(c \to g) \circ \chi_p$, and maps to the terminal n-truss $\bar{\mathbb{T}}_0^n$ otherwise (see Observation 2.3.83). Then the n-mesh bundle $\mathsf{CMsh}\,q$ over $\mathsf{CStr}\,X$, defined inductively using Construction 4.2.45, pulls back along $c \hookrightarrow \mathsf{CStr}\,X$ to $\mathsf{CMsh}\,q|_c$, which coarsening along $c \to g$ to yield $\mathsf{CMsh}\,p$.

Construction 4.2.67 (General classifying mesh bundles)**.** We construct the functor

$$
\mathsf{CMsh} : \mathsf{Trs}_n(\mathsf{Entr}\,g) \to \mathcal{M}esh_n(B, g)
$$

for open cellulable (B, g) and a chosen cellulation triple (g, c, X).

We first construct CMsh on objects. Let p be an n-truss bundle over Entr g. Pass to its compactification $\bar{p}$ (see Definition 4.2.53). (Note that the cellulation $\bar{q}$ of $\bar{p}$ is the compactification of the cellulation q of p.) Construct the closed n-mesh bundle CMsh $\bar{p}$ using Construction 4.2.66. Then pass to a constructible substratifications as determined by the inclusion $p \hookrightarrow \bar{p}$ to define the classifying n-mesh bundle CMsh p.

We next construct CMsh on morphisms. Let $F : p \to p'$ be a base preserving n-truss bundle map over Entr g. Note that F canonically induces a map $G : q \to q'$ (where q and q' are the cellulations of p resp. p'). We may now define CMsh G : CMsh $q \to$ CMsh q' following our previous idea (from Definition 4.2.62), namely by (level-wise) composing the retraction CMsh $q \twoheadrightarrow$ CStr q, with the stratified realization CStr $q \xrightarrow{\text{CStr } G}$ CStr q', with the inclusion CStr $q' \hookrightarrow$ CMsh q'. This map of cellulations CMsh G : CMsh $q \to$ CMsh q' induces (after pulling back along $c \hookrightarrow$ CStr X, and then coarsening along $c \to g$) the required classifying mesh bundle map CMsh F : CMsh $p \to$ CMsh p'. ⌐

Observation 4.2.68 (CMsh as a right inverse). One verifies that Construction 4.2.67 satisfies ETrs $\circ$ CMsh $= \text{id}_{\text{Trs}_n}$ (up to unique natural isomorphism). ⌐

4.2.4.4 Classifying mesh coarsenings

Finally, we discuss a more specific construction of classifying mesh maps in the case of coarsening. Note that the construction will not be needed for the proofs of the main theorems in this and later chapters (but it will be useful for several other results). The construction is motivated by the following observation.

Observation 4.2.69 (Classifying mesh maps of degeneracies and coarsenings). Given an n-truss degeneracy map $F : T \to S$, then the classifying mesh map CMsh F : CMsh $T \to$ CMsh S is in fact a mesh degeneracy, and thus the classifying mesh functor restricts to a functor CMsh : $\text{Trs}_n^{\text{deg}} \to \text{Mesh}_n^{\text{deg}}$, which provides a 'right inverse' to the functor ETrs $\text{Mesh}_n^{\text{deg}} \to \text{Trs}_n^{\text{deg}}$.

However, if F is an n-truss coarsening, then the analogous observation fails: that is, CMsh F need not be an n-mesh coarsening. ⌐

Addressing this asymmetry, in this section we will construct a 'homotopical replacement' CrsMsh F of CMsh F (which is an n-mesh coarsening and homotopic to CMsh F). We call CrsMsh F the 'classifying mesh coarsening' of the n-truss coarsening F. We start with the construction of classifying mesh coarsenings in the case of closed trusses.

Construction 4.2.70 (Classifying mesh coarsenings of closed truss coarsenings)**.** Given closed n-trusses $T = (p_n, ..., p_1)$ and $S = (q_n, ..., q_1)$ and a coarsening $F : T \to S$, we construct an n-mesh coarsening CrsMsh F : CMsh $T \to$ CMsh S with the following two properties. Firstly, components CrsMsh F_i : CMsh $T_i \to$ CMsh S_i are *linear* on each simplex of CMsh $T_i =$ CStr T_i. Secondly, we have ETrs CrsMsh $F = F$.

Arguing inductively in n, assume to have constructed CrsMsh $F_{<n}$: CMsh $T_{<n} \to$ CMsh $S_{<n}$ with the claimed properties. We define the top component CrsMsh F_n : CMsh $T_n \to$ CMsh S_n on vertices $x \in T_n$, by picking any image point CrsMsh $F_n(x)$ in the fiber of the 1-truss bundle CMsh(q_n) over CrsMsh $F_{n-1} \circ$ CMsh$(p_n)(x)$ subject to the following conditions: firstly, CrsMsh $F_n(x) <$ CrsMsh $F_n(x')$ (in the linear order of the fiber) whenever $x \prec x'$ (in the frame order of T_n); secondly, CrsMsh $F_n(x)$ lies in the stratum $F_n(x) \in S_n$ of CMsh S_n. We then extend CrsMsh F_n linearly to all other simplices

in $\mathsf{CMsh}\,T_n = \mathsf{CStr}\,T_n$. One checks that this makes $\mathsf{CrsMsh}\,F_n$ a coarsening, and thus $\mathsf{CrsMsh}\,F$ an n-mesh coarsening as claimed. $\quad\quad\lrcorner$

We next generalize our construction of classifying mesh coarsenings to the case of general trusses. The generalization is based on the observation that 'cubical compactification is functorial on coarsenings of trusses' in the following sense.

Observation 4.2.71 (Cubical compactification is functorial on coarsenings). Given a coarsening of n-trusses $F : T \to S$, there is a unique 'cubically compactified coarsening' $\overline{F} : \overline{T} \to \overline{S}$ between the respective cubical compactifications of T and S, such that the following two squares commute

$$
\begin{array}{ccc}
\overline{T} & \xrightarrow{\ \ \overline{F}\ \ } & \overline{S} \\
\mathsf{cr}\Big\downarrow\Big\uparrow\mathsf{ci} & & \mathsf{cr}\Big\downarrow\Big\uparrow\mathsf{ci} \\
T & \xrightarrow{\ \ F\ \ } & S \quad .
\end{array}
$$

(The construction of $\overline{F}$ proceeds inductively, similar to the sequence of constructions defining cubical compactifications.) $\quad\quad\lrcorner$

Construction 4.2.72 (Classifying mesh coarsenings of truss coarsenings)**.** Given a coarsening of n-trusses $F : T \to S$, we construct an n-mesh coarsening $\mathsf{CrsMsh}\,F : \mathsf{CMsh}\,T \to \mathsf{CMsh}\,S$ with the following properties. Firstly, components $\mathsf{CrsMsh}\,F_i : \mathsf{CMsh}\,T_i \to \mathsf{CMsh}\,S_i$ are coarsenings which are linear each open simplex in $\mathsf{CMsh}\,T_i \hookrightarrow \mathsf{CMsh}\,\overline{T}_i$. Secondly, $\mathsf{ETrs}\,\mathsf{CrsMsh}\,F = F$.

We may apply Construction 4.2.70 and Observation 4.2.71 to construct a coarsening $\mathsf{CrsMsh}\,\overline{F} : \mathsf{CMsh}\,\overline{T} \to \mathsf{CMsh}\,\overline{S}$. This coarsening descends to a coarsening $\mathsf{CrsMsh}\,F : \mathsf{CMsh}\,T \to \mathsf{CMsh}\,S$ along the submeshes $\mathsf{CMsh}\,\mathsf{ci} : \mathsf{CMsh}\,T \hookrightarrow \mathsf{CMsh}\,\overline{T}$ and $\mathsf{CMsh}\,\mathsf{ci} : \mathsf{CMsh}\,S \hookrightarrow \mathsf{CMsh}\,\overline{S}$. One checks that this gives an n-mesh coarsening with the claimed properties. $\quad\quad\lrcorner$

Note that by Remark 4.2.42, since $\mathsf{ETrs}\,\mathsf{CrsMsh}\,F = F = \mathsf{ETrs}\,\mathsf{CMsh}\,F$, we find that the classifying mesh coarsening $\mathsf{CrsMsh}\,F$ and the classifying mesh map $\mathsf{CMsh}\,F$ are in fact homotopic mesh maps, and in this sense $\mathsf{CrsMsh}\,F$ is a 'homotopical replacement' of $\mathsf{CMsh}\,F$, as claimed earlier.

4.2.5   The theorem and its applications We finally prove Theorem 4.2.1 and Theorem 4.2.2, which claimed that the classifying mesh functor and the entrance path truss functor provide a weak equivalence pair between (certain) ∞-categories of meshes and trusses, resp. their bundles.

Proof of Theorem 4.2.1 and Theorem 4.2.2. Recall, we set out to show that the ∞-functors $\mathsf{ETrs} : \bar{\mathcal{M}}esh_n(B,g) \to \overline{\mathsf{Trs}}_n(\mathsf{Entr}\,g)$ and $\mathsf{ETrs} : \mathring{\mathcal{M}}esh_n(B,g) \to \mathring{\mathsf{Trs}}_n(\mathsf{Entr}\,g)$ are weak equivalences of ∞-categories. We argue in the closed case (the argument for the open case is fully analogous). We need to check the following (see [Lur09a, Def. 1.1.3.6]).

1. For each closed n-truss $T \in \overline{\mathsf{Trs}}_n$ there exists a closed n-mesh $f \in \bar{\mathcal{M}}esh_n$ whose entrance path truss $\mathsf{ETrs}\,f$ is weakly equivalent to T.
2. The mapping of hom spaces $\mathsf{ETrs} : \bar{\mathcal{M}}esh_n(f,g) \to \overline{\mathsf{Trs}}_n(\mathsf{ETrs}\,f, \mathsf{ETrs}\,g)$ is a weak equivalence of topological spaces.

194

The first statement is a consequence of the construction of classifying meshes $\mathsf{CMsh}\,T$ (see Construction 4.2.66) which satisfies that $\mathsf{ETrs}\,\mathsf{CMsh}\,T = T$. The second statement follows from the weak faithfulness of the entrance path truss functor (see Proposition 4.2.40) together with the observation that fibers of ETrs are never empty (since all elements in hom sets of $\bar{\mathsf{Trs}}_n$ can be realized as mesh bundle maps by Construction 4.2.66). $\qquad\square$

In particular, it follows that the ∞-category $\bar{\mathcal{M}}esh_n$ (respectively $\mathring{\mathcal{M}}esh_n$) is 1-truncated (and thus it can be thought of as an ordinary 1-category, cf. [Lur09a, Prop. 2.3.4.18]). We next discuss applications of the theorem.

4.2.5.1 Meshes of framed cells and their subdivisions As a first application of the equivalence of meshes and trusses, we discuss the relation of meshes and framed regular cells.

Terminology 4.2.73 (Cell meshes). The composite of the truss integration functor $\mathsf{T}_*\!\int :$ $\mathsf{FlatFrCellCplx}_n \to \bar{\mathsf{Trs}}_n$ with classifying mesh functor $\mathsf{CMsh} : \bar{\mathsf{Trs}}_n \to \bar{\mathcal{M}}esh_n$ will be denoted by $\mathsf{CellMsh} : \mathsf{FlatFrCellCplx}_n \to \bar{\mathcal{M}}esh_n$ and called the 'cell mesh functor'. $\quad\rule[0.1em]{1em}{0.1em}\!\!\rule[0.1em]{0.5em}{0.5em}$

Terminology 4.2.74 (Mesh cells). Meshes in the image of cell mesh functor $\mathsf{CellMsh}$ restricted to the subcategory of n-framed regular cells will be called 'n-mesh cells'. $\quad\rule[0.1em]{1em}{0.1em}\!\!\rule[0.1em]{0.5em}{0.5em}$

Conversely to cell meshes, we define mesh complexes as follows.

Terminology 4.2.75 (Mesh complexes). The composite of the entrance path truss functor $\mathsf{ETrs} : \bar{\mathcal{M}}esh_n \to \bar{\mathsf{Trs}}_n$ with the framed complex translation functor $\mathsf{C}^*\nabla : \bar{\mathsf{Trs}}_n \to$ $\mathsf{FlatFrCellCplx}_n$ will be denoted by $\mathsf{MshCplx} : \bar{\mathcal{M}}esh_n \to \mathsf{FlatFrCellCplx}_n$ and called the 'mesh complex functor'. $\quad\rule[0.1em]{1em}{0.1em}\!\!\rule[0.1em]{0.5em}{0.5em}$

As recorded in Corollary 4.2.6 these functors (regarded as functors of ∞-categories) yield weak equivalences $\mathsf{MshCplx} : \bar{\mathcal{M}}esh_n \simeq \mathsf{FlatFrCellCplx}_n : \mathsf{CellMsh}$. Restricting the equivalence further to subcategories of framed regular cells resp. mesh cell, we also record the following.

Corollary 4.2.76 (Equivalence of flat framed cells and mesh cells)**.** *The cell mesh and mesh complex functors establish a weak equivalence between n-framed regular cells and n-mesh cells.*

The translation of framed regular cells into mesh cells provides a 'geometric realization' of framed regular cell as framed stratified spaces. We can leverage that stratified spaces come with a natural notion of 'refinement map' (while cellular posets do not), to turn this into a definition of 'framed subdivisions' of framed regular cells. We start with unframed case of (combinatorial) regular cell complexes.

Terminology 4.2.77 (Subdivision of regular cells). Let X be a combinatorial regular cell (i.e. a cellular poset with initial object), and Y a combinatorial regular cell complex. A 'subdivision' of X by Y is a stratified coarsening $F : \mathsf{CStr}\,Y \to \mathsf{CStr}\,X$ of between their respective classifying stratifications. Given $y \in Y$, the stratified map F restricts on the cell $\mathsf{CStr}\,(Y^{\geq y}) \hookrightarrow \mathsf{CStr}\,Y$ to the 'cell inclusion' $F|_y : \mathsf{CStr}\,(Y^{\geq y}) \hookrightarrow \mathsf{CStr}\,X$.[26] $\quad\rule[0.1em]{1em}{0.1em}\!\!\rule[0.1em]{0.5em}{0.5em}$

---

[26]Note that the cell inclusions $F|_y$ generally are non-cellular maps (see Definition 1.3.18), and, in this sense, they are not immediately combinatorializable (see Remark 1.3.25).

If $(X, \mathcal{G})$ is a *framed* regular cell, then its classifying stratification $\mathsf{CStr}\, X$ is the underlying space of the cell mesh $\mathsf{CellMsh}(X, \mathcal{G})$, and this allows us to keep track of framing structures: a 'framed subdivision' is a subdivision that restricts on each cell to a framed map, as follows.

Definition 4.2.78 (Framed subdivisions of framed regular cells). For an n-framed regular cell $(X, \mathcal{F})$ and an n-framed regular cell complex $(Y, \mathcal{G})$, a **framed subdivision** F : $\mathsf{CStr}\, Y \to \mathsf{CStr}\, X$ is a subdivision such that, for each $y \in Y$, the cell inclusion $F|_y$: $\mathsf{CellMsh}(Y^{\geq y}, \mathcal{G}|_y) \hookrightarrow \mathsf{CellMsh}(X, \mathcal{F})$ is (the top component of a) framed map of meshes. $\quad\rule{1.5mm}{1.5mm}\,$

Example 4.2.79 (Framed subdivisions of framed regular cells). In Fig. 4.16 we illustrate the framed subdivision F of a framed regular cell by a framed regular cell complex, as well as the condition that F restrict on each framed regular cell to the top component of a framed map of meshes. $\quad\rule{1.5mm}{1.5mm}\,$
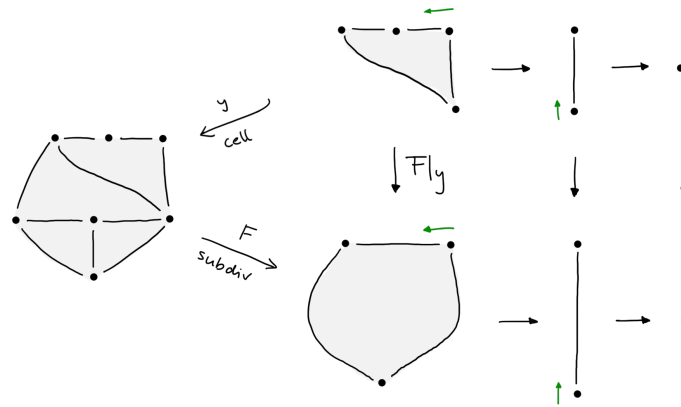

Figure 4.16: Framed subdivisions of framed regular cells

Lemma 4.2.80 (Framed subdivisions are flat). *Given a framed subdivision* $F : (Y, \mathcal{G}) \twoheadrightarrow (X, \mathcal{F})$ *of an n-framed regular cell* $(X, \mathcal{F})$ *by an n-framed regular cell complex* $(Y, \mathcal{F})$, *then* $(Y, \mathcal{G})$ *is a flat.*

The proof follows the description of framed regular cells as 'section' and 'spacer' cell from Section 3.3.2, and we only provide an outline.

Proof. We argue inductively in n. First, assume $(X, \mathcal{F})$ is a section cell. Then all cells in $(Y, \mathcal{G})$ must be section cells, and F induces a framed subdivision $F : (Y, \mathcal{G}_{n-1}) \to (X, \mathcal{F}_{n-1})$ where $(X, \mathcal{F}_{n-1})$ is the projected cell of $(X, \mathcal{F})$, an $(Y, \mathcal{G})_{n-1}$ is the complex of projected cells $(Y^{\geq y}, (\mathcal{G}|_x)_{n-1})$ (see Terminology 3.3.18). By induction, this makes $(Y, \mathcal{G}_{n-1})$ a flat framed cell complex, which implies $(Y, \mathcal{G})$ too is a flat framed cell complex.

Next, assume $(X, \mathcal{F})$ is a spacer cell and denote its lower section cell by $\partial_- X$. Denote by $\partial_- Y \subset Y$ the preimage of $\partial_- X$ under F on entrance path posets (that is, under the map $\mathsf{Entr}\, F : Y \to X$). Note that F restricts to a framed subdivision $F : \mathsf{CStr}\, \partial_- Y \to \mathsf{CStr}\, \partial_- X$ (of the section cell $(\partial_- X, \mathcal{F}|_{\partial_- X})$ by the framed complex $(\partial_- Y, \mathcal{G}|_{\partial_- Y})$). Consider the cell projection $p_n : \mathsf{CStr}\, X \to \mathsf{CStr}\, X_{n-1}$ of X to its projected cell X_{n-1} (note $X_{n-1} \cong \partial_- X$,

196

see Remark 3.3.16). The top 1-mesh bundle in the n-mesh $\mathsf{CellMsh}(X, \mathcal{F})$ endows p_n with the structure of an 1-mesh bundle. We may refine both the domain and codomain of p_n by F to obtain another 1-mesh bundle q_n (with the same fiber 1-framing as p_n) as shown:

$$
\begin{array}{ccc}
\mathsf{CStr}\, Y & \xrightarrow{\ F\ } & \mathsf{CStr}\, X \\
{\scriptstyle q_n}\big\downarrow & & \big\downarrow{\scriptstyle p_n} \\
\mathsf{CStr}\, \partial_- Y & \xrightarrow{\ F\ } & \mathsf{CStr}\, \partial_- X
\end{array}
$$

(The construction of q_n uses that spacer and section cells must be alternatingly 'layered' over cells in the lower section $\mathsf{CStr}\, \partial_- Y \hookrightarrow \mathsf{CStr}\, Y$). Augmenting the inductively constructed $(n-1)$-mesh $\mathsf{CellMsh}(\partial_- Y, (\mathcal{G}|_{\partial_- Y})_{n-1})$ by the 1-mesh bundle q_n yields an n-mesh, which we denote by M: by construction this satisfies $\mathsf{MshCplx}\, M \cong (Y, \mathcal{G})$ and thus $(Y, \mathcal{G})$ is a flat framed regular cell complex as claimed. $\qquad\square$

Having established that all subdividing framed regular cell complexes $(Y, \mathcal{G})$ must be flat (which enables us to construct the cell mesh $\mathsf{CellMsh}(Y, \mathcal{G})$), we may now equivalently characterize framed subdivisions as follows.

Corollary 4.2.81 (Framed subdivision are mesh coarsenings)**.** *For an n-framed regular cell $(X, \mathcal{F})$ and cell complex $(Y, \mathcal{G})$, a stratified coarsening $F : \mathsf{CStr}\, Y \to \mathsf{CStr}\, X$ is a framed subdivision if and only if F is a n-mesh coarsening $F : \mathsf{CellMsh}(Y, \mathcal{G}) \to \mathsf{CellMsh}(X, \mathcal{F})$.* $\quad\square$

Terminology 4.2.82 (Space of subdivisions)**.** For an n-framed regular cell $(X, \mathcal{F})$ and cell complex $(Y, \mathcal{G})$, the space of 'framed subdivision' $\mathsf{SubDiv}(Y, \mathcal{G}; X, \mathcal{F})$ is the space of mesh coarsenings $\mathcal{M}esh_n^{\mathrm{crs}}(\mathsf{CellMsh}(Y, \mathcal{G}), \mathsf{CellMsh}(X, \mathcal{F}))$. $\quad\sqcup\!\!\!\sqcup$

The notion of framed subdivisions of framed regular cells can be 'combinatorialized'.

Notation 4.2.83 (Set of truss coarsening)**.** For an n-block B and a closed n-truss T, denote the set of truss coarsening $T \to B$ by $\mathsf{Trs}_n^{\mathrm{crs}}(T, B)$. $\quad\sqcup\!\!\!\sqcup$

Corollary 4.2.84 (Framed subdivisions are up to homotopy truss coarsenings)**.** *The map* $\mathsf{ETrs} : \mathsf{SubDiv}(Y, \mathcal{G}; X, \mathcal{F}) \to \mathsf{Trs}_n^{\mathrm{crs}}(\mathsf{T}_* \!\!\int (Y, \mathcal{G}), \mathsf{T}_* \!\!\int (X, \mathcal{F}))$ *is a weak homotopy equivalence.*

In other words, up to contractible choice in $\mathsf{SubDiv}(-; X, \mathcal{F})$, framed subdivisions of $(X, \mathcal{F})$ correspond exactly to truss refinements of $\mathsf{T}_* \!\!\int (X, \mathcal{F})$.

Proof. We need to show that entrance path truss functor ETrs, mapping framed subdivisions $\mathsf{SubDiv}(Y, \mathcal{G}; X, \mathcal{F})$ to truss coarsenings $\mathsf{Trs}_n(\mathsf{T}_* \!\!\int (Y, \mathcal{G}), \mathsf{T}_* \!\!\int (X, \mathcal{F}))$, has weakly contractible preimages. This follows from Remark 4.2.42. $\qquad\square$

Remark 4.2.85 (Framed subdivisions can be made piecewise linear)**.** Combining the corollary with our earlier construction of 'piecewise linear' classifying mesh coarsenings (see Construction 4.2.72), it follows that any framed subdivision is homotopic to a piecewise linear framed subdivision. $\quad\sqcup\!\!\!\sqcup$

The corollary not only provides a combinatorialization (up to contractible choice) of framed subdivisions, but the resulting combinatorialization is also computationally tractable: namely, since truss refinements of $\mathsf{T}_*\!\int(X, \mathcal{F})$ can be algorithmically listed, so can (up to contractible choice) framed subdivisions of the framed regular cell $(X, \mathcal{F})$. This stands in further contrast to the classical observation that it is *impossible* to algorithmically decide when a simplicial complex is piecewise linearly homeomorphic to the simplex.

Finally, the fact that framed subdivisions can combinatorialized in terms of morphisms in the category of trusses (living in the same category as faces and degeneracies), is by itself rather remarkable property of framed regular cells. In contrast, for most classes of combinatorial shapes, notions subdivisions cannot be given in terms of combinatorial 'morphism' (for instance, in the case of simplices, a combinatorial description of subdivision requires rather different techniques than the definition of simplicial maps, see [Pac91]).

4.2.5.2   Dualization of meshes As a second, and central, application of Theorem 4.2.1 we now construct the dualization functors between the categories of closed meshes with singular maps and open meshes with regular maps. This proves our earlier Corollary 4.2.9.

Proof of Corollary 4.2.9. The 'mesh dualization' functors

$$\dagger : \bar{\mathcal{M}}esh_n \simeq \mathring{\mathcal{M}}esh_n : \dagger$$

are defined by the respective composites in

$$\bar{\mathcal{M}}esh_n \; \underset{\mathsf{CMsh}}{\overset{\mathsf{ETrs}}{\rightleftarrows}} \; \bar{\mathsf{T}}\mathsf{rs}_n \; \underset{\dagger}{\overset{\dagger}{\rightleftarrows}} \; \mathring{\mathsf{T}}\mathsf{rs}_n \; \underset{\mathsf{ETrs}}{\overset{\mathsf{CMsh}}{\rightleftarrows}} \; \mathring{\mathcal{M}}esh_n$$

where the central arrows (labeled by '$\dagger$') are the dualization functors of trusses, see Observation 2.3.89. Since each functor in the above composite is an equivalence, so are the mesh dualization functors. $\square$

Example 4.2.86 (Meshes and their duals)**.** In Fig. 4.17 we depict meshes M together with their duals $M^\dagger$ (note in particular, this turns open meshes into closed meshes and vice versa). Note that the depicted closed 3-mesh contains *two* 3-cells (corresponding to the two 0-cells in the depicted open 3-mesh); this is a subcomplex of the gluing depicted earlier in Fig. 1.41. $\quad\rightharpoondown$

Remark 4.2.87 (Dualization of flat framed regular cell complexes)*.* While flat framed regular cell complexes don't dualize to flat framed regular cell complexes (they correspond to closed meshes, which to open meshes), they do so 'up to compactification' (see Definition 4.2.53). Namely, given a flat n-framed regular cell complex $(X, \mathcal{F})$ define its 'dual' to be the flat n-framed regular cell complex $\mathsf{C}^*\nabla\,\overline{\mathsf{T}_*\!\int(X, \mathcal{F})^\dagger}$ (that is, the framed complex translation of the compactification of the dual of $\mathsf{T}_*\!\int(X, \mathcal{F})$). This is the 'duality' used in an earlier example: namely, the framed complex in Fig. 1.41 'dualizes' to the framed complex in Fig. 1.40. $\quad\rightharpoondown$

The fact that meshes admit a notion of dualization is fundamental, and provides the key to many ideas in the interplay of framed combinatorial topology and geometric higher category theory (such as those outlined in Section I.5).
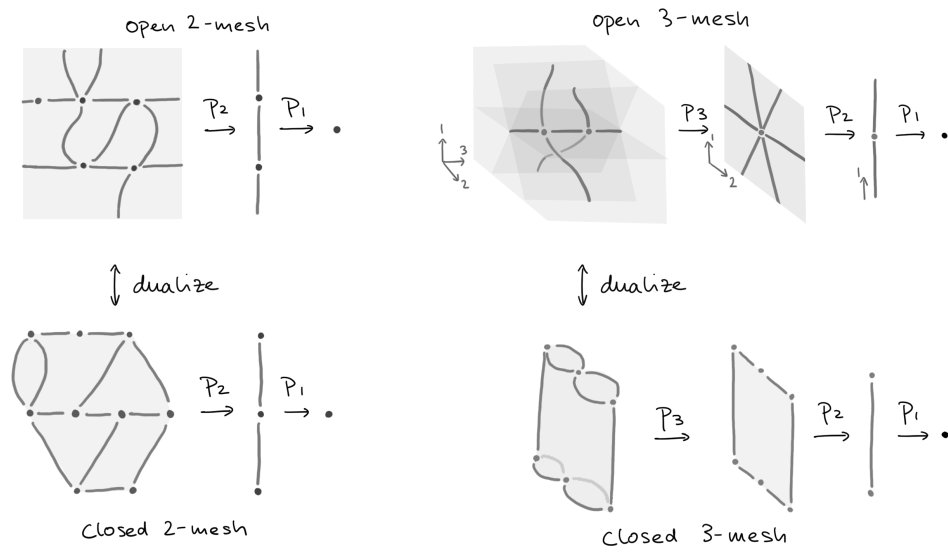
Figure 4.17: Meshes and their duals.

CHAPTER 5

Combinatorializability of flat framed stratifications

In the final chapter of this book we will begin our study of framed combinatorial *stratified* topology, or more precisely, of its model stratifications given by so-called *flat* framed stratifications. Flat n-framed stratifications provide a general class of stratifications defined by the property of being 'meshable', i.e. admitting a framed refinement by some n-mesh. Their 'flat n-framing' is induced by a framed embedding into standard n-framed $\mathbb{R}^n$. Flat framed stratifications provide 'local models' for more general, 'global' framed stratified spaces (much as framed regular cells provide local models for framed regular cell complexes). While we will not discuss 'global' framed stratifications here, the basic machinery of flat framed stratifications developed in this chapter will be tailored towards future applications in that and other related directions.

5.1 Chapter overview

We outline the main results of this chapter. Recall, a stratified map is a 'refinement of its image' if its underlying map of spaces is a homeomorphism. The definition of flat framed stratifications can be formally given as follows.

Definition 5.1.1 (Flat framed stratifications)**.** A **flat n-framed stratification** is a stratification (Z, f) of a subspace $Z \subset \mathbb{R}^n$, for which there exists an n-mesh M whose flat n-framing (as a map $M_n \hookrightarrow \mathbb{R}^n$) induces a stratified coarsening $(M_n, f_n) \to (Z, f)$. ⌊⌋

In Fig. 5.1, we illustrate several examples of flat n-framed stratifications (I^3, f) of the open 3-cube $I^3 \subset \mathbb{R}^3$. All examples can be seen to admit a refinement by the flat 3-framing of an open 3-mesh. (Note that, in general, the underlying space $Z \subset \mathbb{R}^n$ of a flat framed stratification (Z, f) need not be open, as it can be the image of the n-framing of any mesh.)

We will prove that flat framed stratifications, despite their definition in stratified topological terms, have an unexpected combinatorial counterpart. While we've shown in
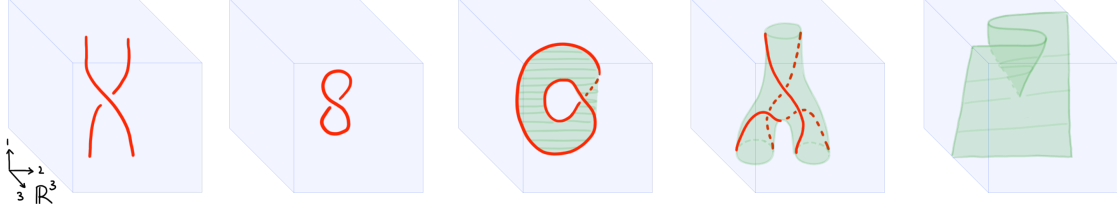
Figure 5.1: Examples of flat 3-framed stratifications of the open 3-cube representing (from left to right) a 'braid', a 'Hopf circle', a 'Mobius band', a 'pair of pants with defects', and a 'cusp singularity'.

the previous chapter that meshes are 'weakly' equivalent to constructible combinatorial objects (namely, to trusses), the above definition of flat framed stratifications makes reference merely to the *existence* of some mesh refinement without singling out a particular such refinement. It therefore remains entirely unclear that flat framed stratifications can themselves be combinatorialized as claimed.

For the combinatorialization of meshes to transfer to that of flat framed stratifications, we will have to understand the class of all refining meshes in combinatorial terms. This will be achieved by the notion of 'stratified trusses'. The notion is a specialization of our earlier definition of 'labeled n-trusses' (see Definition 2.3.44) requiring the labeling to be the characteristic map of a stratification (here, we regard posets as topological spaces via their specialization topology, see Convention B.0.2).

Definition 5.1.2 (Stratified n-truss)**.** A **stratified n-truss** T is a labeled n-truss T whose labeling lbl_T is the characteristic map of a stratification on the total poset T_n of T. ⌐

The correspondence between n-trusses and n-meshes can now be extended to a correspondence between *stratified n-trusses* and *refinements* of flat framed stratifications by n-meshes (up to 'framed stratified homeomorphism'). Namely, given a refinement of a flat framed stratification by an n-mesh, we can first consider the n-truss corresponding to the n-mesh, and then produce a labeling of that n-truss by precomposing the characteristic map of the stratification with the refinement's entrance path poset map. To obtain *unique* combinatorial representatives of flat framed stratifications, we will be interested in the following subclasses of stratified trusses, consisting of those stratified trusses that do not admit a non-identity 'label preserving truss coarsening' (see Terminology 2.3.61 and Terminology 2.3.97).

Definition 5.1.3 (Normalized stratified n-trusses)**.** We say a stratified n-truss T is **normalized** if any label preserving truss coarsening of T is an identity. ⌐

Our main theorem in this chapter will show that flat framed stratifications are indeed 'combinatorializable', by producing a correspondence of flat framed stratifications (up to framed stratified homeomorphism) with normalized stratified trusses (up to 'stratified truss isomorphisms'), as follows.

Theorem 5.1.4 (Normalized stratified trusses classify flat framed stratifications)**.** *Framed stratified homeomorphism classes of flat n-framed stratifications are in correspondence with isomorphisms classes of normalized stratified n-trusses.*

The theorem's proof will be based on the following observation: while, a priori, flat framed stratifications may have many refining meshes, there is, in fact, always one canonical (namely, 'coarsest') refining mesh. The correspondence of the theorem will take a given flat n-framed stratification to the stratified n-truss representing the coarsest n-mesh refinement of that stratification. The proof will therefore hinge on the construction of coarsest refining meshes which, in turn, will be based on a key property of meshes: the existence of so-called 'mesh joins'. As suggested by the name, mesh joins are finest mutual coarsenings of any two given meshes (with identical support). The proof of the existence of mesh joins and the construction of coarsest refining meshes will take up all of Section 5.2. This will provide the technical groundwork for Section 5.3.1 where we will spell out the proof of Theorem 5.1.4.

