Nine short stories about geometric higher categories

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Abstract

We introduce and discuss basic concepts from the emerging area of geometric higher category theory. The exposition aims to be brief and informative but mathematically as self-contained as possible (some familiarity with higher category theory is presumed). Several small exercises, with web-linked solutions, have been included, and plenty of open questions as well as future research directions in the area are discussed.

Contents

Introduction

While the term "geometric higher category" is new, its underlying idea is not: coherences in higher structures can be derived from (stratified) manifold topology. This idea is central to the cobordism hypothesis (and to the relation of manifold singularities and dualizability structures as previously discussed on the n-Café), as well as to many other parts of modern Quantum Topology. So far, however, this close relation of manifold theory and higher category theory hasn't been fully worked out—the theory of geometric higher categories aims to change that, and my goal is to explain the "*how*" in a series of nine short stories.

After a short introduction (*story 1*), the first part of the series (*story 2 to 6*) will explore how manifold stratifications can be made to work in directed space. In doing so, we will essentially spell out what are the 'local models' of geometric higher categories. In the second part of the series (*story 7 to 9*), we will then revisit the same ideas from a global perspective. I should remark that none of the material in these stories is new, all mistakes are mine,

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[*] Please report errors and typos to my current email.

and that an honest attempt was made to improve presentation and accessibility of it (also see the conclusion for further remarks).

Disclaimer. What this work, despite its title, does not contain is an actual, fully general definition of geometric higher categories (only the case of *free* geometric higher categories will be discussed). While such definitions have been given (see for instance [6]), the details of a workable theory of geometric higher categories are still under development at this point in time. The aim here is to provide enough material on the subject in order for the pathway to a general theory becoming a very plausible option in the reader's mind.

Read this first. Originally, this text was envisioned to be a blog post and so in most parts it is written in a rather chatty style. In fact, a (shortened and condensed but much more colorfully illustrated) version of this document did appear as a blog post here—it might be worth reading that article first, and then returning here (and skipping story 1 below).

1 What is "geometric" about geometric higher categories?

I would like to argue that there is a useful categorization of models of higher structures into three categories, but really I will only give one good example of this. The absence of other examples, however, can be taken as a problem that needs to be addressed, and as one of the motivations for studying geometric higher categories! The three categories of models that I want to consider are "geometric", "topological" and "combinatorial" models of higher structures. Really, depending on your taste, different adjectives could have been chosen for these categories: for instance, in place of "combinatorial", maybe you find that the adjectives "categorical" or "algebraic" are more applicable for what is to follow; and in place of "geometric", maybe saying "manifold-stratified" would have been more descriptive. As a further aside, note that I prefer to think of manifolds as being smooth (at least for the purposes of examples and illustrations). In fact, in the presence of framings, smooth manifolds become 'equivalent' to piecewise-linear manifolds, and this will later allow us to formally recover smoothness in several instances.

Let's get to the promised example of how these three categories of models work and how they relate. We start with the archetypical model of a type of higher structure: namely, with topological spaces. Unsurprisingly, topological spaces fall firmly into the category of topological models. The type of higher structure modelled by topological spaces deserves a name: we will refer to it as homotopy types. There is a second well-known model for homotopy types, namely, ∞-groupoids. Unlike spaces, whose theory is based on the continuum $\mathbb{R}^n$, ∞-groupoids are discrete structures whose data is captured by collections of morphisms in each dimension $k \in \mathbb{N}$. This makes ∞-groupoids a prime example of a 'combinatorial' (or, 'algebraic', or, 'categorical') model of a higher structure. (I should point out that I am being vague about concrete definitions of the above named 'models'; for instance, 'spaces' could be, more concretely, taken to mean CW complexes, and ∞-groupoids could be taken to mean Kan complexes.) Despite being rather different in flavour, the higher theory (i.e. the 'homotopy theory') of topological spaces and the higher theory of ∞-groupoids turn out to be equivalent. The two models are related by two important constructions: we can pass from spaces to ∞-groupoids by taking the fundamental categories of spaces (usually referred as their 'fundamental ∞-groupoids'), and, conversely, we can realize ∞-groupoids as spaces.

Now the interesting part: what is the geometric counterpart to the above topological and combinatorial models? In different words, how can we understand the homotopy theory of spaces in terms of a theory of manifold-stratified structures? The answer is given by the theory of cobordisms, or more precisely, *stratified* cobordism. The most well-known instance of how cobordisms and spaces relate in this sense is the classical Pontryagin theorem. The theorem describes the isomorphism $\Omega^{\mathrm{fr}}_m(\mathbb{R}^n) \cong \pi_n(S^{n-m})$ between the cobordism group of smooth (normal) framed m-manifolds in $\mathbb{R}^n$ and the nth homotopy group of the $(n-m)$-sphere. The resulting relation between smooth manifold theory and homotopy theory is incredibly ubiquitous in modern Algebraic Topology (and, relatedly, in Physics) but often implicitly so—in the words of Mike Hopkins, Pontryagin's theorem itself marks the point in time at which Algebraic Topology became 'modern'.

Importantly for us, the theorem generalizes from spheres to arbitrary spaces (or, more precisely, CW complexes): namely, the nth homotopy group of any space can be understood in terms of the framed stratified cobordisms group of certain manifold *stratifications* of $\mathbb{R}^n$ whose singularity types depend on X. The details of this generalization are spelled out in [1, §VII], in a chapter titled "the geometry of CW complexes", but really the basic idea of the construction remains essentially the same (instead of working with regular points of the sphere S^{n-m}, we work with regular 'dual stratifications' of CW complexes). To summarize, we can study the homotopy groups of spaces, or, in combinatorial terms, the higher morphisms in ∞-groupoids, by means of stratified cobordisms in $\mathbb{R}^n$. The relation between the geometry of stratified cobordisms and the homotopy theory of spaces can be thought of as a two-step process of 'dualizing globalization': this comprises *dualization* (which translates stratification data back into cells as we will explain) and *globalization* (which, glues cells to obtain spaces, or, in combinatorial terms, switches from the perspective of individual morphisms back to the 'global' category that these morphisms define).

