A brief introduction to framed combinatorial topology

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Abstract

In this expository note, we give a brief overview of some recently developed ideas in the study of framed combinatorial spaces, and its applications in the definition of manifold diagrams, tame tangles, and the combinatorial study of higher Morse theory.

Contents

Introduction

This note is an attempt to give a *brief* introduction to framed combinatorial topology [DD21] which is a recently started program centered around the study of framed (and thereby *directed*) spaces. The program roots in the observation that certain combinatorial-topological questions have good answers for framed spaces while for unframed spaces they generally do not (see, for instance, Remark 2.2.4 and Theorem 2.4.2). These features of framed combinatorial space turn out to be especially useful when trying to relate the topology of framed spaces to the combinatorics of higher category theory, leading, for instance, to the notion of manifold diagrams [DD22] as a framed topological model of higher-categorical composition (see Section 4.1).

The focus of this note will lie on motivating and sketching some of the main ideas in framed combinatorial topology, and less on a formal presentation of the technical details. The note is structured as follows.

1. Section 1 distils and amalgamates some necessary "foundational ingredients" in the theory of stratified spaces.
2. Section 2 motivates and recalls basic framed combinatorial-topological notions, including framed spaces, framed regular cells and their complexes. The section also begins the study of local ('flat') framed spaces and the interesting ways in which these can be cellulated.
3. Section 3 discusses the theory of meshes and trusses, which are two sides (namely, the topological and the categorical side) of the same idea: the description of flat framed cellulations by an inductive mechanism (but in a uniform larger theory that also encompasses dual cells).
4. Section 4 provides a first look at manifold diagrams (which generalize ordinary string diagrams to higher dimensions), cell diagrams dual to manifold diagrams, and tame tangles. In the context of the latter notion, we briefly discuss combinatorial approach to higher Morse theory and singularity theory, recalling conjectures about the combinatorialization of smooth structure.
5. In the final Section 5 we sketch two directions for future research, towards 'constructive' foundations of manifold and singularity theory and geometric higher category theory.

1 A recollection of stratified spaces

The term *stratified space* usually refers to a topological space equipped with the structure of a stratification, i.e. a decomposition of the space into subspaces called strata. Often strata are 'nice' in some sense (for instance, one may require strata to be manifolds, while the stratified space itself need to be a manifold), and the way that strata are linked together is controlled by additional conditions.

1.1 Stratifications and stratified maps

There are many different definitions of stratifications on spaces: poset-stratifications, stratifications from filtrations, Whitney stratifications, Thom-Mather stratifications, stratifolds, homotopical stratifications, conically smooth stratifications, etc. Most of these definitions share the same "fundamental" features, which may be captured by the following concepts.

Definition 1.1.1 (Prestratifications)**.** A **prestrafication** f on a topological space X is a decomposition of X into disjoint connected subspaces called 'strata' (in the following, strata are denoted by lower-case letters such as s). The **exit path preorder** $\mathsf{Exit}(f)$ of the stratification f is the preorder of strata s with generating arrows $s \to r$ whenever the closure of r intersects s non-trivally. The opposite poset $\mathsf{Entr}(f)^{\mathrm{op}}$ is also called the **entrance path poset** and denoted by $\mathsf{Entr}(f)$. ⌟

Definition 1.1.2 (Stratification)**.** A **stratification** f is a prestratification whose exit path preorder is a poset (in which case one can speak of its 'exit path poset'). ⌟

Given a stratification f on X, there is a map $X \to \mathsf{Exit}(f)$ mapping points $x \in s$ in a stratum s to the stratum s itself, which is sometimes called the **characteristic map** of f, and (abusing notation) denoted by $f : X \to \mathsf{Exit}(f)$. The map need not be continuous in general (unless the stratification is *locally finite*, see Remark 1.1.6); it is, however, finitely continuous, i.e. continuous on the preimage of any finite set.

Example 1.1.3 (Poset-stratifications)**.** Let X be a space, P a poset, and $f : X \to P$ a continuous map[1]: this is usually called a 'P-stratification of X'. This determines a stratification $\mathrm{c}(f)$ of X whose strata are the connected components of the preimages $f^{-1}(x)$, $x \in P$. The map f factors uniquely through the characteristic map $\mathrm{c}(f) : X \to \mathsf{Exit}(\mathrm{c}(f))$ by a poset map $\mathsf{Exit}(\mathrm{c}(f)) \to P$, and this map is conservative[2]. (Such (characteristic,conservative)-factorizations are essentially unique.) ⌟

The example shows that any poset-stratification determines a unique stratification in the sense defined above. The converse does not hold; many poset-stratifications may determine the same stratification.

Example 1.1.4 (Filtered spaces)**.** Any filtered space $X_0 \subset X_1 \subset ... \subset X_n$ in which X_i is a closed subspace of X_{i+1} defines a continuous map $X \to [n] = (0 \to 1 \to ... \to n)$ mapping points in $X_{i+1} \setminus X_i$ to $i \in [n]$, and thus a stratification by the previous example. For instance, the filtration by skeleta of any cell complex defines a stratification 'by cells of the complex' in this way. ⌟

Example 1.1.5 (Trivial stratification)**.** Every topological space U is trivially stratified by its connected components, and we denote this stratification by U itself. Note that $\pi_0 U \cong \mathsf{Exit}(U)$. ⌟

Remark 1.1.6 (Continuity of characteristic map). Let (X, f) be a stratified space. One says that the stratification f is 'locally finite' if each stratum s of f has an open neighborhood in X which only contains finitely many strata. (If (X, f) satisfies the frontier condition,

[1] Here, we work with the 'upward-closure' topology on posets P, which has subbasic opens that are 'upper closures', i.e. full subposets of the form $P^{\geq x} = \{y \in P \mid y \geq x\}$. Note, when working with entrance path posets, the dual 'downward-closure' topology (generated by the 'lower closures' $P^{\leq x}$) is more convenient.

[2] Recall, a conservative functor of categories is a functor that reflects isomorphisms; for functors between posets, it's simply a map of posets whose preimages are discrete!

see next remark, then, equivalently, (X, f) is locally finite iff each point $x \in X$ has an open neighborhood intersecting only finitely many strata.) If f is locally finite, then the characteristic map $f : X \to \mathsf{Exit}(f)$ is a continuous map. ⌟

Remark 1.1.7 (Openness of characteristic map). Let (X, f) be a stratified space. One says the stratification f satisfies the 'frontier condition' (or, as an adjective, that it is 'frontier-constructible') if, for any two strata s, r, whenever the closure $\overline{r}$ intersects s non-trivially then $s \subset \overline{r}$. The stratification f is frontier-constructible iff the characteristic map $f : X \to \mathsf{Exit}(f)$ is an open map. ⌟

Local finiteness and frontier-constructibility are fundamental properties of stratified spaces and it is often very reasonable to assume them.

Definition 1.1.8 (Stratified maps)**.** A map $F : (X, f) \to (Y, g)$ of stratified spaces is a continuous map $F : X \to Y$ which factors through the characteristic maps f and g by a (necessarily unique!) map $\mathsf{Exit}(F) : \mathsf{Exit}(f) \to \mathsf{Exit}(g)$. ⌟

Stratified spaces and their maps form the category **Strat** of stratification. The construction of exit path posets yields a functor $\mathsf{Exit} : \mathbf{Strat} \to \mathbf{Pos}$ (*aside*: under mild conditions, this can be made an $(\infty, 2)$-functor). The functor has a right inverse, as follows.

Remark 1.1.9 (Classifying stratifications of posets). Every poset P has a **classifying stratification** $||P||$ (also called the 'stratified realization' of P), whose underlying space is the classifying space $|P|$ of P (i.e. the realization of the nerve of P), and whose characteristic map is the map $|P| \to P$ that maps points in $|P^{\leq x}| \setminus |P^{<x}|$ to x. Moreover, given a poset map $F : P \to Q$, the realization of its nerve yields a stratified map $||F|| : ||P|| \to ||Q||$. We thus obtain a functor $|| - || : \mathbf{Pos} \to \mathbf{Strat}$, and this is a right inverse to Exit. ⌟

It makes sense to further terminologically distinguish maps of stratifications as follows.

Definition 1.1.10 (Types of stratified maps)**.** Let $F : (X, f) \to (Y, g)$ be a stratified map. The stratified map F is called:

- a **substratification**, if $F : X \to Y$ is a subspace and $\mathsf{Exit}(F)$ is conservative; if, moreover, $X = g^{-1} \circ \mathsf{Exit}(F) \circ f(X)$ then one says the substratification is **constructible**;
- a **coarsening**, if $F : X \to Y$ is a homeomorphism (to emphasize the opposite process, one also calls F a **refinement**);
- a **stratified homeomorphism**, if $F : X \to Y$ is a homeomorphism of spaces and $\mathsf{Exit}(F)$ is an isomorphism of posets. ⌟

The next definition generalizes the usual topological definition of fiber bundles. We first need to introduce products of stratifications.