Remark 5.1.5 (Bundles of flat framed stratifications). Definition 5.1.2, Definition 5.1.3 and Theorem 5.1.4 all have immediate generalizations to the case of bundles; that is, 'bundles of flat framed stratifications' will correspond to 'normalized stratified truss bundles'. We will address this generalization at the end of Section 5.3.1. ⌐⌐

Theorem 5.1.4 has several key corollaries, as we now outline. Most immediately, the theorem implies that (up to homotopically unique homeomorphism) any flat framed stratification may be replaced by a canonical piecewise linear one; here, a 'piecewise linear stratification' means a stratification that admits a triangulation by linear simplices in $\mathbb{R}^n$.

Corollary 5.1.6 (Flat framed stratification are piecewise linear). *Any flat framed stratification is framed stratified homeomorphic to a canonical piecewise linear flat framed stratification.*

The canonical piecewise linear flat framed stratification can be obtained by constructing the 'classifying stratification' of the normalized stratified trusses corresponding to the given flat framed stratification. This corollary, in particular, entails that flat framed stratifications are better behaved than mere topological stratifications: indeed, there are bounded stratifications in $\mathbb{R}^n$ that are not topologically (nonframed) homeomorphism to any piecewise linear stratifications (for instance, take any embedding of the E_8 manifold in euclidean space).

Not only are flat framed stratifications framed stratified homeomorphic to piecewise linear ones, but the notions of 'framed stratified homeomorphism' and 'framed stratified PL homeomorphism' coincide.

Corollary 5.1.7 (Flat framed Hauptvermutung). *Any framed stratified homeomorphism between piecewise linear flat framed stratifications is homotopic to some framed stratified piecewise linear homeomorphism.*

In fact, the homotopy is 'unique up to contractible choice'. As we will discuss, the classical nonframed analog of the statement fails to hold: namely, there are bounded piecewise linear stratifications in $\mathbb{R}^n$ that are topologically stratified homeomorphic but not piecewise linearly stratified homeomorphic.

Finally, we also have the following 'converse' to Corollary 5.1.6.

Proposition 5.1.8 (Piecewise linear stratifications are flat framed). *Any bounded piecewise linear stratification in $\mathbb{R}^n$ is (up to adding an 'ambient' stratum) a flat framed stratification.*

The previous three results will be content of Section 5.3.2. Together, they establish the 'polyhedrality' of flat framed stratifications, and they can be summarized as follows.

Remark 5.1.9 (Framed, framed piecewise linear, and piecewise linear categories). Corollary 5.1.6 shows that the functor from 'piecewise linear flat framed stratifications' into 'flat framed stratifications' is surjective on homeomorphism classes, and Corollary 5.1.7 shows that it is injective on homeomorphism classes. Further, Proposition 5.1.8 shows that the functor from 'piecewise linear flat framed stratifications' to 'piecewise linear stratifications' is surjective on homeomorphism classes. However, observe that that functor is far from being injective since framed stratified PL homeomorphism is a much finer equivalence relation than (nonframed) stratified PL homeomorphism, and the two categories are far from being equivalent. ⌐⌐

A second important area of applications of the combinatorializability of flat framed stratifications concerns their 'computability' properties. We will highlight the following two results.

Corollary 5.1.10 (Canonical mesh refinement are computable). *Given a flat framed stratification in $\mathbb{R}^n$, its coarsest refining mesh can be algorithmically computed.*

The proof of this result will translate the process of 'coarsening mesh refinements' into a combinatorial notion of 'reduction' of stratified trusses. Any chain of reductions of a given stratified truss eventually ends in the same normalized stratified truss (which is a consequence of the existence of coarsest mesh refinements) and this normalized stratified truss can be algorithmically computed. It follows that coarsest refining meshes also can be algorithmically computed.

A second corollary will be the following.

Corollary 5.1.11 (Decidability of framed stratified homeomorphism). *Given two flat framed stratifications, presented as piecewise linear stratifications in $\mathbb{R}^n$, we can algorithmically decide whether they are framed stratified homeomorphic.*

The preceding two results will be discussed in Section 5.3.3.

The results in this chapter provide a first set of core properties of flat framed stratifications, in particular establishing their 'combinatorial and computational tractability'. We will end the chapter in Section 5.4 with a short of overview of further steps in the program of framed combinatorial topology.

5.2 Flat framed stratifications have canonical meshes

Recall from Definition 5.1.1 that a flat n-framed stratification (Z, f) is a stratification of a subspace $Z \subset \mathbb{R}^n$ for which there is a 'refining' n-mesh M, given by a tower of 1-mesh bundles $p_i : (M_i, f_i) \to (M_{i-1}, f_{i-1})$ and n-framing components $\gamma_i : M_i \hookrightarrow \mathbb{R}^i$, such that $\gamma_n : (M_n, f_n) \to (Z, f)$ is a stratified refinement of (Z, f) by (M_n, f_n). Our goal in this section, will be to show that there is a canonical such refining n-mesh: namely, we will construct the 'coarsest' refining n-mesh of a given flat framed stratification. The following notational convention will be helpful.

Notation 5.2.1 (Keeping n-framings implicit). When working with n-meshes M (resp. n-mesh bundles p over a base (B, g)) given by towers of 1-mesh bundles $p_i : (M_i, f_i) \to (M_{i-1}, f_{i-1})$ and framing γ as defined in Construction 4.1.47 (resp. Construction 4.1.54) we will usually identify the space M_i with the image of $\gamma_i : M_i \hookrightarrow \mathbb{R}^i$ (resp. of $\gamma_i : M_i \hookrightarrow B \times \mathbb{R}^i$). We call $M_n \subset \mathbb{R}^n$ resp. $M_n \subset B \times \mathbb{R}^n$ the 'support' of M resp. p. ⎯⎯

In particular, M is a refining mesh of a flat framed stratification (Z, f) exactly if $(M_n, f_n) \to (Z, f)$ is a stratified refinement (whose underlying map is the identity on $M_n = Z \subset \mathbb{R}^n$).

5.2.1 Definition of mesh joins A key step to prove the existence of coarsest mesh refinements will be the construction of so-called 'mesh joins'. A 'join' of two stratifications on the same space is the finest mutual coarsening of both stratifications; in general, this may be a prestratification as the following definition records (recall, a prestratification allows cycles in the formal entrance path relation, see Terminology B.1.3).

Definition 5.2.2 (Joins of stratification). Given stratifications f and g of a space X, the **join** $f \vee g$ is the unique prestratification of X that coarsens both f and g, and such that, for any other prestratification h of X, if h coarsens both f and g then it must be finer that $f \vee g$. ⎯⎯

Explicitly, the join may be constructed as follows.

Construction 5.2.3 (Joins of stratifications). Given a space X, and stratifications $(X, f), (X, g)$, let $\sim$ denote the (transitive closure of the) relation on the union of strata of both f and g given by

$$s \sim t \iff (s \cap t \neq \emptyset)$$

Then the join $f \vee g$ is the prestratification of X given by the decomposition of X into non-empty connected subspaces $\bigcup_{s \in \mathsf{s}} s$, where s is an equivalence class under the relation $\sim$. ⎯⎯

Remark 5.2.4 (Joins as pushouts). Joins of (pre)stratifications (X, f) and (Y, g) may be equivalently defined as pushouts in the category of prestratifications of the span of stratified coarsenings

$$(X, f) \leftarrow (X, \mathsf{discr}(X)) \to (X, g)$$

where $\mathsf{discr}(X)$ is the discrete stratification on X (see Terminology B.1.11). Note that this pushout is preserved when passing to entrance path preorders, yielding a pushout in the category of preorders. ⎯⎯

Notation 5.2.5 (Equivalence classes and strata in joins). Given stratifications (X, f) and (X, g), abusing notation we often denote their strata by $\mathsf{s} \equiv \bigcup_{s \in \mathsf{s}} s$, where s is an equivalence class of strata—that is, we consider s both as an equivalence class of strata (for instance, we write $r \in \mathsf{s}$ to mean a member r of that class) and as a stratum of $f \vee g$ (for instance, we write $x \in \mathsf{s}$ to mean a point in that stratum). Further, we denote by s_f the subclass of s consisting of strata that lie in f, and by s_g the subclass of strata in s that lie in g. ⎯⎯

Note that the disjoint unions $\bigsqcup_{s \in \mathsf{s}_f} s$ and $\bigsqcup_{s \in \mathsf{s}_g} s$ in M_n both equal the stratum s.

Example 5.2.6 (Joins of stratifications)**.** In Fig. 5.2 we depict the joins of two strati-fications. Note that in the second case, the join is in fact a prestratification, and not a stratification (using Remark 5.2.4, this reflects that pushouts of diagrams of posets in the category of preorders need not themselves land in the subcategory of posets). ⌐⌐
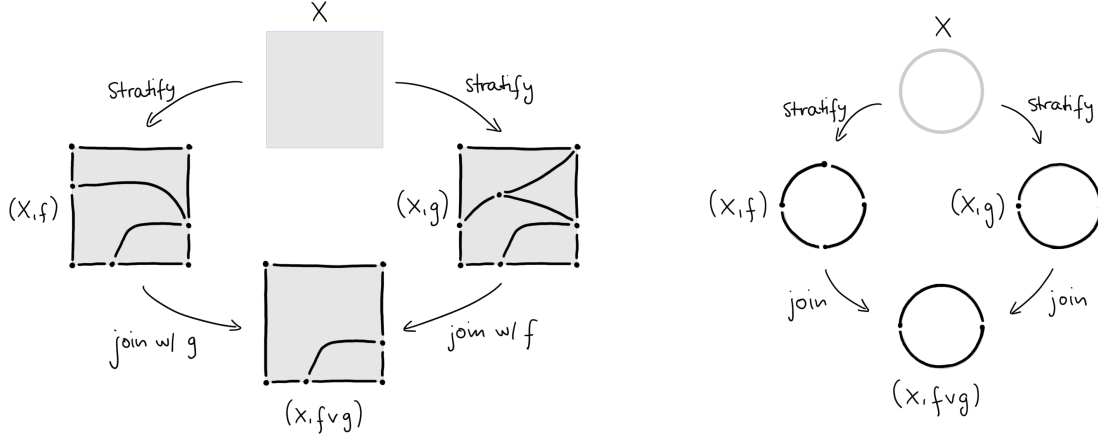

Figure 5.2: Joins of stratifications.

For fixed domain and codomain, the join operations extend to stratified maps as the following observation records.

Observation 5.2.7 (Joins of stratified maps)**.** Given stratifications (X, f), (X, g), (X', f'), (X', g'), and stratified maps $F : (X, f) \to (X', f')$ and $G : (X, g) \to (X', g')$ that are identical as maps of underlying spaces $X \to X'$, then the latter map of spaces also underlies a stratified map $F \vee G : (X, f \vee g) \to (X', f' \vee g')$. We call $F \vee G$ the 'join' of the stratified maps F and G. ⌐⌐

Specifically, we will be interested in the joins of meshes and their bundles. Recall from Notation 5.2.1 that we consider meshes resp. their bundles as (fiberwise) trivialized in $\mathbb{R}^n$.

Definition 5.2.8 (Mesh bundle joins)**.** Given two n-mesh bundles p and p' consisting of 1-mesh bundles $p_i : (M_i, f_i) \to (M_{i-1}, f_{i-1})$ resp. $p'_i : (M'_i, f'_i) \to (M'_{i-1}, f'_{i-1})$ over $(M_0, f_0) = (B, g) = (M'_0, f'_0)$ such that M and M' have identical support in $B \times \mathbb{R}^n$, then the **mesh bundle join** $p \vee q$ is the tower stratified maps $p_i \vee p'_i : (M_i, f_i \vee f'_i) \to (M_{i-1}, f_{i-1} \vee f'_{i-1})$ (where each M_i is a stratified subspace of $B \times \mathbb{R}^i$). ⌐⌐

The definition specializes to a notion of 'mesh joins' $M \vee M'$ of n-meshes M and M' by setting the base to be trivial, i.e. $(B, g) = *$.

Example 5.2.9 (Mesh joins)**.** In Fig. 5.3 we depict the mesh join of two open 2-meshes. Note that mesh join is in fact again a 2-mesh. ⌐⌐

The crucial fact, illustrated by the preceding example, is the following: mesh joins are meshes (automatically endowed with an n-framing by virtue of being subtowers of Π). The proof of this fact will take up the next section.
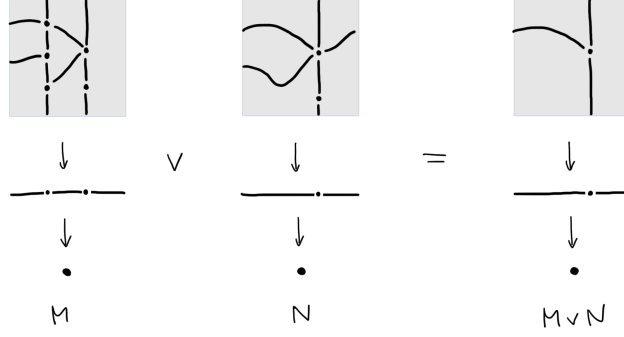
Figure 5.3: The join of two open meshes.

5.2.2 Construction of mesh joins This section will be concerned with the following 'key lemma'.

Key Lemma 5.2.10 (Join stability of meshes)**.** *Given n-meshes M and M' with identical support, then their mesh join $M \vee M'$ is an n-mesh (with the same support).*

Proof of Key Lemma 5.2.10. The proof is by induction in n. Inductively, the mesh join $M_{<n} \vee M'_{<n}$ of the $(n-1)$-truncations $M_{<n}$ and $M'_{<n}$ is an $(n-1)$-mesh. The inductive step requires us to show that the stratified map $p_n \vee p'_n : f_n \vee f'_n \to f_{n-1} \vee f'_{n-1}$ is in fact a 1-mesh bundle; this follows from Lemma 5.2.11 (using that both f_{n-1} and f'_{n-1} are finite frontier-constructible stratifications by Observation 4.1.44 and Observation 4.1.45). □

Lemma 5.2.11 (Join stability of 1-mesh bundles)**.** *Let (B, g) and (B, g') be finite frontier-constructible stratifications. Given 1-mesh bundles $p : (M, f) \to (B, g)$ and $p' : (M, f') \to (B, g')$ that are identical as maps of spaces, then their join $p \vee p' : (M, f \vee f') \to (B, g \vee g')$ is again a 1-mesh bundle.*

Before proving the lemma we will show the following auxiliary results.
1. In Lemma 5.2.12 we show that image of any stratum in $f \vee f'$ under p is exactly a stratum in $g \vee g'$.
2. In Lemma 5.2.16 we show that each stratum s in $f_n \vee g_n$ whose image under p_n is the stratum r in $(f \vee g)_{n-1}$, admits continuous sections $\hat{\gamma}_{\mathsf{s}}^{\pm} : \mathsf{r} \to \mathsf{r} \times \mathbb{R}$ which fiberwise bound s from above and below.

We will then assemble these observations into a proof of Lemma 5.2.11.

Lemma 5.2.12 (Joined strata project onto joined strata)**.** *For 1-mesh bundle $p : (M, f) \to (B, g)$ and $p' : (M, f') \to (B, g')$ with identical underlying maps, the image of any stratum s of $f \vee f'$ under $p \vee p'$ is exactly a stratum in $g \vee g'$.*

Notation 5.2.13 (Identifying bundle maps). We often notationally identify $p \equiv p' \equiv p \vee p'$ (since p, p', and $p \vee p'$ are all identical as maps of spaces). ⌐

Proof of Lemma 5.2.12. First note $p(\mathsf{s})$ is contained in some stratum r of $g \vee g'$ (see Observation 5.2.7). Now, since p is 1-mesh bundle, images of strata in f are exactly strata of g (and similarly for p'). Consequently, there is a subclass $\mathsf{r}_f^{\mathsf{s}}$ of r_f (see Notation 5.2.5) such that the union of strata in $\mathsf{r}_f^{\mathsf{s}}$ is exactly $p(\mathsf{s})$. To proof the lemma, we show $\mathsf{r}_f^{\mathsf{s}} = \mathsf{r}_f$.

If $r_f^s \subsetneq r_f$, then we could find a stratum r' in the class $r_{f'}$ which intersects both r_f^s and $r_f \setminus r_f^s$. Pick a stratum r in r_f^s that intersects r'. By definition of r_f^s, there is a stratum s in the class s_f lying over r. Since M and N have identical support, and since r intersects r', there is at least one stratum s' in f' lying over r' that intersects s. This implies s' is a member of the class s. But since $p(s') = r'$, and $p(s') \subset p(s) = r_f^s$, and r' intersects $r_f \setminus r_f^s$ non-trivially, this constructs a contradiction. $\square$

We next want to show that the following 'upper and lower fiber bounds' are in fact images of continuous sections.

Notation 5.2.14 (Upper and lower fiber bounds). Under the assumptions of Lemma 5.2.12, given a stratum s of $f \vee f'$ for each point $x \in p(s)$ we denote by s_x the restriction of s to the $\mathbb{R}$-fiber of the projection $r \times \mathbb{R} \to r$. We further denote by $\hat{\gamma}_s^-(x)$ resp. $\hat{\gamma}_s^+(x)$ the 'lower resp. upper bound' of the subspace s_x in $\mathbb{R}$. ⌐

Observation 5.2.15 (Upper and lower boundaries lie in singular strata). Using the previous notation, we emphasize the basic but important observation that, for each $x \in p(s)$, the point $\hat{\gamma}_s^\pm(x) \in r \times \mathbb{R}$ either equals $\gamma^\pm(x)$ (where $\gamma^\pm : B \to B \times \mathbb{R}$ are the 1-fiber framing bounds of both the 1-mesh bundles p and p') or the point lies in singular strata of both f and f'. Indeed, it cannot lie in regular strata of either f or f', since regular strata intersect fibers of $B \times \mathbb{R} \to B$ in open intervals. ⌐

Lemma 5.2.16 (Joined strata are bounded by continuous sections). *For 1-mesh bundles $p : (M, f) \to (B, g)$ and $p : (M, f') \to (B, g')$ with finite frontier-constructible base stratifications, then, given any stratum s in $f \vee f'$ lying over $r = p(s)$ in $g \vee g'$, the function $\hat{\gamma}_s^\pm : r \to r \times \mathbb{R}$ mapping x to $\hat{\gamma}_s^\pm(x)$ is continuous.*

Proof. We first observe that, for each stratum v in the equivalence class r, the mapping $\hat{\gamma}_s^\pm$ restricts to a continuous map on v. Indeed, assume $v \in r_f$ (or similarly, $v \in r_{f'}$). Then the intersection $s \cap p^{-1}(v)$ is exactly a union of strata in f, namely those strata $s_v \subset s_f$ which lie over v. The image $\hat{\gamma}_s^\pm(v)$ of $\hat{\gamma}_s^\pm : v \to v \times \mathbb{R}$ is therefore either equal to $\gamma_n^\pm(v)$ or to some singular stratum in f lying over v. In either case, $\hat{\gamma}_s^\pm$ is continuous on v.

The following terminological distinction will be useful. If a stratum $u \in r$ contains a point x at which $\hat{\gamma}^\pm$ is not continuous (that is, $\hat{\gamma}_s^\pm$ is not continuous in any neighborhood of x in the stratum r) we say u is 'bad'; otherwise, we say u is 'good'. Note, by our first observation, bad strata cannot be minimal elements in either r_g or $r_{g'}$ (considered as full subposets of $\mathsf{Entr}(g)$ resp. $\mathsf{Entr}(g')$). In particular, there exist at least some good strata; namely the minimal elements in r_g and $r_{g'}$.

We first show at given a bad stratum u, then $\hat{\gamma}_s^\pm$ is not continuous at any point of u. Assume $u \in r_g$ (the argument is the same if $u \in r_{g'}$). By assumption, there is $x \in u$ at which $\hat{\gamma}_s^\pm$ is not continuous. Since g is finite and frontier-constructible, there must be $\tilde{u} \to u$ (in $\mathsf{Entr}(g)$) such that $\hat{\gamma}_s^\pm$ is not continuous at x when restrict to $\tilde{u} \cup u$. As observed earlier, the subspaces $\hat{\gamma}_s^\pm(u)$ (resp. $\hat{\gamma}_s^\pm(\tilde{u})$) are either singular strata in f or of the form $\gamma^\pm(u)$ (resp. $\gamma^\pm(\tilde{u})$). It follows by constructibility of $p : f \to g$ and continuity of $\gamma^\pm$, that either $\hat{\gamma}_s^\pm(\tilde{u})$ contains *all* or else *none* of $\hat{\gamma}_s^\pm(u)$ in its closure. The latter case must be true, since we assumed that $\hat{\gamma}_s^\pm$ is not continuous on $\tilde{u} \cup u$ at x. We deduce that $\hat{\gamma}_s^\pm$ is discontinuous at all points in u as claimed.

To see that $\hat{\gamma}^{\pm}$ is continuous on all of r we now argue by contradiction. Assume a bad stratum u_0 (say, $u_0 \in \mathsf{r}_g$) exists. Now, let $\mathsf{r}_{f'}^{u_0}$ denote the subclass of strata in $\mathsf{r}_{f'}$ which intersect u_0. Note all strata in $\mathsf{r}_{f'}^{u_0}$ are bad, since they all intersect u_0 at least in one point. Moreover, since good strata exists (and thus $u_0 \subsetneq \mathsf{r}$), the union of strata in $\mathsf{r}_{f'}^{u_0}$ covers u_0 but it cannot cover u_0 exactly. Thus, pick a $u_1 \in \mathsf{r}_{f'}^{u_0}$ which contains points outside of u_0. Denote by $\mathsf{r}_f^{u_0,u_1}$ the strata of r_f which non-trivially intersect $u_0 \cup u_1$. Again, since good strata exist (and thus $u_0 \cup u_1 \subsetneq \mathsf{r}$), the union of strata $\mathsf{r}_f^{u_0,u_1}$ cannot cover $u_0 \cup u_1$ exactly. Thus, pick a stratum $u_2 \in \mathsf{r}_f^{u_0,u_1}$ which contains points outside of $u_0 \cup u_1$. Once more, u_2 is bad. Repeating the argument in this way, we obtain a sequence of strata u_i, such that each u_i has points outside of $u_1 \cup ... \cup u_{i-1}$. The fact that the sequence is non-repeating contradicts the finiteness of the equivalence class r and thus completes the argument. $\square$

We can now proof that the join of 1-mesh bundles is again a 1-mesh bundle.

Proof of Lemma 5.2.11. We first check that $p : f \vee f' \to g \vee g'$ is a stratified bundle whose fibers are 1-meshes. Given a stratum s of $f \vee f'$ lying over a stratum r of $g \vee g'$, denote by $\hat{\gamma}_{\mathsf{s}}^{\pm}(\mathsf{r})$ the respective images of r under the mappings $\hat{\gamma}_{\mathsf{s}}^{\pm} : \mathsf{r} \to \mathsf{r} \times \mathbb{R}$. Constructibility of both $p : f \to g$ and $p : f' \to g'$ implies that either $\hat{\gamma}_{\mathsf{s}}^{\pm}(\mathsf{r}) = \gamma^{\pm}(\mathsf{r})$ or else $\hat{\gamma}_{\mathsf{s}}^{\pm}(\mathsf{r})$ is a union of singular strata in both f and f'. In the latter case, $\hat{\gamma}_{\mathsf{s}}^{\pm}(\mathsf{r})$ is also a stratum in $f \vee f'$. It follows that either (1) $\hat{\gamma}_{\mathsf{s}}^{-}(\mathsf{r}) = \hat{\gamma}_{\mathsf{s}}^{+}(\mathsf{r})$ or else (2) $\hat{\gamma}_{\mathsf{s}}^{-}(\mathsf{r})$ and $\hat{\gamma}_{\mathsf{s}}^{+}(\mathsf{r})$ are disjoint. If (1) holds, s is a 'singular stratum', that is, a section of $p_n : M_n \to M_{n-1}$ (restricted to r in the base). If (2) holds, s is a 'regular stratum' over r, fiberwise bounded by the continuous sections $\hat{\gamma}_{\mathsf{s}}^{\pm}$. This shows that $p_n : f_n \vee g_n \to (f \vee g)_{n-1}$ is stratified bundle with 1-mesh fibers. (The bundle of course trivially inherits its fiber 1-framing from that of p.)

Finally, we observe that $p : f \vee f' \to g \vee g'$ is constructible. Indeed, since both $p : f \to g$ and $p : f' \to g'$ are constructible, one easily checks that closure of singular strata in $f \vee f'$ only contain singular strata, and that lifts of entrance paths in $\mathsf{Entr}(g \vee g')$ to singular strata in $\mathsf{Entr}(f \vee f')$ are unique. $\square$

This completes our construction of mesh joins, and thus the proof of Key Lemma 5.2.10.

We briefly discuss two generalizations of the key Key Lemma 5.2.10. Firstly, the reader may have noted already that our arguments immediately carry over to the case of mesh bundles as well. We record this in the following result.

Lemma 5.2.17 (Key lemma for n-mesh bundles)**.** *Let p and q be n-mesh bundles over finite frontier-constructible base (B, g) resp. (B, g') with the same support in $B \times \mathbb{R}^n$. Then the join $p \vee q$ is a n-mesh bundle over $(B, g \vee g')$.*

Proof. The proof of Key Lemma 5.2.10 applies verbatim after replacing 'meshes' by 'mesh bundles'. $\square$

Secondly, the statement of the key lemma has the following variation that takes into consideration maps between meshes.

Lemma 5.2.18 (Generalized key lemma)**.** *Given n-meshes M and M' with a framed but not necessarily stratified map $F : M \to M'$, then the stratified join $M \vee F^{-1}(M')$ is an n-mesh with the same support as M. (Here, $F^{-1}(M')$ is the tower of substratifications of M' with inclusion map F, and the join is taken as a level-wise join of towers.)*

Proof idea. The proof follows again the same inductive argument as that of the Key Lemma 5.2.10, but is lengthier since it also requires the inductive description of joins with strata of M' that partially intersect the image of F (we omit a detailed verification). $\square$

5.2.3   The coarsest refining mesh We now define and then prove the existence of coarsest refining meshes of flat framed stratifications.

Definition 5.2.19 (Coarsest refining meshes)**.** Given a flat framed stratification (Z, f), a refining n-mesh M, with refinement $(M_n, f_n) \to (Z, f)$, is the **coarsest refining mesh** of (Z, f) if for any other refining n-mesh N, with refinement $(N_n, g_n) \to (Z, f)$, there is a refinement $(N_n, g_n) \to (M_n, f_n)$ factoring the latter refinement (of f by g_n) through the former refinement (of f by f_n). $\rightharpoondown$

The fact that coarsest refining meshes exists can be proven as follows.

Theorem 5.2.20 (Construction of coarsest refining meshes)**.** *Any flat n-framed stratification (Z, f) has a (necessarily unique) coarsest refining n-mesh.*

Proof. Given any two refining n-meshes M and N of (Z, f), note that the mesh join $M \vee N$ (as constructed in Key Lemma 5.2.10) yields another n-mesh refining (Z, f), which is coarser than both M and N. Since meshes are finite stratifications (which excludes the possibility of infinite chains of coarsenings starting at any given mesh) it follows that there must exists a unique coarsest mesh through which all other meshes factor as claimed. $\square$

Importantly, the construction of coarsest refining meshes is also compatible with framed stratified homeomorphisms of flat framed stratifications, in the following sense.[27]

Terminology 5.2.21 (Framed maps)**.** Given subspaces $Z \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^n$, a map $F : Z \to W$ is called a 'framed map' if for each $0 < i < n$, F descends along the standard projection $\pi_{>i} = \pi_{i+1} \circ \dots \pi_{n-1} \circ \pi_n : \mathbb{R}^n \to \mathbb{R}^i$ to a map $F_i : \pi_{>i} Z \to \pi_{>i} W$. $\rightharpoondown$

Terminology 5.2.22 (Framed stratified homeomorphism and mesh isomorphism)**.** A 'framed stratified homeomorphism' $(Z, f) \to (W, g)$ of flat framed stratifications is a stratified homeomorphism whose underlying map $Z \to W$ is framed. Note that, in contrast, we continue to speak of 'n-mesh isomorphisms' to mean isomorphisms in the category $\mathcal{M}esh_n$ (since meshes are not just mere stratifications, but towers of 1-mesh bundles). $\rightharpoondown$

Lemma 5.2.23 (Homeos of flat framed stratifications induce isos of their coarsest refining meshes)**.** *Let (Z, f) and (W, g) be flat n-framed stratifications with coarsest refining meshes M resp. N. If there is a framed stratified homeomorphism $F : f \cong g$ then F induces an n-mesh isomorphism $F : M \to N$ between their respective coarsening refining meshes.*

Proof. Since F is a framed stratified homeomorphism, note that there is a 'push-forward' mesh FM refining g: the mesh FM is determined by an n-mesh isomorphism $M \cong FM$ whose top component is F. Conversely, there is a 'pull back' mesh $F^{-1}N$ refining f determined by $N \cong F^{-1}N$ having top component F^{-1}. Note that, since N is the coarsest refining mesh of g, the refining mesh FM must be finer than N. If FM were

---

[27]Yet more generally, this observation in fact also holds for 'weak equivalences' of flat framed stratifications, i.e. weakly invertible maps of flat framed stratifications, in an appropriate sense.

strictly finer than N (that is, $FM \to N$ is a non-identity coarsening of FM), then $M = F^{-1}FM \to F^{-1}N$ would be a non-identity coarsening of M. This is impossible since M is the coarsest refining mesh of g. We deduce that $FM = N$, and thus we obtain an induced n-mesh isomorphism $M \cong N$ with top component $F : Z \to W$. Abusing notation, we denote this mesh isomorphism again by $F : M \cong N$. $\qquad\square$

Having shown the existence of coarsest refining meshes (and their compatibility with framed stratified homeomorphism of flat framed stratification), we now give several illustrative examples. The slogan for these illustrations is that 'coarsest refining meshes are refining meshes that record exactly where changes in framed stratified homeomorphism type of a flat framed stratification happen'. (Heuristically, they bear visual resemblance with 'critical level set' decompositions from (higher) Morse theory.)

Example 5.2.24 (Coarsest refining meshes of the circle, the braid and Hopf circle). In Fig. 5.4 we depict three flat framed stratifications (together with their respective coarsest refining meshes): these include flat 2-framed stratification of the open 2-cube obtained by an embedded circle and its complement, the flat 3-framed stratification of the open 3-cube given by the braid and its complement, and the flat 3-framed stratification given by the 'Hopf circle' in the 3-cube and its complement. Observe that the last two examples recover the first two examples in Fig. 5.1 up to a rotation of the ambient $\mathbb{R}^3$. $\qquad\lrcorner$
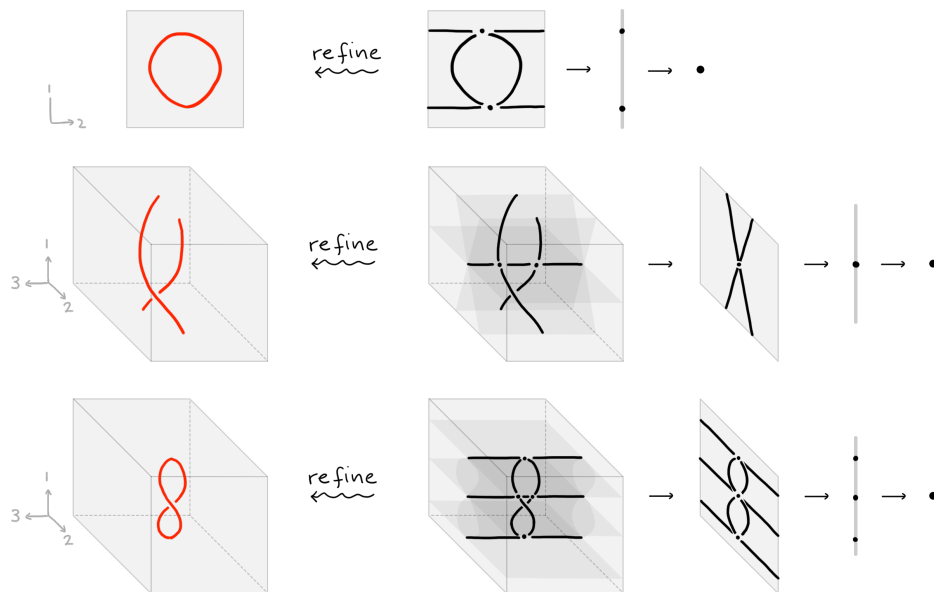

Figure 5.4: Coarsest refining meshes of the circle, the braid, and Hopf circle.

Example 5.2.25 (Non-coarsest refining meshes of the circle). In Fig. 5.5 we depict two refining 2-meshes of a flat framed stratification of the open 2-cube obtained by an embedded circle and its complement. Neither of the two refining meshes is the coarsest refining mesh. $\qquad\lrcorner$

Example 5.2.26 (Coarsest refining mesh of the cusp). In Fig. 5.6 we depict the coarsest refining mesh of the cusp singularity (which appeared as the last example in our earlier Fig. 5.1). $\qquad\lrcorner$
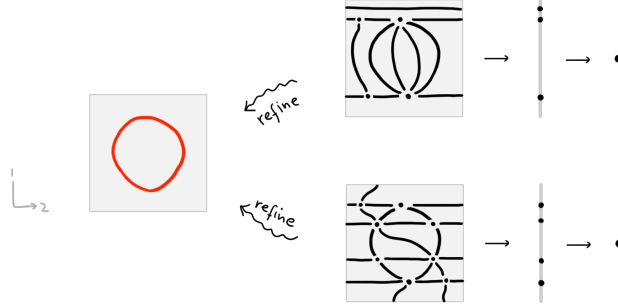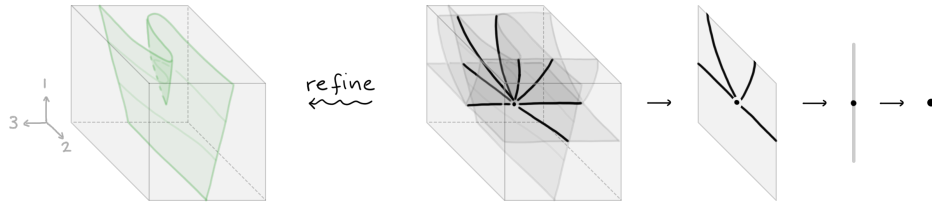
Figure 5.5: Non-coarsest refining meshes of the circle.