The main point I now want to make is that the trilogy of geometric, topological, and combinatorial models exemplified above in the case of homotopy types should also extend to other types of higher structures. In particular, $(\infty, 1)$-categories and (∞, n)-categories should admit both topological models and geometric models—however, these classes of models haven't been much explored so far (remark: in the $(\infty, 1)$-case, there is the theory of d-spaces which I will count as an existing topological counterpart to $(\infty, 1)$-categories even though I'm unsure how much this relation has been formally explored). The situation is summarized in Table 1 below, in which we have filled in precisely some of the missing entries for geometric and topological models of higher structures: to indicate their conceptual nature, names of these models have been kept in quotes. At a first glance, it seems like finding concrete definitions for these models would be a tall order: after all, the theory of stratified cobordisms itself has its mathematical depths, and realizing the step from ∞-groupoids (say, Kan complexes) to concrete definitions of (∞, n)-categories is not necessarily an obvious one.

Now the revelation: in the course of this series of stories, I hope to convince you that the geometric models of higher structures in fact *get easier* when passing to the directed setting of n- or (∞, n)-categories. Moreover, studying the undirected setting (i.e., in combinatorial terms, the case of higher structures with *invertible* morphisms) from the perspective of directed geometric models provides a refined view on the computational intricacies of invertibility as we will learn later.

Of course, in order to be able to tell this story, we will discuss concrete notions that aim to realize precisely the geometric models outlined above. The notions go by the names *manifold diagrams* resp. *tangle diagrams* (the latter being the relevant notion in the undirected case). Both terms have been added to Table 1 in their respective places (note that the (∞, n) case requires us to deal with manifold diagrams in and below dimension n, and with tangle diagrams above dimension n). Importantly, note that I wrote that these notions 'aim to realize', instead of just 'realize', the aforementioned geometric models. Indeed, how do we measure 'correctness' of our definitions of manifold diagrams? Unfortunately, no reasonably-straight-forward benchmark for models of directed geometry exists at this point (actually, the same was true for manifolds when they were discovered, but they were quickly accepted due to their ubiquity in mathematics). Certainly, passing from geometry to combinatorics in Table 1, a comparison to existing models of (∞, n)-categories would provide such a benchmark, but it also requires substantial work (nonetheless, Lukas Heidemann is a PhD student at Oxford who is actively thinking about this question at the time of this writing!). For today, instead of such comparisons, I would like to argue that the geometric models of higher structures deserve your interest for the following set of more elementary reasons.

Geometric Models	Topological Models	Combinatorial Models	Higher structures
(stratified) cobordism	spaces	∞-groupoids	homotopy types
"1-directed cobordism"	1-directed spaces	$(\infty, 1)$-categories	1-directed types
$\vdots$	$\vdots$	$\vdots$	$\vdots$
"n-directed cobordism" *manifold diagrams (dim $\leq n$)* *& tangle diagrams (dim $> n$)*	"n-directed spaces"	(∞, n)-categories	n-directed types
"n-directed n-truncated cobordism" *manifold diagrams*	"n-directed n-truncated spaces" *n-framed spaces*	n-categories *geometric n-categories*	n-directed n-types

dualization combinatorialization

Table 1: Categorizing models of higher structures (with "missing entries", *concrete definitions* and *derived notions*)

1. *Simplicity and ubiquity.* Firstly, the definitions of manifold and tangle diagrams are simultaneously simple and expressive: both definitions succeed in encompassing large classes of known examples, including ordinary string diagrams and surface diagrams, knot and surface-knot diagrams, as well as their respective moves (Reidemeister move and 'movie moves'), and smooth manifold singularities such as Arnold's ADE singularities.
2. *Trilogy of models.* Secondly, and this is a central part of the story, both manifold and tangle diagrams have *canonical* topological and combinatorial representations. This enables a powerful translation between all three columns of Table 1: in the topological column this translation results in a local model of directed space termed *framed spaces*, and in the combinatorial column it yields, you guessed it, *geometric higher categories*. In this combinatorial model, higher-categorical coherences are then naturally related to stratified manifold isotopies!
3. *Application.* Thirdly and lastly, even without higher structures as our primary object of study, manifold and tangle diagrams provide a new tool at the interface of combinatorial higher algebra and differential topology. This leads to interesting questions such as the precise nature of the relation between diagram combinatorics and differential singularities (which I will briefly return to later), and a potentially 'natural' approach to combinatorial encodings of smooth structures.

This completes my ramblings about geometric higher categories, about what makes them 'geometric', and about why they might be interesting. In what follows, I shall try to explain parts of the above story in more mathematical detail!

I Local models

2 Stratifications from a higher-categorical perspective

Just as the top-left entry of Table 1 reads '*stratified* cobordisms', the theory of manifold and tangle diagrams will require us to deal with not just manifolds but stratifications of manifolds. We therefore start the mathematical part of this series by recalling a few facts about stratifications... from a higher-categorical perspective!

At a basic level, stratifications can be thought of as decompositions of topological spaces into disjoint subspaces which are then referred to as 'strata'. However, to a category theorist, one can easily convey much more detail

Directed base structure	Higher structure	Presentation
sets	spaces $\simeq$ “∞-sets” a.k.a. ∞-groupoids	“sets with w.e."
posets	*stratified spaces* $\simeq$ “∞-posets”	posets with w.e.
categories	1-directed spaces $\simeq$ ∞-categories	categories with w.e.

Table 2: Locating stratifications in the landscape of higher structures

about what stratified spaces are: namely, they are a topological model for a type of higher structure which lives ‘between’ types (i.e. homotopy types) and 1-directed types. The correct set of analogies is summarized in Table 2, which warrants a bit more explanation as follows. In the central column we list types of higher structures: in each case we do so by naming both a topological and a combinatorial model (terms are put in quotes if they aren't standard terminology). In the left column, we give the corresponding (combinatorial) structure obtained after truncating all invertible morphisms: in other words, truncating ∞-X's to X's in this way (where X could stand for ‘set’, ‘poset’, ‘category’, or, more generally, (n,r)-category for $n \leq r \leq n+1$) is to say that ∞-X's are precisely the subclass of (∞,∞)-categories $\mathscr{C}$ whose homotopy (n,r)-truncations[1] $\tau_{(n,r)}\mathscr{C}$ are X's such that the induced functor $\mathscr{C} \to \tau_{\leq r}\mathscr{C}$ is conservative. Finally, the right column (which has been added only for completeness and may otherwise be ignored) describes how we can use structures from the left column to present structures in the middle column: here, a “set with weak equivalences” is simply a poset with weak equivalences in which each arrow is marked as a weak equivalence. The analogies in Table 2 should be taken seriously (but not *too* seriously): many constructions relating sets and spaces have analogs for posets and stratifications as we will now see.

