Geometric Higher Categories

TALK NOTES
HAMBURG, 10.01.2023

Outline:

0. Why "geometric"?
1. Stratifications
   - 1.1 As a higher structure
   - 1.2 Basic def's
   - 1.3 Fundamental categories
2. Local geometric models
   - 2.1 Directed space
   - 2.2 Def's
   - 2.3. Examples
3. Combinatorial classification
4. Conclusion

References:

[FCT] "Framed combinatorial topology"

[HA] "Higher Algebra"

[MT] "Manifold diagrams and tame tangles"

[GH] "A geometric approach to homology theories"

"verbal remarks" have been added to these notes in red

0. Why "geometric"?

Higher structures

models of

(Local) Geometric (manifold-like)		Topological (continuous)	Algebraic / Combinatorial (discrete)	
[Stratified] cobordism cf. [GH] §VIII	$\Omega^{fr}_k(\mathbb{R}^n) \simeq \pi_n(S^{n+k})$ (*)	Spaces (CW complexes)	∞-groupoids (Kan complexes)	homotopy types (= types)
specifically, I mean "tangles"		1-directed spaces (d-spaces?)	$(\infty,1)$-categories (quasicategories)	1-directed types
		⋮	⋮	
manifold and tangle diagrams		n-dir. spaces (??)	(∞,n)-categories (...)	n-directed types ↓ simplify
manifold n-diagrams (☆)		n-truncated n-dir spaces	n-categories	n-directed n-types

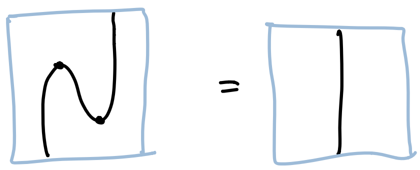
(*) translation is a combination of dualization + globalization

Purpose of talk:

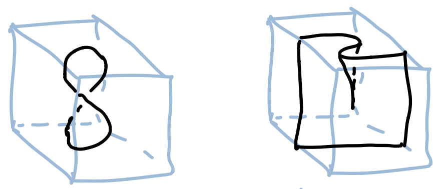
1. show (★) have simple geometric def.!
2. canonical <u>discrete</u> classification!

<u>Examples</u>: some familiar instances of manifold diagrams

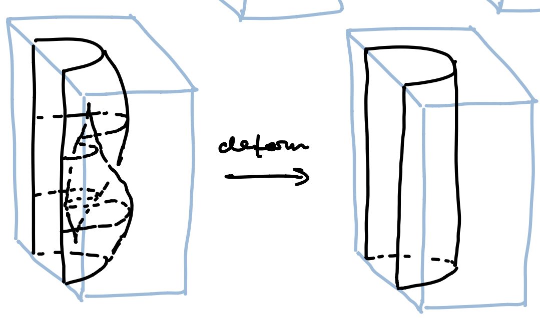
→ <u>2d</u>: string diagrams



→ <u>3d</u>: knot & surface diagrams



→ <u>4d</u>: "surface movies"



→ <u>nd</u>: "higher singularities"

$$A_k \quad , \quad D_k \quad , \quad E_{6,7,8}$$

1. Stratifications

1.1. As a higher structure

(stratifications are topological model for a type of higher structure)

truncation (of invertible structure)	higher structure	presentation
Sets	Spaces ("∞-sets" = ∞-groupoids)	sets w/ we (~~groupoids~~)

posets	stratifications ("∞-posets" = ∞-categories w/ conservative functor to poset)	posets w/ we
category	∞ - categories	categories w/ we

take home message: stratifications model ∞-posets $\simeq$ $(\infty,0,1)$-categories

meta-comment:
[posets $\equiv$ types enriched in (-1)-types $\equiv$ cat. 0-types
$\rightsquigarrow$ strat. are "space-like"]

1.2 Definition

idea: a stratified space is a space "decomposed into strata"