Figure 5.6: Coarsest refining mesh of the cusp.

Remark 5.2.27 (Twisted circle embedding). We revisit the Hopf circle embedding from Example 1.3.57. As a flat framed stratification of the open 3-cube, its coarsest refining mesh is an open 3-mesh M, as shown in Fig. 5.4. As a stratification of the closed cube, its coarsest refining mesh $\overline{M}$ is a closed 3-mesh (and equals, up to isomorphism, the classifying mesh $\mathsf{CMsh}\,\overline{\mathsf{ETrs}\,M}$ of the compactification $\overline{\mathsf{ETrs}\,M}$ of the entrance path truss $\mathsf{ETrs}\,M$ of M). The mesh complex of $\overline{M}$ (see Terminology 4.2.75) recovers exactly the flat framed regular cell complex given earlier in Example 1.3.57. ⌟

5.3 Tractability of flat framed stratifications

In this section we discuss the 'combinatorializability' of flat framed stratifications and its corollaries, as outlined in the beginning of the chapter. We prove the central theorem about the combinatorializability of flat framed stratifications in Section 5.3.1. A first set of corollaries relating to the 'polyhedrality' of flat framed stratifications will be discussed in Section 5.3.2. Finally, the theorem also implies the 'computability' of flat framed stratification as we will explain in Section 5.3.3.

5.3.1 Combinatorializability

5.3.1.1 Stratified trusses Recall from Definition 5.1.2, that a stratified n-truss is a poset labeled n-truss T (that is, a 'underlying' n-truss $\underline{T} = (T_n \to \dots \to T_0)$ together with a 'labeling' $\mathrm{lbl}_T : T_n \to P$ in some poset P) with the condition that the labeling lbl_T is the characteristic map of a stratification on T_n. This last condition can be more concretely rephrased as follows.

Remark 5.3.1 (Characteristic maps are connected-quotient maps). A 'quotient map' of posets is a surjective poset map for which subposets in the codomain are open, i.e.

downward closed, if and only if their preimages are open in the domain. A 'connected-quotient map' of posets is a quotient map of posets whose preimages are connected. A truss labeling $\mathrm{lbl}_T : T_n \to P$ in a poset P is the characteristic map of a stratification on T_n if and only if it is a connected-quotient map. This is shown (in slightly more general form) in Lemma B.1.29. ⌐

To highlight that a labeled truss $T = (\underline{T}, \mathrm{lbl}_T)$ is a *stratified* truss we will usually denote the poset of labels by $\mathsf{Entr}(T)$ (and thus write $\mathrm{lbl}_T : T_n \to \mathsf{Entr}(T)$ for the labeling of T). Before discussing the notion of stratified n-trusses further, let us give a first example.

Example 5.3.2 (Stratified trusses). In Fig. 5.7 we depict a stratified open 2-truss. ⌐

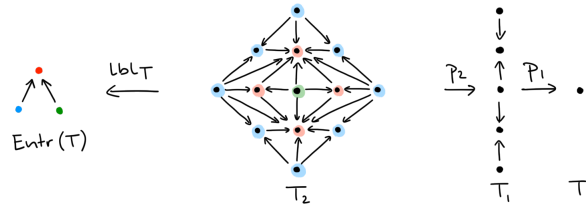
Figure 5.7: A stratified open 2-truss.

Terminology 5.3.3 (Entrance path posets and strata of stratified trusses). Given a stratified n-truss $T \equiv (\underline{T}, \mathrm{lbl}_T : T_n \to \mathsf{Entr}(T))$ we call $\mathsf{Entr}(T)$ the 'entrance path poset' of T. The 'strata' of T are the (by definition, connected) subposets of T_n given by the preimages of lbl_T. ⌐

Note that, up to isomorphism of entrance path posets, any decomposition of T_n into connected subposets fully determines a stratified n-truss T. One may further obtain stratified trusses from 'P-labelings' as follows.

Remark 5.3.4 (Poset-labeled trusses yield stratified trusses). Note that any poset labeled truss $T = (\underline{T}, \mathrm{lbl}_T : T_n \to P)$ yields a stratified n-truss $\tilde{T}$, by defining strata of $\tilde{T}$ to be the connected components of the non-empty preimages of lbl_T. (This 'connected component splitting' construction is formalized, in more general form, in Construction B.1.34.) While we will not formally need this construction here, it turns out to be somewhat useful when illustrating examples: it allows us to reduce the number of colors needed in the depiction of stratified trusses, by replacing a given connected-quotient map with some (non-unique choice of) poset map that has smaller domain and whose connected component splitting recovers the original connected-quotient map.

For instance, in Fig. 5.7 we may replace the labeling by a map to the poset $(0 \to 1)$, mapping both the blue and green stratum to 0, and the red stratum to 1. ⌐

The notion of maps for stratified n-trusses carries over from that of labeled n-trusses, but we may further make the following terminological distinctions.

Terminology 5.3.5 (Stratified maps and coarsenings). A 'stratified map' of stratified n-trusses T and S is a labeled n-truss map $F \equiv (\underline{F}, \mathrm{lbl}_F) : T \to S$ (that is, a map of n-trusses $\underline{F} : \underline{T} \to \underline{S}$, together with a map of labelings $\mathrm{lbl}_F : \mathsf{Entr}(T) \to \mathsf{Entr}(S)$ such that $\mathrm{lbl}_F \mathrm{lbl}_T = \mathrm{lbl}_S F_n$ commutes).

1. We call F a '(truss preserving) label coarsening' if $\underline{F} = \mathrm{id}$ is the identity truss map and lbl_F is a connected-quotient map (see Remark B.1.27).
2. We call F '(label preserving) truss coarsening' if $\underline{F}$ is an n-truss coarsening (see Terminology 2.3.97) and $\mathrm{lbl}_F = \mathrm{id}$.
3. We call F a 'coarsening' if $\underline{F}$ is an n-truss coarsening and $\underline{F}$ is a connected-quotient map. ⌟

Note that the definition of 'label coarsenings' reflects that, in the context of topological stratifications, connected-quotients of entrance path posets to poset describe exactly the coarsenings of a given stratification (see Lemma B.2.12).

Example 5.3.6 (A truss coarsening). We depict a label preserving truss coarsening $F : T \to S$ of stratified 2-trusses S and T in Fig. 5.8. ⌟
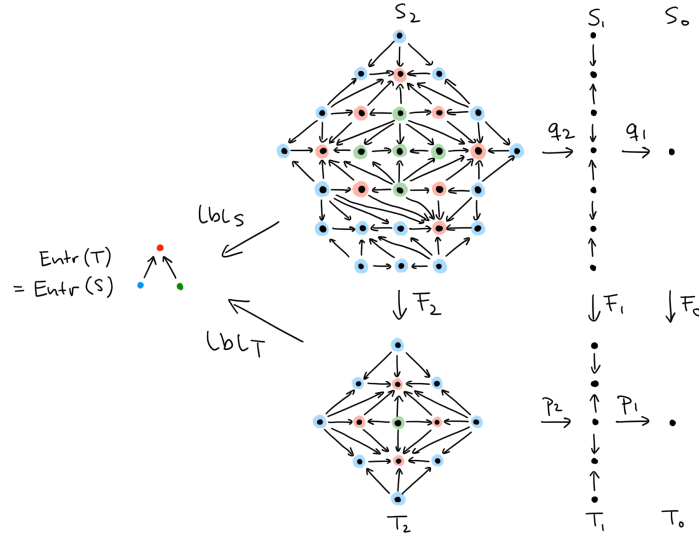

Figure 5.8: A truss coarsening of stratified trusses.

Note that every coarsening can be written both as a unique composite of a truss coarsening followed by a label coarsening, and as a unique composite of a label coarsening followed by a truss coarsening.

5.3.1.2 Stratified meshes Stratified trusses (as well as their maps) can be obtained from flat framed stratifications together with chosen refining meshes. We call the latter structure a 'stratified mesh'.

Definition 5.3.7 (Stratified mesh). A flat n-framed stratification (Z, f) together with a refining mesh M of (Z, f) will be called a **stratified mesh** and denoted by a tuple (M, f) (note that Z may be recovered as the subspace $M_n \subset \mathbb{R}^n$). ⌟

Definition 5.3.8 (Stratified mesh map). A **map of stratified meshes** $F : (M, f) \to (N, g)$ is a mesh map $F : M \to N$ whose top component is a map of stratifications $F_n : f \to g$. ⌟

Terminology 5.3.9 (Maps of stratified meshes and their coarsenings). Given a map of stratified meshes $F : (M, f) \to (N, g)$ we make the following terminological distinction.

1. If $F_n : f \cong g$, and if N is a coarser refining mesh of f than M (i.e. $F : M \to N$ is an n-mesh coarsening) then we call $F : (M, f) \to (N, f)$ a 'mesh coarsening' of stratified meshes.
2. If $F : M \cong N$ and g is coarser than f (i.e. $F_n : f \to g$ is a coarsening), then we call $F : (M, f) \to (N, f)$ a 'stratification coarsening' of stratified meshes.
3. If both $F : M \to N$ and $F_n : f \to g$ are coarsenings, we call $F : (M, f) \to (N, f)$ simply a 'coarsening' of stratified meshes.

⌞

Example 5.3.10 (A mesh coarsening)**.** We depict a mesh coarsening $F : (M, f) \to (N, g)$ of stratified 2-meshes (M, f) (with $M = (q_2, q_1)$) and (N, g) (with $N = (p_2, p_1)$) in Fig. 5.9.
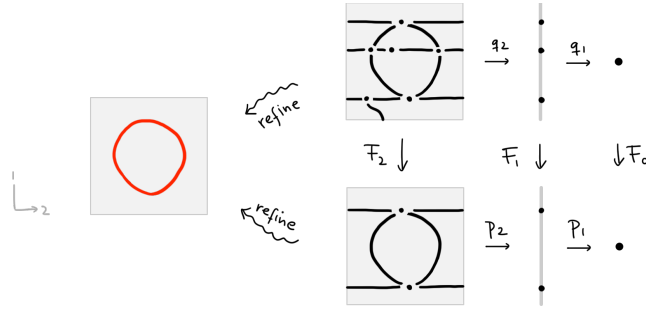
⌞

Figure 5.9: A mesh coarsening of stratified meshes.

5.3.1.3   Stratified entrance path trusses Every stratified mesh has a 'stratified entrance path truss'.

Definition 5.3.11 (Stratified entrance path trusses)**.** Given a stratified n-mesh (M, f) define the **stratified entrance path truss** $\mathsf{ETrs}(M, f)$ to be the stratified n-truss $(\mathsf{ETrs}\, M, \mathrm{lbl}_{M,f})$ whose underlying n-truss is the entrance path truss $\mathsf{ETrs}\, M$ of M, and whose labeling $\mathrm{lbl}_{M,f} : T_n \to \mathsf{Entr}(f)$ is the entrance path poset map $\mathsf{Entr}(f_n \to f) :$ $\mathsf{Entr}\, f_n \to \mathsf{Entr}\, f$ of the stratified coarsening $(M_n, f_n) \to (Z, f)$. ⌞

Definition 5.3.12 (Stratified entrance path truss maps)**.** Given a stratified mesh map $F :$ $(M, f) \to (N, g)$, define the **stratified entrance path truss map** $\mathsf{ETrs}\, F : \mathsf{ETrs}(M, f) \to$ $\mathsf{ETrs}(N, g)$ to be the map of stratified n-trusses whose underlying map of trusses $\mathsf{ETrs}(M \to N)$ is the entrance path truss map of the mesh map $F : M \to N$, and whose label map $\mathsf{Entr}(f \to g)$ is the entrance path poset map of the stratified map $F_n : f \to g$. ⌞

Observation 5.3.13 (Translations of notions of coarsenings)**.** Notions of coarsenings of stratified meshes translate to the corresponding notions of coarsenings of stratified trusses. Namely, given a mesh coarsening, a stratified coarsening, a coarsening of stratified meshes $(M, f) \to (N, f)$, then $\mathsf{ETrs}((M, f) \to (M, g)) : \mathsf{ETrs}(M, f) \to \mathsf{ETrs}(M, g)$ is a truss coarsening, resp. a label coarsening, resp. a coarsening of stratified trusses. ⌞

Example 5.3.14 (Translating mesh coarsenings to truss coarsenings)**.** The mesh coarsening in Fig. 5.9 translates to the truss coarsening in Fig. 5.8. ⌞

5.3.1.4 Stratified classifying meshes The stratified entrance path truss construction has a converse construction as follows. Recall that stratified coarsenings of a given stratification are exactly determined by connected-quotient maps of the stratification's entrance path poset (see Lemma B.2.12).

Definition 5.3.15 (Stratified classifying meshes of stratified trusses)**.** Let T be a stratified n-truss. Its **stratified classifying mesh** $\mathsf{CMsh}\,T$ is the stratified mesh $(\mathsf{CMsh}\,\underline{T}, \mathsf{f}_T)$ whose underlying mesh $\mathsf{CMsh}\,\underline{T}$ is the classifying mesh of the underlying n-truss $\underline{T}$ of T, and whose stratification f_T is the stratification of $Z = M_n \subset \mathbb{R}^n$ obtained by coarsening (M_n, f_n) by the connected-quotient map $\mathrm{lbl}_T : T_n \to \mathsf{Entr}(T)$ on entrance path posets. ⌟

Note in particular, that $(Z = M_n, \mathsf{f}_{T,f})$ is a flat framed stratification. The stratified classifying mesh construction extends to maps of stratified trusses as follows.

Definition 5.3.16 (Stratified classifying mesh maps of stratified trusses maps)**.** Given a map $F = (\underline{F}, \mathrm{lbl}_F) : T \to S$ of stratified trusses $T = (\underline{T}, \mathrm{lbl}_T)$ and $S = (\underline{S}, \mathrm{lbl}_S)$, we define the **stratified classifying mesh map** $\mathsf{CMsh}\,F : \mathsf{CMsh}\,T \to \mathsf{CMsh}\,S$ of stratified n-meshes to be the classifying n-mesh map $\mathsf{CMsh}\,\underline{F} : \mathsf{CMsh}\,\underline{T} \to \mathsf{CMsh}\,\underline{S}$ which, by definition of stratified classifying meshes, can be seen to descend to a stratified map $\mathsf{CMsh}\,F_n : \mathsf{f}_T \to \mathsf{f}_S$ as required. ⌟

The construction are mutually (weakly) inverse in the following sense.

Observation 5.3.17 (Mutual invertibility). Let T be a stratified truss, and (M, f) be a stratified mesh.

1. There is a unique isomorphism $T \cong \mathsf{ETrs}\,\mathsf{CMsh}(M, f)$. This follows since $\underline{T} \cong \mathsf{ETrs}\,\mathsf{CMsh}\,M$ uniquely, and from the definitions of stratified entrance path trusses resp. stratified classifying meshes.
2. There is a (up to contractible choice of homotopy unique) isomorphism of stratified meshes $(M, f) \cong \mathsf{CMsh}\,\mathsf{ETrs}(M, f)$. This follows since there is a (up to contractible choice of homotopy unique) mesh isomorphism $M \cong \mathsf{CMsh}\,\mathsf{ETrs}\,M$, and from the definitions of stratified classifying meshes resp. stratified entrance path trusses. ⌟

Observation 5.3.18 (Translations of notions of coarsenings). Conversely to Observation 5.3.13, notions of coarsenings of stratified trusses translate to the corresponding notions coarsenings of stratified meshes. Namely, given a truss coarsening (resp. label coarsening resp. coarsening) of stratified trusses $T \to S$, then $\mathsf{ETrs}(T \to S) : \mathsf{ETrs}\,T \to \mathsf{ETrs}\,S$ is a mesh coarsening (resp. stratification coarsening resp. coarsening) of stratified meshes. ⌟

While not used for the main result of the chapter, the case of coarsenings will be of separate importance to us, and, as discussed before, it needs additional care (see Observation 4.2.69). Recall that truss coarsenings can be realized as mesh coarsenings, via the 'classifying mesh coarsening' construction (see Construction 4.2.72).

Definition 5.3.19 (Stratified classifying mesh coarsening of stratified trusses coarsening)**.** Given a coarsening of stratified trusses $F : T \to S$, we define the **stratified classifying mesh coarsening** $\mathsf{CMsh}\,F : \mathsf{CMsh}\,T \to \mathsf{CMsh}\,S$ of stratified n-meshes to be the classifying n-mesh coarsening $\mathsf{CrsMsh}\,\underline{F} : \mathsf{CMsh}\,\underline{T} \to \mathsf{CMsh}\,\underline{S}$ which, by definition of stratified

classifying meshes, can be seen to descend to a stratified map $\mathsf{CrsMsh}\, F_n : \mathsf{f}_T \to \mathsf{f}_S$ as required. ⌐⌐

5.3.1.5 Normalization and combinatorializability Recall the definition of 'coarsest refining meshes M' of flat framed stratifications (Z, f): this is precisely a stratified meshes (M, f) to which no non-identity mesh coarsening applies. Combinatorially, this motivates our earlier Definition 5.1.3 of 'normalized' stratified trusses: namely, a normalized stratified truss T is precisely a stratified truss to which no non-identity truss coarsening applies. Our earlier Observation 5.3.17 about the correspondence of stratified meshes and stratified trusses now specializes to a correspondence of coarsest stratified meshes (meaning stratified meshes (M, f) such that M is the coarsest refining mesh of f) and normalized stratified trusses up to isomorphism. This 'almost' proves Theorem 5.1.4, claiming the correspondence of flat framed stratifications and normalized stratified trusses up to isomorphism. The only missing ingredients lies in the comparison of framed stratified homeomorphisms of flat framed stratifications and isomorphisms of coarsest stratified meshes; but this comparison was exactly provided by our earlier Lemma 5.2.23.

Proof of Theorem 5.1.4. Given a normalized stratified truss T, define its corresponding flat framed stratification to be f_T where $(\mathsf{CMsh}\, T, \mathsf{f}_T)$ is the stratified classifying mesh of T. Note that, given another normalized stratified truss S and $T \cong S$ by a label preserving balanced truss isomorphism, then $(\mathsf{CMsh}\, T, \mathsf{f}_T) \cong (\mathsf{CMsh}\, S, \mathsf{f}_S)$ as stratified meshes which descends to a framed stratified homeomorphism $\mathsf{f}_T \cong \mathsf{f}_S$ of flat framed stratifications. Conversely, given a flat framed stratification (Z, f), define its corresponding normalized stratified truss T to be the stratified entrance path truss $\mathsf{ETrs}(M, f)$ where M is the coarsest refining mesh of f. Note that, given another flat framed stratification (W, g), with $f \cong g$, we obtain an isomorphism of stratified meshes $(M, f) \cong (N, g)$ (where N is the coarsest refining mesh of g) by Lemma 5.2.23, and thus a label preserving balanced truss isomorphism $\mathsf{ETrs}(M, f) \cong \mathsf{ETrs}(N, g)$. Now, Observation 5.3.17 implies that both constructions are mutually inverse to each other, which proves the claim of the theorem. □

Corollary 5.3.20 (Normalization vs coarsest mesh refinements)**.** *Let (M, f) be a stratified mesh and T a stratified truss such that $T \cong \mathsf{ETrs}(M, f)$ (or equivalently, $(M, f) \cong \mathsf{CMsh}\, T$). Then T is normalized if and only if M is the unique coarsest mesh refining f.* □

We will use the rest of this section to briefly illustrate the correspondence established by Theorem 5.1.4.

Example 5.3.21 (The normalized stratified truss of the circle)**.** The normalized stratified truss corresponding to the flat framed stratification of the circle in the open 2-cube (see the first example in Fig. 5.4) is the stratified truss given in Fig. 5.7. ⌐⌐

Example 5.3.22 (The normalized stratified truss of the braid)**.** The normalized stratified truss corresponding to the braid (see the second example in Fig. 5.4) is the stratified 3-truss depicted in Fig. 5.10 (note that for readability we depict only generating arrows, see Construction 2.3.69). Note that this fulfils our earlier promises in Preview 2.3.5. ⌐⌐
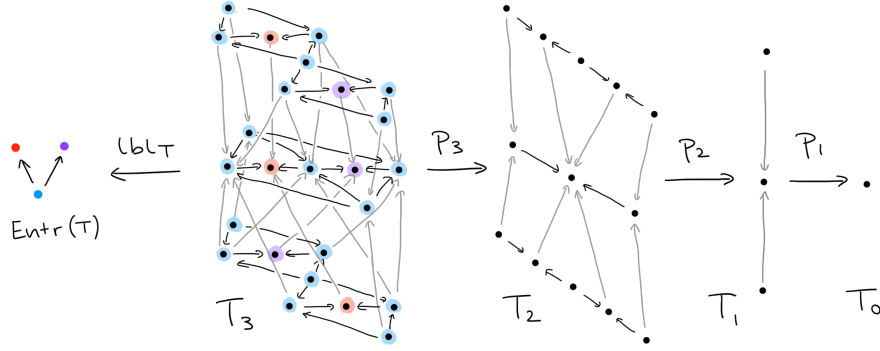
Figure 5.10: The normalized stratified truss of the braid.

Example 5.3.23 (The normalized stratified truss of the cusp)**.** The normalized stratified truss corresponding to the cusp singularity (see Fig. 5.6) is the stratified 3-truss depicted in Fig. 5.11 (note again, that for readability we depict only generating arrows). ⌐
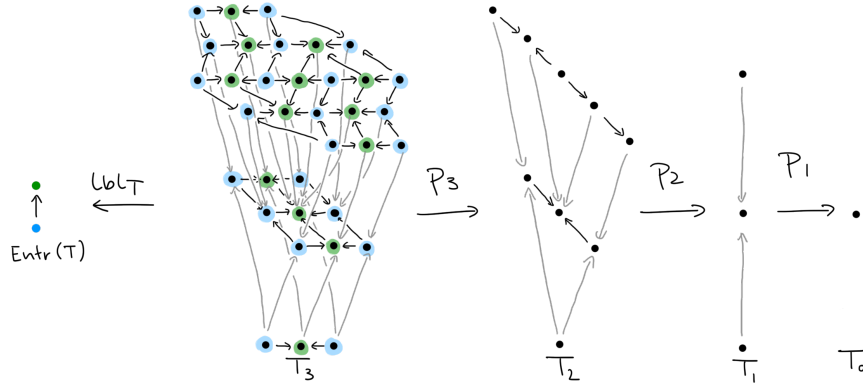

Figure 5.11: The normalized stratified truss of the cusp.

5.3.1.6   The case of bundles We very briefly mention how our combinatorializability results generalize from flat framed stratifications to *bundles* of flat framed stratifications, addressing our earlier Remark 5.1.5.

Firstly, a 'flat framed stratifications over (B, g)' is of course a stratification of a subspace of $B \times \mathbb{R}^n$ that can be refined by an n-mesh bundle over (B, g). Assuming B is sufficiently nice (e.g. finite frontier-constructible), the proofs of Theorem 5.2.20 and Lemma 5.2.23, now using Lemma 5.2.17 in the place of Key Lemma 5.2.10, immediately carry over to case of bundles as follows.

Theorem 5.3.24 (Coarsest refining mesh bundles)**.** *For finite frontier-constructible (B, g), any flat framed stratification bundle (Z, f) over (B, g) has a unique coarsest refining mesh bundle. Moreover, any stratified bundle homeomorphism of flat framed stratification bundles over (B, g) preserves this coarsest refining mesh bundle.* ☐

Secondly, a 'stratified n-truss bundle' is of course a labeled truss n-bundle whose labeling is a characteristic map, which stratifies the total poset of the bundle (with the condition that strata live in the poset fibers of the bundle). A stratified n-truss bundle is

normalized if no truss bundle coarsening applies to it (where a 'truss bundle coarsening' is a label and base preserving truss bundle map that is fiberwise a truss coarsening). Assuming (B, g) is also cellulable, our results from chapter 4 apply (at least, in the case of closed resp. open bundles). Re-using the proof of Theorem 5.1.4 with evident adaptations then yields the following result.

Theorem 5.3.25 (Normalized stratified truss bundles classify flat framed stratification bundles). *Let (B, g) be a finite, open cellulable and frontier-constructible stratification. Framed stratified bundle homeomorphism classes of (closed resp. open) flat n-framed stratification bundles over (B, g) are in correspondence with isomorphism classes of (closed resp. open) normalized stratified n-truss bundles over $\mathsf{Entr}(g)$.* □

Certainly, with further care, the result can also be generalized to the case of general bundles and/or to the case of over non-finite, but locally finite base stratifications.[28] Note that, while the conditions on the base stratifications are lengthy sounding, the conditions are rather mild (e.g. any finite conical PL stratification satisfies them).

5.3.2   Polyhedrality In this section we discuss the relationship of flat framed stratifications and 'piecewise linear stratifications'. We introduce the latter notion below. Note that all simplicial complexes K in this section will be assumed to be compact and 'linearly realized' in $\mathbb{R}^n$, i.e. they come equipped with an embedding $|K| \hookrightarrow \mathbb{R}^n$ of their geometric realization with compact image, and such that the embedding is linear on each simplex (in particular, note that the underlying space of a simplicial complex, as a subspace of $\mathbb{R}^n$, is a polyhedron in the usual sense, see [RS72, Defn. 1.1]).

Definition 5.3.26 (Piecewise linear stratifications). **A piecewise linear stratification** is a stratification (Z, f), $Z \subset \mathbb{R}^n$, for which there exists a 'triangulating' simplicial complex K in $\mathbb{R}^n$ such that each stratum of f is a union of open simplices in K. ⌐⌙

Note that we do not require Z to have the same support as K (as a subspace in $\mathbb{R}^n$). In particular, Z itself may be non-compact. We further record terminology for those piecewise linear stratifications whose underlying space Z is that of a mesh.

Terminology 5.3.27 (Mesh-supported piecewise linear stratifications). A piecewise linear stratification (Z, f) is called 'mesh-supported' if $Z \subset \mathbb{R}^n$ is the space underlying some n-mesh $M \subset \mathbb{R}^n$. ⌐⌙

Note that every piecewise linear stratification (Z, f) can be trivially extended to a mesh-supported one, by the following construction. Pick $r \in \mathbb{R}$ large enough such that the open n-cube $(-r, r)^n$ contains Z and such that $(-r, r)^n \setminus Z$ has one connected component. Since the open cube $(-r, r)^n$ underlies an open n-mesh (for instance, the terminal open n-mesh), we obtain a stratification (Z^+, f^+) where $Z^+ = (-r, r)^n$ and f^+ contains strata of f together with $(-r, r)^n \setminus Z$ as a new 'ambient' stratum. One checks that (Z^+, f^+) is again a piecewise linear stratification (in fact, this also follows from our later Observation 5.3.29). Therefore, 'up to adding an ambient stratum', we may think of all piecewise linear stratification as being mesh-supported. Note, for notational convenience (and 'up to scaling') we will usually assume $r = 1$, and work in the open unit n-cube $\mathbb{I}^n := (-1, 1)^n$.

---

[28]Meaningfully weakening the conditions of cellulability and frontier-constructibility is maybe less obvious.

As a first, and most immediate corollary to the combinatorializability of flat framed stratifications, let us observe that any flat framed stratification is in fact framed stratified homeomorphic to a piecewise linear stratification (which is necessarily 'mesh-supported'). This proves our earlier claim in Corollary 5.1.6.

Proof of Corollary 5.1.6. Given a flat framed stratification (Z, f), let M be its coarsest refining mesh. Then f is framed stratified homeomorphic to the piecewise linear stratification $f_{\mathsf{ETrs}(M,f)}$ obtained from the stratification of the stratified classifying mesh $\mathsf{CMsh}\,\mathsf{ETrs}(M, f)$ (note that the fact that $f_{\mathsf{ETrs}(M,f)}$ is piecewise linear follows from our construction of classifying meshes, which realizes truss posets linearly in $\mathbb{R}^n$, see Definition 4.2.47 and Definition 4.2.59). $\qquad\square$

We will now prove two further corollaries of the combinatorializability of flat framed stratifications in this section. Firstly, we address a 'converse' to Corollary 5.1.6: namely, we will show that every piecewise linear stratification is, in fact (up to ambient extension) a flat framed stratification, which proves the claim of our earlier Proposition 5.1.8. Secondly, we will discuss a variation of a classical question posed by the so-called 'Hauptvermutung': this asks whether topological homeomorphism of piecewise linear structures can be, up to homotopy, replaced by piecewise linear homeomorphism. As we will recall, the conjecture fails to hold in the classical setting of piecewise linear stratifications in $\mathbb{R}^n$. We then show the conjecture does hold when appropriately adapted to the flat framed setting, proving our earlier Corollary 5.1.7.

5.3.2.1 Piecewise linear stratifications are flat framed

We prove the claim that any piecewise linear stratification (up to passing to its 'ambient extension') defines a flat framed stratification in $\mathbb{R}^n$. The proof will rely on the following observations.

Observation 5.3.28 (Intersection refinements). Let K be a finite simplicial complex, and $F : |K| \to \mathbb{R}^n$ a simplex-wise linear (not-necessarily injective) map. By standard results, there exists a simplex-wise linear embedding $G : |L| \hookrightarrow \mathbb{R}^{n-1}$ of some simplicial complex L, such that the image $F\,|x|$ of each simplex x in K is a union of images $G\,|y|$ of simplices y in L (see [RS72, Thm. 2.15]). We call the simplicial complex L (linearly realized in $\mathbb{R}^n$ via G) an 'intersection refinement' of F. $\qquad\lrcorner$

Observation 5.3.29 (Refining meshes of simplicial complexes). Let K be a simplicial complex linearly realized in $\mathbb{R}^n$. Up to scaling, we may assume K lives in the open unit n-cube $\mathbb{I}^n = (-1, 1)^n$. Consider K as a piecewise linear stratification in which each open simplex is a stratum, and pass to its ambient extension K^+ in $\mathbb{I}^n$ (whose strata are open simplices of K, as well as the complement $\mathbb{I}^n \setminus K$). We construct an open n-mesh M that refines K^+. First, restrict the projection $p_n : \mathbb{I}^n \to \mathbb{I}^{n-1}$ to K, obtaining a map $p_n : K \to \mathbb{I}^{n-1}$ and, via Observation 5.3.28, construct an intersection refinement $L \hookrightarrow \mathbb{I}^{n-1}$ for it. Observe that the projection $p_n : K^+ \to L^+$ (with codomain the ambient extension of L in $\mathbb{I}^{n-1}$) is 'almost' a 1-mesh bundle (but generally, fails to be a stratified map): indeed, if we define K_L^+ to be the refinement of K^+ whose strata are connected components of intersections $p_n^{-1}(s) \cap r$ where s and r are strata in L^+ resp. K^+ (note, if r is an open simplex in K, then this intersection is necessarily connected) then, since L is an intersection refinement, one checks that $p_n : K_L^+ \to L^+$ is now a 1-mesh bundle.

Now, arguing inductively, we construct an $(n-1)$-mesh $M_{<n}$ which refines L^+, consisting of 1-mesh bundles $p_i : (M_i, f_i) \to (M_{i-1}, f_{i-1})$ (where $M_i = \mathbb{I}^i$). To construct the n-mesh M, we augment $M_{<n}$ by a 1-mesh bundle $p_n : (M_n, f_n) \to (M_{n-1}, f_{n-1})$ defined as the pullback bundle of the 1-mesh bundle $p_n : K_L^+ \to L^+$ along the refinement $(M_{n-1}, f_{n-1}) \to L^+$. Since $(M_n, f_n) \to K_L^+ \to K^+$ is a refinement, this completes the construction of the mesh M refining K. ⌐

Example 5.3.30 (Piecewise linear stratifications are flat framed). In Fig. 5.12 we depict a simplicial complex K linearly realized in $\mathbb{R}^2$ (and bounded by the 2-cube $\mathbb{I}^2$) together with a 2-mesh M refining it, as it may be constructed via Observation 5.3.29. ⌐
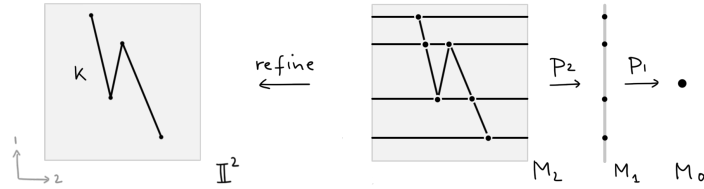

Figure 5.12: A simplicial complex in $\mathbb{R}^2$ with refining mesh.

The preceding observations assemble into a proof of Proposition 5.1.8, which claimed that all piecewise linear stratifications are flat framed, as follows.

Proof of Proposition 5.1.8. Let (Z, f) be a piecewise linear stratification with triangulating simplicial complex K. Up to scaling, we may assume (Z, f) and K lie inside the open unit n-cube $\mathbb{I}^n = (-1, 1)^n$. Using Observation 5.3.29 we can construct an n-mesh M refining the ambient extension K^+, and thus also refining the ambient extension $(\mathbb{I}^n, f^+)$. This shows that (up to ambient extension) (Z, f) is a flat framed stratification as claimed. □

Observation 5.3.31 (Refining meshes of mesh-supported piecewise linear stratification). Let (Z, f) be a piecewise linear stratification. Assume that (Z, f) is mesh-supported (in the sense of Terminology 5.3.27). Then, as in the preceding proof, we can construct a refining mesh M of the ambient extension $(\mathbb{I}^n, f^+)$ of (Z, f) to the open n-cube $\mathbb{I}^n$. Now, we may restrict the n-mesh M (and its tower of 1-mesh bundles $p_i : (M_i, f_i) \to (M_{i-1}, f_{i-1})$) to the constructible substratification (M_n^Z, f_n^Z) of (M_n, f_n) of strata lying in the subspace Z of $\mathbb{I}^n$ (in particular, $M_n^Z = Z$ as subspaces in $\mathbb{I}^n$). The assumption that (Z, f) is mesh supported implies that this defines a tower of 1-mesh bundle, and thus an n-mesh M^Z. This mesh refines (Z, f), which shows that (Z, f) itself is in fact a flat framed stratification (without the need for ambient extension to $\mathbb{I}^n$). ⌐

Finally, the following condition on refining meshes of piecewise linear stratifications will be useful to record. Recall from Definition 4.2.59, that the classifying mesh $\mathsf{CMsh}\,T$ of an n-truss T is obtained as a constructible substratification of the classifying mesh $\mathsf{CMsh}\,\overline{T}$ of its compactification $\overline{T}$, with the latter mesh being isomorphic to the classifying stratification $\mathsf{CStr}\,\overline{T}$. An 'open simplex' in $\mathsf{CMsh}\,T$ will refer to an open simplex in the simplicial complex $|\overline{T}_n|$ (triangulating the stratification $\mathsf{CMsh}\,\overline{T}_n$) that lies in $\mathsf{CMsh}\,T_n \hookrightarrow \mathsf{CMsh}\,\overline{T}_n$.