Inspired by the above discussion, let us define stratifications ‘by analogy with spaces’ as follows. First, make the redundant observation that a topological space is a topological space X together with a quotient map $f : X \to \mathscr{E}_0X$ to a discrete space $\mathscr{E}_0X$ such that preimages of f are connected (of course, these conditions on f force $\mathscr{E}_0X$ to simply be the set π_0X of connected components of X). By analogy, a *stratified space* is a topological space X together with a quotient map $f : X \to \mathscr{E}_0X$ to a Kolmogorov-Alexandroff space $\mathscr{E}_0X$ whose preimages are connected. Recall, Kolmogorov-Alexandroff spaces are spaces in the image of the fully-faithful embedding of posets into topological spaces (by declaring downward closed subset of a given poset to be open; *N.B.*: another convention would be to choose upward closed subsets as open sets). On sets this embedding maps to discrete spaces, and so our definition of stratifications generalizes that of topological spaces!

For a stratification (X, f), we call $f : X \to \mathscr{E}_0X$ the *characteristic map* and the preimages of f the *strata* of (X, f) (of course, if a stratification (X, f) is a space, then the map f is redundant and can be dropped from our notation). Observe that f determines the strata, and, conversely, the strata determine the characteristic map f up to changing $\mathrm{cod}(f) = \mathscr{E}_0X$ by a canonical poset isomorphism. We will say a stratification is *finite* if it has only finitely many strata. For simplicity, let's assume all our stratifications (and posets) to be finite by default! (At least locally, i.e. in small neighborhoods, finiteness is usually a very reasonable assumption.)

As an aside, let me note that our definition of stratified spaces above differs slightly from the definition given by Lurie in [2, App. A]: there, a stratification is defined simply as a space X with a continuous map $f : X \to P$ to some poset P (considered as an Alexandroff space). While the definitions are essentially interchangeable, I much prefer the above definition. One reason is that an arbitrary choice for P can contain loads of additional data that has nothing to with the decomposition of X into strata (this can get a bit in the way when defining maps for instance): the situation becomes more pronounced in the case of spaces, in which the definition would specialize

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[1] A ‘homotopy truncation’, in essence, truncates higher structure by turning equivalences into equalities.

to 'a space is a space with a map to some set'—and this changes the definition of spaces by endowing them with additional structure. A second reason has to do with fundamental categories... we will see it shortly.

There are a few immediate (1-)categorical constructions that can be given. First, the category **Strat** of stratified spaces consists of stratified spaces and *stratified maps* $(X, f) \to (Y, g)$ which are maps $X \to Y$ that factor through the characteristic maps f and g by a (necessarily unique) map $\mathscr{E}_0(F) : \mathscr{E}_0 X \to \mathscr{E}_0 Y$. There is of course a functor $\underline{\ \ } : \mathbf{Strat} \to \mathbf{Spaces}$ that forgets stratifications and passes to the underlying space. There are also functors $\mathscr{E}_0 : \mathbf{Strat} \to \mathbf{Pos}$ and $\|-\| : \mathbf{Pos} \to \mathbf{Strat}$ which are the direct analogs of the classical 'connected components' functor $\pi_0 : \mathbf{Spaces} \to \mathbf{Set}$ and the 'discrete geometric realization' functor $|-| : \mathbf{Set} \to \mathbf{Spaces}$ (more generally, we write $|-| : \mathbf{Pos} \to \mathbf{Spaces}$ for the usual geometric realization of posets). The first functor $\mathscr{E}_0$ takes a stratification (X, f) to the poset $\mathscr{E}_0 X$ which we call the 'fundamental poset' of (X, f) (as a warning: this poset will also be denoted by $\mathscr{E}_0(X, f)$, or $\mathscr{E}_0 f$, depending on the weather). The second functor $\|-\|$ is defined as follows: given a poset P, then its 'stratified realization' $\|P\|$ is the stratified space with underlying space $|P|$, $\mathscr{E}_0\|P\| = P$, and characteristic map $f : |P| \to P$ mapping $x \mapsto p$ if x lies in the interior of a simplex with first vertex p.

In further analogy with the case of spaces, we can now consider fundamental categories of stratified spaces. Given a space X its fundamental ∞-groupoid is described by the simplicial set (Kan complex) with k-simplices $|[k]| \to X$ (where $[k]$ is the poset $(0 \to 1 \to \ldots \to k)$). Given a stratification (X, f), it would be natural to instead consider the simplicial set consisting of k-simplices $\|[k]\| \to (X, f)$ (indeed, observe that this recovers the case of maps $|[k]| \to X$ if (X, f) is a space). Lurie shows in [2, App. A] that in order for this simplicial set to behave in the expected way (namely, as a higher compositional structure, or more concretely in this case, as an instance of a quasi-category), our definition of stratified spaces needs to be amended! To say it more drastically, our higher-categorically-inspired definition is missing an ingredient. And this ingredient is (drumroll) ... *conicality*.

Conicality is an ubiquitious feature when studying stratifications in the wild (for instance, all stratifications in the image of $\|-\|$ are conical), and it is quite easily defined as well. The definition will use the following 'obvious' notions of stratified neighborhoods, products and cones, which we briefly recall for completeness: a *neighborhood* $F : (C, c) \hookrightarrow (A, a)$ is a stratified map such that $F : C \hookrightarrow A$ is an ordinary neighborhood and strata of (C, c) are precisely the connected components of strata of (A, a) intersected with C; a *product* of two stratifications is defined by $(A, a) \times (B, b) = (A \times B, a \times b)$; and a *cone* $(\mathrm{Cone}(A), \mathrm{cone}(a))$ of a stratification (A, a) stratifies the topological cone $\mathrm{Cone}(A) = A \times [0, 1)/A \times \{0\}$ by the stratum $\{0\}$ (the 'cone stratum') and the strata of the product $(A, a) \times (0, 1)$. With these notions at hand, we may now define: a stratification (X, f) is *conical* if it has, around each point $x \in X$, a stratified neighborhood isomorphic to $Z \times (\mathrm{Cone}(L), \mathrm{cone}(l))$, with $x \in Z \times \{0\}$, where Z is some space and (L, l) some stratification (often referred to as a 'link' stratification). Further constraints in the definition of conicality are possible: for instance, note that if we require Z to always be $\mathbb{R}^m$ for some $m \in \mathbb{N}$ then all strata are topological manifolds!