Defn.: A stratification is

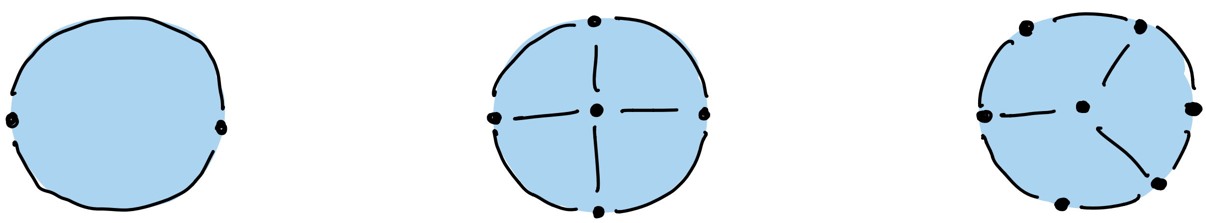
(A) [HA] a space X w/ a continuous $f : X \to P$

$\to$ (B) [FCT] a space X w/ a "conservative", essentially surj. $f : X \to P$

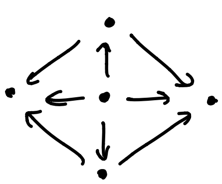
(topologically: quotient w/ connected preimages
categorically: usual meaning)

forces P to be the truncation of Lurie's fundamental category.
Also consider special case of trivial strat. (B) $\Rightarrow$ $P = \pi_0 X$

E.g. strat. of the closed 2-disk:



$\downarrow f$

P 

note: all of these examples are strat. 0-types

Obs: analoguous to:

$$\mathrm{Set} \xrightarrow{|-|} \mathrm{Spaces} \xrightarrow{\pi_0} \mathrm{Set},$$

there is:

$$\mathrm{Pos} \xrightarrow{\|-\|} \mathrm{Strat} \xrightarrow[\text{(Exit)}]{\mathrm{Entr}} \mathrm{Pos}.$$

(Exit) ← -- dual convention, see [HA]

Strat. geom. realization

$$P \longmapsto (|P|,\ |P| \to P)$$

$$x \longmapsto p$$

if x lies in a simplex starting with vertex p

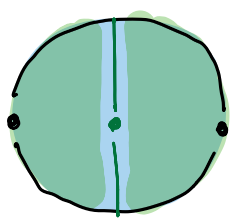
entrance path poset

$$(X,\ f: X \to P) \longmapsto P$$

... from this being a functor you can guess what stratified maps are!

Rmk: $\|P\|$ and $\|P^{op}\|$ are dual stratifications

e.g.



1.3 Fundamental categories (≡ the analogue of Π_∞ for stratifications)

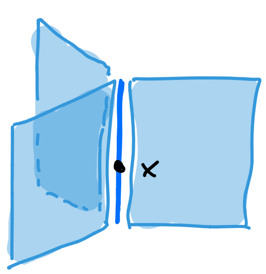
Obs.: for fundamental cat. to exist, we've missed a condition! (see [HA])

Defn.: [HA] A strat. is conical if it locally looks like:

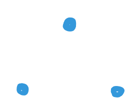
$$Z \times \mathrm{cone}(\text{link-strat})$$

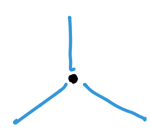
"tangential data" "normal data"

e.g.:



at x:

link-strat = 

cone(link-strat) = 

$Z = \mathbb{R}$

not conical at this point

2. Local geometric models (of n-directed n-types) (thus, manifold diagrams)

Slogan: manifold diagrams = conical stratifications of n-directed space

2.1 Directed space or, for us, framed space

"n-framing = choice of n tangential directions at each point"

Obs (motivational): inner product vspace V. TFAE

(A) orthonormal basis $(v_1, v_2, \dots, v_n)$ of V

(B) surj. $V_i \to V_{i-1}$ of oriented V_i w/ $\dim V_i = i$, $V_n = V$

(analogous to: (A) Morse vfield ∇f (B) $f : M \to \mathbb{R}$)