Terminology 5.3.32 (Linear meshes). An n-mesh M is called a 'linear mesh' if, setting $T = \mathsf{CMsh}\,M$, there is an isomorphism $\mathsf{CMsh}\,T \cong M$ that is linear on each open simplex in $\mathsf{CMsh}\,T$. ⌐

Observation 5.3.33 (Linear refining meshes). Every finite simplicial complex K (and similarly, every piecewise linear stratification) has, up to ambient extension, a linear refining n-mesh M. Indeed, this follows inductively, if in the inductive step of Observation 5.3.29 we chose $M_{<n}$ to be a linear $(n-1)$-mesh. Tracing through the rest of the construction one checks that M as constructed is necessarily linear as well. As an example, note that the refining 2-mesh in Fig. 5.12 is linear. ⎯⎤

5.3.2.2 Flat framed Hauptvermutung We next prove that notions of framed stratified homeomorphisms and piecewise linear framed stratified homeomorphism coincide, as recorded in the 'flat framed Hauptvermutung'. We begin with a brief recollection of the classical Hauptvermutung (see e.g. [RCS$^+$96]).

Disproven Conjecture 5.3.34 (Hauptvermutung). *Every homeomorphism $P \cong Q$ between polyhedra P, Q is homotopic to a PL homeomorphism.*

The Hauptvermutung, and its related conjectures, have been famously disproven in various ways: not only are there polyhedra with different PL structures (where 'PL structure' means 'PL homeomorphism class'), but there are also topological manifolds with different PL structures and even topological manifolds that don't admit a PL structures at all (that is, the inclusion of PL manifolds into topological manifolds is not surjective on homeomorphism classes). The failure of the Hauptvermutung may be understood as a symptom of the 'wildness' of topological homeomorphism which finds no counterpart in combinatorial topology; 'taming' this wildness is possible (making, in particular, the Hauptvermutung a true statement) while usually rather technically cumbersome (cf. [Shi14] [Shi13]). Our goal in this section will be to show that a 'flat framed' variation of the Hauptvermutung holds naturally in the setting of flat framed stratifications. An adequate classical analog to this statement (for the case of stratifications embedded in $\mathbb{R}^n$) may be phrased as follows. Note that we take a 'stratified homotopy' to mean a homotopy of stratified maps that is constant on entrance path posets.

Disproven Conjecture 5.3.35 (Ambient stratified Hauptvermutung). *Every stratified homeomorphism between (mesh-supported) piecewise linear stratifications is stratified homotopic to a piecewise linear stratified homeomorphism.*

Like its non-ambient counterpart the ambient Hauptvermutung fails to hold in general.[29]

Disproof of the ambient Hauptvermutung. We construct homeomorphic mesh-supported piecewise linear stratification f and g by piecewise linear embeddings $P_0 \hookrightarrow \mathbb{I}^n$ and $P_1 \hookrightarrow \mathbb{I}^n$ of polyhedra (where $\mathbb{I}^n$ is the open unit cube, stratified by P_i and its complement in each case) as follows.

Step 1: Take simplicial complexes K and L that are both PL homotopy 5-tori but not PL homeomorphic. Then both K and L are homeomorphic to the 5-torus T^5. (*References for step 1*: Hsiang and Shaneson showed that there are non-standard PL homotopy 5-tori [HS70]. Later, Hsiang and Wall showed that all PL homotopy 5-tori are in fact homeomorphic to the 5-torus [HW69].)

Step 2: PL embed both K and L in $\mathbb{I}^n$; the image of the embeddings define the polyhedra $P_0 \hookrightarrow \mathbb{I}^n$ and $Q_0 \hookrightarrow \mathbb{I}^n$. We can assume these embeddings are locally flat.

---

[29]We are indebted to Mark Powell for outlining the given counter-example to us.

(*References for step 2*: Rourke and Sanderson construct the required embeddings in [RS72, Theorem 5.5]; the following remark says they are locally flat. In the case of homotopy 5-tori the construction requires $n = 10$.)

Step 3: If we enlarge to $n = 12$, then the topological embeddings $P_0 \hookrightarrow \mathbb{I}^n$ and $Q_0 \hookrightarrow \mathbb{I}^n$ become ambient isotopic by a compactly supported isotopy since P_0 are Q_0 are both the 5-torus up to homeomorphism. (*References for step 3*: This follows from [DV09, Thm 4.4.2]; see also Corollary 4.4.3 loc. cit. This states homotopic topological embeddings of high codimension into a PL manifold, with 1-LCC images, are compactly supported ambient isotopic. Here 1-LCC stands for '1-local-co-connected' and is implied by the embeddings being locally flat (see Prop. 1.3.1 loc. cit.).)

Step 4: The last time slice of the isotopy provides the topological homeomorphism of pairs. A PL homeomorphism of pairs cannot exist by initial choice of PL structures. □

In contrast, the 'flat framed' variation of the Hauptvermutung is true. Recall, as stated in Corollary 5.1.7, this claims that any framed stratified homeomorphism between piecewise linear flat framed stratifications is stratified homotopic to some framed piecewise linear homeomorphism. In fact we will see that the homotopy is unique up to contractible choice (among stratified homotopies of framed stratified homeomorphisms).

Proof of Corollary 5.1.7. Let (Z, f) and (W, g) be piecewise linear flat framed stratification that are framed stratified homeomorphic by a framed stratified homeomorphism $F : (Z, f) \cong (W, g)$. By definition of piecewise linearity, (Z, f) and (W, g) have triangulations K resp. L. Using Observation 5.3.31 construct n-meshes M and N that refine the triangulations K resp. L (restricted to the subspaces Z resp. W). In particular, M and N refine (Z, f) resp. (W, g). By Observation 5.3.33, we may choose M and N to be linear n-meshes, that is, $\mathsf{CMsh}\, T \cong M$ and $\mathsf{CMsh}\, S \cong N$ piecewise linearly where $T = \mathsf{ETrs}\, M$ and $S = \mathsf{ETrs}\, N$. Let Q be the coarsest refining mesh of (Z, f), and set $R = \mathsf{ETrs}\, Q$. Since coarsest refining meshes are compatible with framed stratified homeomorphism (see Lemma 5.2.23) it follows that $F : Q \to FQ$ yields the coarsest refining mesh FQ of (W, g). In particular, Q is coarser than M, and FQ is coarser than N, and thus R is coarser than both T and S by the coarsenings $\mathsf{ETrs}(M \to Q)$ resp. $\mathsf{ETrs}(N \to FQ)$. Recall the construction of 'classifying mesh coarsenings' (see Construction 4.2.72) and note that these are linear on each open simplex in their domain. This allows us to construct piecewise linear coarsenings $\mathsf{CrsMsh}(T \to R)$ and $\mathsf{CrsMsh}(S \to R)$. We obtain the maps in the upper row of the following diagram

$$
\begin{array}{ccccccccc}
(M_n, f_n) & \longleftarrow & \mathsf{CMsh}\, T_n & \longrightarrow & \mathsf{CMsh}\, R_n & \longleftarrow & \mathsf{CMsh}\, S_n & \longrightarrow & (N_n, g_n) \\
\downarrow & & & & \sim & & & & \downarrow \\
(Z, f) & & & & \xrightarrow{\quad\quad\quad F \quad\quad\quad} & & & & (W, g) \quad .
\end{array}
$$

This is a zig-zag of piecewise linear homeomorphisms (namely, the outer two arrows) and piecewise linear coarsenings (namely, the inner two arrows). Since all maps are piecewise linear homeomorphisms of underlying spaces, the upper row defines piecewise linear homeomorphism $G : Z \to W$. By construction, the map also induces a stratified homeomorphism $G : (Z, f) \to (W, g)$ which therefore provides a piecewise linear stratified homeomorphism between the two stratifications.

Finally, note that F and G induce n-mesh maps $F, G : Q \to FQ$ on the respective minimal meshes of (Z, f) and (W, g) which are identical on entrance path trusses. Thus (by faithfulness of the entrance path functor construction, see Proposition 4.2.40 and Remark 4.2.42) we deduce that F and G are in fact homotopic as n-mesh maps (in fact, uniquely so up to contractible choice). It follows that they are homotopic as stratified maps $F, G : (Z, f) \to (W, g)$ as well (in fact, again uniquely so up to contractible choice). This proves that every framed stratified homeomorphism of piecewise linear stratification is homotopic to a piecewise linear framed stratified homeomorphism as claimed. $\qquad \square$

Remark 5.3.36 (Equivalence of all triangulations). Let (Z, f) be a piecewise linear stratification with two different triangulations K and L. Running the preceding proof (setting $(W, g) = (Z, f)$ and $F = \mathrm{id}$) shows that all triangulations are piecewise linearly equivalent. $\qquad \dashv$

Remark 5.3.37 (Contractibility of framed structure groups). The fact that the Hauptvermutung fails in the nonframed setting but holds in the framed setting is underpinned by the following intuition: while the automorphism groups $\mathrm{Aut}_{\mathrm{TOP}}(\mathbb{R}^n)$ and $\mathrm{Aut}_{\mathrm{PL}}(\mathbb{R}^n)$ differ, the groups of framed automorphisms $\mathrm{Aut}^{\mathrm{fr}}_{\mathrm{TOP}}(\mathbb{R}^n)$ and $\mathrm{Aut}^{\mathrm{fr}}_{\mathrm{PL}}(\mathbb{R}^n)$ (see Terminology 5.2.21) are both contractible and thus trivially equivalent. In fact, one would expect the same should hold using classical notions of framings, that is, the notions of 'framed topological' and 'framed PL' homeomorphisms (up to details of their definitions) should coincide. $\qquad \dashv$

5.3.3 Computability Finally, we briefly address two basic computability questions about the theory of flat framed stratifications. First, we will see that 'coarsest refining meshes' may in fact be algorithmically constructed. Secondly, as a corollary we will find that it can always be decided whether or not two framed stratifications are framed stratified homeomorphic or not. The idea is to reduce both problems to combinatorial problems by passing to entrance path posets.

Proposition 5.3.38 (Truss coarsenings are confluent)**.** *Let T be a stratified n-truss. Any chain of non-identity truss coarsenings of T eventually ends in the same normalized n-truss $[\![T]\!]$, and the truss coarsening $T \to [\![T]\!]$ is unique.*

Terminology 5.3.39 (Normal forms). We call $[\![T]\!]$ the 'normal form' of T. $\qquad \dashv$

Proof. Let $F_1 : T \to T_1$ and $F_2 : T \to T_2$ be non-identity truss coarsenings of T. The underlying maps of trusses are coarsenings $\underline{F}_i : \underline{T} \to \underline{T}_i$ of $\underline{T}$. Construct the classifying mesh coarsenings $\mathsf{CrsMsh}\,\underline{F}_i : \mathsf{CMsh}\,\underline{T} \to \mathsf{CMsh}\,\underline{T}_i$ using Construction 4.2.72 (up to pulling back $\mathsf{CMsh}\,\underline{T}_i$ along $\mathsf{CrsMsh}\,\underline{F}_i$ we will assume that $\mathsf{CrsMsh}\,\underline{T}_i$ and $\mathsf{CrsMsh}\,\underline{T}$ all have the same support in $\mathbb{R}^n$). By Key Lemma 5.2.10, we may take the join of $\mathsf{CMsh}\,\underline{T}_1$ and $\mathsf{CMsh}\,\underline{T}_2$ obtain the mesh $\mathsf{CMsh}\,\underline{T}_1 \vee \mathsf{CMsh}\,\underline{T}_2$, and abbreviate it by M. Construct the stratified classifying mesh $(\mathsf{CMsh}\,\underline{T}, \mathsf{f}_T)$ of T. Note that, since F_1 and F_2 are truss coarsenings, both $\mathsf{CMsh}\,\underline{T}_1$ and $\mathsf{CMsh}\,\underline{T}_2$ refine f_T. Thus their join M refines f_T as well. Passing to stratified entrance path trusses, this yields a diagrams of truss coarsenings

$$
\begin{array}{ccc}
 & T_i & \\
T \overset{F_1}{\underset{F_2}{\rightrightarrows}} & & S \\
 & T_2 &
\end{array}
$$

where $T_i = \mathsf{ETrs}(\mathsf{CMsh}\,\underline{T}_2, \mathsf{f}_T)$ and $S := \mathsf{ETrs}(M, \mathsf{f}_T)$. Since any chain of truss coarsenings of T must ends eventually in some normalized truss, to above implies all chains must end in the same normalized n-truss $[\![T]\!]$. Note also that the truss coarsening $T \to [\![T]\!]$ is unique (otherwise, if there were two truss coarsening F_1 and F_2, repeat the above steps to contradict that $[\![T]\!]$ is normalized). $\qquad\square$

Remark 5.3.40 (Mesh joins as truss pushouts). While the preceding proof of Proposition 5.3.38 crucially relies on our earlier construction of mesh joins, one may also prove the statement in purely combinatorial terms. Namely, the constructed diagram of truss coarsening in the proof of Proposition 5.3.38 is a pushout of the n-truss coarsenings $\underline{F}_1 : \underline{T} \to \underline{T}_1$ and $\underline{F}_2 : \underline{T} \to \underline{T}_2$ in the category of n-trusses and their maps, and these pushouts can be shown to exist by purely combinatorial arguments. The construction of pushouts of truss coarsenings may be regarded as a combinatorial counterpart to the construction of joins of meshes (see Remark 5.2.4). $\qquad\dashv$

Observation 5.3.41 (Normal forms are computable). Given a stratified truss T, there is a finite set of label preserving surjective truss maps $F : T \to S$ from T to another stratified truss S. Searching this set for truss coarsenings, and selecting the truss coarsening with smallest codomain, provides an algorithm to compute the normal form coarsening $T \to [\![T]\!]$.[30] $\qquad\dashv$

Recall Corollary 5.1.10, which stated that coarsest refining meshes of flat n-framed stratifications (Z, f) can be algorithmically computed.

Proof of Corollary 5.1.10. Let (Z, f) be a flat framed stratification. Since it is flat framed it must have some refining mesh M (we exclude the non-constructive existence of refining meshes). Compute its entrance path truss $\mathsf{ETrs}(M, f)$ and the resulting normal form $[\![\mathsf{ETrs}(M, f)]\!]$. This yields a coarsening of M (determined by the coarsening $\mathsf{ETrs}(M, f) \to [\![\mathsf{ETrs}(M, f)]\!]$ on entrance path posets), that coarsens M to the coarsest refining mesh of (Z, f). $\qquad\square$

Immediately, we deduce that the question of framed stratified homeomorphism between flat framed stratifications is decidable, as stated Corollary 5.1.11.

Proof of Corollary 5.1.11. Given flat framed stratifications (Z, f) and (W, g) construct their coarsest refining meshes M and N. (Z, f) and (W, g) are framed stratified homeomorphic if and only if the corresponding stratified trusses $\mathsf{ETrs}(M, f)$ and $\mathsf{ETrs}(N, g)$ are related by a label preserving balanced isomorphism (note it must be unique if it exists). Since the latter isomorphism problem can be algorithmically solved (e.g., by brute force) so can the former homeomorphism problem. $\qquad\square$

The above results establish the 'computational tractability of flat framed stratifications'. Note that the existence of coarsest refining meshes (which, conceptually, is 'opposite' to the classical quest for constructing mutually refining triangulations in the classical Hauptvermutung) plays a most fundamental role in this story.

---

[30]Note that there are better algorithms than the brute-force search described here.

5.4 Towards transversality, tangles, singularities, and stability

In this final section, among a large pool of further topics that deserve discussion, we select and briefly discuss two future directions in the program of framed combinatorial topology, which we plan to address in subsequent work.

1. In Section 5.4.1 we will describe a class of flat framed stratifications whose strata are 'transversal'. We call such stratifications manifold diagrams. Manifold diagrams solve the long-standing problem of generalizing string diagrams and surface diagrams to higher dimensions.

2. In Section 5.4.2 we describe a class of flat framed stratification that model tame tangles. We discuss how these tangles may be 'perturbed', and how 'perturbation-stable' tangles singularities provide a combinatorial model for singularity theory.

In Section 5.4.3 we mention four specific problem for potential future research.

Remark 5.4.1 (Geometric higher category theory). We also emphasize that there is (at least) one central omission in our discussion of future work here. One may of course 'globalize' our discussion of flat framed structures to more general (non-flat) framed combinatorial spaces and stratified spaces. This direction of investigation provides novel geometric models for various objects endowed with 'directed higher structure'. The fundamental idea is that we can think of truss blocks and presheaves on blocks in a categorical manner: k-blocks are shapes of k-morphisms, and presheaves on blocks (or more precisely 'sheaves' in an appropriate sense) can be used to model higher categories. This model of higher categories has some striking differences from other presheaf models [Lei04] [CL04] [Ber20] [Had20]. Traditional 'filler' or 'contractibility' conditions are absent in this model, and moreover the model allows us to work with 'small examples' of higher categories. For instance, the block model provides an n-category that has a single generating 2-endomorphism on an object: higher coherence data is encoded in the geometry of pastings of cell diagrams. At the same time, due to the tight correspondence of combinatorics and stratified topology built into our model, this approach allows the formalization of several various folklore intuitions about the interaction of higher category theory and stratified topology. ⌐

5.4.1 Transversality and manifold diagrams Strata in flat n-framed stratifications may inherit different types of framings by restricting the ambient flat framing of $\mathbb{R}^n$ to a given stratum (the resulting framing may of course include 'singularities'). For an unordered 1-simplex, consider linear embeddings of the 1-simplex in $\mathbb{R}^2$ as shown in Fig. 5.13 on the left. Requiring such an embedding to be a framed realization uniquely determines a 2-embedded frame on the 1-simplex (see Definition 1.1.38). While there are two 2-embedded frames of the 1-simplex (with frame label '1' resp. '2' as indicated), only one of them is 'generic' in the sense that it determines an open dense subset in the space of all linear embeddings. This genericity may be expressed in more familiar terms as a *transversality* condition: a linear embedding $e : |S| \hookrightarrow \mathbb{R}^2$ of a 1-simplex S is 'transversal' if it is transversal to the fibers of the projection $\pi_1 : \mathbb{R}^2 \to \mathbb{R}^1$. This transversality condition can alternatively be phrased as requiring $\pi_1 \circ e : |S| \to \mathbb{R}^1$ to be a homeomorphism onto its image.

The idea of transversality applies more generally as follows. We may say a 1-stratum in $\mathbb{R}^2$ is 'transversal' at a point if the projection $\pi_1 : \mathbb{R}^2 \to \mathbb{R}^1$ restricts, in a small
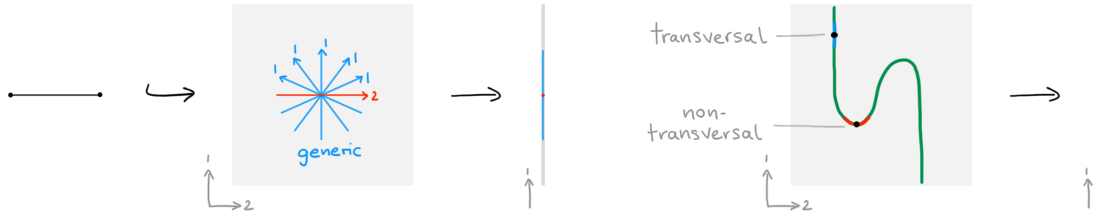
Figure 5.13: Genericity of framings of 1-strata in $\mathbb{R}^2$.

neighborhood of the point in the stratum, to a homeomorphism onto its image. We illustrate this on the right in Fig. 5.13 for two points in a 1-stratum; at one point the stratum is transversal while at the other it is not. More generally, we call a k-dimensional stratum s in $\mathbb{R}^n$ 'transversal at a point' $p \in s$ if the projection $\pi_{>k} : \mathbb{R}^n \to \mathbb{R}^k$ restricts to a homeomorphism from a neighborhood of p in the stratum s onto its image.

In Fig. 5.14 we illustrate two flat 3-framed stratifications: the left one (consisting of the green 2-dimensional stratum and the two connected components of its complement) is refined by the right one. While the right stratification is transversal at all points, the left stratification fails to be transversal. Indeed, the above transversality condition is not satisfied at any point that, after passing to the refinement, lies in either the 1- or 0-dimensional strata (indicated in blue and red).

A *manifold n-diagram* is a flat framed stratification of the open unit n-cube $\mathbb{I}^n$ in which every stratum is transversal, i.e. transversal at all points.[31] The flat framed stratification depicted on the right in Fig. 5.14 is an example of a manifold 3-diagram. This definition
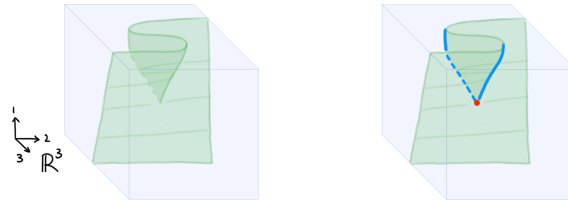

Figure 5.14: A non-transversal stratum and its transversal refinement.

of manifold n-diagrams in terms is given in terms of topological transversality condition, but manifold diagrams admit a fully combinatorial representation. This representation is based on the correspondence of flat framed stratifications to normalized stratified trusses, together with the observation that transversality of strata in flat framed stratifications can be phrased as a combinatorial condition on strata in the corresponding stratified trusses. Alltogether, we obtain a correspondence between manifold n-diagrams and so-called normalized *transversally* stratified trusses. Note further, the combinatorial condition of transversality is computably verifiable, so we can algortihmically recognize transversally stratified trusses among all stratified trusses.

---

[31] Technically, we also want manifold n-diagrams to be *conical* stratifications, with the additional condition that conical neighborhoods $C \times U$ of strata s can be chosen such that the stratum s is 'transversal' to the cone C. We defer a more detailed discussion of this condition and the resulting notion of 'framed conicality' to future work.

This formalization of manifold n-diagrams in all dimensions resolves a long-standing open problem. Previously, stratified topological definitions of 'manifold diagrams' had only been given in low dimensions, initially in dimension 2 [JS91], later in dimension 3 [Hum12], [BMS12] (see also [Tri99]), and in restricted form in dimension 4 [CKS96] [CS98]. The importance of manifolds diagrams derives in part from their dual connection to cellular pastings and from their usefulness for 'diagrammatic reasoning' via geometric deformations of the underlying stratifications [Sel10]. The approach outlined here provides formal underpinnings for such techniques in all dimensions. In particular, it allows us to formalize manifold n-diagram deformations as specific instances of manifold $(n + 1)$-diagrams (namely, those that do not have point singularities).

The relationship of 'manifold' and 'cellular' geometry relies on the dualization operations on meshes and trusses described in previous chapters. Given a manifold n-diagram (Z, f), By our results on flat framed stratifications (see Theorem 5.2.20), we may first pass to its coarsest refining open mesh M, obtaining a stratified mesh (M, f) (see Definition 5.3.7). Dualizing the mesh M, we obtain a closed mesh $M^\dagger$, and the stratification f induces a stratification $(M^\dagger, f^\dagger)$ on this mesh. (Formally, this stratification is the stratified classifying mesh of the dual $\mathsf{ETrs}(M, f)^\dagger$ of the labeled truss $\mathsf{ETrs}(M, f)$.) We call $(M^\dagger, f^\dagger)$ a 'pasting diagrams of cells with degeneracies'. (Note, in particular, that $M^\dagger$ is a framed regular cell complex, see Terminology 4.2.75.) To obtain a 'traditional pasting diagram' (without degeneracies) corresponding to the manifold diagram (Z, f), we may further 'quotient' $M^\dagger$ by collapsing to a single cell all the cells in each stratum of $f^\dagger$. The process is illustrated in Fig. 5.15.
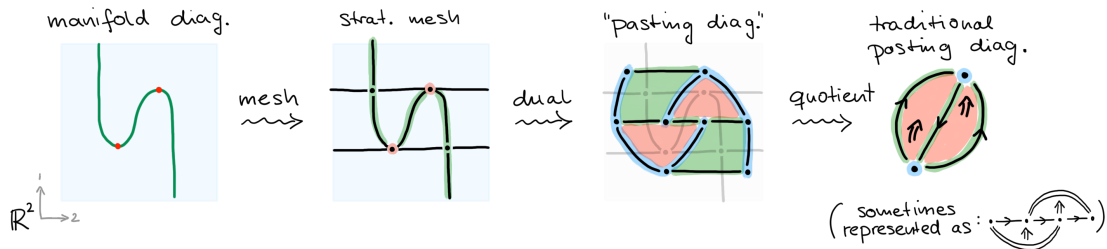

Figure 5.15: The pasting diagram dual to a manifold diagram.

5.4.2 Tangles, perturbative stability, and singularity theory Having discussed transversality and the class of 'manifold n-diagrams', we now introduce another important and related class of flat framed stratifications, namely 'tame tangles'. Tame tangles provide a combinatorializable counterpart to the classical notion of tangles given by manifolds with corners embedded in the closed unit cube. Formally, we define a *tame k-tangle in codimension* $(n - k)$ to be a flat framed stratification of the open unit n-cube $\mathbb{I}^n$ consisting of an open k-manifold M in $\mathbb{I}^n$ (with strata being the connected components of M and $\mathbb{I}^n \setminus M$) that can be refined by an n-manifold diagram. As in the case of both flat n-framed stratifications and manifold n-diagrams, tangles may be captured in purely combinatorial terms by their normalized entrance path trusses, which we refer to as *tangle trusses*.[32] We illustrate two (tame) tangles together with their coarsest refining

---

[32]It is, moreover, reasonable to ask that tangles $M \subset \mathbb{I}^n$ to not just be topological manifolds but combinatorial manifolds with piecewise linear structure induced by their combinatorialization as

manifold diagram in Fig. 5.16. Note that, to obtain a 'classical' tangle as a manifold
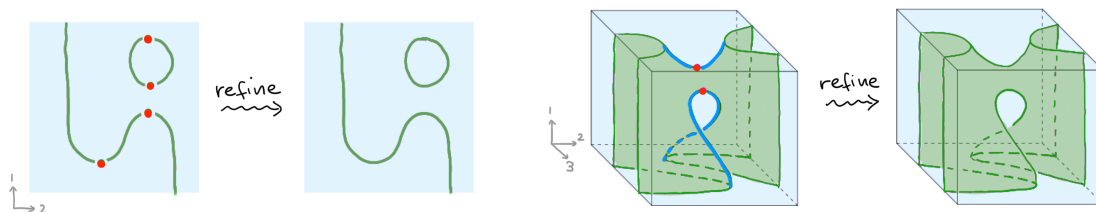

Figure 5.16: Tangles and their refining manifold diagrams.

with corners we may always compactify the open cube stratification to a closed cube stratification. (Formally, this compactification process uses our earlier combinatorial definition of 'cubical compactifications', see Definition 4.2.53; compactification establishes an equivalence between 'open' and 'closed' tame tangles, and we may work with either perspective.)

If the coarsest refining manifold diagram of a tangle contains a point stratum, then we call a small open neighborhood around this point (on which the tangle restricts to a 'subtangle') a 'tangle singularity'. For instance, the tangle depicted on the right of Fig. 5.16 has two (red) point singularities in its coarsest refining manifold diagram; passing to the 'subtangles' around those points yields the tangle singularities depicted in Fig. 5.17.
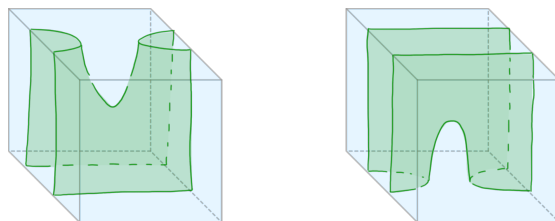
Figure 5.17: Tangle singularities.

Not all tangles are created equal, and the space of all tangles carries a natural stratification. This stratification is, roughly speaking, a measure of the complexity of a tangle's singularities. More precisely, there is a notion of 'perturbing a tangle S to a tangle T': using the machinery of truss bundles, we may more define this perturbation by considering 'bundles of tangles' over the 1-simplex $[1] = (0 \to 1)$ whose generic fiber (over 0) is the tangle T, and whose special fiber is the tangle S. We may then say that a tangle singularity S is 'perturbation-stable' if, for any perturbation to another tangle T, the resulting singularities in T look no less 'complex' than the singularities in S that they perturb. (Owing to the combinatorial nature of tangles, the 'complexity' of a singularity can be measured in elementary combinatorial terms, for instance by counting the size of the singularity's normalized stratified entrance path truss). We illustrate two instances of perturbations of tangle singularities in Fig. 5.18. In the first case, a 'monkey saddle' (see [Mil63]) is perturbed into two saddles, which are strictly 'less complex'; thus the monkey saddle is not perturbation-stable. In contrast, in the second case, a saddle is

---

stratified trusses; this avoids 'wildness' phenomena of [Can79].

perturbed into three tangle singularities (two saddles and a maximum) and these have the same complexity as the saddle we started with; indeed, the saddle is a perturbation-stable tangle singularity.
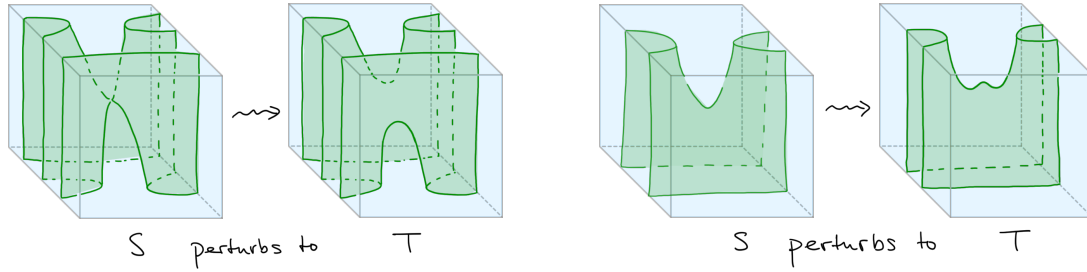

Figure 5.18: Perturbations of tangle singularities.

Perturbation-stable tangle singularities will be referred to as 'elementary tangle singularities'. The notion of perturbation-stability moreover extends from tangle singularities to general tangles, and the resulting subspace of 'perturbation-stable tangles' forms the deepest stratum in the stratification of the space of tangles, In particular, such tangles can only contain elementary singularities. Roughly speaking, the rest of the stratification may then be constructed by defining its entrance paths to be complexity-reducing perturbations.

Elementary tangle singularities turn out to be a particularly interesting class of tangles, and are intimately related to classical singularity theory. In Fig. 5.19 we depict *all* elementary 2-tangles in codimension-1.[33] It happens that these 2-tangle singularities
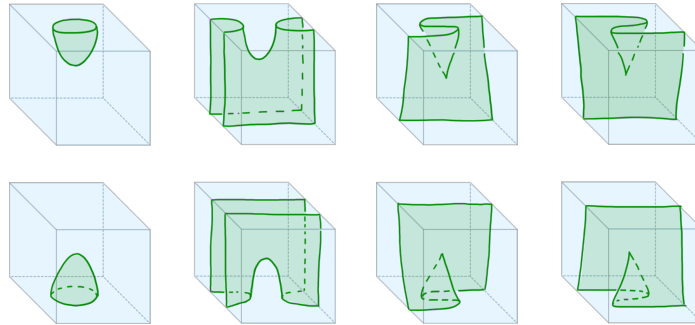

Figure 5.19: Perturbation stable 2-tangles (in codimension-1).

exactly recover a collection of classical 2-dimensional singularities, namely namely the Morse-type singularities (saddles, minima, and maxima) and Cerf-type singularities ('cusps' of 1-parameter families of 1-variable functions). For codimension-1 k-tangle singularities with $k > 2$, the parallel with classical singularities initially appears to continue into higher dimensions. Some elementary tangle singularities maybe be associated to known classical singularities, including some of the 'elementary singularities' that were described in the work of Thom [TF18] and Arnold [Arn75], such as the 'swallowtail' singularities

---

[33]Here, the fact that we emphasize tangles in codimension-1 reflects the fact that classical singularity theory often studies singularities in (r-parameter families of) n-variable functions $\mathbb{R}^n \to \mathbb{R}$, whose graphs (resp. families of graphs) yield hypersurfaces in $\mathbb{R}^{n+1}$ (resp. in $\mathbb{R}^{n+r+1}$).

(succeeding the cusp in Arnold's list of A_k singularities) whose corresponding elementary tangle singularity was depicted already in Fig. I.6.

However, with increasing number of parameters, the classical machinery of smooth singularity theory eventually breaks down (namely, for parameter dimensions above 5, see [PS14, §7.6]): the 'dimensions of jet spaces outgrow the dimensions of the structure groups', which causes the smooth equivalence relation of singularities to become too fine, leading one to encounter uncountably many equivalence classes of singularities. In stark contrast, the combinatorial machinery of elementary tangle singularities works in all dimensions, and this leads to a profound discrepancy between the two approaches in higher dimensions. We mention one concrete example of the divergence between the two approaches: among the elementary singularities in Thom's classification one finds the so-called 'elliptic umbilic' singularity (in Arnold's classification, it is called the D_4^- singularity); in combinatorial terms, this umbilic singularity is no longer 'elementary', in that it may be perturbed into a tangle consisting of less complex tangle singularities. The question of how to classify the combinatorial patterns underlying elementary tangle singularities appears fundamental, and will be revisited in the next section.