To summarize, for conical stratifications (X, f) we can construct their fundamental categories $\mathscr{E}(X, f)$ in a natural way, namely, by considering the mappings $\|[k]\| \to (X, f)$. Observe that there is a natural functor $\mathscr{E}(X, f) \to \mathscr{E}_0(X, f)$ which is conservative and exhibits $\mathscr{E}_0$ as the truncation of $\mathscr{E}$ in the sense of Table 2. Since $\mathscr{E}(X, f)$ is also referred to as the *entrance path ∞-category* of (X, f), calling $\mathscr{E}_0(X, f)$ the *entrance path poset* is an obvious choice (*N.B.*: switch 'entrance' for 'exit' when working with dual conventions). Personally, for its consistency with the general concept of fundamental categories (and in analogy with the term 'fundamental ∞-groupoid'), I like speaking of the 'fundamental ∞-poset $\mathscr{E}(X, f)$' resp. 'fundamental poset $\mathscr{E}_0(X, f)$', which we saw appear earlier already.

This almost completes our brief review of stratifications, except for one feature: duality. As we are only interested in one simple case, we will keep things short: the *dual stratification* of $\|P\|$ is $\|P^{\mathrm{op}}\|$. Indeed, this recovers the usual intuition of dual cell complexes in the sense of Poincaré duality. For instance, given a simplicial complex K triangulating a PL manifold W, the face poset $F(K)$ (which orders faces by inclusion) of K has a dual $F(K)^{\mathrm{op}}$: while $\|F(K)^{\mathrm{op}}\|$ recovers the triangulation K of W, $\|F(K)\|$ is the cell complex (in fact, the regular cell complex) that dualizes this triangulation. (There is more too be said here: namely, regular cell complexes bijectively correspond to so-called cellular posets... but let us not get distracted too much.)

Mini-exercises 2.1 *Draw a picture of the stratified space* $\|[2]\|$. *Convince yourself that it is conical. Find a non-conical stratification. Take a your favorite regular cell complex* f *(e.g. a pasting diagram), stratify it by taking strata to be the individual cells, and compute its entrance path poset* $P := \mathscr{E}_0(f)$. *Show that* $\|P\|$ *recovers the regular cell complex* f *you started with. Compare this to* $\|P^{\mathrm{op}}\|$. *(Link to Solutions)*

3 Manifold and tangle diagrams

Recall from Table 1 that manifold diagrams ought to be some sort of directed version of stratified cobordisms. We can now make this precise. The simple (and, in fact, quite accurate) slogan for this story will be: manifold n-diagrams are compactly triangulable, conical stratifications of n-directed euclidean space. We are already familiar with the notion of conical stratifications from the previous section, so the only two things that remain to be done is to talk about n-directed space and to be a bit more explicit about what we mean by 'compact triangulability' for stratifications in directed euclidean space. *Remark:* in this section, all manifolds are topological manifolds without boundary.

We will infuse our spaces with directions by means of framings: recall, classically, a framing is something akin to a 'choice of tangential directions' at all points of a given space (usually a manifold, as otherwise it may be hard to talk about 'tangential directions'). Our use of the term 'framing' will be a somewhat non-standard variation of this idea. For motivation, we start with the observation that given a real n-dimensional inner product space V, the following two structures on V are equivalent: firstly, the choice of an orthonormal framing of V, i.e. an ordered sequence $(v_1, v_2, ..., v_n)$ with $\langle v_i, v_j \rangle = \delta_{ij}$, and, secondly, a chain of linear surjections $V_i \to V_{i-1}$ of orientied i-dimensional V_i's starting at $V_n = V$. A correspondence between these structures can be produced by defining $V_i = \mathrm{span}(v_1, ..., v_i)$ with $V_i \to V_{i-1}$ forgetting v_i. To get a bit of intuition about what's going on here, let's consider the following analogous and hopefully familiar situation: given a Riemannian manifold M there is a correspondence between (smooth) gradient vector fields on M and (smooth) functions $M \to \mathbb{R}$ up to shifting functions by a constant. (To ensure the analogy is clear: the vector field, where it is non-zero, plays the role of a 1-frame v_1, whereas the corresponding function $M \to \mathbb{R}$ plays the role of a linear surjection $V \to V_1$ of tangent spaces at these points.) So why would we want to shift perspectives from vectors to surjections in this way? The secret reason is that 'orthonormality' ceases to exist in absence of inner products (or, in the given analogy, in absence of Riemannian metrics), but the notion of linear surjections does not. Put differently, by basing our notion of framings on surjections rather than vectors we can emulate some form of orthnormality even in the absence of inner products (cf. [3, App. A]). Somehow, this is very important for the story of manifold diagrams. Let's go ahead and spell it out.

While it is possible to use the above idea to define framed spaces in quite some generality, following our slogan in the beginning we will only be interested in the euclidean case (and this case is, unsurprisingly, in some sense a 'local model' for more general framed spaces). The *standard n-framing* of $\mathbb{R}^n$ is the chain of oriented $\mathbb{R}$-fiber bundles $\pi_i : \mathbb{R}^i \to \mathbb{R}^{i-1}$ ($1 \leq i \leq n$) with π_i defined to be the map that forgets the last coordinate of $\mathbb{R}^i$ (and fibers carry the standard orientation of $\mathbb{R}$ after identifying $\mathbb{R}^i = \mathbb{R}^{i-1} \times \mathbb{R}$). When considering $\mathbb{R}^n$ we tacitly always think of it as 'standard framed $\mathbb{R}^n$' and, thus, we stop mentioning the standard framing as an explicit structure all-together. Indeed, more important than defining the standard n-framing is to define the maps that preserve it: a *framed map* $F : \mathbb{R}^n \to \mathbb{R}^n$ is a map for which there exist (necessarily unique) maps $F_j : \mathbb{R}^j \to \mathbb{R}^j$

$(0 \leq j \leq n)$ with $F_n = F$ such that $\pi_i \circ F_i = F_{i-1} \circ \pi_i$ with F_i preserving orientations of fibers of π_i (here, 'orientation preserving' can be taken to mean strictly monotonous; another version of the definition asks for non-strictly monotonous).