Remark 1.1.11 (Products). Given stratification (X, f) and (Y, g) their **product** is the stratification of $X \times Y$ with characteristic map $f \times g$. ⌟

Definition 1.1.12 (Stratified fiber bundles)**.** A **stratified fiber bundle** (or simply a 'stratified bundle') is a stratified map $p : (X, f) \to (Y, g)$ such that for each stratum s of the 'base' stratification (Y, g), each point x in s has a neighborhood V (*inside* the stratum s) over which the map p trivializes to a stratified 'projection' map $V \times (Z, h) \to V$ (here, V is trivially stratified). ⌟

Remark 1.1.13 (Idea of constructibility). One often imposes additional 'constructibility' conditions on stratified bundles in order to control how fibers behave when transitioning between strata in the base. Roughly speaking, the term 'constructible' indicates that something can be reconstructed, up to equivalence, only from categorical data associated to the 'fundamental categories' of stratifications—we will turn to the construction such fundamental categories in the next two sections. ⌟

1.2 Regularity

For many purposes (for instance, the construction of fundamental categories) the basic definition of stratified space outlined above is too wildly behaved. One therefore imposes regularity condition to tame this behaviour. With a view towards defining higher fundamental categories of stratifications, we mention "conicality" and "regularity".

1.2.1 Conical stratification

Definition 1.2.1 (Cones)**.** The **open cone** $(\mathsf{cone}(X), \mathsf{cone}(f))$ of a stratification (X, f) is a stratification that stratifies the topological open cone $\mathsf{cone}(X) = X \times [0, 1)/X \times \{0\}$ by the product $(X, f) \times (0, 1)$ away from the cone point $\{0\}$ (here, the open interval $(0, 1)$ is trivially stratified), and by setting the cone point $\{0\}$ to be its own stratum. To define the **closed cone** $(\overline{\mathsf{cone}}(X), \overline{\mathsf{cone}}(f))$ replace '1)' by '1]'. ⌟

Definition 1.2.2 (Conical stratification, [Lur12, App. A])**.** A **conical** stratification (X, f) is a stratification in which each point $x \in X$ has a neighborhood (i.e. an open substratification) that is a stratified product $U \times (\mathsf{cone}(Z), \mathsf{cone}(l))$ with $x \in U \times \{0\}$. ⌟

1.2.2 Regular stratifications

Definition 1.2.3 (Regular stratifications)**.** A stratification (X, f) is **regular** if it admits a refinement $||P|| \to (X, f)$ by the stratified realization of some poset P. ⌟

(NB: a choice of refinement in particular endows the underlying space X with a triangulation, and thus a PL structure; a variation of the definition requires X to have a PL structure to begin with, and the refinement to be compatible with that structure.)

1.3 Fundamental ∞-posets

Let us make a heuristic observation about the place stratifications in the bigger landscape of higher category theory: stratifications are spatial models of '∞-posets', and conversely every stratification should have a fundamental ∞-poset. This is illustrated in Fig. 1.1, and can be further explained as follows.

sets $\equiv (0,0)$-categories	∞-sets $\simeq$ spaces	sets with w.e.
posets $\equiv (0,1)$-categories	∞-posets $\simeq$ **stratifications**	posets with w.e.
categories $\equiv (1,1)$-categories	∞-categories	categories with w.e.

∞-fication (left column $\to$ middle column) $\qquad$ ∞-localization (right column $\to$ middle column)

Figure 1.1: Stratifications in the categorical landscape

Intuitively (and without going into any technically details), an '∞-X' is to be understood as an (∞,∞)-category which admits a conservative functor to an X, where X can e.g. stand for 'set', 'poset', or 'category'. Yet more generally, X can be an (n,k)-category for $n,k < \infty$ (note that, if $n < k$, then for $n < m \leq k$ it is convention to require that there is at most one m-arrow between any two $(m-1)$-arrows; in particular, posets and preorders are $(0,1)$-categories by this convention). A 'set with weak equivalences' means a poset with weak equivalences in which each arrow is a weak equivalence. The left column is related to the middle column by an '∞-zation functor' (which simply interprets 1-structures as ∞-structures), and the middle and right columns are related by an '∞-localization functor' (which should be a weak equivalence).

Special cases of the translation between columns can be sketched as follows:

- Given a conical stratification (X,f), then [Lur12, App. A] one finds a construction of the **entrance path ∞-poset** $\mathcal{E}\mathrm{ntr}(f)$ as a *quasicategory*: the k-simplices of the quasicategory $\mathcal{E}\mathrm{ntr}(f)$ are precisely stratified maps $||[k]|| \to (X,f)$, where $[k] = (0 \to 1 \to ... \to k)$. This translates "stratified spaces" into "∞-posets" in the above table.
- Given a regular stratification (X,f) and a refinement $F : ||P|| \to (X,f)$, one can construct a **presented entrance path ∞-poset** $\mathcal{P}\mathcal{E}\mathrm{ntr}(f)$ as a *category with weak equivalences*, whose presenting category is the poset P and whose weak equivalence are $\mathsf{Exit}(F)^{-1}(\mathrm{id})$. This translates "stratified spaces" into "posets with weak equivalences" in the above table.

Remark 1.3.1. One reason for the given construction of $\mathcal{P}\mathcal{E}\mathrm{ntr}(f)$ being reasonable is the observation that, firstly, any stratified realization $||P||$ of a poset is conical (if the poset is locally finite at least), and that, secondly, $\mathcal{E}\mathrm{ntr}||P||$ is a 0-truncated ∞-category. ⌟

2 First definitions in framed combinatorial topology

Framed combinatorial topology studies framed spaces; intuitively, the reason for why framed spaces are an interesting starting point for a theory of higher homotopical structures is that they are in some ways *foundationally much simpler* than their classical (unframed) counterparts—a basic observation that relates to this claim is that the space of topological automorphisms of standard euclidean space $\mathbb{R}^n$ is complicated, but the space of framed topological automorphisms of standard framed $\mathbb{R}^n$ is (in homotopical terms) maximally simple: it is contractible. This theme of 'foundational simplicity' is reflected in many parts of the theory of framed combinatorial spaces, and allows one to overcome several fundamental obstructions to classification and computability questions that one usually encounters the classical study of combinatorial-topological phenomena.

2.1 Framed spaces

The definition of framed spaces in framed combinatorial topology is based on a non-standard notion of framings: the notion is motivated by the following observation about 'metric-free orthonormal' frames.

Remark 2.1.1 (Motivation: frames from projections). Let V be an n-dimensional vector space with an inner product g. The following structures on V are equivalent.

1. An orthonormal frame of V, i.e. an ordered sequence of vectors v_i, $1 \leq i \leq n$, such that v_i is normalized and orthogonal to all v_j with $j \neq i$.
2. A sequence of projections $V_i \to V_{i-1}$, $1 \leq i \leq n$, where V_i is an oriented i-dimensional vector space (and $V_n = V$).

The two structures are related by setting v_i to be the unit vector spanning $\ker(V_i \to V_{i-1})$ such that $V_{i-1} \oplus v_i$ recovers the orientation of V_i (note that all V_i canonically embed in V as the orthogonal complement of the kernel of the composite projection $V \to V_i$).

In the absence of inner products, one cannot speak of orthonormal frames any longer. However, sequences of projections can still be defined, and may be regarded as playing the role of 'metric-free orthonormal' frames. (A vaguely analogous line of thinking is that a Morse function $M \to \mathbb{R}$ provides useful 'direction' information on M, e.g. for the construction of handlebodies, that is ultimately independent from any chosen metric on M; see Section 4.4 for a related discussion.) Moreover, this approach offers room for generalization by varying the length of the projection sequence and the vector space dimensions (leading to notions such as "partial" or "embedded" frames, see [DD21, App. A] for details). ⌟

Example 2.1.2 (Standard orthonormal frame)**.** The standard orthonormal frame of n-dimensional euclidean space $\mathbb{R}^n$ consists for the ordered sequence of vectors $e_1 = (1, 0, ..., 0)$, $e_2 = (0, 1, 0, ..., 0)$, ..., $e_n = (0, ..., 0, 1)$. By the previous remark, this orthonormal frame is equivalently described by the sequence of projections $\pi_i : \mathbb{R}^i = \mathbb{R}^{i-1} \times \mathbb{R} \to \mathbb{R}^{i-1}$ (each $\mathbb{R}^i$ being endowed with standard orientation). ⌟

The idea above is "local", but can be extended to a "global" one. First note, when forgetting the base point of an inner product space (i.e. thinking of it as an affine space), then a sequence of projections in the previous sense endows each point of the affine space with an orthonormal frame (by considered the associated vector space at that point). One may use this observation to define framings of spaces globally (i.e. choices of frames at every point), by using the affine, standard framed $\mathbb{R}^n$ as a local model as follows.

Definition 2.1.3 (Framed space)**.** Let X be a (convenient) topological space. Fix $n \in \mathbb{N}$ and denote by $\pi_{>i}$ the projection $\mathbb{R}^i \times \mathbb{R}^{n-i} \to \mathbb{R}^i$.