5.4.3   Open problems Wrapping up the discussion of the previous sections, we list four areas of open problems.

5.4.3.1   The classification of elementary singularities In the last section we introduced 'elementary', i.e. perturbation-stable, tangle singularities. We saw that for 2-tangles in codimension-1 there are exactly eight such singularities. In higher dimensions, elementary tangle singularities quickly become more complicated.

Problem 5.4.2 (Classifying elementary singularities). For tangles of a given dimension and codimension, can we list all elementary tangle singularities?   ⌐⌐

It is moreover reasonable to expect a *finite* list of elementary tangle singularities in each dimension and codimension. As explained, while elementary tangle singularities reproduce singularities from classical singularity theory in low dimensions, in general, there is a discrepancy between the two approaches, and a more precise quantification of this discrepancy would be desirable. One can further consider tangles with structure (for instance, by tangentially framing tangles embedded in $\mathbb{R}^n$), and similarly ask how the classification of tangle singularities changes in the presence of these additional structure. Via the generalized tangle and cobordism hypotheses (see [BD95] [Lur09b]), such classification problems are closely related to understanding '(structured) dualizability laws' of objects in higher category theory. A better understanding of elementary tangle singularities may therefore provide insights both for the study of classical singularity theory as well as higher dualizable structures.

Let us also mention a related problem about the recognition of general (not necessarily perturbation-stable) tangles.

Problem 5.4.3 (Recognizing tangle trusses). Can we recognize tangle trusses among all stratified trusses?   ⌐⌐

Even if we require tangles to be combinatorial manifolds (with piecewise linear structure induced by their combinatorialization as stratified trusses), it is not immediately clear how a recognition algorithm could work: such an algorithm would require checking

the sphericality of certain links of strata in the stratified truss, which, as discussed in Remark 1.3.43, is undecidable in general. However, links in stratified trusses are better behaved than general simplicial complexes, and thus the unrecognizability of spheres need not necessarily imply the unrecognizability of tangle trusses; for instance, tangle trusses *would* be algorithmically recognizable if it turned out that all links of tangle singularities were *shellable* (see Terminology 3.4.3). Other algorithms are also conceivable (which, for instance, could exploit a finite classification of elementary tangle singularities).

5.4.3.2 The classification of elementary homotopies Given a manifold n-diagram we may 'continuously deform' it: namely, a 'homotopy' of manifold n-diagrams is a manifold $(n + 1)$-diagram which itself does not contain any 0-dimensional strata. Examples of such deformations were given earlier: the 'braid' (depicted as the first example in Fig. 5.1) is a homotopy that continuously deforms a manifold 2-diagram consisting of two point singularities by rotating them around each other; one dimension up, we have also seen the 'Reidemeister III move' (depicted in Fig. I.6) which continuously deforms an arrangement of three braids.

Homotopies of manifold diagrams encode coherences laws of higher algebraic structures. For instance, the 'braid' witnesses the commutativity law of elements in 2nd (and higher) homotopy groups of spaces. In fact, the braid moreover is the simplest homotopy of manifold 2-diagrams in the following sense: other homotopies, such as the 'triple braid', can be perturbed slightly into a composite of braids; in contrast, the braid itself cannot be perturbed any further. We call the braid an 'elementary homotopy' and, more generally, we say that a homotopy is 'elementary' if it cannot be perturbed into simpler homotopies. (The formalization of this notion mirrors our earlier discussion of elementary singularities.) Analogous to the question of classifying elementary tangle singularities, we may now ask the following.

Problem 5.4.4 (Classification of elementary homotopies). Can we list all the elementary homotopies of manifold n-diagrams?⌐

It is tempting to conjecture that the list of elementary homotopies is finite in any given dimension. However, in reality, little is known about elementary homotopies in higher dimensions (and codimension). In low dimension, such elementary deformations have been considered in the context of 'knotting surfaces' [CRS97] [CS98]. More generally, elementary homotopies relate to the question of how 'Gray categories' (a notion of 3-categories with natural geometric semantics [GPS95] [Hum12]) can be generalized to higher dimensions (a discussion of dimension 4 can be found in [Cra00] [BV17]).

5.4.3.3 Higher Morse theory Morse theory is a powerful tool in the study of manifolds [Mil63]; more generally but less ubiquitously, stratified Morse theory is a similarly powerful tool in the study of (sufficiently nice) stratified spaces [GM88]. Both theories allow for the introduction of invariants associated to manifolds (resp. stratified spaces) based on the decomposition of space that is encoded by a (stratified) 1-Morse function. This leads, for instance, to the construction of Morse homology and intersection homology. Classical (stratified) Morse functions are, however, only the 'first degree' in a more general picture which studies manifolds and stratified spaces via 'generic' functions to $\mathbb{R}^n$, that is, via 'higher Morse functions'. The idea of 2-Morse functions has been largely realized in dimension 2 (see [Cer70]) and used, for instance, in the classification of extended field

theories in dimension 2 in terms of generators and relations [SP09]. However, defining n-Morse functions for $n > 2$ turns out to be a technically difficult task—at least, when attempting to do so in traditional differential terms.

Using the notions of flat n-framed stratifications developed here, the idea of stratified n-Morse function finds a simple instantiation as follows. A flat framed stratifications comes with an embedding into $\mathbb{R}^n$, and, post-composing this embedding with the projections $\mathbb{R}^n \to \mathbb{R}^m$, yields functions to $\mathbb{R}^m$, $m < n$. We may think of the resulting maps as *instances* of 'm-Morse functions'. Note that increasing m gradually, each m-Morse function contains more information than its 'predecessor' $(m-1)$-Morse function. While this discussion of higher Morse functions relies on 'embedded framings' $M \hookrightarrow \mathbb{R}^n$, it may be possible to leverage our notion of 'partial framings' to obtain a more direct description of such functions (as an illustrative example see the earlier Fig. 1.17).

Problem 5.4.5 (Formalization and application of n-Morse theory). Can we use the combinatorial framework developed here (including the notion of partial framings) to formalize a combinatorial theory of n-Morse functions? Can we derive stronger manifold invariants from n-Morse decompositions than from 1-Morse decompositions? ⌐⌐

One may expect the answer to the second question to be 'yes' for the following heuristic reason. Classical Morse functions 'miss' exotic m-spheres from a singularity-theoretic point of view, in that exotic spheres may have 1-Morse functions with a single maximum and a single minimum (just as the standard sphere); instead the information of the smooth structure is hidden in the exotic diffeomorphism of fiber spheres. However, the same cannot be true for n-Morse functions, $n \geq m$, of exotic m-spheres: since now fibers are discrete, no 'exotic' automorphisms of fibers are possible. Instead, the exotic structure needs to be recorded purely in terms of (elementary) tangle singularities and homotopies.

5.4.3.4   The relation of framed piecewise linear and smooth topology Our final question concerns a more precise relation of framed piecewise linear and smooth topology. We phrase this in two conjectures as follows. (For consistency with earlier definitions, note that we identify $\mathbb{R}^n$ with the open unit cube $\mathbb{I}^n$ by some framed homeomorphism.)

Conjecture 5.4.6 (Framed stratified homeomorphisms implies diffeomorphism). *From any two smooth embeddings $e : M \hookrightarrow \mathbb{R}^n$ and $e' : M \hookrightarrow \mathbb{R}^n$ that correspond to the same normalized tangle truss in $\mathbb{R}^n$, we can produce a diffeomorphism $M \cong M'$.*

A proof of this conjecture could possibly use a 'smooth' version of our earlier proof of conservativity of the entrance path truss functors (see Proposition 4.2.22) to produce a smooth 'ambient' isomorphism of the respective coarsest refining meshes of M and M', which would then restrict to a diffeomorphism $M \cong M'$.

Conjecture 5.4.7 (Smooth embedded manifolds are generically tame tangles). *Given a compact smooth k-manifold M, any smooth embedding $e : M \hookrightarrow \mathbb{R}^n$ has an arbitrarily small perturbation such that the image of the perturbed embedding $e' : M \hookrightarrow \mathbb{R}^n$ defines a tame k-tangle in $\mathbb{R}^n$.*

In particular, since any smooth manifold can be embedded in $\mathbb{R}^n$, the two conjectures together imply that any smooth structure on a compact manifold M can be represented combinatorially by a tangle truss. Again, heuristically, this claim is not unexpected since,

by arguments similar to those outlined in the previous section, combinatorializable 'n-Morse embeddings' should be able to detect smooth structures.

If these conjectures hold, the resulting correspondence of 'framed combinatorial manifolds M' (i.e. tame tangles with underlying manifold M) and smooth structures on M would realize a goal similar to that of MacPherson's program of 'combinatorial differential manifolds' [Mac91], namely, the faithful combinatorial representation of smooth structures, and the ability to work 'smoothly' without direct reference to the continuum.

APPENDIX $\mathbf{A}$

Linear and affine frames

We discuss classical notions of frames and trivializations of both linear and affine spaces, as well as various notions of 'generalized' frames and trivializations.

In Appendix A.1 we will start with the following two fundamental observations about linear frames. Firstly, we will see that orthonormal frames have reformulations both in terms of sequences of linear subspaces (yielding a notion of 'indframes') and in terms of sequences of linear projections (yielding a notion of 'proframes'); importantly, while 'orthonormality' requires euclidean structure on the vector spaces, the definitions of indframes and proframes do not, and will therefore allow us to define a notion of 'orthoequivalence' for general linear frames. Secondly, the notion of linear frames itself generalizes as follows: instead of considering trivializations by isomorphisms $V \xrightarrow{\sim} \mathbb{R}^m$, we may also consider projections $V \twoheadrightarrow \mathbb{R}^k$ (yielding a notion of 'partial trivialization') or subspaces $V \hookrightarrow \mathbb{R}^n$ (yielding a notion of 'embedded trivialization'), or, yet more generally, general linear maps $V \to \mathbb{R}^n$ (yielding a notion of 'embedded partial trivialization').

In Appendix A.2 we will then see that notions of linear frames on vector spaces, as well as their generalizations, can be adapted to notions of 'affine frames' on affine spaces. This perspective will allow us to describe our earlier definition of combinatorial frames in purely linear algebraic terms.

A.1   Linear frames

Linear trivializations, frames, indframes, and proframes We introduce notions of 'linear indframes' and 'linear proframes'. For euclidean vector spaces, both are equivalent to the ordinary notion of 'orthonormal frames'. In the case of general vector spaces, they provide a notion of 'orthoequivalence' of linear frames. The idea underlying both notions (translated appropriately into combinatorial terms) will be pervasive throughout our work, and in particular will play a crucial role in classifying framed combinatorial structures in later chapters. For consistency, let us first (re-)introduce linear trivializations and linear frames.

Definition A.1.1 (Linear trivializations). A **linear trivialization** of an m-dimensional vector space V is a linear isomorphism $V \xrightarrow{\sim} \mathbb{R}^m$. ⌐

Preimages of the standard basis vectors $e_i \in \mathbb{R}^n$ under the linear trivialization map define an ordered list of 'frame vectors' $v_i \in V$. Every linear trivialization therefore determines and is determined by a linear frame in the following sense.

Definition A.1.2 (Linear frames). A **linear frame** of an m-dimensional vector space V is an ordered list of linearly independent vectors $\{v_1, v_2, \ldots, v_m\} \subset V$. ⌐

We now want to compare the structure of linear trivializations (and equivalently of linear frames) on vector spaces to the following two structures.

Definition A.1.3 (Linear indframes). A **linear indframe** of a vector space $V = V_m$ is a sequence of inclusions of vector spaces V_i, with $\dim(V_i) = i$:

$$0 = V_0 \hookrightarrow V_1 \hookrightarrow V_2 \hookrightarrow \cdots \hookrightarrow V_{m-1} \hookrightarrow V_m = V.$$ ⌐

Definition A.1.4 (Linear proframes). A **linear proframe** of a vector space $V = V^m$ is a sequence of projections of vector spaces V^i, with $\dim(V^i) = i$:

$$V = V^m \twoheadrightarrow V^{m-1} \twoheadrightarrow V^{m-2} \twoheadrightarrow \cdots \twoheadrightarrow V^1 \twoheadrightarrow V^0 = 0.$$ ⌐

Observation A.1.5 (Equivalence of indframes and proframes). Note that linear indframes and proframes define the same structure on a vector space. For a linear indframe $\{V_i \hookrightarrow V_{i+1}\}_{0 \leq i < m}$ on V, the corresponding proframe is determined by the cokernels of the sequence of inclusions into the total vector space: $(V \twoheadrightarrow V^{m-i}) = \operatorname{coker}(V_i \hookrightarrow V)$. Conversely, for a linear proframe $\{V^i \twoheadrightarrow V^{i-1}\}_{0 < i \leq m}$ on V, the corresponding indframe is determined by the kernels of the sequence of projections from the total vector space: $(V_i \hookrightarrow V) := \ker(V \twoheadrightarrow V^{m-i})$. ⌐

There are two important standard instances of indframes and proframes.

Terminology A.1.6 (The standard euclidean indframe). The 'standard euclidean indframe' of $\mathbb{R}^n$ is the sequence of subspace inclusions

$$0 \hookrightarrow \mathbb{R} \hookrightarrow \mathbb{R}^2 \hookrightarrow \cdots \hookrightarrow \mathbb{R}^{n-1} \hookrightarrow \mathbb{R}^n$$

where $\mathbb{R}^{i-1} \hookrightarrow \mathbb{R}^i$ includes into the last $i - 1$ coordinates of $\mathbb{R}^i$. ⌐

Terminology A.1.7 (The standard euclidean proframe). The 'standard euclidean proframe' of $\mathbb{R}^n$ is the sequence of projections

$$\mathbb{R}^n \twoheadrightarrow \mathbb{R}^{n-1} \twoheadrightarrow \mathbb{R}^{n-2} \twoheadrightarrow \cdots \twoheadrightarrow \mathbb{R}^1 \twoheadrightarrow \mathbb{R}^0$$

where $\mathbb{R}^i \to \mathbb{R}^{i-1}$ forgets the last component of vectors in $\mathbb{R}^i$. ⌐

Observe that the complement of the image of each standard indframe inclusion $\mathbb{R}^{i-1} \hookrightarrow \mathbb{R}^i$ has two components $\mathbb{R}^i \setminus \mathbb{R}^{i-1} = \epsilon_i^- \sqcup \epsilon_i^+$, where the 'negative' component ϵ_i^- resp. 'positive' component ϵ_i^+ consist of points with ith negative resp. positive coordinate. We call the assignment of positive and negative signs to those components an 'orientation structure' on the standard indframe, and more generally introduce the following notions.

Terminology A.1.8 (Oriented indframe). An 'oriented indframe' on a vector space V is an indframe $\{V_i \hookrightarrow V_{i+1}\}$ together with an association of signs to the connected components of the complement of the image of each inclusion: $V_i \setminus V_{i-1} = \nu_i^- \sqcup \nu_i^+$.

An orientation structure on an indframe is equivalent to requiring each V_i to be an oriented vector space.

Observe similarly that the fiber $\pi_i^{-1}(0)$ over $0 \in \mathbb{R}^{i-1}$ of each projection $\pi_i : \mathbb{R}^i \to \mathbb{R}^{i-1}$ is $\mathbb{R}$, and thus, $\pi_i^{-1}(0) \setminus 0$ has again a 'negative' component $\epsilon_-^i = \mathbb{R}_{<0} \subset \pi_i^{-1}(0)$ and a 'positive' component $\epsilon_+^i = \mathbb{R}_{>0} \subset \pi_i^{-1}(0)$. We call this an 'orientation structure' on the standard proframe, and more generally have the following notion.

Terminology A.1.9 (Oriented proframe). An 'oriented proframe' on a vector space V is a proframe $\{p_i : V^i \twoheadrightarrow V^{i-1}\}$ together with an association of signs to the connected components of the complements $p_i^{-1}(0) \setminus 0 = \nu_-^i \sqcup \nu_+^i$.

An orientation structure on a proframe is equivalent to requiring each V^i to be an oriented vector space. Our earlier correspondence of indframes $\{V^i \hookrightarrow V^{i-1}\}$ and proframes $\{p_i : V^i \twoheadrightarrow V^{i-1}\}$ on an m-dimensional vector space V extends to the oriented case: writing $p_{>i}$ for the composite projection $p_{i+1} \circ ... \circ p_{n-1} \circ p_m : V \twoheadrightarrow V^i$, an orientation structure on the indframe determines an orientation structure on the corresponding proframe, and vice versa, by setting $\nu_\pm^i = p_{>i}(\nu_i^\pm)$, and conversely $\nu_i^\pm = p_{>i}^{-1}(\nu_\pm^i)$.

Any indframe on a vector space V can be obtained by 'pulling back' the standard indframe of $\mathbb{R}^n$ along a trivialization $V \xrightarrow{\sim} \mathbb{R}^m$. This similarly applies to the standard proframe in $\mathbb{R}^n$. With a view towards later generalizations of our discussion here, it will be useful to formalize this using the following more general constructions.

Terminology A.1.10 (Pullback sequences). Given a sequence of j vector space inclusion $\{W_i \hookrightarrow W_{i+1}\}$ and a map $F : V \to W_j$ as shown below, then we obtain a 'pullback' sequence of vector space inclusions $\{V_i \hookrightarrow V_{i+1}\}$ by iterated pullback as shown:

$$
\begin{array}{ccccccccccc}
0 & \hookrightarrow & V_0 & \hookrightarrow & V_1 & \hookrightarrow & V_2 & \hookrightarrow & \cdots & \hookrightarrow & V_{j-1} & \hookrightarrow & V_j = V \\
 & \searrow & \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow & \lrcorner & \cdots & & \downarrow & \lrcorner & \downarrow{\scriptstyle F} \\
 & & 0 & \hookrightarrow & W_1 & \hookrightarrow & W_2 & \hookrightarrow & \cdots & \hookrightarrow & W_{j-1} & \to & W_j & .
\end{array}
$$

Terminology A.1.11 (Restriction sequences). Given a sequence of vector space projections $\{W^i \twoheadrightarrow W^{i-1}\}$ and a map $F : V \to W_j$ as shown below, then we obtain a 'restriction sequence' of vector space projections $\{V^{i+1} \twoheadrightarrow V^i\}$ constructed by iterated image factorization as shown

$$
\begin{array}{ccccccccccc}
V & \dashrightarrow & V^j & \dashrightarrow & V^{j-1} & \dashrightarrow & V^{j-2} & \dashrightarrow & \cdots & \dashrightarrow & V^1 & \dashrightarrow & V^0 = 0 \\
 & {\scriptstyle F}\searrow & \uparrow\downarrow & & \uparrow\downarrow & & \uparrow\downarrow & & \cdots & & \uparrow\downarrow & & \uparrow\downarrow \\
 & & W^j & \twoheadrightarrow & W^{j-1} & \twoheadrightarrow & W^{j-2} & \twoheadrightarrow & \cdots & \twoheadrightarrow & W^1 & \twoheadrightarrow & 0 & .
\end{array}
$$

Observation A.1.12 (Trivializations induce oriented indframes and proframes). A trivialization $F : V \xrightarrow{\sim} \mathbb{R}^n$ induces both an oriented indframe and a oriented proframe on V as follows. The indframe on V is defined as the pullback sequence of the standard indframe of

236

$\mathbb{R}^n$ by F. A proframe on V is defined as the restriction sequence of the standard proframe of $\mathbb{R}^n$ by F. Note that the resulting proframe corresponds to the resulting indframe on V. Thus, to endow both with orientation it suffices to endow the indframe with an orientation ν, which is canonically determined by requiring $F(\nu_i^{\pm}) = \epsilon_i^{\pm}$. ⌐

Note well that different linear trivializations (and thus linear frames) on V may induce the same oriented indframe resp. proframe. This leads to the following equivalence relation on trivializations (or frames).

Terminology A.1.13 (Orthoequivalence). We say two linear trivializations (or, equivalently, two linear frames) on V are 'orthoequivalent' if they induce the same indframe (and thus the same proframe). ⌐

The name of this equivalence relation is derived from the following observation.

Observation A.1.14 (Orthonormal frames, oriented indframes, and oriented proframes are equivalent). If V is an m-dimensional euclidean vector space, then an 'orthonormal frame' $(v_1, v_2, ..., v_m)$ is a linear frame consisting of orthonormal vectors. Orthonormal frames determine exactly those trivializations $V \xrightarrow{\sim} \mathbb{R}^n$ which are isometries. Note that any oriented indframe (and similarly, any oriented proframe) on V is induced by exactly one isometry $F : V \xrightarrow{\sim} \mathbb{R}^n$, namely the one determined by the property that $F(\nu_i^{\pm}) = \epsilon_i^{\pm}$. Thus, orthonomal frames (resp. isometric trivializations) are in correspondence with oriented indframes and proframes. ⌐

In other words, each orthoequivalence classes of trivializations of an euclidean vector spaces has a unique orthonormal representative. In Fig. A.1 we depict an orthonormal frame (v_1, v_2) in an euclidean vector space V, together with its corresponding oriented indframe (via Observation A.1.14).

Figure A.1: An orthonormal frame and its corresponding indframe.

The preceding discussion suggests the following.

Remark A.1.15 (Orthoequivalence as generalization of orthonormality). The notion of 'trivializations up to orthoequivalence' can be regarded as a generalization of the notion of 'orthonormal frames'. In the absence of euclidean structure, we may work with the former structure in place of the latter structure. ⌐

Partial and embedded trivializations and frames We now generalize the notions of linear trivializations and frames. Namely, instead of considering linear isomorphisms $V \xrightarrow{\sim} \mathbb{R}^m$, we will consider projections $V \twoheadrightarrow \mathbb{R}^k$, subspaces $V \hookrightarrow \mathbb{R}^n$, and general linear maps $V \to \mathbb{R}^n$. This leads to a more subtle interplay of 'trivializations' and 'frames'.

The ideas discussed here (translated into appropriate combinatorial form), especially the notion of 'embedded' frames, will later be crucial to relate and 'glue' frames of objects of different dimensions.

We start with the case of 'partial' trivializations and frames.

Definition A.1.16 (Linear partial trivializations). A **linear k-partial trivialization** of a vector space V is a linear projection $V \twoheadrightarrow \mathbb{R}^k$. ⌐

Definition A.1.17 (Linear partial frames). A **linear k-partial frame** of a vector space V is an ordered list $\{v_1, v_2, \ldots, v_k\}$ of k linearly independent vectors in V. ⌐

Remark A.1.18 (Relation of partial frames and trivializations). If V is euclidean then linear partial trivializations and frames define the same structure on V. Indeed, any linear projection $V \twoheadrightarrow \mathbb{R}^k$ has a canonical splitting $\mathbb{R}^k \hookrightarrow V$ (obtained as the inverse of the restriction of $V \twoheadrightarrow \mathbb{R}^k$ to its orthogonal kernel complement), whose images on standard basis vector e_i determine a linear k-partial frame. Conversely, a linear k-partial frame $\{v_1, v_2, \ldots, v_k\}$ determines an inclusion $\mathbb{R}^k \hookrightarrow V$, mapping v_i to e_i, which splits to give a k-partial trivialization $V \twoheadrightarrow \mathbb{R}^k$. However, in the absence of euclidean structure, we say that a linear k-partial frame $\{v_1, v_2, \ldots, v_k\}$ is 'compatible' with a linear k-partial trivialization $V \twoheadrightarrow \mathbb{R}^m$ if the projection is split by the inclusion $\mathbb{R}^k \hookrightarrow V$ of the frame. In general, however, this compatibility relation does not yield a 1-to-1 correspondence. ⌐

Next, instead of projections $V \twoheadrightarrow \mathbb{R}^k$, we may consider an 'embedded trivialization' given rather by an inclusion $V \hookrightarrow \mathbb{R}^n$.

Definition A.1.19 (Linear embedded trivialization). A **linear n-embedded trivialization** of an m-dimensional vector space V is an inclusion $V \hookrightarrow \mathbb{R}^n$. ⌐

Definition A.1.20 (Linear embedded frame). A **linear n-embedded frame** of an m-dimensional vector space V is an ordered list of n vectors, $\{v_1, v_2, \ldots, v_n\} \subset V$, exactly m of which are nonzero, and such that all nonzero vector are linearly independent. ⌐

Remark A.1.21 (Relation of embedded frames and trivializations). As before we can translate embedded frames into embedded trivializations by mapping frame vectors to standard basis vectors: that is, every linear n-embedded frame $\{v_1, v_2, \ldots, v_n\} \subset V$ induces a linear n-embedded trivialization $V \hookrightarrow \mathbb{R}^n$, defined to map nonzero vectors v_i to e_i. However, the translation is far from bijective (certainly not all linear embedded trivialization are induced by an embedded frame in this way). ⌐

Finally, there is a common generalization of partial trivializations and embedded trivializations, and an analogous notion of frames, as follows.

Definition A.1.22 (Linear embedded partial trivializations). A **linear embedded partial trivialization** of a vector space V is a linear map $V \to \mathbb{R}^n$. ⌐

Definition A.1.23 (Linear embedded partial frames). A **linear n-embedded k-partial frame** of V is an ordered list of n vectors $\{v_1, v_2, \ldots, v_n\} \subset V$, exactly k of which are nonzero, and such that all nonzero vectors are linearly independent. ⌐

Remark A.1.24 (Relation of embedded partial frames and trivializations). As in the previous two cases, the structures do not correspond 'one-the-nose'. ⌐

Importantly, the failure of correspondence of trivialization and frame structures that we observed in the preceding three remarks can be remedied by working with 'orthonormal' frames and trivializations up to 'orthoequivalence'. As in the case of ordinary linear frames, orthoequivalence will be based on (now generalized) notions of indframes and proframes. Given a partial trivialization $V \twoheadrightarrow \mathbb{R}^k$, an n-embedded trivialization $V \hookrightarrow \mathbb{R}^n$, or an n-embedded partial trivialization $V \to \mathbb{R}^n$, we may form the pullback sequence of the standard euclidean indframe along the trivialization. Sequences obtained in this way respectively describe notions of 'partial', 'embedded', and 'partial embedded' indframes on V.

Definition A.1.25 (Linear partial, embedded, and embedded partial indframes). A **linear k-partial indframe** on V is a sequence of the form (where $\dim(V_i) = i$):

$$0 \hookrightarrow V_{m-k} \hookrightarrow V_{m-k+1} \hookrightarrow \cdots \hookrightarrow V_{m-1} \hookrightarrow V_m = V.$$

A **linear n-embedded indframe** on V is a sequence of the form (where $\dim(V_{m_i}) = m_i$, and, for each i, either $m_{i+1} = m_i + 1$ or $m_{i+1} = m_i$):

$$0 = V_0 = V_{m_0} \hookrightarrow V_{m_1} \hookrightarrow V_{m_2} \hookrightarrow \cdots \hookrightarrow V_{m_{n-1}} \hookrightarrow V_{m_n} = V_m = V.$$

A **linear n-embedded k-partial indframe** is a sequence of the form (where $\dim(V_{k_i}) = k_i$, and for each i, either $k_{i+1} = k_i + 1$ or $k_{i+1} = k_i$):

$$0 \hookrightarrow V_{m-k} = V_{k_0} \hookrightarrow V_{k_1} \hookrightarrow V_{k_2} \hookrightarrow \cdots \hookrightarrow V_{k_{n-1}} \hookrightarrow V_{k_n} = V_m = V.$$

One further defines 'orientations' ν as before by associating signs $V_{k_i} \setminus V_{k_{i-1}} = \nu_{k_i}^- \sqcup \nu_{k_i}^+$ to the connected components of complements of images of the above inclusions when these complements are non-empty (and similarly in the other two cases of indframes). ⌐⌐

Similarly, given a partial trivialization $V \twoheadrightarrow \mathbb{R}^k$, an n-embedded trivialization $V \hookrightarrow \mathbb{R}^n$, or an n-embedded partial trivialization $V \to \mathbb{R}^n$, we may form the restriction sequence of the standard euclidean proframe along the trivialization. Sequences obtained in this way respectively describe notions of 'partial', 'embedded' and 'partial embedded', proframes on V.

Definition A.1.26 (Linear partial, embedded, and embedded partial proframes). A **linear k-partial proframe** on V is a sequence of the form (where $\dim(V^i) = i$)

$$V = V^m \twoheadrightarrow V^k \twoheadrightarrow V^{k-1} \twoheadrightarrow V^{k-2} \twoheadrightarrow \cdots \twoheadrightarrow V^0 = 0.$$

An **linear n-embedded proframe** on V is a sequence of the form (where $\dim(V^{m_i}) = m_i$, and, for each i, either $m_{i-1} = m_i - 1$ or $m_{i-1} = m_i$)

$$V = V^m = V^{m_n} \twoheadrightarrow V^{m_{n-1}} \twoheadrightarrow V^{m_{n-2}} \twoheadrightarrow \cdots \twoheadrightarrow V^{m_1} \twoheadrightarrow V^{m_0} = 0.$$

An **linear n-embedded k-partial proframe** is a sequence of the form (where $\dim(V^{k_i}) = k_i$, and, for each i, either $k_{i-1} = k_i - 1$ or $k_{i-1} = k_i$)

$$V = V^m \twoheadrightarrow V^k = V^{k_n} \twoheadrightarrow V^{k_{n-1}} \twoheadrightarrow V^{k_{n-2}} \twoheadrightarrow \cdots \twoheadrightarrow V^{k_1} \twoheadrightarrow V^{k_0} = 0. \qquad ⌐⌐$$

Observation A.1.27 (Correspondence of indframes and proframes). In each of the above three cases (adding the adjectives 'partial', 'embedded', or 'embedded partial'), the notions of indframes and proframes define identical structures on a vector space V: that is, one can be constructed from the other as before by taking cokernels and conversely kernels. ⌐

Observation A.1.28 (Trivializations induce corresponding indframes and proframes). By passing to pullback sequences resp. restriction sequences (and adding the adjectives 'partial', 'embedded', or 'embedded partial'), trivializations of a vector space V induce oriented indframes resp. oriented proframes of V. Given a trivialization its induced oriented indframe corresponds to its induced oriented proframe. ⌐

Terminology A.1.29 (Orthoequivalence). Two (partial, embedded, or embedded partial) trivializations are 'orthoequivalent' if they induce the same oriented indframe, or equivalently, the same oriented proframe. ⌐

Orthoequivalence allows us to relate trivializations and 'orthonormal' frames; in the following, a list of vectors is called 'orthonormal' if all its nonzero vectors are orthonormal.

Observation A.1.30 (Orthoequivalence classes of orthonormal embedded frames). Let V be an m-dimensional euclidean vector space. The following auxiliary notion will be helpful: a 'standard isometric' n-embedded trivialization $V \hookrightarrow \mathbb{R}^n$ is an n-embedded trivialization which is an isometry containing m standard vectors in its image. Note that, orthonormal n-embedded frames induce (in the sense of Remark A.1.21) exactly standard isometric n-embedded trivialization, which yields a 1-to-1 correspondence between the two notions. We will show each orthoequivalence classes of an embedded trivialization contains exactly one orthonormal embedded frame.

Consider an oriented n-embedded indframe $\{V_{m_i} \hookrightarrow V_{m_{i+1}}\}$ of V with orientation ν. Note there is a unique isometry $F : V \xrightarrow{\sim} \mathbb{R}^m$ such that $F(\nu_{m_i}^\pm) = \epsilon_{m_i}^\pm$. Define an isometric inclusion $\mathbb{R}^m \hookrightarrow \mathbb{R}^n$ by mapping basis vectors e_{m_j} to e_j if $m_j > m_{j-1}$. Composing the two maps yields a standard isometric n-embedded trivialization $V \hookrightarrow \mathbb{R}^n$ and thus determines an orthonormal n-embedded frame. Thus orthonormal embedded frames (resp. their standard isometric embedded trivializations) are in correspondence with oriented embedded indframes and thus with orthoequivalence classes of embedded trivializations as claimed. ⌐

In Fig. A.2 we depict a 3-embedded orthonormal frame (v_1, v_2, v_3) on a 2-dimensional euclidean vector space V, together with its corresponding oriented 3-embedded indframe (via Observation A.1.30): note that the embedded indframe determines exactly and orthoequivalence class of embedded trivializations (namely, those embedded trivializations that pull back the oriented standard indframe of $\mathbb{R}^3$ to the depicted oriented embedded indframe of V).

We may further generalize the correspondence to the case of *partial* embedded frames as follows.

Observation A.1.31 (Orthoequivalence classes of orthonormal embedded partial frames). Let V be a euclidean vector space. The following auxiliary notion will be helpful: a 'standard isometric' n-embedded k-partial trivializations is a subspace $W \hookrightarrow V$ together with a standard isometric embedded trivialization $W \hookrightarrow \mathbb{R}^n$ in the sense of Observation A.1.30. Given an orthonormal n-embedded k-partial frame define $W \hookrightarrow V$ to be the subspace spanned by its nonzero vectors. This canonically splits by a projection $V \twoheadrightarrow W$, which

$(v_1 = 0, v_2, v_3) \in V$ $V_0 \hookrightarrow V_0 \hookrightarrow V_1 \hookrightarrow V_2 = V$ $\mathbb{R}^3$
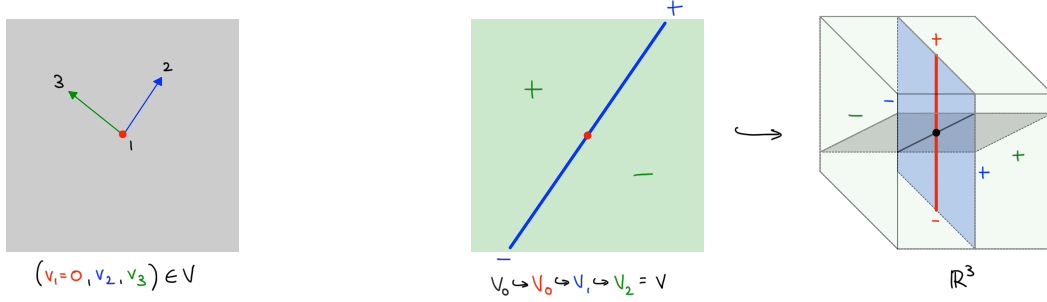
Figure A.2: An 3-embedded orthonormal frame and its corresponding 3-embedded ind-frame.

maps the orthonormal n-embedded k-partial frame of V to an orthonormal n-embedded frame of W. Thus, by Observation A.1.30, orthonormal n-embedded k-partial frames are in correspondence with standard isometric n-embedded k-partial trivializations. Adapting the arguments of Observation A.1.30, we observe that each orthoequivalence classes of an embedded partial trivialization contains exactly one orthonormal embedded partial frame. ⌐

In other words, orthonormal embedded (partial) frames provide unique representatives orthoequivalence classes of embedded (partial) trivializations of an euclidean vector spaces. Importantly, in the absence of euclidean structure, we do not have access to a notion of 'orthonormality' of frames any more, but we still do have access the notion of 'orthoequivalence' of trivializations; this suggests the the following.