Mini-exercises 3.1 *For $n = 1$ and $n = 2$, write down a framed map $\mathbb{R}^n \to \mathbb{R}^n$. Also write down a map that is not framed. What about the case $n = 3$? Show that the space $\mathrm{Aut}_{\mathrm{fr}}(\mathbb{R}^n)$ of framed automorphisms is contractible. (Link to Solutions)*

We may now combine framed and stratified notions: for instance, a framed stratified map $(\mathbb{R}^n, f) \to (\mathbb{R}^n, g)$ is a stratified map whose underlying map $\mathbb{R}^n \to \mathbb{R}^n$ is framed. Moreover, when working with products $(\mathbb{R}^k, f) \times (\mathbb{R}^{n-k}, g)$ we will identitfy $\mathbb{R}^n \cong \mathbb{R}^k \times \mathbb{R}^{n-k}$ in the standard way; and, when working with cone stratifications $(\mathrm{Cone}(S^{n-1}), \mathrm{cone}(l))$, we will standard embed $S^{n-1} \hookrightarrow \mathbb{R}^n$ and identify $\mathrm{Cone}(S^{n-1}) \cong \mathbb{R}^n$ by mapping $(x \in S^{n-1}, \lambda \in [0,1))$ to $\frac{\lambda}{1-\lambda} x \in \mathbb{R}^n$. (The resulting cone stratifications of $\mathbb{R}^n$ could be called '$\mathbb{R}_{>0}$-cones': they are precisely those stratifications in which the origin $\{0\} \subset \mathbb{R}^n$ is its own stratum and all other strata are closed under multiplication by a positive scalar.) With these conventions at hand, we may now also combine framedness and conicality as follows: a stratification $(\mathbb{R}^n, f)$ is *framed conical* if each point $x \in \mathbb{R}^n$ it has a framed stratified neighborhood (framed) isomorphic to $\mathbb{R}^k \times (\mathrm{Cone}(S^{n-k-1}), \mathrm{cone}(l))$ with $x \in \mathbb{R}^k \times \{0\}$, where $0 \leq k \leq n$ and (S^{n-1}, l) is some stratification. Compare this to our earlier definition of conicality—framed conicality is really just a 'framed' version of conicality!

The last concept from our earlier slogan that needs to be addressed is the notion of compact triangulability for stratifications of (standard framed) euclidean space. To begin, let me remark that imposing a compact triangulability condition is a *reasonable* thing to do: indeed, in our earlier discussion of Pontryagin's theorem we similarly met cobordisms of manifolds embedded in $\mathbb{R}^n$ that had 'compact support'—'compact triangulability' should be thought of as a generalization of this situation (adapted to the setting of directed spaces). The condition can be succinctly formulated by starting in the PL category: a *compactly-defined triangulation* K of $\mathbb{R}^n$ is a finite stratification of $\mathbb{R}^n$ by open disks whose closures are the images of linear embeddings $\Delta^k \times \mathbb{R}^l_{\geq 0} \hookrightarrow \mathbb{R}^n$ $(k + l \leq n)$. This now translates to the framed stratified case as follows: a stratification $(\mathbb{R}^n, f)$ is *framed compactly triangulable* if it admits a framed stratified subdivision $(\mathbb{R}^n, K) \to (\mathbb{R}^n, f)$ of f by a compactly-defined triangulation K. (I believe the word subdivision is self-explanatory, but in case it is not: a stratified subdivision is a stratified map whose underlying map is a homeomorphism.)

I most happy to tell you that we have now arrived at the central notion of this series (I am also very happy that you are still reading). A *manifold n-diagram* is a stratification $(\mathbb{R}^n, f)$ that is both framed compactly triangulable and framed conical. As simple as that! Now, let us think a bit about the consequences of this definition.

Mini-exercises 3.2 *Consider the case $n = 2$. Observe the product stratification* $\mathbb{R}^2 \times (\mathrm{Cone}(S^{-1}), \mathrm{cone}(l))$ *is just* $\mathbb{R}^2$ *(since $S^{-1} = \emptyset$ we find* $\mathrm{Cone}(S^{-1}) = \{0\} = \mathbb{R}^0$*). Observe S^0 only admits one stratification. Thus, there is only one choice for the product stratification* $\mathbb{R}^1 \times (\mathrm{Cone}(S^0), \mathrm{cone}(l))$ *of* $\mathbb{R}^2$*. In contrast, observe that there are many choices of cone stratifications* $(\mathrm{Cone}(S^1), \mathrm{cone}(l))$ *of* $\mathbb{R}^2$*. Combining these observations, draw some pictures of manifold 2-diagrams and convince yourself that this case recovers ordinary string diagrams. What about the case $n = 3$? Can you come up with an example of a framed conical stratification of* $\mathbb{R}^3$ *that is not framed compactly triangulable? (Link to Solutions)*

Glancing back at Table 1, how can we describe undirected topological behaviour from the lens of directed geometry? In the spirit of Pontryagin's theorem and the cobordism hypothesis, the answer involves replacing directed manifold strata by embedded cobordisms, a.k.a. tangles. This yields a variation of the definition of manifold diagrams which we call tangle diagrams: intuitively, while manifold strata in a manifold diagrams must respect the direction of the ambient directed euclidean space, strata in tangles can (in a controlled way) fail to do so.

Let us briefly address the notion of tangle diagrams in more precise terms. To begin, note that any embedding of a manifold $W \hookrightarrow X$ into a space X defines a stratification of X whose strata are the connected components of W and the connected components of the complements $X \setminus W$—any manifold embedding will be tacitly treated as a stratification in this way. Note, given an embedding $S^k \hookrightarrow X$ we can construct the manifold embedding $\mathrm{Cone}(S^k \hookrightarrow X) = \mathrm{Cone}(S^k) \hookrightarrow \mathrm{Cone}(X)$ (since the topological cone construction is functorial). Combining this, we define: an *m-tangle n-diagram* ($m < n$) is an m-manifold embedding $W \hookrightarrow \mathbb{R}^n$ that is framed compactly triangulable, and, for all points $x \in W$, we may choose $k_x \in \mathbb{N}$ such that x has a framed stratified neighborhood isomorphic to $\mathbb{R}^{k_x} \times \mathrm{Cone}(S^{m-k_x-1} \hookrightarrow S^{n-k_x-1})$ with $x \in \mathbb{R}^{k_x} \times \{0\}$ and $k_y > k_x$ for all $y \in U \setminus \mathbb{R}^{k_x} \times \{0\}$. (Technically, we also require choices of dimensions k_x to be *maximal*: that is, the condition fails to hold for any other choice $k'_x > k_x$.)