1. An 'n-framed chart' (U, γ) in X is an embedding $\gamma : U \hookrightarrow \mathbb{R}^n$ of a subspace $U \subset X$. If $U' \subset U$, then γ restricts to the 'restricted chart' (U', γ).
2. Given two n-framed charts (U, γ), (V, ρ) in spaces X resp. Y, a map $F : X \to Y$ is said to be 'framed' on the given charts, if it restricts to a map $F : U \to V$ such that, for all i, there is a continuous map $F_i : \pi_{>i}(U) \to \pi_{>i}(V)$ making $\pi_{>i} \circ \rho \circ F = F_i \circ \pi_{>i} \circ \gamma$ commute.
3. Two n-framed charts (U, γ), (V, γ') in X are 'framed compatible' if $\mathrm{id} : X = X$ is a framed map on the restricted charts $(U \cap V, \gamma)$ resp. $(U \cap V, \gamma')$.

An 'n-framing structure' on X is an 'atlas' $\mathcal{A}$ of framed compatible charts $\{(U_i, \gamma_i)\}$ such that the subspaces U_i are a locally finite compact cover of X. An **n-framed space** is a space endowed with an n-framing structure. ⌟

(The reason for working with locally finite compact atlases is that it is a convenient choice for cell complexes, as discussed shortly.)

Example 2.1.4 (Circle with framing poles)**.** Take two intervals $I_1, I_2 \cong [0, 1]$ and glue them together at their endpoints to get the space S^1 (homeomorphic to the circle). Denote the inclusion $[0, 1] \subset \mathbb{R}$ by ϕ. Then S^1 has a 1-framing structure $\mathcal{A}$ given by the charts (I_1, ϕ) and (I_2, ϕ). (How to think about this: think of $\mathbb{R}$ as having its standard framing, i.e. frames at each point pointing in positive direction; via ϕ these restrict to $[0, 1]$; as a result, one may think of $(S^1, \mathcal{A})$ as a version of the vector flow of the standard Morse function $S^1 \to \mathbb{R}$ with two critical points.) ⌟

Definition 2.1.5 (Maps of framed spaces)**.** Given spaces with n-framing structure $(X, \mathcal{A}) \to (Y, \mathcal{B})$ then a map $F : X \to Y$ is said to be 'framed' if for each point $x \in X$, denoting by $(U_i, \gamma_i) \in \mathcal{A}$ the charts containing x and by $(V_j, \rho_j) \in \mathcal{B}$ the charts containing $F(x)$, the map $F : (U_i \cap F^{-1}(V_j), \gamma_i) \to (V_j, \rho_j)$ is framed (for all i, j). ⌟

2.2 Framed combinatorial spaces

Framed spaces as defined above may still, to some extent, contain 'wild' behaviour. To 'tame' this behaviour, one can control it by framed *combinatorial* structures (analogous to passing from topological spaces to piecewise linear spaces). The appropriate classical combinatorial-topological analogs for this purpose turn out to be *regular cell complexes*: a regular cell complex is a cell complex in which the closure of each open cell is a closed ball (also referred to as a 'face' of the complex). Of particular importance for the definition of framings on regular cells is the following generalization of the notion of simplicial degeneracy maps.

Definition 2.2.1 (Cell projections)**.** Given regular cells X and Y of dimension k respectively $k-1$, a **cell projection** $X \twoheadrightarrow Y$ is a map which on the open interior Y° of Y restricts to a closed interval bundle $Y^\circ \times \mathbf{I} \to Y^\circ$ and which, inductively in cell dimension k, restricts on each proper face $X' \hookrightarrow X$ of X either to a cell homeomorphism $X' \cong Y'$ or to another cell projection $X' \twoheadrightarrow Y'$ (where $Y' \hookrightarrow Y$ is a face of Y). ⌟

Example 2.2.2 (Simplicial projections)**.** Cell projections of cellular simplices are precisely the (geometric realizations of) degeneracy maps. ⌟

Definition 2.2.3 (Framed regular cells)**.** A **framed regular cell** X is a regular cell $X \equiv X_n$ together with a sequence of maps $p_i : X_i \to X_{i-1}$ such that p_i is either a cell homeomorphism or a cell projection together with a choice an orientation of the $\mathbb{R}$-fiber in each such cell projection. (These choices of orientations play a role exactly analogous to the orientation choices in Remark 2.1.1). ⌟

The underlying spaces of an n-framed regular cell has a natural n-framed structure. This structure can be represented by an atlas with a single chart: namely, by any embedding of the cell into $\mathbb{R}^n$ with the property that, when post-composed with the projections $\pi_{>i} : \mathbb{R}^n \to \mathbb{R}^i$, the embedding factors through the cell maps $p_{i+1} \circ ... \circ p_n : X \to X_i$ by an embedding $X_i \hookrightarrow \mathbb{R}^i$ (with the condition that, if $X_i \to X_{i-1}$ happens to be a cell projection, then the chosen orientations of fibers of this projection are preserved by the resulting embedding into standard oriented $\mathbb{R}$-fibers of the projection $\pi_i : \mathbb{R}^i \to \mathbb{R}^{i-1}$).

Remark 2.2.4 (Framed regular cells are combinatorial). Classical regular cells are really 'semi-combinatorial' structures, as their definition must refer to the topological sphere (or the topological ball). In fact, classical regular cells are not classifiable in any constructive way (while for any $n, k \in \mathbb{N}$, there are only finitely many regular n-cells with k faces, there can be no algorithm that lists all such cells for general n, k!). In contrast, framed regular cells are constructively classifiable in combinatorial terms. Their classification is based on the combinatorial theory of trusses (see Section 3). ⌟

Framed regular cells organize into a category, generated by *non-degenerate framed* cell projections and face inclusions of framed regular cells (here, 'framed' is to be understood as in Definition 2.1.5, and 'non-degenerate' means that whenever p_i is a cell homeomorphism then its image $\mathrm{im}(p_i)$ too is a cell homeomorphism). Via the previous remark, this category can be described in purely combinatorial terms (see Definition 3.3.5). Framed combinatorial spaces are, in essence, presheafs on this category, though one may want to impose additional properties, such as 'regularity' (meaning non-degenerate cells include by monomorphisms, see e.g. [DD21, Sec. 2.3.3]) or 'sheafiness' (meaning presheafs preserve certain pullbacks). Without going into details of these properties, let us record the following 'polymorphic' phrasing for their definition.

Definition 2.2.5 (Framed combinatorial spaces)**.** A **framed combinatorial space** is a presheaf, potentially 'with properties', on the category of framed regular cells. A combinatorial map of two framed combinatorial spaces is a presheaf map. ⌟

One concrete instantiation of this polymorphic definition is obtained by imposing regularity: the notion of 'framed *regular* combinatorial spaces', also called 'framed regular cell complexes', is simply that of a regular cell complex in which all cells are framed, and these framings are mutually compatible.

Depending on details in the definitions, framed combinatorial spaces should be, in particular, framed spaces. The next observation is an instance of this claim for the case of framed regular combinatorial spaces. Recall from Example 1.1.4 that cell complexes are stratifications, and that stratifications can be locally finite (see Remark 1.1.6): taken together, a locally finite cell *regular* complex is a regular cell complex in which each cell is the face of finitely many other cells.

Observation 2.2.6 (Framed regular combinatorial spaces are framed spaces)*.* Every locally finite framed regular cell complex is in particular a framed space: a locally finite compact cover by charts can be obtained from the closed cells of the framed combinatorial space. ⌟

Without imposing regularity, the types of 'framing singularities' that may appear in framed combinatorial spaces are more general than those described by Definition 2.1.3—however, one may weaken the definition (e.g. by weakening the condition of covers being 'locally finite compact') to accommodate these more general framing singularities.

2.3 Flat framed spaces and tame stratifications

Of particular interest in the study of framed combinatorial space is the local case of 'flat' framings.

Definition 2.3.1 (Flat framed space)**.** A **flat n-framed space** $(X, \mathcal{A})$ is an n-framed space which admits a framed embedding into standard framed $\mathbb{R}^n$. ⌟

In particular, any (locally compact) subspace of $\mathbb{R}^n$ can be thought of as a flat framed space, and one speaks of a 'framed subspace' to highlight this. Conversely any flat framed space is framed homeomorphic to framed subspace of $\mathbb{R}^n$. Moreover, observe that framed maps $F : X \to Y$ of framed subspace $X, Y \subset \mathbb{R}^n$ are particularly simple to define: these are continuous maps that factor through all projections $\mathbb{R}^i \times \mathbb{R}^{n-i} \to \mathbb{R}^i$.

Similarly, a 'flat n-framed regular cell' complex is an n-framed regular cell complex that has a framed embedding into $\mathbb{R}^n$. As a special case one obtains the following notion of 'nicely bounded' flat framed regular cell complexes. Denote, as before, by π_i the projection $\mathbb{R}^{i-1} \times \mathbb{R} \to \mathbb{R}^{i-1}$.