Remark A.1.32 (Orthoequivalence as generalization of orthonormality). We may regard the notion of (embedded partial) 'trivializations up to orthoequivalence' as a generalization of the (embedded partial) 'orthonormal frames', and work with the former structure in place of the latter structure when euclidean structure is absent. ⌐

A.2   Affine frames

Affine trivializations, frames, indframes, and proframes In the previous sections we discussed the correspondence of frames, indframes, and proframes in the setting of vector spaces. We now discuss how these ideas carry over to the case of affine spaces. This case is important as a parallel to the affine combinatorics of framed simplices.

An affine space V is a space of 'points' freely and transitively acted upon by a vector space $\vec{V}$ (the 'associated vector space'); the vectors of $\vec{V}$ are called 'translations'. A map of affine spaces $F : V \to W$ is a continuous map such that for a necessarily unique linear map $\vec{F} : \vec{V} \to \vec{W}$ (the 'associated vector space map') we have $\vec{F}(v - v') = F(v) - F(v')$. Denote the category of affine spaces and affine maps by Aff. This comes with an 'associated vector space' functor $\vec{-} : \mathsf{Aff} \to \mathsf{Vect}$ to the category of vector spaces.

Remark A.2.1 (Simplices realize to affine spaces). Given an m-simplex S, its geometric realization $|S|$ is the subspace of the free vector space $\mathbb{R}\langle S \rangle$ (on the set of vertices of S) that consists of affine combinations of the standard basis. Therefore, the realization $|S|$ is contained in an affine hyperplane $V(S)$ of $\mathbb{R}\langle S \rangle$, and carries 'partial' affine structure

('partial' in the sense that the action by translations in $\vec{V}(S)$ is partial). Given another affine space W, an 'affine map' $|S| \to W$ is a map that is the restriction of an affine map $V(S) \to W$.[34]

Notation A.2.2 (Standard geometric simplices). We denote the geometric realizations of the 'standard' m-simplex S with vertices $\{0, 1, ..., m\}$ by Δ^m, and refer to it as the 'standard geometric m-simplex'. ⌐

Notation A.2.3 (Space of affine vectors). Given an affine space V, the space of affine embeddings $e : \Delta^1 \hookrightarrow V$ of the realized 1-simplex Δ^1 into V is the 'space of affine vectors' in V, and is denoted by $\hat{V}$. Note that the space of affine vectors is itself an affine space (it has an action by $\vec{V} \oplus \vec{V}$). A 'basepoint forgetting' map that takes an affine vector $e : \Delta^1 \hookrightarrow V$ to the translation $e(1) - e(0)$; we denote this map by $\mathsf{unbase} : \hat{V} \to \vec{V}$. Note that any affine map $F : V \to W$ induces (by postcomposition) a map of affine vectors $\hat{F} : \hat{V} \to \hat{W}$; this induces an 'affine vector' functor $\hat{-} : \mathsf{Aff} \to \mathsf{Aff}$. ⌐

Note that the notation similarly applies to vector spaces V (which, by acting on themselves, are in particular affine spaces with $\vec{V} \equiv V$).

The notions of frames, indframes, and proframes carry over to the affine case, as follows.

Definition A.2.4 (Affine trivializations, frames, indframes, and proframes)**.** An **affine trivialization** resp. an **affine frame** of an affine space V is a linear trivialization resp. a linear frame of its associated vector space $\vec{V}$. Similarly an **affine indframe** resp. an **affine proframe** is defined as a linear indframe resp. proframe of $\vec{V}$. ⌐

The definitions of affine trivializations, frames, indframes, and proframes can further be translated into 'affine terms', i.e. expressed in terms of structures on/maps of affine spaces and not of their associated vector spaces. Importantly, we will encounter that the definitions of indframes and proframes behave unequally under this translation. We start with the case of affine trivializations (and, equivalently, frames).

Remark A.2.5 (Affine perspective on affine trivializations and frames). Given an affine space V and a trivialization $\vec{V} \xrightarrow{\sim} \mathbb{R}^m$, precomposing with $\mathsf{unbase} : \hat{V} \to \vec{V}$ yields a map $\hat{V} \to \mathbb{R}^m$ that trivializes 'vector spaces of affine vectors based at x' for any $x \in V$. In particular, preimages of standard vectors $e_i \in \mathbb{R}^n$ under this map provide a set of frame vectors $v_i^x : \Delta^1 \to V$ at each point $v_i^x(0) = x$ of V. ⌐

Now, to express affine indframes and proframes in affine terms, the following observation needs to be taken into account.

Observation A.2.6 (Asymmetry of affine projections and subspaces). Given an affine space V, and a projection of associated vector spaces $\vec{V} \twoheadrightarrow \vec{W}$, then this canonically induces an 'affine projection' $V \twoheadrightarrow W$ where W is constructed as the quotient of V by the action of $\ker(\vec{V} \twoheadrightarrow \vec{W})$ (the associated vector space of W is $\vec{W}$). In contrast, given an inclusion $\vec{W} \hookrightarrow \vec{V}$ then there is no canonical corresponding injective affine map $W \hookrightarrow V$. In particular, given an affine projection $V \twoheadrightarrow W$ there is no canonical choice of 'affine kernel' $U \hookrightarrow V$ corresponding to the linear kernel $(\vec{U} \hookrightarrow \vec{V}) = \ker(\vec{V} \twoheadrightarrow \vec{W})$. ⌐

---

[34]In the main text, an 'affine map' $|S| \to W$ is, abusing terminology, referred to as a 'linear map'.

The observation leads to the following asymmetry in the translation of affine proframes and indframes.

Remark A.2.7 (Affine data of affine proframes). Given an affine space V and a proframe $\vec{V} = \vec{V}^m \twoheadrightarrow \vec{V}^{m-1} \twoheadrightarrow ... \twoheadrightarrow \vec{V}^0$ on $\vec{V}$, we obtain a sequence of affine surjective maps

$$V = V^m \twoheadrightarrow V^{m-1} \twoheadrightarrow V^{m-2} \twoheadrightarrow ... \twoheadrightarrow V^1 \twoheadrightarrow V^0 = 0$$

where the map $V \to V^m$ is the quotient of V induced by the projection $\vec{V} \to \vec{V}^m$. Applying the affine vector functor $\hat{-}$, the above sequence becomes a sequence of affine spaces of affine vectors, which factors through the original proframe on $\vec{V}$ by the basepoint forgetting maps $\mathsf{unbase} : \hat{V}^i \to \vec{V}^i$. $\quad\quad\quad\quad\quad\quad\lrcorner$

Remark A.2.8 (Affine data of affine indframes). Given an affine space V and an indframe $0 = \vec{V}^0 \hookrightarrow \vec{V}^1 \hookrightarrow ... \hookrightarrow \vec{V}^m = \vec{V}$ on $\vec{V}$, this sequence of linear subspaces does *not* canonically induce a sequence of affine injective maps ending in V. Instead, we may think 'affinely' of the indframe by pulling it back along $\mathsf{unbase} : \hat{V} \to \vec{V}$ to a filtration of $\hat{V}$ (which simply 'bases' a copy of the indframe at all points of V). $\quad\quad\quad\quad\quad\quad\lrcorner$

In other words, affine proframes can be canonically expressed in terms of sequences of affine projections; however, affine indframes cannot be canonically expressed as sequence of affine subspaces. This observation has an analog in affine combinatorics of simplices; indeed, in Section 1.1.1 we saw see that 'simplicial subspaces' cannot be expressed canonically as simplicial maps while 'simplicial projections' can (namely, in terms of degeneracies).

Simplicial trivializations and framed realizations Let us now relate the notion of framed realizations of framed simplices to our discussion of (linear and affine) trivializations here. Recall framed realizations (introduced for (partial) framed simplices in Definition 1.1.25 and Definition 1.1.25, and generalized to the (partial) *embedded* case by Definition 1.1.38 resp. Definition 1.1.44) are affine maps from simplices to $\mathbb{R}^n$ such that simplicial vectors with frame label i are mapped to the ith positive component ϵ_i^+ of the standard indframe of $\mathbb{R}^n$. As we now explain, framed realizations can be understood as orthoequivalence classes of 'simplicial' trivializations (which capture trivialization of affine simplicial space that, up to orthoequivalence, can be obtained from frames of simplicial vectors).

Terminology A.2.9 (Simplicial embedded trivialization). For an m-simplex S, an injective affine map $r : |S| \hookrightarrow \mathbb{R}^n$ (see Remark A.2.1) is a 'simplicial embedded trivialization' if the linear n-embedded trivialization $\vec{r} : \vec{V}(S) \hookrightarrow \mathbb{R}^n$ is orthoequivalent to a trivialization of $\vec{V}(S)$ determined by some n-embedded frame $\{\vec{v}_i\}$ of vectors v_i in S (see Remark A.1.21). We call the ordered list of n vectors $\{v_i\}$ an 'r-compatible frame'. Two such maps $r, r' : |S| \hookrightarrow \mathbb{R}^n$ are 'orthoequivalent' if they are orthoequivalent as linear embedded trivializations $\vec{r}, \vec{r}' : \vec{V}(S) \hookrightarrow \mathbb{R}^n$. If further $m = n$, we speak of 'simplicial trivializations' instead. $\quad\quad\quad\quad\quad\quad\lrcorner$

Observation A.2.10 (Simplicial embedded trivialization determine embedded frames). Given a simplicial embedded trivialization $r : |S| \hookrightarrow \mathbb{R}^n$ and an r-compatible frame $\{v_i\}$, then exactly m of the vectors v_i in S must be nonzero. The choice of nonzero vectors v_i becomes unique if we require them to form a set of spine vectors under some identification $S \cong [m]$. Moreover, if v_i is a zero vector then it is exchangeable for any other zero vector,

and thus choices of zero vectors are redundant: we may replace zero vectors v_i simply by the affine zero 0^{aff}. This results in a unique ordered list $\{v_i\}$ of n vectors in S, containing m nonzero vectors forming the spine of $S \cong [m]$, and $(n - m)$ zeros 0^{aff}. This defines a simplicial n-embedded framed simplex $(S \cong [m], \mathcal{F})$ with $v_i = \mathcal{F}(j)$ iff v_i is the jth nonzero vector in the list $\{v_i\}$. ⌣

Observation A.2.11 (Framed realizations are simplicial trivialization classes). Given an n-embedded framed m-simplex $(S \cong [m], \mathcal{F})$, note that any framed realization is a simplicial embedded trivialization. In fact, the set of framed realizations of $(S \cong [m], \mathcal{F})$ is exactly the orthoequivalence class of simplicial embedded trivialization of S which determine the embedded framed simplex $(S \cong [m], \mathcal{F})$. This observation of course specializes to the non-embedded case $n = m$. ⌣

Let us next consider the partial case: again, framed realizations of a partial embedded framed simplex can be understood as describing an orthoequivalence class of partial embedded trivializations of the underlying simplex, as follows.

Terminology A.2.12 (Simplicial embedded partial trivialization). Given an m-simplex S, we say that an affine map $r : |S| \to \mathbb{R}^n$ is 'simplicial embedded partial trivialization' if it factors by affine maps $|S| \twoheadrightarrow |T| \hookrightarrow \mathbb{R}^n$ such that the first map realizes a degeneracy and the second is a simplicial embedded partial trivialization. Two such maps $r, r' : |S| \to \mathbb{R}^n$ are 'orthoequivalent' if they are orthoequivalent as linear embedded partial trivializations $\vec{r}, \vec{r}' : \vec{V}(S) \to \mathbb{R}^n$. If $m = n$, we speak of a 'simplicial partial trivialization' instead. ⌣

Observation A.2.13 (Simplicial embedded partial trivializations determined frames). Any simplicial embedded partial trivialization $r : |S| \to \mathbb{R}^n$ factors uniquely by maps $|S| \twoheadrightarrow |T| \hookrightarrow \mathbb{R}^n$; since the map $|T| \hookrightarrow \mathbb{R}^n$ determines an n-embedded frame $(T \cong [k], \mathcal{F})$, it follows that r determines a n-embedded k-partial framed $(S \twoheadrightarrow T \cong [k], \mathcal{F})$. ⌣

Observation A.2.14 (Framed realizations are simplicial trivialization classes). Given an n-embedded k-partial framed m-simplex $(S \twoheadrightarrow T \cong [m], \mathcal{F})$, note that any framed realization is a simplicial embedded partial trivialization. In fact, the set of framed realizations of $(S \twoheadrightarrow T \cong [m], \mathcal{F})$ is exactly the orthoequivalence class of simplicial embedded partial trivializations of S which determine embedded partial framed simplex $(S \twoheadrightarrow T \cong [m], \mathcal{F})$. The observation specializes to the non-embedded case $n = k$. ⌣

In summary, we may understand combinatorial frames in affine algebraic terms as follows.

Remark A.2.15 (Combinatorial frames describe simplicial trivialization classes). Combinatorial (embedded partial) frames correspond exactly to orthoequivalence classes of (embedded partial) trivializations of simplices. In light of the role of orthoequivalence (see Remark A.1.15 and Remark A.1.32), they thus provide a combinatorial description of 'generalized orthonormal frames' of simplices (realized as affine spaces). ⌣

A.3 Towards globally framed spaces

Taking affine standard proframed euclidean spaces of the previous section as 'local models', we will define more general framed spaces. Based on this definition, we then construct a geometric realization functor for framed regular cell complexes.

A.3.1   Framed spaces We start with a definition of framed spaces, which will then provide the 'target' for geometric realization of framed combinatorial structures in the next section. Our definition of framed spaces will use a non-standard notion of framings: it inspired by the relation of 'frames' and 'proframes' in the previous section. We recall, once more, the rough idea of this relation.

Remark A.3.1 (Frames from projections). Let V be an euclidean vector space (i.e. finite dimensional real vector space with an inner product). The following structures on V are equivalent.

1. A **classical orthonormal frame** of V, i.e. an ordered sequence of vectors v_i, $1 \leq i \leq n$, such that v_i is normalized and orthogonal to all v_j with $j \neq i$.
2. A **proframe** of V, i.e. a sequence of projections $V_i \to V_{i-1}$, $1 \leq i \leq n$, where V_i is an oriented i-dimensional vector space (and $V_n = V$).

The two structures are related by setting $V_i = \mathrm{span}(v_1, v_2, ..., v_i)$, with $V_i \to V_{i-1}$ the map that forgets v_i and $(v_1, v_2, ..., v_i)$ being a positively oriented basis in V_i.   ⌞

Example A.3.2 (Standard orthornomal framing of euclidean space). The standard orthonormal frame of the n-dimensional euclidean vector space $\mathbb{R}^n$ consists for the ordered sequence of vectors $e_1 = (1, 0, ..., 0)$, $e_2 = (0, 1, 0, ..., 0)$, ..., $e_n = (0, ..., 0, 1)$. By the previous remark, this orthonormal frame is equivalently described by the sequence of projections $\pi_i : \mathbb{R}^i = \mathbb{R}^{i-1} \times \mathbb{R} \to \mathbb{R}^{i-1}$ (each $\mathbb{R}^i$ being endowed with standard orientation).   ⌞

When forgetting the base point of an inner product space, i.e. thinking of it as an *affine* space, then a proframe in the previous sense endows each point of the affine space with an orthonormal frame, by considered the associated vector space at that point. One may use this observation to define framings of spaces (i.e. choices of frames at every point), by using 'framed bounded patches' of affine, standard framed $\mathbb{R}^n$ as a local model as follows.

Terminology A.3.3 (Patches). An 'n-framed patch' $U \subset \mathbb{R}^n$ is a framed bounded subspace of $\mathbb{R}^n$ (in the sense of Definition 1.2.27).   ⌞

Terminology A.3.4 (Framed maps and partial maps of patches). Given two n-framed patches U and V, a *(partial) n-framed patch map* $F : U \to V$ is a (partial) continuous map which descends along π_n to a (partial) $(n-1)$-framed patch map $F_{n-1} : \pi_n(U) \to \pi_n(V)$ such that mappings of fibers $F : \pi_n^{-1}(x) \to \pi_n^{-1}(F_{n-1}(x))$ are monotone.   ⌞

Note that n-framed patches are contractible spaces.

Example A.3.5 (The closed cube). The standard example of an n-framed patch is the closed n-cube $\mathbf{I}^n = [-1, 1]^n \subset \mathbb{R}^n$. In general, however, n-framed patches need not be 'n-dimensional': for instance, the 0th slice $\{0\} \times [-1, 1]^{n-1}$ of the n-cube is itself an n-framed patch.   ⌞

Globally, framings of spaces can be defined by giving an 'atlas of charts' of framed patches of that space.

Definition A.3.6 (Framed spaces). Let X be a topological space. Fix $n \in \mathbb{N}$.
1. An *n-framed chart* (U, γ) in X is an embedding $\gamma : U \hookrightarrow \mathbb{R}^n$ of a subspace $U \subset X$ whose image $\mathrm{im}(\gamma)$ is an n-framed patch.

2. Two n-framed charts (U, γ), (V, ρ) in X are *compatible* if $\rho \circ \gamma^{-1} : U \to V$ is a locally framed (partial) map.

An n-**framed space** is a space X together with an 'atlas' $\mathcal{A}$ of compatible n-framed charts $\{(U_i, \gamma_i)\}$ such that $\{U_i\}$ are a locally finite cover of X. $\quad\rule[0pt]{0pt}{0pt}\lrcorner$

Definition A.3.7 (Maps of framed spaces). Given spaces with n-framing structure $(X, \mathcal{A}) \to (Y, \mathcal{B})$, a **framed map** $F : X \to Y$ is a map such that, for charts $(U, \gamma) \in \mathcal{A}$ and $(V, \rho) \in \mathcal{B}$, F yields a locally framed (partial) map $\rho \circ F \circ \gamma^{-1} : U \to V$. $\quad\lrcorner$

Note that asking for 'locally finite' covers is convenient as it mirrors the situation of locally finite cell complexes. The definition has several variations and generalization.

Example A.3.8 (Standard framings of compact subspaces). Any n-framed patch $U \hookrightarrow \mathbb{R}^n$ is a canonically n-framed space with a single chart $(U, U \hookrightarrow \mathbb{R}^n)$. $\quad\lrcorner$

Example A.3.9 (Open standard framed space). Standard $\mathbb{R}^n$ has a standard framing with charts $[-l, l]^n \subset \mathbb{R}^n$, $l \in \mathbb{N}$. Similarly, the open n-cube $\mathbb{I}^n = (-1, 1)^n$ has a standard framing with charts $[-1 + \frac{1}{l}, 1 - \frac{1}{l}]^n \subset \mathbb{I}^n \subset \mathbb{R}^n$. Framed maps $\mathbb{R}^n \to \mathbb{R}^n$ (resp. $\mathbb{I}^n \to \mathbb{I}^n$) are precisely maps that factor through the standard projections $\mathbb{R}^n = \mathbb{R}^i \times \mathbb{R}^{n-i} \to \mathbb{R}^i$. Note that $\mathbb{R}^n$ and $\mathbb{I}^n$ are framed homeomorphic. $\quad\lrcorner$

Example A.3.10 (Circle with framing poles). Take two intervals $I_1, I_2 \cong [0, 1]$ and glue them together at their endpoints to get the space X (homeomorphic to the circle). Denote the inclusion $[0, 1] \subset \mathbb{R}$ by ϕ. Then X has a 1-framing structure given by the charts (I_1, ϕ) and (I_2, ϕ). $\quad\lrcorner$

A.3.2 Framed geometric realization and combinatorial structures on framed spaces With our previous definitions, it is now straight-forward to construct geometric realizations of framed regular cell complexes.

Terminology A.3.11 (Local finiteness). A poset P is called 'locally finite' if any point $x \leq P$ has finite lower closure $P^{\leq x} = y \leq x \subset P$. Geometrically, this is equivalent to the statement that any point in the classifying stratification $\mathsf{CStr}\, P$ has a neighborhood that intersects only finitely many strata (see Remark B.3.7). $\quad\lrcorner$

Definition A.3.12 (Framed geometric realization). Given a locally finite n-framed regular cell complex $(X, \mathcal{F})$, its **framed realization** $|X, \mathcal{F}|$ is the framed space $(|X|, \mathcal{A}_{(X, \mathcal{F})})$ where $|X|$ is the geometric realization of the poset X and $\mathcal{A}_{(X, \mathcal{F})}$ is the atlas containing a chart $(|X^{\geq x}|, r_x)$ for each cell $X^{\geq x} \hookrightarrow X$ with chosen framed realization $r_x : |X, \mathcal{F}| \hookrightarrow \mathbb{R}^n$ of that cell (see Definition 1.2.24; in particular, note that such framed realizations exists since framings on cells are flat). The fact that this defines a n-framed atlas will follow from the equivalence of framed and proframed regular cells, discussed in Chapter 3.

Similarly, given a framed cellular map $F : (X, \mathcal{F}) \to (Y, \mathcal{G})$ of framed regular cell complexes, its **framed realization** $|F| : |X, \mathcal{F}| \to |Y, \mathcal{G}|$ ist the framed map which on underlying spaces is given by the geometric realization $|F| : |X| \to |Y|$. $\quad\lrcorner$

Note that framed realizations of framed regular cells automatically endow the underlying space of the cell with PL structure (in fact, this is always the standard PL ball).

It is useful to consider such framed realizations 'up to framed PL homeomorphism' (i.e. framed maps which are PL as maps of underlying PL spaces). In essence, this means that underlying combinatorial topological spaces will be treated in the usual piecewise-linear manner, while the framing structure will be governed by the combinatorics of framed regular cell complexes and their maps.

Definition A.3.13 (Framed re-cellulation)**.** A 'framed subdivision' $|X, \mathcal{F}| \to |X', \mathcal{F}'|$ is a framed PL homeomorphism which is stratified refinement of cell stratifications. A **framed re-cellulation** $|X, \mathcal{F}| \rightleftarrows |Y, \mathcal{G}|$ is a zig-zag of framed subdivisions. ⌐

For the next definition, recall that cellular maps of 'geometrically realized' (as opposed to 'combinatorial') regular cell complexes are continuous maps that preserve cell closures, see Definition 1.3.18.

Definition A.3.14 (Framed combinatorial maps)**.** A **framed combinatorial map** $F : |X, \mathcal{F}| \to |Y, \mathcal{G}|$ is a framed map which, after framed re-cellulation and framed cellular PL homeomorphism of its domain and codomain, is the framed realization of a framed cellular map. ⌐

Remark A.3.15 (Generalizing maps)*.* Weakening this definition, it may also be useful to think of framed realizations as defined up to 'cellular framed homotopy', i.e. up to homotopy through cell-preserving framed homemorphisms (instead of framed *PL* homeomorphism. Indeed, locally, i.e. in standard framed $\mathbb{R}^n$, framed homeomorphism and framed PL homeomorphism are equivalent as both $\mathrm{Aut}^{\mathrm{fr}}(\mathbb{R}^n)$ and $\mathrm{Aut}^{\mathrm{fr}}_{\mathrm{PL}}(\mathbb{R}^n)$ are contractible. Thus, generalizing the definition given above, one may want to call a framed map 'combinatorial' if, after re-cellulation, it is framed cellular homotopic to the framed realization of a framed cellular map. ⌐

Definition A.3.16 (Framed combinatorial spaces)**.** Given a framed space $(Z, \mathcal{A})$, a **framed cellulation** is a framed homeomorphism $\alpha : (Z, \mathcal{A}) \cong |X, \mathcal{F}|$ for $(X, \mathcal{F})$ a framed regular cell complex. Two such cellulation α and β of $(Z, \mathcal{A})$ are said to be 'framed equivalent' if there is a framed combinatorial homeomorphism ρ such that $\rho \circ \alpha = \beta$. A 'framed combinatorial structure' on $(Z, \mathcal{A})$ is a framed equivalence class of framed cellulations, and, equipped with such a structure, $(Z, \mathcal{A})$ becomes a **framed combinatorial space**. ⌐

There are many conceivable variations of the above definitions—our goal here is to merely give a first sketch of how one could interpret framed combinatorial structures in topological models.

Stratified topology

The notion of 'stratified space' (or 'singular space') refers to a decomposition of a space into 'strata', usually ordered by dimension or 'depth'. Frequently, such order is enforced by working with filtrations $X_0 \subset X_1 \subset ... \subset X_{k-1} \subset X_k$ of spaces $X = X_k$ where X_{i-1} is required to be closed in X_i. Equivalently, and more concisely, such a filtration may be expressed by a continuous function $f : X \to [k]^{\mathrm{op}}$ (where $[k]^{\mathrm{op}}$ is the poset $(0 \leftarrow 1 \leftarrow ... \leftarrow k)$ topologized such that downward closed subposets are open sets) which allows us to recover X_i as the preimage $f^{-1}[i]^{\mathrm{op}}$, $i \leq k$. This has been generalized by defining stratifications as continuous maps of spaces to any poset, yielding, for instance, definitions of '$\mathscr{S}$-filtered spaces' in [GM88, §III.2.2.1] and of 'P-stratifications' in [Lur12, Defn. A.5.1]. Note, however, posets in the domain of such continuous maps may contain information that is unrelated to the decomposition of the underlying space, even when the map is surjective. In this appendix, we develop a notion of stratification which is similarly general, but in which the role of posets faithfully represents topological information about the stratification. The different definitions coincide in many cases, for instance for locally finite stratifications. We use the following conventions.

Notation B.0.1 (Convenient spaces). We denote by TOP the category of *all* topological spaces, by Top the subcategory of compactly generated spaces, and by kTop the subcategory of compactly generated weak Hausdorff spaces (see [Str09]). The category Top is cartesian closed with internal hom denoted by $\mathrm{Map}(-,-)$, and this internal hom is inherited by Top. All our spaces will be assumed to be compactly generated. ⌐

Convention B.0.2 (Specialization topology). Given a preorder $(P, \leq)$ we regard it as a topological space with the *specialization topology*, declaring the open subsets to be those that are downward closed; a subset U is downward closed if $x \leq y$ and $y \in U$ implies that $x \in U$.[35] ⌐

Remark B.0.3 (Specialization order). Given a topological space $X \in k$Top, we denote by $\mathrm{Spcl}\, X$ its *specialization order*: this is the preorder whose objects are the objects of the underlying set X, and whose morphisms $x \to y$ are given whenever y is contained in the closure of x. Note that for a poset P, $\mathrm{Spcl}\, P = P$. Note that the specialization

---

[35]We frequently write the relation $x \leq y$ as $x \to y$, interpreting preorders and posets as categories.

topology provides a functor, which is an adjoint equivalence between finite preorders and finite topological spaces. The inverse functor is given by the specialization order functor Spcl. ⌐

B.1   Stratified spaces

Entrance paths, stratifications, and characteristic functions In this section we introduce the basic notions of 'stratified spaces', their 'entrance path posets' and their 'characteristic functions'.

A robust definition of stratified spaces is obtained by letting the topological decomposition of a space into strata determine the corresponding poset structure, in terms of the existence of so-called entrance paths between strata, as follows.

Definition B.1.1 (Entrance path). Given a space X and two subspaces X_r and X_s, an **entrance path** from X_r to X_s is a path $\alpha : [0,1] \to X$ with $\alpha(x < 1) \subset X_r$ and $\alpha(1) \in X_s$. ⌐

Here, the path is thought of as 'entering' from the former subspace X_r into the latter subspace X_s.

Definition B.1.2 (Formal entrance path). Given a space X and two subspaces X_r and X_s, we say there exists a **formal entrance path** from X_r to X_s, denoted $r \to s$, when the closure of X_r has nonempty intersection with X_s. ⌐

In contrast to entrance paths, note that the structure of formal entrance paths is boolean: either there exists a formal entrance path between subspaces or there doesn't. If there is an entrance path from a subspace X_r to a subspace X_s of a space X this implies the existence of a formal entrance path, but the converse need not hold unless additional conditions are imposed.

Terminology B.1.3 (Formal entrance path relation of a decomposition). Given a decomposition $X = \bigsqcup_{s \in \mathsf{Dec}} X_s$ of a space X into a disjoint collection of subspaces, the 'formal entrance path relation' on the set Dec of subspaces (the 'decomposition set') is the relation that has an arrow $r \to s$ exactly when there is a formal entrance path from X_r to X_s. ⌐

Note that the formal entrance path relation of a decomposition is reflexive, but need not be antisymmetric or transitive. Stratifications are exactly those decompositions for which this relation has no cycles, that is for which it is a directed acyclic graph.

Definition B.1.4 (Prestratifications and stratifications). A **prestratification** (X, f) of a topological space X is a disjoint decomposition $f = \{X_s \subset X\}_{s \in \mathsf{Dec}(f)}$ of X into nonempty connected subspaces indexed by a set $\mathsf{Dec}(f)$. The subspaces X_s are called **strata** of (X, f). A **stratification** (X, f) is a prestratification such that the formal entrance path relation on the decomposition set $\mathsf{Dec}(f)$ has no cycles. ⌐

Notation B.1.5 (Shorthand for (pre)stratifications). We frequently abbreviate a (pre)stratification (X, f) simply by f, referring to f as a '(pre)stratification on X'. We will often abbreviate a stratum $X_s \subset X$ simply by its index $s \in \mathsf{Dec}(f)$. ⌐

Observe that, given a stratification (X, f), the transitive closure of the formal entrance path relation on the decomposition set $\mathsf{Dec}(f)$ is a partially ordered set, which has an arrow $r \to s$ exactly when there is a chain of formal entrance paths beginning at r and ending at s. Note that the equivalence classes of the resulting transitive relation remain exactly the decomposition set $\mathsf{Dec}(f)$. This does not hold true if (X, f) is merely a prestratification, in which case $\mathsf{Dec}(f)$ obtains the structure of a preordered set.

Definition B.1.6 (Entrance path preorder and poset)**.** For a prestratification (X, f), the **entrance path preorder** $\mathsf{Entr}(f)$ is the decomposition set of the stratification together with the transitive closure of the formal entrance path relation. If (X, f) is a stratification, then we refer to $\mathsf{Entr}(f)$ as the **entrance path poset** of (X, f). ⌟

Remark B.1.7 (Exit paths and the exit path preorder)**.** Given a prestratification (X, f), the opposite preorder of the entrance path preorder is called the exit path preorder, and is denoted $\mathsf{Exit}(f) = \mathsf{Entr}(f)^{\mathrm{op}}$. An exit path from X_s to X_r is a path $p : [0, 1] \to X$ with $p(0) \in X_s$ and $p(x > 0) \subset X_r$; the path is 'exiting' from the stratum X_s into the stratum X_r. Whether to focus on entrance or exit paths is a matter of convention and convenience; in this book, we find that entrance paths have more natural functoriality dependencies and so we work entirely with them. ⌟

Example B.1.8 (Entrance path poset)**.** Fig. B.1 shows a stratification of the open 2-disk into five strata, along with its entrance path poset (shown on the right, together with an indication of which poset elements correspond to which strata). ⌟

Figure B.1: A stratification and its entrance path poset.

Example B.1.9 (Entrance path poset requiring transitive closure)**.** Fig. B.2 depicts a stratification of the open interval, into one open interval and two half-open interval strata, together with its entrance path poset. ⌟

Figure B.2: A stratification with entrance path poset as the transitive closure of the entrance path relation.

Example B.1.10 (Entrance path preorder)**.** In Fig. B.3 we depict a decomposition of the circle that is not a stratification but a prestratification, because the formal entrance path relation has a cycle. ⌟

Terminology B.1.11 (Discrete and indiscrete stratifications)**.** Every space X has an 'indiscrete stratification' whose strata are the connected components of X. The entrance path preorder of the indiscrete stratification is the set of connected components of X.

Figure B.3: A decomposition that is not a stratification but a prestratification, and its formal entrance path relation.

Conversely, every space also has a 'discrete prestratification', such that each point becomes its own stratum. The entrance path poset of the discrete stratification of X is the specialization order $\operatorname{Spcl} X$ (in particular, the definition of specialization orders can be recovered from the definition of entrance path preorders). ⎯⎦

Unless indicated otherwise, a bare topological space is implicitly given the indiscrete stratification.

Terminology B.1.12 (Finite (pre)stratifications). We call a (pre)stratification (X, f) 'finite' if its entrance path preorder $\mathsf{Entr}(f)$ is finite, and call it 'infinite' otherwise. ⎯⎦

Definition B.1.13 (Characteristic function)**.** Given a prestratification (X, f), we refer to the function $X \to \mathsf{Entr}(f)$ sending each point $x \in X_r$ to its corresponding stratum $r \in \mathsf{Entr}(f)$, as the **characteristic function** of the prestratification; we denote the characteristic function of a prestratification (X, f) by $f : X \to \mathsf{Entr}(f)$. ⎯⎦

A fundamental property of characteristic functions is that they are 'finitely continuous', as follows.