(B) allows for "inner-product-free" orthonormal frames

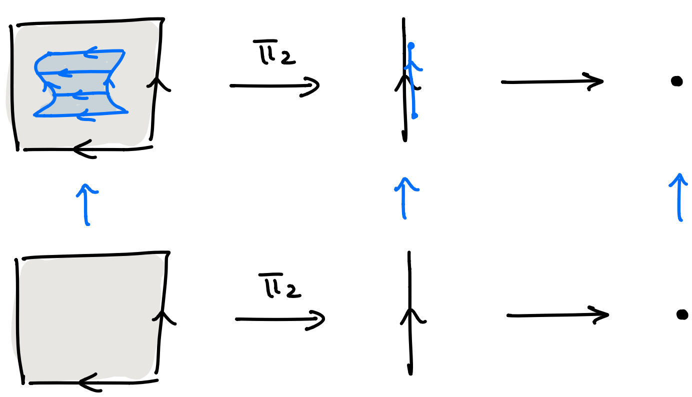
Defn: A map $F : \mathbb{R}^n \to \mathbb{R}^n$ preserves the standard framing of $\mathbb{R}^n$ ($\equiv$ F is framed) if it factors as

$$
\begin{tikzcd}
\mathbb{R}^n \arrow[r, "\pi_n"] \arrow[d, "F = F_n"'] & \mathbb{R}^{n-1} \arrow[r, "\pi_{n-1}"] \arrow[d, "F_{n-1}"'] & \cdots \arrow[r, "\pi_2"] & \mathbb{R}^1 \arrow[r, "\pi_1"] \arrow[d, "F_1"'] & \mathbb{R}^0 \arrow[d, "F_0"] \\
\mathbb{R}^n \arrow[r, "\pi_n"] & \mathbb{R}^{n-1} \arrow[r, "\pi_{n-1}"] & \cdots \arrow[r, "\pi_2"] & \mathbb{R}^1 \arrow[r, "\pi_1"] & \mathbb{R}^0
\end{tikzcd}
$$

(where π_i projects $\mathbb{R}^i \cong \mathbb{R}^{i-1} \times \mathbb{R}$ to $\mathbb{R}^{i-1}$)

Such that F_i preserves orientation of $\mathbb{R}$-fibers.

e.g. $n = 2$



Ex.: $f : \mathbb{R}^k \times \mathbb{R}^{n-k} \longrightarrow \mathbb{R}^n$ framed $\not\Rightarrow$ $f \circ s_{k,n-k}$ framed

$$\begin{array}{c} \mathbb{R}^k \times \mathbb{R}^{n-k} \\ s \big\downarrow \; s_{k,n-k} \\ \mathbb{R}^{n-k} \times \mathbb{R}^k \end{array}$$

the order of components matters!

Ex.: $\mathrm{Aut}_{fr}(\mathbb{R}^n)$ is contractible

this should hold for other notions of framings as well!

2.2. Definition

Defn: A manifold n-diagram $(\mathbb{R}^n, f)$ is a strat. of framed $\mathbb{R}^n$ that locally looks like

$$\mathbb{R}^k \times \mathrm{Cone}(f)$$

Rmk: omitted a "finiteness" (or "tameness" [MT]) condition

→ f has framed compact support iff

∃ finite triangulation of $\mathbb{R}^n$ that refines f by a framed homeorphism

allow vertices at ∞ :

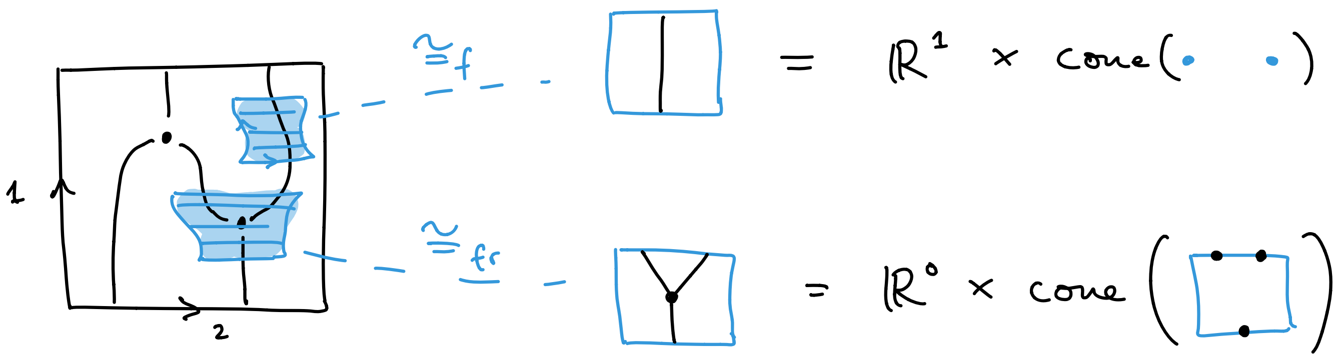
Rmk.: A bit more convenient to work with the open cube $(-1,1)^n$ ($\cong_{fr} \mathbb{R}^n$) for the purpose of "framed compactification" (yielding strat. of the closed cube $[-1,1]^n$)