Terminology 2.3.2 (Framed bounded). Inductively in n, a compact subspace $U \hookrightarrow \mathbb{R}^n$ is called 'framed bounded' if the projection $\pi_n : U \to \pi_n(U)$ has fibers of the form $[\gamma_-(u), \gamma_+(u)]$ for continuous sections $\gamma_\pm : \pi_n(U) \to U$, and the projected subspace $\pi_n(U) \hookrightarrow \mathbb{R}^{n-1}$ is itself framed bounded. ⌟

Definition 2.3.3 (Closed meshes)**.** A **closed n-mesh** M is a framed regular cell complex, that admits a framed embedding into $\mathbb{R}^n$ with framed bounded image. ⌟

Note, if a framed regular cell complex is a closed n-mesh, then any framed embedding of it into $\mathbb{R}^n$ will be framed bounded, and the space of all such embeddings is contractible—so, homotopically, being a closed n-mesh is a *property* even if you tried to think of embeddings as a structure. For convenience, we usually fix a choice of framed embedding for any given mesh, and refer to the image of that embedding as the 'support' of the mesh in $\mathbb{R}^n$.

The adjective 'closed' in the previous definition indicates that there are further types of meshes. Indeed, closed meshes have a *dual version* (in the sense of Poincaré duality), referred to as 'open' meshes. Closed and open meshes can be defined uniformly as part of a yet more general theory of meshes, and we will do so in Section 3. To avoid this lengthier, more general approach for the moment, one can instead define open meshes in the following way, which emphasizes the duality of open and closed meshes.

Definition 2.3.4 (Open meshes)**.** An **open n-mesh** M is a stratification of the open n-cube $\mathbb{I}^n \hookrightarrow \mathbb{R}^n$, obtained by first taking a closed n-mesh supported on the closed n-cube $\mathbf{I}^n \hookrightarrow \mathbb{R}^n$ and then removing all cells lying in the boundary $\partial\mathbf{I}^n$ of the cube, subject to the additional condition that there must exist **dual closed mesh** $M^\dagger$, together with an identification of entrance path posets $\dagger : \mathsf{Exit}(M^\dagger) \cong \mathsf{Exit}(M)^{\mathrm{op}}$, and a 'dualizing' framed embedding $M^\dagger \hookrightarrow M$ such that images of strata s of $M^\dagger$ intersect their dual strata $s^\dagger$ in M in a single point. ⌟

Meshes, whether open or closed, should be thought of as providing a sort of triangulation (or rather, a 'cellulation') for the certain types of stratifications as follows.

Definition 2.3.5 (Tame stratifications)**.** An (open resp. closed) **tame stratification** is a stratification of a flat framed space $U \hookrightarrow \mathbb{R}^n$ that has a framed refinement by an (open resp. closed) n-mesh M—that is, there is a stratified refinement $M \to U$ whose underlying map is a framed map. ⌟

(Note, in particular, that tame stratifications have framed bounded support.)

Remark 2.3.6 (Tame maps)*.* One similarly defines "tame maps" of tame stratifications to be framed stratified maps which, after passing to framed refinements of their domain and codomain by meshes, descend to framed stratified maps of these meshes. ⌟

The reason why framed regular cells (as opposed to, say, framed simplices) are a natural choice of combinatorial structure for the purposes of framed combinatorial topology, and why they are in particular the "right" choice for defining meshes, roots in the following theorem.

Theorem 2.3.7 (Canonical meshes)**.** *Every tame stratification is refined by an (up to framed stratified homeomorphism) unique coarsest mesh, which is coarser than all other refining meshes.*

Proof. Proven in detail in [DD21, Ch. 5], and recalled in some detail in [DD22, Sec. 1]. □

Since meshes are framed regular cell complexes, and since framed regular cells are combinatorial (see Remark 2.2.4), this theorem is really the bedrock connecting the topology and combinatorics of tame stratifications. It thus takes a central role in framed combinatorial constructions (e.g. in the theory of manifold diagrams in [DD22]).

A central source of tame stratification are piecewise linear (PL) stratifications.

Terminology 2.3.8 (PL stratification). A 'PL stratification' (V, f) of a subspace $V \hookrightarrow \mathbb{R}^n$ is a stratification that can be "chopped up" into linear pieces; formally, take this to mean that there is a closed subspace $\tilde{V} \hookrightarrow \mathbb{R}^n$ with $V \subset \tilde{V}$, and a simplicial complex $(\tilde{V}, g)$ supported on $\tilde{V}$ (with each simplex linearly embedded in $\mathbb{R}^n$), such that each simplex s of g is either fully contained in a stratum of f or lies in the complement $\tilde{V} \setminus V$. ⌟

Example 2.3.9 (PL stratifications are tame)**.** All piecewise linear (PL) stratifications of the closed cube $\mathbf{I}^n$, or any other framed bounded set, in $\mathbb{R}^n$ are tame (a proof can be found in [DD21, Ch. 5]). ⌟

Example 2.3.10 (Other familiar examples)**.** Further examples of classes of tame stratifications with (maybe) familiar names are the following.

- *String diagrams*, in the sense of Joyal-Street, are tame stratifications.
- *Manifold diagrams* are tame stratifications: they generalize string diagrams to higher dimension, and we will meet them in Section 4.1.
- *Tame tangles* are tame stratifications: in essense, tame tangles are tangles which admit a finite stratification by the 'types of their critical values'. We will discuss them in more detail in Section 4.3. ⌟

Non-Example 2.3.11 ('Wild' stratifications are not tame)**.** Consider the E_8 manifold, and embed it in the closed n-cube $E_8 \hookrightarrow \mathbf{I}^n$ (for sufficiently large n). This defines a stratification of the closed cube, whose strata are the image of the embedding and its complement. No such stratification can be tame, as the E_8 manifold is not triangulable, but every tame stratification is triangulable—in fact, all tame stratification have canonical PL structures, as we will learn in the next section. ⌟

Remark 2.3.12 (Framed homeomorphism preserves tameness). Given two stratifications (U_1, f_1), (U_2, f_2) of flat framed spaces $U_i \hookrightarrow \mathbb{R}^n$, if (U_1, f_1) is tame and (U_2, f_1) is framed stratified homeomorphic to (U_1, f_1), then (U_2, f_2) is tame. ⌟

2.4 Framed TOP vs framed PL structures (locally)

In the local case (i.e. the case of *flat* framings), one finds that the "framed topology" and "framed PL topology" of tame stratifications are closely related.

Definition 2.4.1 (Framed PL structures)**.** Given a stratification (U, f) of a framed subspace $U \hookrightarrow \mathbb{R}^n$, a 'framed triangulation' of (U, f) is a framed stratified homeomorphism $\alpha : (U, f) \cong (V, g)$ to a PL stratification (V, g). Two framed triangulations $\alpha : (U, f) \cong (V, g)$, $\beta : (U, f) \cong (W, h)$ are 'framed equivalent' if there is a framed stratified PL homeomorphism $\rho : (V, g) \cong (W, h)$ such that $\rho \circ \alpha = \beta$. A **framed PL structure** is a framed equivalence class of framed triangulations. ⌟

One easily shows that every tame stratification has some framed PL structure. Moreover, we have the following.

Theorem 2.4.2 (Flat framed stratified Hauptvermutung)**.** *If two tame PL stratifications are framed homeomorphic as stratifications then they are framed PL homeomorphic as stratifications.*

Proof. Proven in detail in [DD21, Ch. 5], and recalled in some detail in [DD22, Sec. 2]. □

Remark 2.4.3 (Failure of non-framed stratified Hauptvermutung). In contrast, non-framed topological stratifications and PL stratifications (even if the stratifications are nice, say, a torus embedded in a closed n-cube) aren't compatible in this way: given a topological stratification, one may find several inequivalent PL structures for it or none at all. ⌟

Corollary 2.4.4 (Uniqueness of framed PL structures)**.** *Every tame stratification has a unique framed PL structure.* □

When working with stratifications whose strata are smooth manifolds, one may further ask about the relation of smooth structures and framed topological structures: an interesting relation with smooth structures can be conjectured, and will be recalled later in Section 4.4

3 The theory of meshes and trusses

In the last section we touched upon the duality of closed and open meshes; since individual framed regular cells are in particular closed meshes, they they too have a 'dual version', to which one may refer to as 'cocells'. There's a unifying theory of such cells and cocells, and the objects central to this story are called *meshes* and *trusses*. (This notion of 'meshes' will, in particular, generalize the 'closed' and 'open' case that we have met already in the last section.)

3.1 Meshes

Meshes are stratified topological structures that arise as models for 'local (co)cell structures' in framed combinatorial topology. Their definition, modelled on our earlier idea of defining framings by towers of projections (see Remark 2.1.1), is *inductive*: one first defines 1-meshes, then 1-mesh bundles, and then n-meshes as towers of 1-mesh bundles.

1-Meshes are rather 'simple' framed stratified manifolds as follows.