Terminology B.1.14 (Finitely continuous maps). A function of topological spaces $F : X \to Y$ is called 'finitely continuous' for each finite subspace $Q \subset Y$ the function restricts to a continuous map $F : F^{-1}(Q) \to Q$. ⎯⎦

Lemma B.1.15 (Finite continuity in prestratifications)**.** *Characteristic functions are finitely continuous.*

Proof. Consider a prestratification (X, f) with characteristic function $f : X \to \mathsf{Entr}(f)$. Consider a finite subposet $Q \subset \mathsf{Entr}(f)$, and let $U \subset Q$ be a downward closed subposet. Arguing by contradiction, assume $f^{-1}(U) \subset f^{-1}(Q)$ is not open. Then there is a point $p \in f^{-1}(U)$ such that each neighborhood of p intersects a preimage $f^{-1}(q)$ of some $q \in Q \setminus U$ not in U. Since Q is finite, we can pick a $q \in Q \setminus U$ such that $f^{-1}(q)$ intersects *all* neighborhoods of p. This means p lies in the closure $f^{-1}(q)$ which entails there is an arrow from q into some object of U, contradicting downward closure of the latter subposet. Thus, $f^{-1}(U) \subset f^{-1}(Q)$ is open, showing finite continuity of f. $\qquad\square$

In the case of *finite* prestratifications, this of course implies that their characteristic functions are continuous in the usual sense.

Definition B.1.16 (Locally finite stratifications)**.** A stratification (X, f) is **locally finite** if every stratum s has an open neighborhood $s \subset N(s)$ which contains only finitely many strata. ⎯⎦

Terminology B.1.17 (Locally finite posets)**.** A poset$(P, \leq)$ is 'locally finite' if all downward closures $P^{\leq x} = \{y \mid y \leq x\}$ are finite. ⌐

Remark B.1.18 (Frontier-constructibility and local finiteness of entrance path posets)**.** A 'frontier-constructible' stratification (X, f) is a stratification in which the closure $\bar{s}$ of each stratum s can be written as a union of strata in f (in Lemma B.2.10 we show this to hold iff $f : X \to \mathsf{Entr}(f)$ is an open map). For frontier-constructible (X, f) the following are equivalent:

1. (X, f) is locally finite,
2. any point $x \in X$ has a neighborhood intersecting only finitely many strata,
3. The entrance path poset $\mathsf{Entr}(f)$ is locally finite.

The 'frontier-constructbility' condition is also sometimes referred to as the 'frontier' condition. ⌐

Notation B.1.19 (Covering relation)**.** Given a poset $(P, \leq)$ its covering relation is usually defined as follows: we say $x \in P$ 'covers' $y \in P$, written $y <^{\mathrm{cov}} x$, if $y < x$ is non-refinable (that is, for any $y < z < x$ we have either $y = z$ or $z = x$). ⌐

Lemma B.1.20 (Locally finite characteristic functions are continuous)**.** *If (X, f) is a locally finite stratification, then its characteristic function $f : X \to \mathsf{Entr}(f)$ is continuous.*

Proof. We need to show that for each $s \in \mathsf{Entr}(f)$ the downward closure $\mathsf{Entr}(f)^{\leq s}$ of s has open preimage in X under f. For $r \in \mathsf{Entr}(f)^{\leq s}$, let $Q_r \subset \mathsf{Entr}(f)^{\leq s}$ be the full subposet containing only r and the elements $r' <^{\mathrm{cov}} r$ that it covers. The assumption that (X, f) is locally finite implies that Q_r is finite, and that $f^{-1}(Q_r)$ contains an open neighborhood of r. It follows that $f^{-1}(\mathsf{Entr}(f)^{\leq s})$ is open as required. □

Remark B.1.21 (Characteristic maps)**.** Whenever a characteristic function is continuous we usually refer to it as a characteristic *map*. Note, characteristic functions of general infinite (pre)stratifications need not be continuous, as the next example illustrates. ⌐

Example B.1.22 (Infinite characteristic functions can be discontinuous)**.** In Fig. B.4 we depict a stratification of the closed interval with non-continuous characteristic function. In particular, the stratification is not locally finite. ⌐

●▬▬▬▬▬▬●▬▬▬▬●▬▬●▬●▬●⋯●

Figure B.4: A stratification with non-continuous characteristic function.

As we will see in Lemma B.1.29, there is a precise characterization of those functions $f : X \to P$ from a space to a finite poset which are characteristic maps of stratifications.

From now on, we will focus most of our attention on stratifications instead of working in the more general context of prestratifications. In particular, most definitions will be given for stratifications only—however, the reader will notice that many immediately generalize if one replaces 'stratifications' by 'prestratifications', and 'posets' by 'preorders'.

Poset structures and quotient maps In this section we relate our definitions of stratifications with general 'poset structures' on spaces.

Definition B.1.23 (Poset structures)**.** Given a poset P, a P-**structured** space (X, f) is a space X together with a continuous map $f : X \to P$. ⌐

We will first show that characteristic maps of finite stratifications can be understood as a certain class of poset structures. Later we will show that, conversely, every poset structure can be universally 'split' into a stratification.

Recall, a surjective continuous map $f : X \to Y$ of spaces is a quotient map if for each subset $U \subset Y$ we have that U is open if and only if $f^{-1}(U)$ is open. If Y is the specialization topology of a poset, we call f a 'poset quotient'. Poset quotients (to finite posets) admit the following useful characterization.

Lemma B.1.24 (Quotient maps to finite posets)**.** *For a space X, a finite poset P, and a surjective continuous map $f : X \to P$, the following are equivalent:*

1. *f is a quotient map,*
2. *for all covers $p <^{\mathrm{cov}} p'$ in P there is a formal entrance path from $f^{-1}(p)$ to $f^{-1}(p')$.*

Remark B.1.25 (A quotient of posets is a map that is surjective on objects and on covers). In the lemma, if X is itself the specialization topology on a poset Q, then the lemma simplifies to saying that $f : Q \to P$ is a quotient map if and only if f is surjective on objects and on covers. ⌐

Proof of Lemma B.1.24. For $p \in P$, define K_0^p to be the preimage $f^{-1}(p)$. Set $I_0^p = \{p\}$. Let I_1^p be the set of $q \in P$ such that $f^{-1}(q)$ intersects the closure $\overline{K}_0^p$ of K_0^p, and define K_1^p to be the union of preimages $f^{-1}(q)$ of $q \in I_1^p$. Set I_2^p to be the set of $q \in P$ such that $f^{-1}(q)$ intersects the closure $\overline{K}_1^p$ of K_1^p, and define K_2^p to be the union of preimages $f^{-1}(q)$ of $q \in I_2^p$. Repeating this process, since P is finite, we find an index j with $I_j^p = I_{j+1}^p$ and $K_j^p = K_{j+1}^p = \overline{K}_j^p$. Denote these sets by I^p and K^p respectively.

First, assume f is a quotient map. Consider a cover $p < p'$. We claim it is impossible that $p' \notin I^p$: indeed, the complement $X \setminus K^p$ is the preimage of $P \setminus I^p$. Since $X \setminus K^p$ is open and since f is a quotient map, it follows that $P \setminus I^p$ is open which contradicts the assumption that $p < p'$ and $p' \notin I^p$. Thus assume $p' \in I^p$. This implies $f^{-1}(p')$ intersects K^p (and thus $f^{-1}(p') \subset K^p$). Then there is a sequence of arrows $p = p_0 < p_1 < ... < p_k = p'$ with $p_i \in K_i^p$. Since $p < p'$ is a cover we must have $k = 1$. Thus $f^{-1}(p')$ intersects the closure of $f^{-1}(p)$.

Next, assume f satisfies that for any cover $p < p'$ in P, the preimage $f^{-1}(p')$ intersects the closure of the preimage $f^{-1}(p')$. Let $Q \subset P$ be a subposet. Let $I^{P \setminus Q}$ and $K^{P \setminus Q}$ be the respective unions of all I^p and K^p for each $p \in P \setminus Q$. If Q is open then $f^{-1}(Q)$ is open by continuity of f. If $f^{-1}(Q)$ is open, then must be disjoint from $K^{P \setminus Q}$ (by construction of $K^{P \setminus Q}$). Thus $I^{P \setminus Q} = P \setminus Q$. Since $I^{P \setminus Q}$ is upward closed, it follows that Q is downward closed, i.e. open as required. □

A central role will be played by poset quotients whose 'equivalence classes are connected' in the following sense.

Definition B.1.26 (Connected-quotient maps)**.** For a space X and a finite poset P, a continuous map $f : X \to P$ is called a **connected-quotient map** if it is a poset quotient map whose preimages of points $p \in P$ are connected. (Note, we take 'connected' to also entail 'non-empty'.) ⌐

Remark B.1.27 (Connected-quotient maps between posets). A connected-quotient map $f : Q \to P$ where Q is a poset (endowed with specialization topology) is a poset quotient whose preimages are connected subposets of Q. ⌐

Example B.1.28 (Connected-quotient map). In Fig. B.5 we depict three maps from the circle to three different posets (color-coding images and preimages in the same color). The first map is a connected quotient map; the second maps fails to be a quotient map despite have connected preimages, the third map is a quotient map but fails to have connected preimages. ⌐
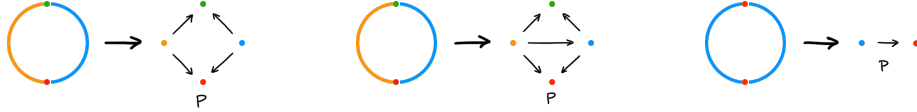
Figure B.5: A connected-quotient map and non-examples.

We now characterize stratifications among P-structures.

Lemma B.1.29 (Characteristic maps are connected-quotient maps). *For a space X, a finite poset P, and a P-structure $f : X \to P$, the following are equivalent:*
1. *f is the characteristic map of a stratification (that is, the decomposition of X into preimages of f is a stratification with characteristic map f and entrance path poset $\mathsf{Entr}(f) = P$),*
2. *f is a connected-quotient map.*

Proof. If f is a characteristic map then, by definition, it has connected preimages and satisfies the second condition in Lemma B.1.24. Thus f is a connected-quotient map.

Conversely, if f is a connected-quotient map, then f defines a stratification by decomposing X into the preimages of f (which are connected by Definition B.1.26). By Lemma B.1.24 the map $f : X \to P$ is exactly the characteristic map of this stratification. □

The correspondence of characteristic maps and connected-quotient maps may further be generalized to the context of infinite stratifications, characterizing characteristic functions as 'finitely connected-quotient' maps (analogous to the notion of 'finite continuity' in Lemma B.1.15), but we forego a discussion of the infinite case here. We end with the following observation.

Observation B.1.30 (Connected-quotient maps compose). Using the definition of connected-quotient maps one verifies that, given a connected-quotient map $X \to P$ (of some stratification on X) and a connected-quotient map $P \to Q$ (of some stratification of P), their composite $X \to Q$ yields another connected-quotient map. ⌐

In particular, in the case of finite stratifications (and, under appropriate conditions, in the infinite case as well), we find that compositions of characteristic functions are again characteristic functions. As we will see, compositions of characteristic functions describe coarsenings (see Lemma B.2.12).

Factoring poset structures into stratifications and labelings In this section we show that any poset structure factors into a stratification followed by a 'labeling'. A labeling of a stratification in a category C functorially associates data in C to formal entrance paths in that stratification. We have seen conceptually similar notions of labelings in our discussions of labeled trusses (see Definition 2.3.44), and in their relation to stratified meshes (see Definition 5.3.7).

Terminology B.1.31 (Labelings). Let C be a category, and (X, f) a (pre)stratification. A 'C-labeling' (or simply a 'labeling') of (X, f) in C is a functor $L : \mathsf{Entr}(f) \to \mathsf{C}$. If C is a poset, we also call L a 'poset labeling'. ⊐

There are other notions of entrance path categories (such as entrance path ∞-*categories*, which we will meet later in Definition B.3.9) that can be considered in place of entrance path preorders. Working only with preorders provides the '0-categorical' base case of such fundamental category structures.

Example B.1.32 (Specialization labelings)**.** Let (X, f) be a finite (pre)stratification. The 'specialization labeling' of X associated to f is the labeling of the discrete prestratification of $X \to \mathrm{Spcl}\, X$ given by the functor $\mathrm{Spcl}\, f : \mathrm{Spcl}\, X \to \mathsf{Entr}(f)$ (obtained by applying the specialization topology functor to the continuous map $f : X \to \mathsf{Entr}(f)$). ⊐

We now show that any P-structure canonically factors as a stratification with a discrete labeling on that stratification. This factorization is referred to as the P-structure's 'connected component splitting'. Discreteness of the labeling will mean the following.

Terminology B.1.33 (Discrete map). A map of posets $F : Q \to P$ is called a 'discrete map' if its preimages are discrete, that is, for each $q \in Q$ the preimage $F^{-1}(q)$ contains no non-identity arrows. Note that the condition is equivalent to requiring F to be a conservative functor of categories $Q \to P$. ⊐

Construction B.1.34 (Connected component splittings)**.** For a P-structure $f : X \to P$, the **connected component splitting** of f is the factorization

$$f = (X \xrightarrow{\mathsf{char}(f)} \mathsf{cmpnt}(f) \xrightarrow{\mathsf{discr}(f)} P)$$

defined as follows. The map $\mathsf{char}(f)$ is the characteristic function of the stratification decomposing X into the connected components of preimages of f; note that the formal entrance path graph cannot have cycles since P is assumed to be a poset and f to be continuous. The map $\mathsf{discr}(f) : \mathsf{cmpnt}(f) \to P$ maps a given connected component X of a preimage $f^{-1}(p)$ back to p. ⊐

Note that even if $f : X \to P$ is continuous, the characteristic function $\mathsf{char}(f)$ need not be continuous (see Example B.1.36). We point out three universal properties of connected component splittings: universality among connected-quotient factorizations, universality among discrete map factorizations, and uniqueness among connected-quotient and discrete map factorizations.

Lemma B.1.35 (Universality of connected component splitting)**.** *Let $f : X \to P$ be a P-structure. Assume f factors into maps $g : X \to Q$ and $b : Q \to P$, where g is*

continuous and b a map of posets. Consider the following diagram:

$$
\begin{array}{c}
X \xrightarrow{\ g\ } Q \xleftarrow{\ b\ } \\
\end{array}
$$
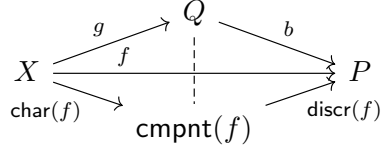
1. Characteristic map universality*: If g is characteristic, then there is a unique poset map $Q \to \mathsf{cmpnt}(f)$ making the above diagram commute,*
2. Discrete map universality*: If b is a discrete map, then there is a unique poset map $\mathsf{cmpnt}(f) \to Q$ making the above diagram commute*
3. Combined universality*: If g is characteristic and b is a discrete map, then there is a unique poset isomorphism $Q \cong \mathsf{cmpnt}(f)$ making the above diagram commute.*

Proof. We first prove statement (1). Since g is characteristic it has connected preimages. Thus its preimages must lie in the connected components of preimages of f. The map $\mathsf{cmpnt}(f) \to Q$ is the inclusion of strata of g into strata of $\mathsf{char}(f)$.

We next prove statement (2). We first show that preimages of g are unions of strata of $\mathsf{char}(f)$ (i.e. connected components of preimages of f). Let Z be a connected component of a preimage of f. Let $\{q_i^Z\}_{i \in I}$ be the set of objects in Q whose preimages $r_i^Z = g^{-1}(q_i^Z)$ intersect Z. Note that, since b is assumed to be a discrete map, there are no arrows between any q_i^Z in Q. Let Q_i^Z be the downward closure of q_i^Z in Q. Since g is assumed continuous, we have a disjoint open cover $\sqcup_i g^{-1}(Q_i^Z) \cap Z$ of Z. Since Z is connected, the indexing set I must be of cardinality 1. This shows that preimages $g^{-1}(q)$ of g are unions of connected components Z of preimages of f. The map $\mathsf{cmpnt}(f) \to Q$ can then be defined by mapping the strata $Z \subset g^{-1}(q)$ back to q.

The final statement (3) follows from combining statements (1) and (2). $\qquad\square$

Example B.1.36 (Translating P-structures into stratifications)**.** In Fig. B.6 we depict a stratification of the circle on the left. To its right, we depict two P-structures (by color-coding images and preimages in the same color). Both P-structures recover the stratification on the left after connected component splitting. In particular, there are many P-structures with the same 'underlying stratification'. In Fig. B.7 we depict another P-structure; it's connected component splitting recovers the stratification from Fig. B.4 with non-continuous characteristic function. ⌐

Figure B.6: Poset structures with the same underlying stratification.

Using the above constructions, we describe relations of our notion of stratifications with two other frequently used definitions of stratifications, namely to 'P-stratifications' and '$\mathscr{S}$-filtered spaces'.

Figure B.7: A non-continuously splitting poset structure.

Remark B.1.37 (Relation of stratifications and 'P-stratifications'). Our notion of a 'P-structure', given by a continuous maps from X to P, is also known as a 'P-stratification' (see [Lur12, Def. A.5.1]). Indeed, by the above results, given a P-stratification $f : X \to P$ we can think of it as the stratification with characteristic map $\mathsf{char}(f)$ obtained by connected component splitting (note that there may be many different P-stratifications $f : X \to P$ that lead to the same stratification in this way). Conversely, every locally finite stratification (X, f) arises as the connected component splitting of a P-stratification; indeed, by Lemma B.1.20, we can simply set $P = \mathsf{Entr}(f)$ and the characteristic map $f : X \to P$ will be continuous. ⌐

Remark B.1.38 (Relation of stratifications and '$\mathscr{S}$-filtered spaces'). Given a poset $\mathscr{S}$ with unique maximal element $\top$, a '$\mathscr{S}$-filtration' of a space X is a collection of closed subset X_s, $s \in \mathscr{S}$, such that $X_\top = X$ and $X_s \subset X_t$ whenever $t \leq s$ in $\mathscr{S}$ (see [GM88, §III.2.2.1]). This defines a continuous map $f_\mathscr{S} : X \to \mathscr{S}$, mapping points in the subspace $X_t \setminus \bigcup_{s>t} X_s$ to $t \in \mathscr{S}$. The characteristic function $\mathsf{char}(f_\mathscr{S})$ of the connected component splitting of $f_\mathscr{S}$ yields a stratification in our sense. Conversely, every stratification (X, f) with continuous characteristic map $f : X \to \mathsf{Entr}(f)$ yields an $\mathsf{Entr}(f)^\triangleright$-filtration of X by setting $X_s = f^{-1}(\mathsf{Entr}(f)^{\geq s})$ (here, $\mathsf{Entr}(f)^\triangleright$ is the poset obtained by adjoining a new top element $\top$ to $\mathsf{Entr}(f)$, and $\mathsf{Entr}(f)^{\geq s}$ is the upper closure of an element s in $\mathsf{Entr}(f)$).[36] ⌐

Classifying stratifications of posets In this section we discuss that the classifying space of any poset P itself carries canonically the structures of a stratification, called the 'classifying stratification' of P. While we introduced the concept already in Terminology 1.3.11, here we will revisit its construction in more concrete terms.

Remark B.1.39 (Nerves of posets). Recall the nerve NP of a poset $(P, \leq)$ is the simplicial set whose m-simplices S are the length-m strings of composable arrows in P; in other words, an m-simplex is a map of posets $S : [m] \to P$. The simplex $S : [m] \to P$ is called nondegenerate if it is injective. ⌐

Remark B.1.40 (Classifying space of posets). Recall the 'classifying space' $|P|$ of a poset P is the geometric realization of the nerve of P. (Abusing terminology, we sometimes refer to the classifying space of a poset itself as the 'geometric realization' of the poset.) Explicitly, $|P|$ is the space of functions $w : \mathrm{obj}(P) \to \mathbb{R}_{\geq 0}$ whose support $\mathrm{supp}(w) \subset \mathrm{obj}(P)$ is the object set of a nondegenerate simplex in P, and whose total weight is fixed, i.e. $\sum_{p \in \mathrm{obj}(P)} w(p) = 1$. We think of such a function w as an 'affine combination' of objects of the poset. ⌐

---

[36]In fact, in the case of finite stratifications, we can always recover (X, f) from an $\mathbb{N}$-filtration of X. Define a 'depth map' $\mathrm{depth} : \mathsf{Entr}(f) \to \mathbb{N}^{\mathrm{op}}$, mapping $s \in \mathsf{Entr}(f)$ to k if chains in $\mathsf{Entr}(f)$ starting at s have maximal length $(k + 1)$ (e.g. greatest elements have depth 0). Define the filtration $X_0 \subset X_1 \subset ... \subset X_{k_{\max}} = X$ (where $k_{\max}$ is the maximal depth of elements in $\mathsf{Entr}(f)$) by setting X_i to be the preimage of $[0, i]$ under the composite $\mathrm{depth} \circ f$.

Construction B.1.41 (Classifying stratifications of posets). The classifying space $|P|$ of a poset P has a stratification $\mathsf{CStr}\, P$, called the **classifying stratification**, with entrance path poset P itself, constructed as follows. The characteristic function of this stratification sends an affine combination w of objects of the poset to the minimal object $\min(\mathrm{supp}(w))$ (in P) of the support of that affine combination:

$$\mathsf{CStr}\, P : |P| \to P \quad , \quad w \mapsto \min(\mathrm{supp}(w)) \in P.$$

The stratum corresponding to the object $p \in P$ is denoted $\mathsf{str}(p) \subset \mathsf{CStr}\, P$; it consists of all affine combinations w of objects weakly greater than p, whose value at p is nonzero.

In fact, the classifying stratification construction is functorial (as we will see in Construction B.2.14).

B.2 Stratified maps

Maps, coarsenings, and substratifications

Definition B.2.1 (Map of stratifications). A **map of stratifications** $F : (X, f : X \to \mathsf{Entr}(f)) \to (Y, g : Y \to \mathsf{Entr}(g))$, also called a 'stratified map', is a continuous map $F : X \to Y$ for which there exists a (necessarily unique) map of posets $\mathsf{Entr}(F) : \mathsf{Entr}(f) \to \mathsf{Entr}(g)$ such that $\mathsf{Entr}(F) \circ f = g \circ F$.

Notation B.2.2 (Shorthand for stratified maps). In analogy to Notation B.1.5, we often abbreviate stratified maps $F : (X, f) \to (Y, g)$ by $F : f \to g$.

Example B.2.3 (Map of stratifications). In Fig. B.8 we depict a stratified map on the left and a non-stratified map on the right. In both case, the underlying map of topological spaces is given by vertical projection.
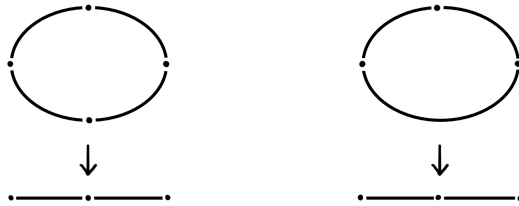
Figure B.8: A stratified map and a non-stratified map.

Definition B.2.4 (Coarsenings and refinements of stratifications). A map of stratifications $F : (X, f) \to (Y, g)$ is a **coarsening** of (X, f) to (Y, g), or, synonymously, a **refinement** of (Y, f) by (X, f), if the underlying map of spaces $F : X \to Y$ is a homeomorphism.

Note that we use coarsening and refinement as synonyms of 'dual flavor', i.e. describing dual processes: a coarsenings 'coarsens' the domain, while a refinement, in opposite direction, 'refines' the codomain.

Example B.2.5 (Coarsening and refinement). In Fig. B.9 we illustrate a coarsening of stratifications on the circle, along with the corresponding the refinement indicated by a dashed arrow in opposite direction.
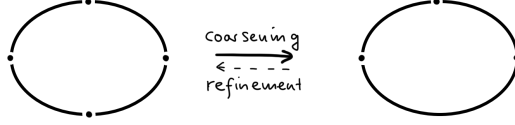
Figure B.9: A coarsening and its corresponding refinement of stratifications.

Definition B.2.6 (Substratification)**.** A stratified map $(X, f) \to (Y, g)$ is a **substratification** if the underlying map $X \subset Y$ is an inclusion and if every stratum s of f is a connected component of $X \cap t$ for some stratum t of g. ⌟

By extension we refer to stratified maps that are not literally inclusions, but whose underlying map is injective and a stratified homeomorphism onto a substratification, also as 'substratifications'.

Example B.2.7 (Substratification)**.** In Fig. B.10 we depict two stratified maps: the first is a substratification, which though is not injective on entrance path posets; the second is a stratified map whose underlying map is injective, but which is not a substratification.

⌟

Figure B.10: Two stratified maps one of which is a substratification.

Terminology B.2.8 (Restricting stratifications)**.** Given a stratification (Y, g) and a subspace $X \subset Y$, the 'restriction' $(X, g|_X)$ is the substratification of (Y, g) whose strata are the connected components of intersections $X \cap t$ for all strata t of g. ⌟

Definition B.2.9 (Constructible substratifications)**.** A substratification $(X, f) \to (Y, g)$ is **constructible** if every stratum of (X, f) is exactly a stratum of (Y, g). ⌟

Recall from Remark B.1.18, a 'frontier-constructible' stratification is a stratification (Y, g) in which the topological closure $\bar{s}$ of each stratum s yields a constructible substratification $(\bar{s}, g|_{\bar{s}})$ of g (by restricting g to $\bar{s}$). Frontier-constructibility has an alternative description purely in terms of characteristic functions as follows.

Lemma B.2.10 (Frontier-constructible stratifications are those with open characteristic function)**.** *A stratification* (X, f) *is frontier-constructible if and only if the characteristic function* $f : X \to \mathsf{Entr}(X)$ *is an open map.*

Proof. Assume f is frontier-constructible. Let $U \subset X$ be an open subset. We need to show that $f(U) \subset \mathsf{Entr}(X)$ is open, which in the specialization topology means that $f(U)$ is downward closed. It suffices (because the entrance path poset is generated by formal entrance paths) to check that given an element $s \in f(U)$ and a formal entrance path $r \to s$, we have $r \in f(U)$. The existence of the formal entrance path $r \to s$ means $s \cap \partial r \neq \emptyset$; frontier-constructibility then implies that $s \subset \partial r$. As $s \in f(U)$, there is some

point of the stratum s that is in U, and because U is open, there must be a point of the stratum r that is in U. Thus $r \in f(U)$ as required.

Conversely, assume $f : X \to \mathsf{Entr}(f)$ is open. It suffices to show that if there is a formal entrance path $r \to s$, i.e. $s \cap \partial r \neq \emptyset$, then $s \subset \partial r$. Suppose there is such an entrance path but by contrast there is a point $p \in s \setminus \partial r = s \setminus \overline{r}$. Then we can chose an open neighborhood $p \in U \subset X$ disjoint from the closure $\overline{r}$. By assumption it follows that the image $f(U)$ is open, which is to say downward closed; thus $s \in f(U)$ and $r \to s$ implies that $r \in f(U)$, contradicting the fact that the neighborhood U does not intersect even the closure of r. $\qquad\square$

We can characterize substratifications and coarsenings in terms of entrance path poset maps, as follows.

Lemma B.2.11 (Substratification from discrete maps)**.** *A map of finite stratified spaces* $F : (X, g) \to (Y, f)$ *is a substratification if and only if* $F : X \to Y$ *is a subspace inclusion and* $\mathsf{Entr}(F) : \mathsf{Entr}(g) \to \mathsf{Entr}(f)$ *is a discrete map.*

Proof. By definition every substratification is a subspace inclusion of underlying spaces. The fact that $\mathsf{Entr}(F)$ is a discrete map follows since strata of substratifications are defined as connected components of the intersection of the subspace X with strata of f, and since $\mathsf{Entr}(f)$ is a poset.

Conversely, assume the stratified map F is a subspace inclusion and that $\mathsf{Entr}(F)$ is a discrete map. Note that the substratification $(X, f|_X)$ of f can be obtained by connected component splitting of the restriction of $f : Y \to \mathsf{Entr}(f)$ to $X \subset Y$. Since g is a continuous characteristic map, and since $\mathsf{Entr}(F)$ is a discrete map, statement (3) of Lemma B.1.35 (applied to $f : Y \to \mathsf{Entr}(f)$ restricted to $X \subset Y$) shows that g is a substratification of f as claimed. $\qquad\square$

Lemma B.2.12 (Coarsenings from connected-quotient maps)**.** *Let* (X, f) *be a finite stratification. Coarsenings of* f *(up to isomorphism) are canonically in bijection with connected-quotients of* $\mathsf{Entr}(f)$: *namely, the bijection takes coarsenings* F *to their entrance path poset maps* $\mathsf{Entr}(F)$.

Proof. Let $F : (X, f) \to (X, g)$ be a coarsening. Then $\mathsf{Entr}(F)$ is a connected-quotient map since its preimages are connected and it satisfies condition (2) in Lemma B.1.24.

Conversely, let $H : \mathsf{Entr}(f) \to P$ be a connected-quotient map. Define a stratification (X, g) whose strata are unions of those strata in f that are mapped to the same object in P under H. Since preimages of H are connected, these unions are connected subspaces of X and thus define a prestratification. Since H is a connected-quotient map to a poset P, this prestratification is in fact a stratification with entrance path poset $\mathsf{Entr}(g) = P$. The resulting coarsening $(X, f) \to (X, g)$ is the identity on the underlying space X, and maps entrance path posets by H. $\qquad\square$

The category of stratifications Having defined stratified spaces and maps, we now obtain the category of stratifications.

Terminology B.2.13 (The ordinary category of stratifications)**.** Denote by Strat the category of stratifications and their stratified maps. $\qquad\lrcorner$

Posets faithfully embed into stratifications by the classifying stratification functor as follows (recall the construction of classifying stratification from Construction B.1.41).

Construction B.2.14 (Classifying stratification functor)**.** Given a map of posets $F : P \to Q$, the induced map on classifying spaces, namely the map $\mathsf{CStr}\, F : \mathsf{CStr}\, P \to \mathsf{CStr}\, Q$ mapping $\mathsf{CStr}\, F(w)(q) = \sum_{p \in F^{-1}(q)} w(p)$, is a stratified map. This yields the 'classifying stratification' functor

$$\mathsf{CStr} - : \mathsf{Pos} \to \mathsf{Strat}$$

from the category of posets to the category of stratifications. ⌐

Conversely, the entrance path poset construction previously described yields a functor from the category of stratifications to the category of posets.

Construction B.2.15 (Entrance path poset functor)**.** The association of the entrance path poset $\mathsf{Entr}(f)$ to the stratification (X, f), and of the map of posets $\mathsf{Entr}(F)$ to the map of stratifications $F : (X, f) \to (Y, g)$ provides the 'entrance path poset' functor

$$\mathsf{Entr} : \mathsf{Strat} \to \mathsf{Pos}$$

from the category of stratifications to the category of posets. ⌐

Observation B.2.16 (Entrance paths invert classifying stratifications)*.* The preceding functors form a section-retraction pair: namely, $\mathsf{Entr} \circ \mathsf{CStr} = \mathrm{id}$. ⌐

We can further promote the entrance path poset functor to a functor of ∞-categories. (Here we use the term '∞-category' to refer to a category enriched in the category $k\mathsf{Top}$ of compactly generated spaces topological spaces, see Convention B.0.2.) We first define the ∞-category of stratifications.

Notation B.2.17 (The ∞-category of stratifications)*.* Denote by $\mathcal{S}trat$ the ∞-category of stratified spaces and their stratified maps: hom spaces $\mathcal{S}trat((X, f), (Y, g))$ are topologized as subspaces of the internal hom $\mathrm{Map}(X, Y)$ in $k\mathsf{Top}$. ⌐

We next define the ∞-category of posets. It will be convenient to assume local finiteness at this point (though it is possible to generalize the discussion below to other cases as well). We first record the following useful observations.

Observation B.2.18 (Properties of locally finite posets)*.* Let P and Q be locally finite posets.
 1. Local finiteness of P implies that P is core compact, and thus exponentiable (see [ELS04, Def. 2.8 & Thm. 2.9]).
 2. By probing P with compact Hausdorff probes $\left|P^{\leq x}\right| \to P^{\leq x} \hookrightarrow P$ one verifies that P is compactly generated (see [ELS04, Def. 3.1]).
 3. The internal hom $\mathrm{Map}(P, Q)$ in $k\mathsf{Top}$ is the k-ification of the space $C_0(P, Q)$, defined as the set $\mathsf{Pos}(P, Q)$ with subbasic opens $M(U, V) = \{F \mid F(U) \subset V\}$ for finite open $U \subset P$ and open $V \subset Q$ (see [ELS04, Prop. 5.11 & Thm. 5.15]). ⌐

Notation B.2.19 (The ∞-category of locally finite posets)*.* The ∞-category $\mathcal{P}os$ of locally finite posets is the ∞-category obtained by topologizing hom sets $\mathsf{Pos}(P, Q)$ using the internal homs $\mathrm{Map}(P, Q)$ in $k\mathsf{Top}$. ⌐

Remark B.2.20 (The case of finite posets). For finite posets P and Q, the space $\mathrm{Map}(P,Q)$ is exactly the hom poset $\mathsf{Pos}(P,Q)$ of posets functors and natural transformations endowed with specialization topology. ⊐

Construction B.2.21 (Entrance path poset ∞-functor). Let $\mathcal{Strat}_{\mathrm{elf}}$ denote the full subcategory of $\mathcal{Strat}$ consisting of stratifications with locally finite entrance path poset. The entrance path poset functor $\mathsf{Strat} \to \mathsf{Pos}$ induces a functor of ∞-categories

$$\mathsf{Entr} : \mathcal{Strat}_{\mathrm{elf}} \to \mathcal{Pos} \quad .$$ ⊐

The continuity of the functor Entr on hom spaces follows from standard arguments. We need to check that $\mathsf{Entr} : \mathcal{Strat}_{\mathrm{elf}}((X,f),(Y,g)) \hookrightarrow \mathrm{Map}(X,Y) \to \mathcal{Pos}(\mathsf{Entr}\, f, \mathsf{Entr}\, g)$ is continuous. Recall $\mathcal{Pos}(\mathsf{Entr}\, f, \mathsf{Entr}\, g) = kC_0(\mathsf{Entr}\, f, \mathsf{Entr}\, g)$ (see Observation B.2.18). Since k-ification is a right adjoint, it suffices to check continuity of Entr when mapping into $C_0(\mathsf{Entr}\, f, \mathsf{Entr}\, g)$. Pick a subbasic $M(U,V)$ for the latter space. Since U is finite, we can pick a finite compact set $K = \{p_s \in f^{-1}(s) \mid s \in U\}$. Since $\mathsf{Entr}\, g$ is assumed locally finite, the stratification g is locally finite, and the characteristic map $g : Y \to \mathsf{Entr}\, g$ is continuous. Thus $W = g^{-1}(V)$ is open in Y. Then $M(K,W) = \{F : X \to Y \mid F(K) \subset W\}$ is an open subset of $\mathrm{Map}(X,Y)$, and its intersection with $\mathcal{Strat}_{\mathrm{elf}}((X,f),(Y,g))$ equals $\mathsf{Entr}^{-1} M(U,V)$, showing the latter is open as required. ⊐

We finally also mention a 'tensoredness' property of stratifications.