Some further explanation of the last condition in the definition is warranted, as it contains a novel idea when compared to the earlier definition of manifold diagrams. First, note that the definition implicitly requires the k_x's to be picked inductively in the descending order for $k_x = m, m-1, ..., 0$ for all $x \in \mathbb{R}^n$: indeed, we must have picked all k_y's before k_x in order for the last condition to make sense! In [4, §3] we called k_x the 'transversal dimension' of W at x. Intuitively, when everything is smooth, you may think of this number as being the argmax of the equation $\mathbb{R}^{k_x} \cong \ker(T_x W \to \mathbb{R}^{k_x})^{\perp}$ where $T_x W$ is the 'tangent space at x', and $T_x W \to \mathbb{R}^{k_x}$ is the linear surjection induced by the standard projection $\mathbb{R}^n \to \mathbb{R}^{k_x}$ which forgets the last $(n - k_x)$ coordinates (of course, the complement '$(-)^{\perp}$' only makes sense if there would be a Riemannian metric!). Now, the last condition of the definition of tangle diagrams says that tranversal dimensions can never spontaneously increase, and this can be understood as a type of *genericity* condition. Let's visualize it!

Mini-exercises 3.3 *Draw examples and non-examples of 1-tangle 2-diagrams, 1-tangle 3-diagrams and 2-tangle 3-diagrams. Mark points of transversal dimension 1 and 0 (resp. 2, 1 and 0 for your 2-tangles). Convince yourself that the Reidemeister moves are 2-tangle 4-diagrams (note, some are also manifold* 4*-diagrams!). (Link to Solutions)*

Now, you ask, wasn't the passage from 'cobordism' to 'stratified cobordism' a central point of the earlier story about geometric models of homotopy types? You are right. And similarly, one should consider *stratified tangle diagrams*. But the idea is mostly parallel to the simpler case of ordinary tangle diagrams above: in place of $W \hookrightarrow X$, one now works with *manifold filtrations* $W_q \hookrightarrow W_{q+1} \hookrightarrow ... \hookrightarrow W_{q+r} = X$ of X, where $W_j \setminus W_{j-1}$ is a (potentially empty) j-manifold embedded in X ($q \leq j \leq q + r$; when $j = q$, set W_{q-1} to be empty). As before, one observes that if a manifold filtration starts with $W_q = S^k$ then applying $\mathrm{Cone}(-)$ to it yields another manifold filtration. This can be use to define stratified tangles analogous to the ordinary case. However, I haven't thought much about stratified tangle diagrams yet: most of the questions I have about the notion already appear in the case of ordinary tangle diagrams. This in particular concerns the role of *framed codimension-1* tangles (where $m = n - 1$) ... an intruiging case from which important classes of stratified tangles in higher-codimensions naturally arise—we shall revisit it shortly.

Let me end by pointing out the formal relation between manifold diagrams and tangle diagrams. *Any tangle n-diagram has a universal (namely, coarsest) subdivision by a manifold n-diagram.* This was shown in [4, Thm. 3.1.32]. The subdivision precisely partitions the tangle into strata of the same transversal dimension. You may think of the subdiving manifold diagram as 'detecting the critical points (and, more generally, critical submanifolds)' of the tangle diagram. It's worth emphasizing that the fact that this works is rather remarkable and owed to working in framed euclidean space: after all, we have made no explicit mention of smooth structure (which is usually required in order to be able to talk about 'critical points') at all but instead formulated all of the above definitions in terms of topological manifolds. To get a better feeling for what's going on, let's draw a few examples.

Mini-exercises 3.4 *For your earlier examples of tangle diagrams, find and draw the coarsest subdividing manifold diagrams. (Link to Solutions)*

4 Combinatorialization theorems

We ended the last story with the cool observation that manifold diagrams can universally subdivide tangle diagrams into their 'critical submanifolds'. The deeper fact that can be used to formally prove this observation is that there is a natural translation of manifold and tangle diagrams into certain purely combinatorial structures. This 'combinatorialization' mechanism is rather special: in classical combinatorial topology we may associate, say, to a compact PL manifold a large and computationally intractable class of combinatorial objects (say, the class of all of compatible triangulations of the PL manifold), but there is usually no one *canonical* combinatorial object (i.e. a single distinguished triangulation) to represent that manifold—in contrast, in the framed stratified setting, canonical combinatorial representations emerge.

Let us set this translation in formal mathematical stone. We will need two pieces of terminology. Firstly, we say that a conical stratification (X, f) is a *stratified* 0*-type* if its fundamental ∞-poset $\mathscr{E}(X, f)$ is equivalent to a poset (in other words, the canonical functor $\mathscr{E}(X, f) \to \mathscr{E}_0(X, f)$ is an equivalence; in yet other words, (X, f) lies in the essential image of $\|-\|$ considered as an ∞-functor into the ∞-category of stratifications). Secondly, a stratified map $F : (X, f) \to Y$ is called a *stratified fiber bundle*, if for each $x \in Y$ there exists a neighborhood U of x over which $F : F^{-1}(U) \to U$ factors as $F^{-1}(U) \cong U \times (Z, h) \to U$ (the second map is the obvious projection; (Z, h) should be thought of as the 'stratified fiber' of the bundle). Similarly, a stratified map $F : (X, f) \to (Y, g)$ (note that now the base is stratified too!) is called a stratified fiber bundle if it is so over each stratum of (Y, g). We say a stratum of (X, f) has 'fiber dimension k' if it is restricts in any fiber of F to a k-manifold. Note, in this case, the fiber dimension is constant across all fibers that the stratum intersects.

The combinatorialization theorem reads as follows. Any manifold or tangle n-diagram $(\mathbb{R}^n, f)$ admits a canonical (namely, coarsest) subdivision $(\mathbb{R}^n, M) \to (\mathbb{R}^n, f)$ by a stratified 0-type M subject to the following two conditions:

(1) There exist (necessarily unique) stratified 0-types $(\mathbb{R}^i, M_i)$ $(1 \leq i \leq n)$ with $M_n = M$, fitting into a tower of stratified bundles $p_i : (\mathbb{R}^i, M_i) \to (\mathbb{R}^{i-1}, M_{i-1})$, whose underlying maps are the standard projections π_i.