Definition 3.1.1 (1-Meshes)**.** A **1-mesh** M is a framed contractible k-manifold, $k \leq 1$, together with a stratification on M whose strata are open l-disks, $l \leq 1$. ⌟

Note that we keep the stratification of M tacit, without introducing another letter for it (it will usually be clear from context when we want M to be considered as a stratification).

Let us next define 1-mesh bundles. Given a stratified bundle $p : E \to B$ (see Definition 1.1.12) whose stratified fibers have manifold strata, and given a stratum $s \in E$, let us write $\mathrm{fibdim}(s)$ for the 'fiber dimension' of s, which is the dimension of the manifold to which the stratum s restricts in fibers $F_x = p^{-1}(x)$ over points $x \in p(s)$ (note the dimension is independent of the choice $x \in p(s)$).

Definition 3.1.2 (1-Mesh bundles)**.** A **1-mesh bundle** $p : M \to B$ is a stratified bundle together with a choice of 1-mesh structure M^b for each stratified fiber $p^{-1}(b)$, $b \in B$, and with the following compatibility condition between fibers.

- There exists a bundle embedding $\gamma : M \hookrightarrow B \times \mathbb{R}$ into the trivial bundle $B \times \mathbb{R} \to B$, which on each 1-mesh fiber $M^b = p^{-1}(b)$ restricts to a framed embedding $\gamma^b : M^b \hookrightarrow \mathbb{R}$ with bounded image $[\gamma^b_-, \gamma^b_+] \subset \mathbb{R}$, such that the mapping $b \mapsto \gamma_\pm(b) := (b, \gamma^b_\pm)$ is a continuous map $B \to B \times \mathbb{R}$.
- For a stratum s in M with $\mathrm{fibdim}(s) = 0$, any arrow $p(s) \to u$ in the entrance path poset $\mathsf{Entr}(B)$ has a unique lift $s \to t$ in the poset $\mathsf{Entr}(M)$ along the poset map $\mathsf{Entr}(p)$, and that lift is such that $\mathrm{fibdim}(t) = 0$.[3] ⌟

There are further variations of the definition: importantly, the second condition is a '0-categorical constructibility' condition, which guarantees that mesh bundles can be classified by functors on the entrance path poset of their (sufficiently nice) base stratification. The condition can be *categorified*, such that bundles are classified by functors on the entrance path ∞-category of the base stratification (see [DD21] for details). The first condition, which is sort of a 'trivializability' condition, also has variations.

[3] One of the reasons that both [DD21] and [DD22] work with *entrance* path posets rather than *exit* path posets is precisely that this condition is a bit more natural to visualize in terms of entrance paths!

Definition 3.1.3 (n-Meshes)**.** An n**-mesh** M is a tower of 1-mesh bundles

$$M_n \to M_{n-1} \to ... \to M_1 \to M_0 = *. \qquad \lrcorner$$

We may sometimes identify M only with its top space $M \equiv M_n$. Note, M_n is naturally a framed space: we may pick any locally finite cover of charts $(U \subset M_n, U \hookrightarrow \mathbb{R}^n)$ that are embeddings which factor through the maps $\mathbb{R}^n \to \mathbb{R}^i$ and $M_n \to M_i$ by a map $M_i \hookrightarrow \mathbb{R}^i$ (with the obvious condition that the framing of 1-mesh fibers of the projection $M_i \to M_{i-1}$ is preserved by the resulting embedding into standard framed $\mathbb{R}$-fibers of $\pi_i : \mathbb{R}^i \to \mathbb{R}^{i-1}$).

The mesh is said to be 'closed' if all fibers of the bundles $M_i \to M_{i-1}$ are compact spaces, and 'open' if all fibers the bundles $M_i \to M_{i-1}$ are open intervals. These special cases of the definition recover (up to an appropriate notion framed equivalence) the definitions of closed and open meshes given earlier in Section 2.3.

n-Meshes too may be easily considered in bundles, as follows.

Definition 3.1.4 (n-Mesh bundles)**.** An n**-mesh bundle** p over a 'base' stratification B is a tower of 1-mesh bundles $p_i : M_i \to M_{i-1}$, $1 \leq i \leq n$, ending in $M_0 = B$. ⌟

Just as stratifications have entrance path posets (or '∞-posets'), n-meshes have fundamental-categorical structures associated to them as well; these structures are called n*-trusses* and we will meet them in the next section. Getting ahead of ourselves slightly, let us remark that n-truss bundles are classified by a category $\mathfrak{T}^n$, and one can show (by establishing an equivalence between n-trusses and n-meshes) that n-mesh bundles are classified by functors on entrance path poset of their base into that category! In this sense, n-mesh bundles are 'constructible' stratified bundles (see Remark 1.1.13).

3.2 Trusses

Trusses are combinatorial structures that arise in framed combinatorial topology, where they yield a combinatorial description of framed regular cells and cocells. The definition is, analogous to that of meshes, inductive. One starts with the definition of 1-dimensional trusses, then studies their bundles, and then defines n-dimensional trusses as towers of bundles of 1-trusses.

Definition 3.2.1 (1-Trusses)**.** A **1-truss** is a finite poset $(T, \leq)$ together with:

- a full and conservative functor $\dim : (T, \leq) \to [1]^{\mathrm{op}}$,
- a second order $(T, \preceq)$ which is total, and whose generating arrows $t \prec s$ satisfy either $t < s$ or $s < t$. ⌟

Given a truss T, denote by $T_{(i)}$ the dimension-i objects of T.

To define bundles of 1-trusses, the language of 'Boolean profunctor' is useful: one may think of a boolean profunctor $H : \mathsf{C} \nrightarrow \mathsf{D}$ as an ordinary profunctor whose values are either the initial set $\emptyset \equiv \bot$ or the terminal set $* \equiv \top$. If C and D are discrete, then such a profunctor H is simply a relation of sets (in this case, we call the profunctor H a 'function' if it is a functional relation or a 'cofunction' if the dual profunctor H^{op} is a function). For any map of posets $F : P \to Q$, the fiber $F^{-1}(x \to y)$ over an arrow $x \to y$ of Q defines a Boolean profunctor $F^{-1}(x) \nrightarrow F^{-1}(y)$ by mapping (a, b) to $\top$ iff $a \to b$ is an arrow in P.

Definition 3.2.2 (Category of 1-truss bordisms)**.** Given 1-trusses T and S, a **1-truss bordism** $R : T \nrightarrow S$ is a Boolean profunctor $T \nrightarrow S$ satisfying the following:

(A) R restricts to a function $R_{(0)} : T_{(0)} \nrightarrow S_{(0)}$ and a cofunction $R_{(1)} : T_{(1)} \nrightarrow S_{(1)}$.

(B) Whenever $R(t, s) = \top = R(t', s')$, then either $t \prec t'$ or $s' \prec s$ but not both.

Given 1-truss bordisms $R : T \nrightarrow S$ and $Q : S \nrightarrow U$, their composite profunctor $R \circ Q$ (composed as ordinary profunctors) is again a 1-truss bordism.[4] This gives rise to the **category $\mathfrak{T}^1$ of 1-trusses and their bordisms**. Forgetting truss structures yields the 'forgetful' functor $\mathfrak{T}^1 \to \mathbf{Prof}$ to the category of profunctors. ⌟

Definition 3.2.3 (1-Truss bundles)**.** A **1-truss bundle** over a poset P is a poset map $q : T \to P$ in which each fiber $q^{-1}(x)$, $x \in P$, is equipped with the structures of a 1-truss T^x, and for each arrow $x \to y$ in P, the fiber $q^{-1}(x \to y)$ is a 1-truss bordisms $T^x \nrightarrow T^y$. ⌟

Definition 3.2.4 (n-Trusses)**.** An **n-truss** T is a tower of 1-truss bundles

$$T_n \xrightarrow{q_n} T_{n-1} \xrightarrow{q_{n-1}} \dots \xrightarrow{q_2} T_1 \xrightarrow{q_1} T_0 = *.$$ ⌟

Definition 3.2.5 (n-Truss bundles)**.** An **n-truss bundle** q over a 'base' poset P is a tower of 1-truss bundles $q_i : T_i \to T_{i-1}$, $1 \leq i \leq n$, ending in $T_0 = P$. ⌟

Definition 3.2.6 (Maps of trusses, cellular and cocellular)**.** A **map of trusses** $F : T \to S$ consists of poset maps $F_i : T_i \to S_i$ that commute with the 1-truss bundle maps $T_i \to T_{i-1}$ and $S_i \to S_{i-1}$ in T respectively in S. (A completely analogous definition can be given for truss bundles.) We say a truss map is 'cellular' if on each 1-truss fiber bundle maps $T_i \to T_{i-1}$ and $S_i \to S_{i-1}$ it preserves dimension-0 objects; dually, we say it is 'cocellular' if it preserves dimension-1 objects in this way. ⌟

n-Truss bundles have a classifying category $\mathfrak{T}^n$ (as constructed e.g. in FCT-citation), i.e. bundles over a poset P correspond to functor $P \to \mathfrak{T}^n$ (up to some notion of equivalence). One particularly concise characterization of this category (pointed out to us by Lukas Heidemann) is the following.