Terminology B.2.22 (Products of stratifications). Given two stratifications (X,f) and (Y,g), the product stratification is simply $(X \times Y, f \times g)$ where $f \times g$ is the characteristic function $X \times Y \to \mathsf{Entr}(f) \to \mathsf{Entr}(g)$ obtained by taking the product of characteristic functions $f : X \to \mathsf{Entr}(f)$ and $g : Y \to \mathsf{Entr}(g)$. One further defines products of stratified maps by taking products of their underlying continuous maps. This yields a topological 'product' functor

$$(- \times -) : \mathcal{Strat} \times \mathcal{Strat} \to \mathcal{Strat}.$$ ⊐

Observation B.2.23 (Fiberwise $k\mathsf{Top}$-tensoredness of $\mathcal{Strat}$). Taking products with topological spaces provides a 'fiberwise $k\mathsf{Top}$-tensor' on the category of stratified spaces as follows. Let (X,f) and (Y,g) be finite locally compact Hausdorff stratifications and $F : \mathsf{Entr}(f) \to \mathsf{Entr}(g)$ a map of their entrance path posets. Denote by $\mathcal{Strat}(f,g)_F$ the preimage of F of the map $\mathsf{Entr} : \mathcal{Strat}(f,g) \to \mathcal{Pos}(\mathsf{Entr}(f), \mathsf{Entr}(g))$. Using cartesian closedness of $k\mathsf{Top}$, identify $\mathrm{Map}(Z, \mathrm{Map}(X,Y)) \cong \mathrm{Map}(Z \times X, Y)$ (for $Z \in k\mathsf{Top}$); in particular, we obtain a homeomorphism

$$\mathrm{Map}(Z, \mathcal{Strat}(f,g)_F) \cong \mathcal{Strat}(Z \times f, g)_F$$

where the right hand side denotes the space of stratified maps $Z \times (X,f) \to (Y,g)$ whose underlying map of entrance path posets is F (noting $\mathsf{Entr}(Z \times (X,f)) \cong \mathsf{Entr}(f)))$. ⊐

Stratified bundles A 'stratified bundle' is a stratified map that is locally trivial along each stratum of the base. The notion generalizes the ordinary notion of 'fiber bundles' of topological spaces.

Definition B.2.24 (Stratified bundles)**.** A stratified map $p : (Y, g) \to (X, f)$ is a **stratified bundle** if for each stratum s of f and each point $x \in s$, there is a neighborhood $U_x \subset s$ inside the stratum s, such that there is a stratification (Z, h) together with an isomorphism of stratifications $T : U_x \times h \cong (p^{-1}(U_x), f)$ for which $p \circ T : U_x \times h \to U_x$ is the projection. The stratification (Z, h) is called the fiber of F over the stratum s. ⌐

Note that every fiber bundle is naturally a stratified bundle with indiscrete stratifications on both base and total space. We will usually further assume that all the fibers of a stratified bundle are non-empty, in other words that the underlying map of spaces is surjective. In this case the stratification of the total space determines the stratification of the base space.

Observation B.2.25 (The base stratification is determined by the total stratification). Suppose $(Y, f) \to (X, g)$ and $(Y, f) \to (X, g')$ are stratified bundles with the same underlying surjective map $F : Y \to X$. Then the stratifications g and g' are equal. ⌐

Just as fiber bundles can be pulled back along continuous map, stratified bundles can be pulled back along stratified maps. Here, a 'pullback' of stratified maps means the following.

Terminology B.2.26 (Pullbacks of stratifications). Given stratifications (X, f), (Y, g), and (Z, h) and maps $F : f \to h$ and $G : g \to h$, the 'pullback stratification' $(X, f) \times_{(Z,h)} (Y, g)$ is the stratification $(X \times_Z Y, f \times_h g)$, where $X \times_Z Y$ is the pullback of spaces and $f \times_h g$ is the restriction $f \times g|_{X \times_Z Y}$ of the product stratification $f \times g$ to the pullback space $X \times_Z Y \subset X \times Y$. ⌐

Example B.2.27 (Pullback stratification need not be finite or have continuous characteristic function)**.** In Fig. B.11 we depict a pullback of finite stratifications that is not finite and does not have continuous characteristic function. ⌐
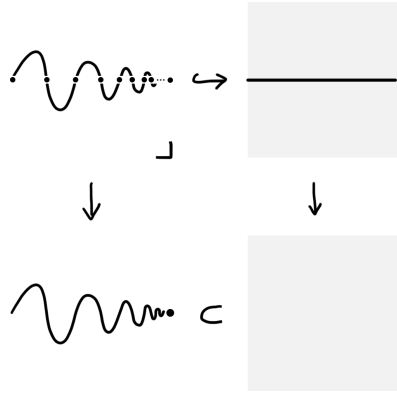
Figure B.11: Pullbacks of stratifications need not preserve finiteness.

Observation B.2.28 (Pullbacks of stratified bundles). Given a stratified bundle $p : (X, f) \to (Y, g)$ and a stratified map $F : (Y', g') \to (Y, g)$, then the pullback map $(X \times_Y Y', f \times_g g') \to (Y', g')$ is a stratified bundle itself, usually denoted by $F^*p : F^*f \to g'$. This follows since stratified maps map strata into strata, which allows us to 'pull back' trivializations of p over neighborhoods in strata of g to trivialization of F^*p over neighborhoods in strata of g'. ⌐

Remark B.2.29 (Constructible bundles). There is a natural strengthening of the notion of stratified bundles, namely to so-called 'constructible bundles'. As discussed in Chapter 4, 'constructibility' requires that bundles can be reconstructed up to isomorphism from functorial data associated to fundamental categories of their base stratifications, such as entrance path posets, or entrance path (∞-)categories (see also Remark 4.1.33). There are several approaches to making this precise (see [AFR15, §6] [CP20]). ⌐

B.3 Conical and cellulable stratifications

We recall notions of conical and cellulable stratifications. Both are 'niceness' conditions on stratifications. Lurie shows in [Lur12, App. A] that conical stratifications have 'entrance path ∞-categories', providing a natural generalization of 'fundamental ∞-groupoids' of spaces. We will provide a similar (but simpler) construction in the case of cellulable stratifications.

Conical stratifications Many of the stratifications in this book satisfy an additional regularity condition called 'conicality'. This condition requires neighborhoods of strata to locally look like a cone 'normal' to the stratum and an open set 'tangential' to the stratum. Let us first formalize the operation of taking cones on stratifications.

Terminology B.3.1 (Cones of stratifications). Given a stratification (X, f), we can define its 'open cone' $f^{\triangleright}$ to be the stratification $(X^{\triangleright}, f^{\triangleright} : X^{\triangleright} \to \mathsf{Entr}(f)^{\triangleright})$ as follows. The space $X^{\triangleright}$ is the usual open cone $X \times (0, 1] \cup_{X \times \{1\}} \top$, where $\top$ is the space with the single point $\top$. The poset $\mathsf{Entr}(f)^{\triangleright}$ is obtained from $\mathsf{Entr}(f)$ by adding a new top element $\top$. The map $f^{\triangleright}$ sends the cone point $\top$ to $\top \in \mathsf{Entr}(f)^{\triangleright}$ and is otherwise given by $X \times (0, 1) \xrightarrow{\pi_X} X \xrightarrow{f} \mathsf{Entr}(f)$. ⌐

Definition B.3.2 (Conical stratification). A **tubular neighborhood** of a point x of a stratification (X, f) is a neighborhood U_x of x, together with a stratified space (Y_x, l_x), called a **link** at x, a connected topological space Z_x, called the **tangential neighborhood**, and a stratified homeomorphism

$$Z_x \times l_x^{\triangleright} \cong (U_x, f|_{U_x})$$

sending $z \times \top$ to x, for some $z \in Z_x$. (Here $\top$ is the cone point in the cone $l_x^{\triangleright}$.) A stratification is **conical** if it has a tubular neighborhood at every point. ⌐

Example B.3.3 (Conical and non-conical stratifications). In Fig. B.12 we illustrate an example of a conical stratification, together with an illustration of a tubular neighborhood as a product of a cone and a space. By contrast, in Fig. B.13 we depict a stratification (of the same space, but now decomposed into only two strata) which is not a conical stratification. ⌐

Remark B.3.4 (Topological stratification). A conical stratification in which every stratum is a topological manifold is usually called a 'topological stratification'. Note that in that situation, the tangential spaces Z_x can always be chosen to be euclidean spaces. The conical stratification shown in Fig. B.12 is a topological stratification. An instance of a conical stratification (with two strata) that is not a topological stratification is depicted in Fig. B.14. ⌐
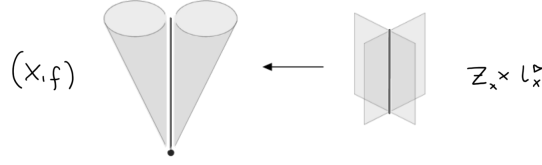
Figure B.12: A conical stratification with a tubular neighborhood.
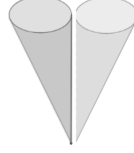
Figure B.13: A non-conical stratification.
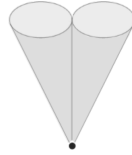
Figure B.14: A conical stratification that is not topological.

Observation B.3.5 (Constructible substratifications inherit conicality). If the stratification (X, f) is conical and $(Y, g) \hookrightarrow (X, f)$ is a constructible substratification (Definition B.2.9), then the stratification (Y, g) is also conical. ⌐

Observation B.3.6 (Coarsening need not preserve conicality). If the stratification (X, f) is conical and $(X, f) \to (Y, g)$ is a coarsening, then the stratification (Y, g) need not be conical. ⌐

Remark B.3.7 (Conical implies frontier-constructible). Every conical stratification is frontier-constructible (see Remark B.1.18). In particular, a conical stratification is locally finite if and only if its entrance path poset is. ⌐

Proposition B.3.8 (Locally finite classifying stratifications are conical). *The classifying stratification* $\mathsf{CStr}\, P$ *(see Construction B.1.41) of any locally finite poset* P *is conical.*

Proof. For any poset element $p \in P$, and any point $x \in \mathsf{str}(p)$ of the corresponding stratum, we construct a tubular neighborhood, which is in fact independent of the point x. The points of the classifying stratification $\mathsf{CStr}\, P$, the stratum $\mathsf{str}(p)$ and the link $\mathsf{link}(p)$ are given by certain affine combinations w, $w_{\mathsf{str}(p)}$ and $w_{\mathsf{link}(p)}$ of objects in P, $P^{\geq p}$ and $P^{<p}$ respectively. Since $P^{\geq p} \hookrightarrow P$ (respectively $P^{<p} \hookrightarrow P$) we may trivially extend $w_{\mathsf{str}(p)}$ (respectively $w_{\mathsf{link}(p)}$) to affine combination of objects in P. The inclusion $\mathsf{str}(p) \hookrightarrow \mathsf{CStr}\, P$ now extends to a tubular neighborhood $\mathsf{str}(p) \times \mathsf{cone}(\mathsf{link}(p)) \hookrightarrow \mathsf{CStr}\, P$ by setting (cf. [Lur12, Prop. A.6.8])

$$\mathsf{str}(p) \times \mathsf{link}(p) \times (0, 1) \to \mathsf{CStr}\, P$$

$$(w_{\mathsf{str}(p)}, w_{\mathsf{link}(p)}, t) \mapsto \left(w(q) := t \cdot w_{\mathsf{str}(p)}(q) + (1 - t) \cdot w_{\mathsf{link}_p^P}(q) \right)$$

away from the conepoint $\mathsf{link}(p) \times (0, 1) \hookrightarrow \mathsf{cone}(\mathsf{link}(p))$. This verifies conicality of $\mathsf{CStr}\, P$ as claimed. □

A central reason for considering conical stratifications is the availability of a good notion of '(higher) entrance path homotopies' between entrance paths. This leads to a definition of entrance path ∞-category ('∞-category' abbreviating '$(\infty, 1)$-category'). Modeled in quasicategories, this can be defined as follows.

Definition B.3.9 (Entrance path ∞-category, [Lur12, Rmk. A.6.5])**.** The **entrance path ∞-category** $\mathcal{E}ntr(f)$ of a conical stratification (X, f) is the ∞-category with underlying simplicial set having as m-simplices the maps from the stratified m-simplex $\mathsf{CStr}\,[m]$ to f; that is $\mathcal{E}ntr(f)_m := \mathsf{Strat}(\mathsf{CStr}\,[m], f)$. ⌐

Remark B.3.10 (The class of entrance path ∞-categories)*.* The class of ∞-categories that are (up to equivalence) the entrance path ∞-categories of a conical stratifications have been described as ∞-categories with a conservative functor to a poset (see [BGH18, §2.1]). ⌐

Lemma B.3.11 (Entrance paths in classifying stratifications)**.** *Given a locally poset P, the entrance path ∞-category of its classifying stratification $\mathsf{CStr}\,P$ is equivalent to P, that is,*
$$\mathcal{E}ntr\,\mathsf{CStr}\,P \simeq NP.$$
In particular, $\mathcal{E}ntr\,\mathsf{CStr}\,P$ is 0-truncated (see [CL18]).

Proof. Recall that an ∞-category is called 0-truncated if its hom spaces are (-1)-types, meaning they are either empty or contractible. We first check that $\mathcal{C} \equiv \mathcal{E}ntr\,\mathsf{CStr}\,P$ is 0-truncated. It suffices [CL18, Prop. 3.12] to show that any sphere $\partial\Delta[m] \to \mathcal{C}$, for $m > 1$, has a filler. Given such a map $\phi : \partial\Delta[m] \to \mathcal{C}$, by the definition of $\mathcal{E}ntr$, ϕ is represented by a continuous map $|\phi| : |\partial\Delta[m]| \to |P|$. Pick $x \in P$ such that $|\phi|\,(0) \in \mathsf{str}(x)$. Then $\mathrm{im}\,|\phi|$ lies in the closure of $\mathsf{str}(x)$, i.e. in $\left|P^{\geq x}\right| \subset |P|$. Note that $\left|P^{\geq x}\right| = \mathsf{cone}\,|P^{>x}|$. Similarly, identify $\Delta[m] \cong \mathsf{cone}(\partial\Delta[m]) = \left(\partial\Delta[m] \times [0, 1]\right)\big/_{\partial\Delta[m] \times \{0\}}$. Then define the filler $\psi : \Delta[m] \to \left|P^{\geq x}\right|$ by mapping $(q, t) \mapsto t \cdot |\phi|\,(q)$. By construction, ψ sends the interior of $\Delta[m]$ to the stratum $\mathsf{str}(x)$ and thus is a stratified map as needed.

Since $\mathcal{C}$ is 0-truncated, it is equivalent to $N(\mathrm{ho}(\mathcal{C}))$ [CL18, Prop. 3.8]. Furthermore, the homotopy category of any 0-truncated ∞-category has a skeleton that is a poset [CL18, Prop. 3.10]. Let Q denote a posetal skeleton of $\mathrm{ho}(\mathcal{C})$. Observe that Q must be isomorphic to P: we have bijections $Q \rightleftarrows P$ identifying q to x iff $q \in \mathsf{str}(x)$. We arrive at the desired equivalence:
$$\mathcal{E}ntr\,\mathsf{CStr}\,P \equiv \mathcal{C} \simeq N(\mathrm{ho}(\mathcal{C}))) \simeq NQ \cong NP.$$

Note that the composed equivalence is the 'canonical' one: it takes 0-simplices q of $\mathcal{E}ntr\,\mathsf{CStr}\,P$ to 0-simplices x in NP iff $q \in \mathsf{str}(x)$. □

Definition B.3.12 (0-truncated stratifications)**.** A conical stratification (X, f) is called **0-truncated** if ⌐

In particular, the preceding lemma shows that classifying stratifications of posets are 0-truncated. Let us end with the following heuristic at play.

Remark B.3.13 (Analogy to sets and spaces). Posets are to (sufficiently nice) stratifications what sets are to spaces. Classifying spaces of sets (thought of as 0-categories) are discrete spaces; their fundamental ∞-groupoid are 0-truncated. Similarly, the above shows that entrance path ∞-categories of classifying stratifications of posets are 0-truncated. ⌐

Cellulable stratifications In this final section we discuss cellulable stratifications. Recall, a regular cell complex is a stratification in which strata are open disks ('open cells') whose closures are closed disks ('closed cells').

Definition B.3.14 (Cellulable stratifications)**.** The class of **cellulable stratifications** is the smallest class of stratifications containing regular cell complexes which is closed under the following:
   1. If $(X, f) \to (Y, g)$ is a coarsening and (X, f) is cellulable then (Y, g) is cellulable.
   2. If $(Y, g) \subset (X, f)$ is a constructible substratification and (X, f) is cellulable then (Y, g) is cellulable. ⌐

Remark B.3.15 (Local finiteness assumption). Sometimes we may want to also assume that our cellulable stratifications have locally finite entrance path posets. One way to ensure this is, is to require in the above definition that regular cell complexes are locally finite and coarsenings $F : (X, f) \to (Y, g)$ are 'open finite' (i.e. $\mathsf{Entr}(F)$ is open with finite preimages). ⌐

Since constructible substratifications and coarsenings commute (i.e. any constructible substratification of a coarsening is a coarsening of a constructible substratification), the definition of cellulable stratifications can be simplified as follows. Recall the notion of regular cell complex from Definition 1.3.1.

Definition B.3.16 (Regular cell stratifications)**.** A stratification (X, f) is called an **(open) regular cell stratification** if there exists a regular cell complex $(\overline{X}, \overline{f})$ and a stratified inclusion $(X, f) \hookrightarrow (\overline{X}, \overline{f})$ making (X, f) an (open) constructible substratification of $(\overline{X}, \overline{f})$. ⌐

Terminology B.3.17 (Cellulations). A refinement of a stratification by an regular cell stratification is called a 'cellulation'. ⌐

Observation B.3.18 (Cellulable stratifications). A cellulable stratification is precisely a stratification that can be obtained by coarsening a locally finite regular cell stratification. ⌐

The following terminology will also be useful.

Terminology B.3.19 (Open cellulable stratification). A cellulable stratification will be called 'open cellulable' if it can be refined by an open regular cell stratification. ⌐

   To better understand the definition of cellulable stratifications better let us discuss two properties of locally finite regular cell complexes. We begin with the property of conicality. Note that the stratification of general (non-regular) cell complexes need not be conical in general, even when the complex is finite (i.e. has only finitely many cells). In contrast, in the regular case, we have the following.

Proposition B.3.20 (Regularity implies conicality)**.** *Locally finite regular cell complexes are conically stratified.*

Proof. In Proposition 1.3.16 we have seen that there is a stratified homeomorphism $X \cong \mathsf{CStr}\,\mathsf{Entr}X$ of X with the classifying stratification of its entrance path poset. From Proposition B.3.8 it then follows that locally finite regular cell complexes are conically stratified. $\square$

Observation B.3.21 (Regular cell stratifications are conical). Combining Proposition B.3.20 with Observation B.3.5 it follows that locally finite regular cell stratifications are conical. ⌐

Note, by Observation B.3.6, cellulable stratifications themselves need not be conical.

Let us next discuss the entrance path structure of locally finite regular cell complexes.

Observation B.3.22 (Regular cell complexes are 0-truncated). Given a locally finite regular cell complex (X, f), we have a canonical equivalence $\mathcal{E}ntr(f) \simeq N\,\mathsf{Entr}(f)$ that takes points x to the stratum s that they live in. This follows from Lemma B.3.11 since locally finite regular cell complexes are classifying stratifications of locally finite posets. ⌐

Observation B.3.23 (Regular cell stratifications are 0-truncated). Given a regular cell stratification (X, f), we have a canonical equivalence $\mathcal{E}ntr(f) \simeq N\,\mathsf{Entr}(f)$ that takes points x to the stratum s that they live in. This follows by appropriately restricting the equivalence to subcategories resp. subposets determined by constructible substratifications ⌐

An advantage of working with cellulable stratifications (in comparison to, say, conical stratifications) is that constructions can exploit the 0-truncatedness of regular cell complexes. For instance, entrance path ∞-categories of cellulable stratifications are easy to define, by presenting them as 'posets with weak equivalences'.

Construction B.3.24 ($\mathcal{E}ntr$ for cellulable stratifications)**.** Let (X, f) be a cellulable stratification. We can find a locally finite regular cell stratification (Y, g) which refines (X, f). Denote by $F : \mathsf{Entr}(g) \to \mathsf{Entr}(f)$ the characteristic map of the refinement. Denote by $W_F = \{\alpha \mid F(\alpha) = \mathrm{id}\}$ the 'weak equivalences' given by those entrance paths in g that become invertible paths in f. The entrance path ∞-category $\mathcal{E}ntr(f)$ of f is the ∞-category presented by the tuple $(\mathsf{Entr}(g), W_F)$, i.e. by a category with weak equivalences. ⌐

Note that an explicit ∞-category $\mathcal{E}ntr(f)$ can be obtained by localizing $\mathsf{Entr}(g)[W_F^{-1}]$ (here, we use the idea of 'localization' without reference to a concrete model; but concrete models could be given [DK80b] [DK80a]). Note that localizing in the traditional setting of 1-categories, yields the homotopy category $\mathrm{ho}(\mathcal{E}ntr(f))$.

Example B.3.25 (Cellulating the circle)**.** Let $(X, f) \equiv S^1$ be the trivially stratfied circle. Choose the minimal regular cell refinement $(Y, g) \to S^1$ (this has two 1-cells, glued together at their boundaries). The 1-category $\mathrm{ho}(\mathcal{E}ntr(f))$ is equivalent to the category with a single object and a single generating non-identity automorphism (without further relations); this in turn is equivalent to the fundamental 1-groupoid $\pi_{\leq 1}S^1$. ⌐

Remark B.3.26 (Well-definedness of $\mathcal{E}ntr$). Given a cellulable stratification f, our construction of $\mathcal{E}ntr(f)$ above depends on a choice of cellulation of f. Showing that the resulting ∞-category $\mathcal{E}ntr(f)$ is independent of this choice is non-trivial in general. However, in the case of *conical* cellulable stratifications one can compare $\mathcal{E}ntr(f)$ with the construction in Definition B.3.9, and show that the two constructions are equivalent. ⌐

List of Figures

Bibliography

[AFR15]   David Ayala, John Francis, and Nick Rozenblyum. A stratified homotopy hypothesis. *arXiv preprint arXiv:1502.01713*, 2015.

[Aki69]   Ethan Akin. Manifold phenomena in the theory of polyhedra. *Transactions of the American Mathematical Society*, 143:413–473, 1969.

[Arn75]   Vladimir I Arnol'd. Critical points of smooth functions and their normal forms. *Russian Mathematical Surveys*, 30(5):1, 1975.

[BD95]   John C Baez and James Dolan. Higher-dimensional algebra and topological quantum field theory. *Journal of Mathematical Physics*, 36(11):6073–6105, 1995.

[Bén00]   Jean Bénabou. Distributors at work. *Lecture notes written by Thomas Streicher*, 11, 2000.

[Ber20]   Julia E Bergner. A survey of models for (∞, n)-categories. *Handbook of Homotopy Theory, edited by Haynes Miller, Chapman & Hall/CRC*, pages 263–295, 2020.

[BGH18]   Clark Barwick, Saul Glasman, and Peter Haine. Exodromy. *arXiv preprint arXiv:1807.03281*, 2018.

[Bjö84]   Anders Björner. Posets, regular cw complexes and bruhat order. *European Journal of Combinatorics*, 5(1):7–16, 1984.

[BMS12]   John W Barrett, Catherine Meusburger, and Gregor Schaumann. Gray categories with duals and their diagrams. *arXiv preprint arXiv:1211.0529*, 2012.

[Bry02]   John L Bryant. Piecewise linear topology. *Handbook of geometric topology*, pages 219–259, 2002.

[BV06]   John Michael Boardman and Rainer M Vogt. *Homotopy invariant algebraic structures on topological spaces*, volume 347. Springer, 2006.

[BV17]   Krzysztof Bar and Jamie Vicary. Data structures for quasistrict higher categories. In *2017 32nd Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pages 1–12. IEEE, 2017.

[Can79]   James W Cannon. Shrinking cell-like decompositions of manifolds. codimension three. *Annals of Mathematics*, pages 83–112, 1979.

[Cer70]   Jean Cerf. La stratification naturelle des espaces de fonctions différentiables réelles et le théoreme de la pseudo-isotopie. *Publications Mathématiques de l'IHÉS*, 39:5–173, 1970.

[CKS96]   J Scott Carter, Louis H Kauffman, and Masahico Saito. Diagrammatics, singularities, and their algebraic interpretations. In *in"10th Brazilian Topology Meeting, Saˆ o Carlos, July 22 26, 1996,"Mathematica Contemporaˆ nea*. Citeseer, 1996.

[CL04]   Eugenia Cheng and Aaron Lauda. Higher-dimensional categories: an illustrated guide book. *Preprint*, 2004.

[CL18]   Alexander Campbell and Edoardo Lanari. On truncated quasi-categories. *arXiv preprint arXiv:1810.11188*, 2018.

[CP20]   Justin Curry and Amit Patel. Classification of constructible cosheaves. *Theory and Applications of Categories*, 35(27):1012–1047, 2020.

[Cra00]   Sjoerd Crans. On braiding, syllapses and symmetries. *Cahiers de topologie et géométrie différentielle catégoriques*, 41(1):2–74, 2000.

[CRS97]   J Scott Carter, Joachim H Rieger, and Masahico Saito. A combinatorial description of knotted surfaces and their isotopies. *advances in mathematics*, 127(1):1–51, 1997.

[CS98]   J Scott Carter and Masahico Saito. *Knotted surfaces and their diagrams*. Number 55. American Mathematical Soc., 1998.

[DK80a]   William G Dwyer and Daniel M Kan. Calculating simplicial localizations. *Journal of Pure and Applied Algebra*, 18(1):17–35, 1980.

[DK80b]   William G Dwyer and Daniel M Kan. Simplicial localizations of categories. *Journal of Pure and Applied Algebra*, 17(3):267–284, 1980.

[Don83]   Simon K Donaldson. An application of gauge theory to four-dimensional topology. *Journal of Differential Geometry*, 18(2):279–315, 1983.

[Dor18]   Christoph Dorn. Associative n-categories. *Oxford University Research Archives*, 2018.

[DV09]   Robert J Daverman and Gerard Venema. *Embeddings in manifolds*, volume 106. American Mathematical Soc., 2009.

[Edw80]   Robert D Edwards. The topology of manifolds and cell-like maps. In *Proceedings of the International Congress of Mathematicians (Helsinki, 1978)*, pages 111–127. Citeseer, 1980.

[ELS04]   Martín Escardó, Jimmie Lawson, and Alex Simpson. Comparing cartesian closed categories of (core) compactly generated spaces. *Topology and its Applications*, 143(1-3):105–145, 2004.

[Fre82]   Michael Hartley Freedman. The topology of four-dimensional manifolds. *Journal of Differential Geometry*, 17(3):357–453, 1982.

[GM88]   Mark Goresky and Robert MacPherson. Stratified morse theory. In *Stratified Morse Theory*, pages 3–22. Springer, 1988.

[GPS95]   Robert Gordon, Anthony John Power, and Ross Street. *Coherence for tricategories*, volume 558. American Mathematical Soc., 1995.

[Gro97]   Alexandre Grothendieck. Esquisse d'un programme. *London Mathematical Society Lecture Note Series*, pages 5–48, 1997.

[Had20]   Amar Hadzihasanovic. Diagrammatic sets and rewriting in weak higher categories, 2020.

[HM74]   Morris W Hirsch and Barry Mazur. *Smoothings of piecewise linear manifolds*. Princeton university press, 1974.

[HS70]   Wu-chung Hsiang and J Shaneson. Fake tori. In *Topology of Manifolds, Proc. 1969 Georgia Topology Conference, Markham Press*, pages 18–51, 1970.

[Hum12]   Benjamin Taylor Hummon. *Surface diagrams for gray-categories*. PhD thesis, UC San Diego, 2012.

[HW69]   Wu Chung Hsiang and C Terry C Wall. On homotopy tori ii. *Bull. London Math. Soc*, 1:341–342, 1969.

[JS91]   André Joyal and Ross Street. The geometry of tensor calculus, i. *Advances in mathematics*, 88(1):55–112, 1991.

[Ker60]   Michel A Kervaire. A manifold which does not admit any differentiable structure. *Comment. Math. Helv*, 34(1):257–270, 1960.

[KS69]   Robion C Kirby and Laurence C Siebenmann. On the triangulation of manifolds and the hauptvermutung. *Bull. Amer. Math. Soc*, 75(4):742–749, 1969.

[KS77]   Robion C Kirby and Laurence C Siebenmann. *Foundational essays on topological manifolds, smoothings, and triangulations*. Number 88. Princeton University Press, 1977.

[Lei04]   Tom Leinster. *Higher operads, higher categories*, volume 298. Cambridge University Press, 2004.

[Lor21]   Fosco Loregian. *(Co) end Calculus*, volume 468. Cambridge University Press, 2021.

[Lur09a]   Jacob Lurie. *Higher topos theory*. Princeton University Press, 2009.

[Lur09b]   Jacob Lurie. On the classification of topological field theories. *Current developments in mathematics*, 2008:129–280, 2009.

[Lur12]   Jacob Lurie. Higher algebra, 2012.

[LW69]   Albert T. Lundell and Stephen Weingram. *The topology of CW complexes*. The University Series in Higher Mathematics. Van Nostrand Reinhold Co., New York, 1969.

[Mac91]   Robert D MacPherson. Combinatorial differential manifolds. In *Topological Methods in Modern Mathematics," Proc. of a Symposium in Honor of John Milnor's Sixtieth Birthday, SUNY Stony Brook*, pages 203–221, 1991.

[Man16]   Ciprian Manolescu. Pin (2)-equivariant seiberg-witten floer homology and the triangulation conjecture. *Journal of the American Mathematical Society*, 29(1):147–176, 2016.

[Mar58]   Andrei Andreevich Markov. The insolubility of the problem of homeomorphy. In *Doklady Akademii Nauk*, volume 121, pages 218–220. Russian Academy of Sciences, 1958.

[Mil56]   John Milnor. On manifolds homeomorphic to the 7-sphere. *Annals of Mathematics*, pages 399–405, 1956.

[Mil61]   John Milnor. Two complexes which are homeomorphic but combinatorially distinct. *Annals of Mathematics*, pages 575–590, 1961.

[Mil63]   John Milnor. Morse theory. *Annals of Mathematics Studies*, 51, 1963.

[Nab95]   Alexander Nabutovsky. Einstein structures: existence versus uniqueness. *Geometric & Functional Analysis GAFA*, 5(1):76–91, 1995.

[NW03]   Alexander Nabutovsky and Shmuel Weinberger. The fractal nature of riem/diff i. *Geometriae Dedicata*, 101(1):1–54, 2003.

[Pac91]   Udo Pachner. Pl homeomorphic manifolds are equivalent by elementary shellings. *European journal of Combinatorics*, 12(2):129–145, 1991.

[Poo14]   Bjorn Poonen. Undecidable problems: a sampler. *Interpreting Gödel: Critical Essays*, pages 211–241, 2014.

[PS14]   Tim Poston and Ian Stewart. *Catastrophe theory and its applications*. Courier Corporation, 2014.

[RCS+96]   AA Ranicki, AJ Casson, DP Sullivan, MA Armstrong, CP Rourke, and GE Cooke. The hauptvermutung book. *Collection of papers by Casson, Sullivan, Armstrong, Cooke, Rourke and Ranicki, K-Monographs in Mathematics*, 1, 1996.

[RS71]   Colin Patrick Rourke and Brian Joseph Sanderson. δ-sets i: Homotopy theory. *The Quarterly Journal of Mathematics*, 22(3):321–338, 1971.

[RS72]   C. P. Rourke and B. J. Sanderson. *Introduction to piecewise-linear topology.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 69. Springer-Verlag, New York-Heidelberg, 1972.

[Sel10]   Peter Selinger. A survey of graphical languages for monoidal categories. In *New structures for physics*, pages 289–355. Springer, 2010.

[Shi13]   Masahiro Shiota. O-minimal Hauptvermutung for polyhedra II. *arXiv preprint arXiv:1310.5489*, 2013.

[Shi14]   Masahiro Shiota. O-minimal Hauptvermutung for polyhedra I. *Inventiones mathematicae*, 196(1):163–232, 2014.

[SP09]   Christopher John Schommer-Pries. *The classification of two-dimensional extended topological field theories.* University of California, Berkeley, 2009.

[Str01]   Ross Street. Powerful functors. *Unpublished expository notes*, 2001. `http://web.science.mq.edu.au/~street/Pow.fun.pdf`.

[Str09]   Neil P Strickland. The category of cgwh spaces. *preprint*, 12:90, 2009.

[TF18]   René Thom and David H Fowler. *Structural Stability and Morphogenesis: An Outline of a General Theory of Models.* CRC press, 2018.

[Tri99]   Todd Trimble. Surface diagrams. *nLab:Surface diagrams*, 1999.

[VKF74]   Igor'Aleksandrovich Volodin, VE Kuznetsov, and Anatoly Timofeevich Fomenko. The problem of discriminating algorithmically the standard three-dimensional sphere. *Russian Mathematical Surveys*, 29(5):71, 1974. With an appendix by S. P. Novikov.

[Wei04]   Shmuel Weinberger. Computers, rigidity, and moduli: The large-scale fractal geometry of riemannian moduli space, 2004.