(2) Abbreviating the poset map $\mathscr{E}_0(p_i) : \mathscr{E}_0(M_i) \to \mathscr{E}_0(M_{i-1})$ by $q_i : T_i \to T_{i-1}$, and denoting by $\mathsf{fdim} : T_i \to \{0, 1\}$ the map that assigns fiber dimensions to strata, we require: (a) fdim induces a poset map $T_i \to [1]^{\mathrm{op}}$, (b) $q_i : T_i \to T_{i-1}$ is exponentiable, (c) $q_i : \mathsf{fdim}^{-1}(1) \to T_{i-1}$ is a fibration, (d) $q_i : \mathsf{fdim}^{-1}(0) \to T_{i-1}$ is an opfibration.

We briefly elaborate on the last condition. Recall, an exponentiable functor is a functor whose fibers over morphisms in the image look like profunctors, a fibration has fibers over morphism that look like opposite functors, and an opfibration has fibers over morphisms that look like functors. However, the situation is much simpler than what these notions describe in full generality: morphism fibers in (b) are always automatically **Bool**-enriched profunctors (since we are dealing with poset maps), and maps in both (c) and (d) always turn out to be *discrete* (op)fibrations (can you see why? Hint: fibers too must be conical stratifications.).

We take a deep breath, and maybe re-read the theorem—why is this a combinatorialization theorem? It is so because of fact that the stratified 0-types in the theorem can be fully reconstructed from their 'fundamental combinatorial data'. Before discussing this further, let us first give names to the objects appearing in the theorem. An *open n-mesh* M is a tower of stratified bundles $p_i : (\mathbb{R}^i, M_i) \to (\mathbb{R}^{i-1}, M_{i-1})$ $(1 \leq i \leq n)$ of stratified 0-types as in **(1)**, such that the maps p_i satisfy condition **(2)** above. Similarly, an *open n-truss* T is a tower of poset maps $q_i : T_i \to T_{i-1}$ $(1 \leq i \leq n)$ whose fibers $F_x = p_i^{-1}(x)$ are endowed with total orders called 'orientation orders', for which there exists a 'classifying mesh' $M = \{p_i\}$ with $q_i \cong \mathscr{E}_0(p_i)$ such that orientations of fibers of p_i are compatible with orientation orders of fibers of q_i. This makes it sound like trusses, supposedly combinatorial objects, require meshes, framed stratified topological objects, in order to be defined. But this is not true: all the data of trusses can be defined in purely combinatorial terms. In fact, trusses have a rich combinatorial theory, without any reference to topology, to which I devoted (way too) many pages in my PhD thesis (a more concise treatment is [3, §2], and here is a blog post about trusses; a super concise treatment is in [6]). This theory, however, is a separate story. (I will remark that the adjective 'open' hints at the fact that there are yet more general notions of meshes and trusses; in particular, we will meet 'closed' meshes and trusses in the next section!)

Mini-exercises 4.1 *For $n = 1, 2$, unwind the above defintitions and give examples of (open) n-meshes and n-trusses. Give an example of a 1-tangle 2-diagram which is not a stratified 0-type, find its coarsest subdividing mesh and double-check that the latter is indeed a stratified 0-type. (Link to Solutions)*

Meshes and trusses are equivalent in the following sense. Both meshes and trusses organize into categories by defining morphisms to be maps of towers (parallel to our earlier definition of framed maps!). In the case of trusses, this yields a 1-category (or, better yet, a 2-poset a.k.a. **Pos**-enriched category). In the case of meshes, it yields a **Top**-enriched category (or, better yet, a **Strat**-enriched category) and this may therefore be understood as an ∞-category (or, better yet, an ∞-2-poset). Modulo details of the categorical set-up, these categories of trusses and meshes should now be equivalent: in [4, §4] we in particular show an ∞-equivalence between the 1-category of open trusses and the ∞-category of open meshes (in the 2022 version of the book, this was further extended to include the case of mesh and truss *bundles*).

As a consequence of this equivalence, open meshes can be reconstructed from their fundamental trusses: the functor that performs this reconstruction is called the 'classifying mesh' functor $C\mathsf{Msh}$, while its inverse is called the 'fundamental truss' functor $\mathscr{E}\mathsf{Trs}$. Combined with our theorem above, this implies that manifold diagrams $(\mathbb{R}^n, f)$ (and similarly tangle diagrams) can be represented by trusses T together with information about the framed stratified subdivision $(\mathbb{R}^n, C\mathsf{Msh}\,T) \to (\mathbb{R}^n, f)$. But this last bit of information is, in fact, combinatorial as well (up to framed homeomorphism). Indeed, this can be seen already in the non-framed case: any stratified subdivision $F : (X, g) \to (X, f)$ is determined (up to homeomorphism of X) by the poset map $\mathscr{E}_0(F)$. Thus, in summary, framed stratified homeomorphism classes of manifold resp. tangle diagrams can be represented by trusses together with poset maps which encode a stratified subdivision as just explained (these maps are also referred to as 'labelings', or more precisely '(combinatorial) stratifications', of the trusses). A full proof of the combinatorialization theorem for the general case of framed compactly triangulable stratifications (which manifold and tangle diagrams are then a special cases of) is given in [4, §5].

Let us end by mentioning a few direct consequences of the combinatorialization theorem. The first concerns the re-emergence of PL and smooth structures. The fact that any manifold diagrams can, up to framed stratified homeomorphism, be obtained by geometrically realizing a canonical labeled trusses can be easily seen to imply that strata of the diagram carry canonical triangulations. Moreover, the framed conicality condition (which, too, has a combinatorial counterpart) implies that this triangulation represents a PL manifold. Finally, this PL manifold also carries a canonical tangential framing inherited from the ambient framed Euclidean space. Together, this implies (by the equivalence of framed PL and framed DIFF) that any stratum in a manifold diagram may be canonically regarded as a smooth manifold! The story of tangle diagrams is, at first, parallel, as all tangle diagrams may be canonically triangulated. However, a priori, this triangulation need not represent a PL manifold but only a *weak PL manifold* (meaning all links are homeomorphic to spheres, but not necessarily PL homeomorphic; the two notions are equivalent if the smooth Poincaré conjecture in dimension 4 is true). Still, more can be said about how tangle diagrams do encode smooth structures—we shall return to this matter in story 6.