Remark 3.2.7 (Classifying n-truss bundles). Applying the profunctorial Grothendieck construction to the forgetful functor $\mathfrak{T}^1 \to \mathbf{Prof}$, yields an exponentiable fibration $E\mathfrak{T}^1 \to \mathfrak{T}^1$. By general nonsense, the composition of pullback $\mathbf{Cat}_{/\mathfrak{T}^1} \to \mathbf{Cat}_{/E\mathfrak{T}^1}$ and forgetful functor $\mathbf{Cat}_{/E\mathfrak{T}^1} \to \mathbf{Cat}$ has a right adjoint $\mathbf{Cat} \to \mathbf{Cat}_{/\mathfrak{T}^1}$; this adjoint is of the form

[4] In stark contrast, composites of Boolean profunctors, composed as ordinary profunctors, in general need not themselves be Boolean.

$\mathsf{C} \mapsto (\mathfrak{T}^1(\mathsf{C}) \to \mathfrak{T}^1)$ giving rise to a endo-functor $\mathfrak{T}^1 : \mathbf{Cat} \to \mathbf{Cat}$. Applying this functor n-times to the terminal category $*$ yields a category $\mathfrak{T}^n \equiv \mathfrak{T}^n(*)$. This is the classifying category of n-truss bundles. ⌟

More explicit constructions of $\mathfrak{T}^n$ can be found e.g. in [DD22] (and also in [DD21], where it is denoted TBord^n).

Finally, and importantly, trusses can be endowed with 'labels' by functorially associating data in some category (or higher category) to their objects and morphisms.

Definition 3.2.8 (Labelings)**.** For a category C, a C-labeled n-truss (T, f) is an n-truss T together with a 'labeling' functor $f : T_n \to \mathsf{C}$. ⌟

Aside: one may fully analogously define C-labeled n-truss bundles—such bundles are classified by the category $\mathfrak{T}^n(\mathsf{C})$ (obtained by applying the functor $\mathfrak{T}^1$ n times to the category C).

A particular type of labeling structure is a stratification: a 'stratified n-truss' is a labeled n-truss whose labeling functor is the characteristic map of a stratification on the poset T_n (where one regards a poset P as a topological space whose subbasic open are downward closures $P^{\leq x} = y \leq x \subset P$ of elements $x \in P$).

Labeling structures on trusses 'normalize'. The normalization steps are given by so-called coarsenings (which are named so, because they correspond to actual stratified coarsenings in the case when labelings are stratifications after geometric realization of trusses as meshes).

Definition 3.2.9 (Truss coarsenings)**.** Given C-labeled n-trusses (T, f) and (S, g), a truss coarsening $F : (T, f) \to (S, g)$ is a truss map $F : T \to S$ such that:

- F commutes with f and g (this makes F a 'labeled truss map'),
- F_i maps dimension-1 objects in fibers of $T_i \to T_{i-1}$ to dimension-1 objects in fibers of $S_i \to S_{i-1}$, and it preserves the dimension of the endpoints of these fibers. ⌟

The crucial observation about normalization of labeling structures (which, in some sense, is the combinatorial counterpart to our earlier Theorem 2.3.7) is now the following.

Observation 3.2.10 (Normalization). A C-labeled n-truss is **normalized** if no non-trivial truss coarsening applies to it. Every C-labeled n-truss has a truss coarsening to an essentially unique normalized C-labeled n-truss. (This was shown in [Dor18, Thm. 5.2.2.11] in the special case of 'open' n-trusses, also called 'singular n-cubes' back then, but the argument given there can be easily reused for the case of general trusses as well. The statement is shown by different arguments for stratified trusses in [DD21, Ch. 5]). ⌟

3.3 Classification of meshes and framed regular cells

We now briefly remark how trusses are the 'framed entrance path posets' of meshes.

Remark 3.3.1 (Relation to meshes). There is a 'fundamental truss' functor from the category of meshes to the category of trusses. The functor takes an *n-mesh* M given by a tower

$$M_n \to M_{n-1} \to \dots \to M_1 \to M_0 = *$$

of 1-mesh bundles, and, by applying Entr, sends it to the tower of entrance path poset maps

$$\mathsf{Entr}(M_n) \to \mathsf{Entr}(M_{n-1}) \to \dots \to \mathsf{Entr}(M_1) \to \mathsf{Entr}(M_0) = *$$

whose fibers can be canonically endowed with the structure of 1-trusses, thus yielding an *n-truss* (this translate the framings of 1-mesh fibers into the total order required in the definition of 1-trusses). With appopriate care, the fundamental truss functor is, in fact, a weak equivalence between the ∞-category of meshes and the 1-category of truss. ⌟

As we've seen, meshes generalize closed meshes, and closed meshes generalize framed regular cells. The combinatorial analog of these notions are the following.

Definition 3.3.2 (Closed n-trusses)**.** A **closed n-truss** T is an n-truss in which the endpoints of fibers of 1-truss bundles $q_i : T_i \to T_{i-1}$ are of dimension 0 ⌟

Definition 3.3.3 (n-Truss block)**.** An **n-truss block** T is a closed n-truss such that the poset T_n has an initial object. ⌟

Observation 3.3.4 (Classification of framed regular cells). n-Framed regular cells (up to framed cell-preserving homeomorphism) are in correspondence with n-truss blocks (up to structure-preserving isomorphism). ⌟

Finally, we can return to an earlier claim, that the category of framed regular cells has a purely 'combinatorial' definition: here it is.

Definition 3.3.5 (Category of framed regular cells)**.** The **'combinatorial' category of framed regular cells** is the category of n-truss block and their cellular maps. ⌟

4 Definitions of manifold diagrams and tangles

Manifold diagrams generalize string diagrams [JS91] to higher dimensions. Just as string diagrams can be regarded the geometric duals (in the sense of Poincare duality) to commutative diagrams in 2-categories, manifold diagrams can be regarded as the geometric duals of commutative diagrams in n-categories. As such, manifold diagrams are 'directed diagrams' (with directions corresponding to those of morphisms in higher categories). Framed combinatorial provides a natural framework for describing such diagrams. It similarly provides a natural framework for describing the closely related notion of tangles [BD95] (or, more precisely, of 'tame' tangles, which are tangles in which there exists a 'finite stratification by critical value types').

4.1 Manifold diagrams

To begin with, a very brief remark on the history of the 'problem': the idea of manifold diagrams in higher category has been around for many decades and was known, for instance, to Joyal and Street at the time of writing their seminal paper on string diagram calculus. There have been several approaches of defining manifold diagrams in dimension 3. For a long time the case of general dimensions n remained unsolved. One reason for this may have been the difficulty of formalizing 'deformations by isotopy' of manifold strata in manifold diagrams (such isotopies arise only in dimension 3 and above, and get more and more complex in higher dimensions). With our framed combinatorial-topological tools developed in the previous sections, a useful definition of manifold diagrams will be quite easy to give!

4.1.1 Formalization on the open cube We first want to have a 'framed' analogue of conical stratifications (see Definition 1.2.2). This will use the following notion of 'framed' cones.

Remark 4.1.1 (Framed cones). Denote by $\mathbb{I}^n = [-1,1]^n$ be the open n-cube, by $\mathbf{I}^n = [-1,1]^n$ be the closed n-cube, and by $\partial\mathbf{I}^n = \mathbf{I}^n \setminus \mathbb{I}^n$ the n-cube's boundary. We identify the open cone $\mathsf{cone}(\partial\mathbf{I}^n) = \partial\mathbf{I}^n \times [0,1)/\partial\mathbf{I}^n \times \{0\}$ with the open cube $\mathbb{I}^n$ by mapping $(x,\lambda) \in \partial\mathbf{I}^n \times [0,1)$ to $\lambda x \in \mathbb{I}^n$. Similarly we identify the closed cone $\overline{\mathsf{cone}}(\partial\mathbf{I}^n)$ with $\mathbf{I}^n$. Via the standard embedding of cubes in $\mathbb{R}^n$, these identification endow the cones of the cube's boundary with the structure of a flat n-framed spaces.

A stratification $(\partial\mathbf{I}^n, f)$ of the n-cube's boundary is called a 'cubical link' (or simply a 'link' in the present context). Given a cubical link $(\partial\mathbf{I}^k, f)$, the 'framed open cone' $(\mathbb{I}^k, \mathsf{cone}(g))$ (resp. the 'framed closed cone' $(\mathbf{I}^k, \overline{\mathsf{cone}}(l))$) is simply the open (resp. closed) stratified cone of f using the previous identifications. We say l is a 'tame link', if its open cone $(\mathbb{I}^n, \overline{\mathsf{cone}}(l))$ is a tame stratification (equivalently, this may use the closed cone). ⌟

Next, we may introduce a 'framed' conicality condition as follows. The definition uses the trivial observation that, given a flat n-framed space $X \hookrightarrow \mathbb{R}^n$, its product $\mathbb{I}^k \times X$ is also a flat $(n+k)$-framed space (simply by taking the product of $\mathbb{I}^k \hookrightarrow \mathbb{R}^k$ with $X \hookrightarrow \mathbb{R}^n$).