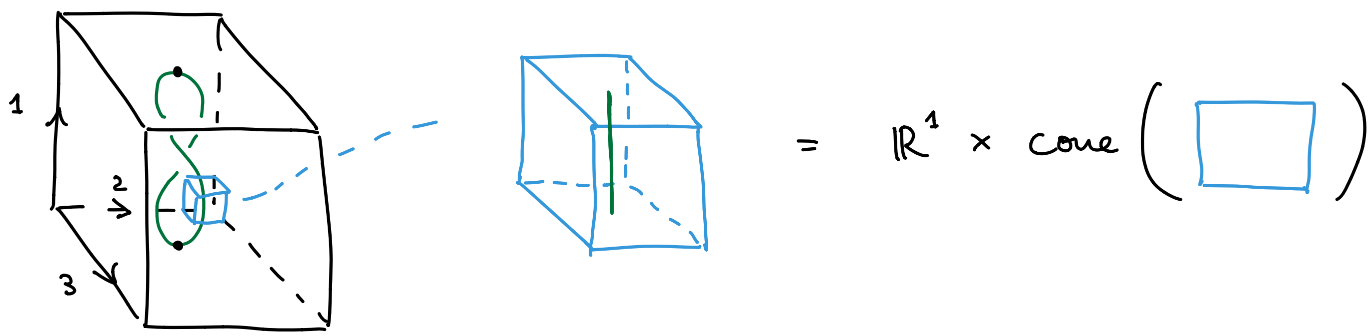
Defn.: An m-tangle n-diagram $(\mathbb{R}^n, f)$ is a strat of framed $\mathbb{R}^n$ locally looking like

$$\mathbb{R}^k \times \widetilde{\text{cone}}\left(S^{m-k-1} \hookrightarrow S^{n-k-1}\right)$$

↳ cone not separate stratum (part of lowest-dim stratum)

2.3 Examples





Ex.: Convince yourself that the Reidemeister moves are manifold 4-diagrams

3. Combinatorial classification

these are interesting

Slogan: $\exists$ universal framed subdiv. by strat 0-types

(for strat. w/ framed compact supp.)

Thm.: Given a mfld. diag. f (or tangle f), there exists a universal (namely, coarsest) framed subdiv. by a strat 0-type $(\mathbb{R}^n, M)$ fitting into a tower of bundles

$$(\mathbb{R}^n, M = M_n) \xrightarrow{p_n} (\mathbb{R}^{n-1}, M_{n-1}) \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_1} (\mathbb{R}^0, M_0)$$

such that $\mathrm{Entr}(p_i)$ is

* exponentiable ($\equiv$ fibers over mor. are profunctors),
* an opfibration on fiber-dimension-0 objects, ($\equiv$ fibers are functors)
* a fibration on fiber-dimension-1 objects. ($\equiv$ fibers are cofunctors)

... these conditions lead to a rich combinatorial theory, on which I wrote a 500-page thesis :)

Term.:

- a tower like $\{p_i\}$ is called an n-mesh
- a tower like $\{\mathrm{Entr}(p_i)\}$ is called an n-truss

Main Obs: mfld. diag.'s (resp. tangles) are classified by certain stratified n-trusses!

stratifications of finite posets (considered as Alexandroff spaces) are finite data.

4. Conclusion

Pay-offs:

- formalization of directed TP
- finite combinatorial representation of (compact) smooth structures
- Combinatorial approach to singularity theory

(vague)
questions:

- Where is the 4D weirdness hiding?
- Classification of singularities? ADE?
  (⇝ Enumerability of manifolds)
- Comparison to other (∞, n)-defs?