Definition 4.1.2 (Framed conical stratifications)**.** A stratification (X, f) of a flat framed space $X \hookrightarrow \mathbb{R}^n$ is **framed conical at** $x \in X$ if there exists a link $(\partial\mathbf{I}^{n-k}, l_x)$ and a framed stratified neighborhood $\phi : \mathbb{I}^k \times (\mathbb{I}^{n-k}, \mathsf{cone}(l_x)) \hookrightarrow (\mathbb{I}^n, f)$ such that $x \in \mathbb{I}^k \times \{0\}$, where 0 is the cone point of $\mathsf{cone}(l_x)$. If the link can be chosen tame, and the framed stratified map ϕ can be chosen to be a tame map, then we say (X, f) is **tame framed conical at** $x \in X$. We say (X, f) is **(tame) framed conical** if it is (tame) framed conical at all $x \in X$. ⌟

The definition of manifold diagrams takes the following simple form.

Definition 4.1.3 (Manifold diagrams)**.** A **manifold n-diagram** is a tame stratification $(\mathbb{I}, f)$ of the open n-cube that is tame framed conical. ⌟

Moreover, in [DD22, Sec. 2.1], it is claimed that this definition is equivalent to defining manifold diagrams as tame stratifications that are framed conical (thus, omitting the requirement of links being tame themselves!).

The almost immediate pay-off of working in the setting of framed combinatorial topology is that we can translate a purely topological phrasing of the definition (which, indeed, uses only tameness, i.e. refinability by meshes and therefore certain towers of constructible stratified bundles, and a framed conicality condition) and obtain a constructive combinatorial counterpart (here, 'constructive', as before, in particular means 'algorithmically tractable'). We only record the existence of such a counterpart briefly in the next remark and refer to [DD22, Sec. 2.2 & 2.3] for details.

Remark 4.1.4 (Combinatorial classification of manifold diagrams). Framed stratified homeomorphisms classes of manifold diagrams are classified by certain normalized stratified open trusses (cf. Observation 3.2.10). ⌟

4.1.2 Formalization on the closed cube While it is in some sense more natural to define manifold diagrams on the open cube than on the closed cube (due to the duality of 'open' and 'closed' structures, and manifold diagrams being 'dual' to closed cell diagrams, see below), one can nonetheless also write down a reasonable definition of manifold diagrams on the closed cube.

Remark 4.1.5 (Corner neighborhoods). Let $\mathfrak{P} = \{\emptyset, -1, +1\}$, and recall that $\mathbb{I} = (-1, 1)$ and $\mathbf{I} = [-1, 1]$. For $\sigma \in \mathfrak{P}$, denote by $\mathbb{I}^\sigma$ the (open or half-open) interval $\mathbb{I} \cup \sigma$ (which is a subinterval of $\mathbf{I}$). Now let $\sigma = (\sigma_1, \sigma_2, ..., \sigma_k) \in \mathfrak{P}^k$ be a $\mathfrak{P}$-valued k-tuple. Denote by $\mathbb{I}^\sigma$ the 'σ-corner' obtained as the k-fold product $\mathbb{I}^{\sigma_1} \times \mathbb{I}^{\sigma_2} \times ... \times \mathbb{I}^{\sigma_k}$. ⌟

Definition 4.1.6 (Compact framed conicality)**.** Generalizing our earlier definition of framed conicality, one says (X, f) is **compact framed conical** at $x \in X$ if there exists a link $(\partial \mathbf{I}^{n-k}, l_x)$ and a framed stratified neighborhood $\phi : \mathbb{I}^\sigma \times (\mathbb{I}^{n-k}, \mathsf{cone}(l_x)) \hookrightarrow (\mathbb{I}^n, f)$ (for some $\sigma \in \mathfrak{P}^k$) such that $x \in \mathbb{I}^\sigma \times \{0\}$. If this holds for all $x \in X$, we say X is compact framed conical. The adjective 'tame' may be added as before. ⌟

Definition 4.1.7 (Compact manifold diagrams)**.** A **compact manifold n-diagram** is a tame stratification $(\mathbf{I}^n, f)$ of the closed n-cube that is tame compact framed conical. ⌟

An equivalent definition is obtained by weakening 'tame compact framed conical' to 'compact framed conical'. Another observation made in [DD22] is that 'tameness' may be replaced by 'piecewise linearity' in the compact case.

Remark 4.1.8 (PL vs tame). Recall from Example 2.3.9 that, in fact, all PL stratifications of $\mathbf{I}^n$ are tame. Therefore, given a PL stratification $(\mathbf{I}^n, f)$ that is compact framed conical then it is a compact manifold diagram. ⌟

An analogous observation allows us to replace 'tameness' by 'piecewise linearity' in the case of open manifold diagrams $(\mathbb{I}^n, f)$, but this requires a bit more care around the cube's boundaries (roughly speaking, a 'framed collar' needs to be imposed to avoided parts of the stratification running of to the cubes sides in places which are disallowed by tameness).

The relation of open and compact manifold diagrams is a bit subtle as our compact definition allows for the cube's boundary to contain non-trivial information (to take-away: really, open manifold diagrams are the way to go). One can may summarize the relation as follows.

Remark 4.1.9 (Open vs compact manifold diagrams). Every open manifold diagram $(\mathbb{I}^n, f)$ can be compactified (in a universal way!) to a compact manifold diagrams $(\mathbf{I}^n, \overline{f})$. Every compact manifold diagram $(\mathbf{I}^n, g)$ has an 'interior' stratification, but this is only an open manifold diagram $(\mathbb{I}^n, g^\circ)$ under additional conditions (again, these are 'collar-like' conditions that ensure that nothing runs off to the cube's sides). Every compactification of an open diagram satisfies these conditions, and we have $(\mathbb{I}^n, f) = (\mathbb{I}^n, \overline{f}^\circ)$. In particular, the image of the compactification operation yields a subclass of compact manifold diagrams that are in 1-to-1 correspondence with open manifold diagrams (up to framed stratified homeomorphism). ⌟

4.2 Duality of manifold diagrams and cell diagrams

Any manifold diagram can be turned into a diagram of directed cells by a process of 'geometric dualization'. We here only sketch the process; for details see [DD22, Sec. 2.4].

Remark 4.2.1 (Geometric dualization to commutative diagrams). Given a manifold n-diagram $(\mathbb{I}^n, f)$, by Theorem 2.3.7 it has a coarsest refining mesh $M \to f$. This mesh is an open mesh and thus has a dual closed mesh $M^\dagger$. The entrance path poset map $\mathsf{Entr}(M \to f) : \mathsf{Entr}(M) \to \mathsf{Entr}(f)$ dualizes to a map $\mathsf{Entr}(M \to f)^{\mathrm{op}} : \mathsf{Entr}(M^\dagger) \to \mathsf{Entr}(f)^{\mathrm{op}}$. This map determines a refinement $M^\dagger \to f^\dagger$ for some stratification $f^\dagger$ (with the same underlying space as $M^\dagger$). The framed regular cell complex, together with the stratification $f^\dagger$ that it refines, can be presented by a point-and-arrow diagram as follows: draw 0-cells in $M^\dagger$ as points, and draw k-cells c as k-arrows (the direction of the arrows is determined by the framing); draw these arrows as equalities if and only if the stratum s of $f^\dagger$ that contains the cell c contains cells c' over strictly lower dimension than c. ⌟

The geometric dualization allows use to interpret manifold diagram as 'classical' commutative diagrams of directed cells—but the kinds of cells that result from this dualization operation are of very general shape, and may be deserving of the name 'weak computadic cell shapes'. We again refer to [DD22, Sec. 2.4] for further discussion.

4.3 Tame tangles

Next let us define tangles in the setting of framed combinatorial topology. In their simplest form, tangles are just m-manifolds embedded in some space X. First note that any such embedding $W \hookrightarrow X$ determines a stratification of X whose strata are the connected components of W and the connected components of the complement $X \setminus W$. In the following, we will always tacitly interpret the given embeddings as stratifications in this way. Let us denote by S^d the d-dimensional topological sphere, by $\mathbb{I}^n = (-1,1)^n$ the open n-cube, and by $\mathbf{I}^n = [-1,1]^n$ the closed n-cube.

Definition 4.3.1 (Framed transversal stratifications)**.** Let W be a topological m-manifold and X a flat framed space $X \hookrightarrow \mathbb{R}^n$. A stratification $f : W \hookrightarrow X$ is said to be **framed transversal at** $x \in W$ if there is $k \leq m$, a cubical link $l_x = (S^{m-k} \hookrightarrow \partial \mathbf{I}^{n-k})$, and a framed stratified neighborhood $\phi : \mathbb{I}^k \times \mathsf{cone}(l_x) \hookrightarrow f$ such that $x \in \mathbb{I}^k \times \{0\}$ (where 0 is the cone point of the framed cone $\mathsf{cone}(l_x)$). If the link can be chosen tame, and the framed stratified map ϕ can be chosen to be a tame map, then we say f is **tame framed transversal at** x. We say (X, f) is **(tame) framed transversal** if it is (tame) framed transversal at all $x \in X$. ⌟

Definition 4.3.2 (Tame tangles)**.** A **tame m-tangle in dimension** n is a tame stratification $W \hookrightarrow \mathbb{I}^n$ of the open n-cube $\mathbb{I}^n$ by embedding an m-manifold W, which is tame framed transversal. ⌟

In [DD22, Sec. 3.1] it is claimed that this definition is equivalent to defining tame tangles as tame stratifications that are framed conical (thus omitting the requirement of links being tame).

Example 4.3.3 (Recovering the case of ordinary tangles)**.** The definition recovers the case of ordinary 1-tangles in the 3-cube (i.e. unions of 1-manifolds suspended at their boundaries on opposing sides of the 3-cube), *except* that tameness enforces that there are *finitely many braid crossings* under the projection $\mathbb{I}^3 \to \mathbb{I}^2$, and that each 1-manifold has *finitely many 'handle singularities'*, i.e. points at which the projection $\mathbb{I}^3 \to \mathbb{I}$ restricted to that 1-manifold is not a local homeomorphisms. ⌟

Remark 4.3.4 (Compact tame tangles). As in the case of manifold diagrams, any tame tangle $W \hookrightarrow \mathbb{I}^n$ an be compactified, in a universal way, to a stratification $\overline{W} \hookrightarrow \mathbf{I}^n$. The resulting 'compact tame tangle' can be understood to embed a 'manifold with corners' $\overline{W}$ in the closed n-cube $\mathbf{I}^n$. Such compact tame tangles can also be described more directly, by replacing the open cube $\mathbb{I}^n$ with corner cubes as in Section 4.1. ⌟

Again analogous to the case of manifold diagrams, tame tangles have a combinatorial counterpart—we make this a remark, which however has an important novelty that may be of particular interest to the reader familiar with the intricacies of the computability theory of manifolds.

Remark 4.3.5 (Combinatorial classifications). Framed homeomorphism classes of tame tangles are classified by certain normalized stratified open trusses, see [DD22, Sec. 3.1] for details. However, this combinatorialization is no longer 'constructive'; that is, the precise class of normalized stratified open trusses referred to here is not algortihmically describable. The reason for this lies in the fundamental fact that it is provably impossible to algorithmically detect manifolds (whether manifolds are presented as simplicial complexes, algebraic sets, or as handlebodies using 1-Morse functions). Interestingly, [DD22] outlines a path towards a resolution: if we can constructively understand the perturbation stable singularities of tangles (i.e. higher analogs of 'saddle points', 'cusps', etc.) which, as we touch upon in the next section, becomes a *combinatorial* question in the setting of tame tangles, then this would allow us to understand the notion of tame tangles in constructive combinatorial terms as well. We return to this point at the end of the next section. ⌟

4.4 Higher Morse and singularity theory

Earlier in this document we claimed that tame tangles are, 'in essence, tangles which admit a finite stratification by their critical value types'. Example 4.3.3 illustrated this claim in the case of 1-tangles in dimension 3: in this case, tameness guarantees that there were only finitely many braid crossings and handle singularities. In general dimensions, the claim can be intuitively understood by regarding *coarsest* refining meshes as a framed-topological representation of critical values of tame stratifications. In fact, this intuition can be exploited to study 'singularities of functions on manifolds' from a fully combinatorial perspective. Some of the details of this approach can be found in [DD22, Sec. 3.4]. We here only recall the connection to Morse, Morse-Cerf, and n-Morse theory. The discussion is purely heuristic, and mostly not formal at all.

To begin, let us understand how the notion of framing used in framed combinatorial topology (see Remark 2.1.1) is conceptually related to the idea of higher Morse functions. A Morse function of a smooth manifold M is in particular a map $f : M \to \mathbb{R}$. A 2-Morse function, or Morse-Cerf function, is in particular a path of functions $f_t : M \to \mathbb{R}$, $t \in \mathbb{R}$, which fail to be Morse functions only for finitely many 'critical values' $t_i \in \mathbb{R}$. Equivalently, such a family f_t can be encoded in a single map $f : \mathbb{R} \times M \to \mathbb{R}^2$ mapping (x, t) to (t, f_t). Yet more abstractly, one may consider maps $f : W \to \mathbb{R}^2$ on some manifold W with the property that, for all but finitely many $t \in \mathbb{R}$, the restricted map $f|_t : f^{-1}(\{t\} \times \mathbb{R}) \to \mathbb{R}$ is a Morse function on a manifold $M_t = f^{-1}(\{t\} \times \mathbb{R})$. This now inductively generalizes. An n-Morse function should in particular be a map $f : W \to \mathbb{R}^n$ such that, for all but finitely many critical values $t \in \mathbb{R}$, the restriction $f|_t : f^{-1}(\{t\} \times \mathbb{R}^{n-1}) \to \mathbb{R}^{n-1}$ is an $(n-1)$-Morse function. Of course, one would want to impose further properties for the definition of an actual n-Morse function in order to guarantee for it to be 'generic' in an appropriate sense (in classical differential terms this turns out to be problematic, since singularities with more than 5 parameters are generally unstable; this is one of the advantages of a combinatorial approach, see [DD22, Sec. 3.4.1]).

In comparison, in framed combinatorial topology we study tame tangles $W \hookrightarrow \mathbb{R}^n$. Such tangles can be canonically refined by coarsest mesh M as shown by our earlier Theorem 2.3.7.

The tower of projections $W \hookrightarrow M_n \to M_{n-1} \to ... \to M_1$ provides a framed-topological counterpart to the tower $\mathbb{R}^n \to \mathbb{R}^{n-1} \to ... \to \mathbb{R}^1$ that arose inductively in our discussion of n-Morse functions above. In particular, 0-strata of the 1-mesh M_1 record precisely the 'critical values' t, on which W may not restrict to a tame tangle itself. To get to the bottom line, this turns out to be a great framework for starting to think about higher Morse functions from a combinatorial perspective. The approach reproduces familiar classical smooth singularities in low dimensions.[5]

There is, potentially, even a bit more to the story, as higher Morse function may genuinely capture more information than their 1-Morse counterpart. Namely, 1-Morse singularities (i.e. 'handles') fail detect smooth structures: for instance, exotic smooth spheres can be given the same handlebodies as standard smooth spheres. We claim that this very likely changes when studying higher Morse singularities (and an argument for this has been sketched in [DD22, Sec. 3.4.4]; at the root of the argument is an induction by dimension using that at the 'boundary of each tame m-tangle singularity there is a tame $(m-1)$-tangle'). Using framed topology of tame tangles as the underlying set-up for higher Morse functions as outlined above, one way of phrasing the conjecture is the following. (Let us call a tame tangle $W \hookrightarrow \mathbb{I}^n$ 'smooth' if W is smooth and its embedding into $\mathbb{I}^n$ is smooth.)

Conjecture 4.4.1 (Presenting smooth structures combinatorially)**.** *If two tame smooth tangles are* framed homeomorphic *as stratifications then their tangle manifolds are* diffeomorphic *as manifolds.*

Due to the combinatorializability of the framed-topological set-up, the conjecture would imply a sort of 'combinatorialization of smooth structures': namely, it would become possible to present (compact) smooth structures on manifolds by the framed homeomorphism classes of a tame tangles, and thus by finite combinatorial data. Note also that the conjecture can be sharpened quite substantially: instead of all tame tangles we may only want to work with the 'generic' ones, i.e. those that are 'perturbation stably' embedded in $\mathbb{I}^n$ (and, arguably, the term 'Morse' should only really be used in this case!). It might be even possible to list all the singularities that such generic tame tangles may contain (such as saddles, cusps, swallowtails, etc.), which would yield constructive combinatorial foundations for the study of generic tame tangles.

5 Future work

Framed combinatorial topology aims lay the 'right' foundations for many phenomena at the intersection of higher algebra, stratified topology, and singularity theory. There are several long-term goals of the program, which can be roughly summarized into two directions as follows.

[5]Note, the idea is also suitable to study maps $W \to \mathbb{R}^n$ which are not embeddings, but depending on the details of the chosen approach this may involve some semi-combinatorial structures.

- *Manifold theory*: framed combinatorial topology is meant to link combinatorics and smooth structures. In particular, it enables a combinatorial approach to studying singularities. This approch is still in its infancy, but, on the horizon, lies a better understanding of higher Morse theory and potentially new tools for our understanding of exotic smooth structures.
- *Higher algebra*: framed combinatorial topology links stratified manifolds and higher category theory, providing geometric models of the latter in the former (in particular, interpreting coherences as isotopies). In the future, this may provide new (constructive combinatorial) insights into other deep geometric-categorical questions, such as cobordism hypothesis.

(While work towards both of these goals is underway, the authors do regard the program as an 'all-hands-on-deck' type of situation!)

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