Another crucial point (and one that you may have already thought about) concerns boundaries. Aren't tangles usually considered as living in the closed cube, so that we can easily refer to the 'in-going' and 'out-going' boundaries, and 'corners' etc.? If we define tangles as manifolds embedded in $\mathbb{R}^n$, can we retrieve those beloved boundaries and corners somehow? It turns out we can, and universally so. Indeed, based on the combinatorial theory of labeled trusses, one easily define compactifications as a universal construction (see [4, §2.2.3]). Porting this combinatorial construction back to the framed topological setting, we find universal compactifications of tangle (and similarly manifold) diagrams. Thus, the above notions, in fact, have analogs in terms of stratifications of the closed n-cube!

A final important point concerns links. Readers familiar with the theory of conical stratifications will recognize the statement 'links are not well-defined': it means that in our earlier definition of conical stratifications non-homeomorphic choices for 'links' (L, l) at the same point x are possible. In the framed setting this problem goes away fully (see [4, §2.3.4]): in a framed conical (and framed compactly triangulable) stratification there are essentially unique choices for all links, and this, too, is a consequence of the combinatorialization theorem.

5 Dualizing globalization revisited

Let me briefly describe the passage from manifold diagrams to framed spaces in Table 1: earlier, we referred to this as *dualizing globalization*. This will also foreshadow some of the ideas appearing in part II of this series. First, we answer a different (but very much related) question, which may have been on your mind for a while now: how do manifold diagrams actually relate to classical pasting diagrams? That's an excellent question! A rigorous answer can be given by combining framed topological thinking with the dualization of stratifications that we met at the end of story 2.

Let's get started. What is a 'pasting diagram' really? Well, it is a type of cell complex in which cells represent morphisms (and, like morphisms, cells should be given a 'direction'). Ignoring directions for a moment, an immediate objection to this statement would be that the term 'cell complexes' is a bit vague (as Jim Davis pointed out to us after uploading [3]) or, at least, it is much more general than we need it to be. Indeed, we can do better: since morphisms always run between other morphisms (even though these may be identities), closures of morphism cells will be unions of other morphism cells. In the absence of identities, these closures will be balls. There is a topological term for this: regular cell complex. So a better thing to say would be that pasting diagrams are (quotients of) regular cell complexes (where quotients should be thought of as degenerating identity cells).

A couple of years back I learned that regular cell complexes are really very nice. Indeed, we don't need any topology at all to understand them! This works as follows. Any cell complex (whether regular or not) can be thought of as a stratification (X, f) whose strata are the open cells of the complex. Consequently, given a cell complex (X, f), we can compute its fundamental poset $\mathscr{E}_0 f$. *Regular cell complexes* (X, f) are exactly though cell complexes such that $(X, f) \cong \|\mathscr{E}_0 f\|$ by a stratified homeomorphism that is the identity on fundamental posets. Two important observations follow from this: firstly, regular cell complex are stratified 0-types, and, secondly, regular cell complexes are exactly those stratifications in the image of a certain class of posets $\mathcal{CP}$ under the functor $\|-\|$. This class $\mathcal{CP}$ is the class of so-called cellular posets, which can be described as follows: a poset P is called *cellular* if, for all $x \in P$, the realization $|P^{>x}|$ is a sphere. (As an aside, cellular posets cannot be algorithmically recognized among posets ... this will follow from the discussion in story 6.)

So, back to our original question, how can we now relate manifold diagrams to classical pasting diagrams? Using the combinatorialization theorem from the last section, this turns out to be a natural construction. Let's be concrete and consider a manifold n-diagram $(\mathbb{R}^n, f)$. We've seen that there exists a unique coarsest open n-mesh $M = \{p_i : M_i \to M_{i-1}\}$ such that $(\mathbb{R}^n, M_n)$ subdivides $(\mathbb{R}^n, f)$. We have also seen that this data can be combinatorially encoded: namely, by an open n-truss $T = \{q_i : T_i \to T_{i-1}\}$ obtained by setting $T = \mathscr{E}\mathsf{Trs}(M)$ together with a labeling poset map $g = \mathscr{E}_0(M_n \to f)$ (where $M_n \to f$ is the stratified subdivision of f by M_n). This data can now be dualized!

The *dual* truss of T is the tower of poset maps $T^\dagger = \{q_i^{\mathrm{op}} : T_i^{\mathrm{op}} \to T_{i-1}^{\mathrm{op}}\}$ (but with the same orientation orders as before). The labeling also transfers: while l was a map $T_n \to \mathscr{E}_0 f$, we obtain $l^{\mathrm{op}} : T_n^{\mathrm{op}} \to \mathscr{E}_0 f^{\mathrm{op}}$ as a labeling for $T^\dagger$. What did we just do? By dualizing the (labeled) open truss T, we created a (labeled) *closed* truss $T^\dagger$. While this truss does not represent an open mesh, it does represent the topological dual of an open mesh which, you guessed it, is called a closed mesh. In fact, the 'classifying closed mesh' $M^\dagger = C\mathsf{Msh}(T^\dagger)$ of $T^\dagger$ can simply be constructed as the stratified fiber bundle tower consisting of realizations $\|q_i^{\mathrm{op}}\|$ (plus compatible fiber orientations). Everything you need to know now is: closed meshes turn out to be towers of maps of regular cell complexes, in other words, T_i^{op} were cellular posets to begin with! Moreover, orientations of fibers in these towers of maps encode 'directions' on cells in the cell complexes, so really, closed meshes are towers of 'directed regular cell complexes' (or, in the terminology of [3], 'framed regular cell complexes').

We have almost arrived at a 'classical' notion of cellular pasting diagram. The last thing that is left to be done is 'quotienting out the degeneracies' (but, in the exercises below we'll also learn that this quotient forgets vital information). The way this quotient works is encoded in the labeling map l^{op}, and there's a concise technical way of defining it, which would, however, require us to dive more in the combinatorial theory of labeled trusses. Instead, let's describe a more intuitive approach. First, compose the characteristic map of the mesh $M_n^\dagger = \|T_n^{\mathrm{op}}\|$, which is a map $\left|T_n^{\mathrm{op}}\right| \to T_n^{\mathrm{op}}$, with the map l^{op}, which is a map $T_n^{\mathrm{op}} \to \mathscr{E}_0 f$. One can show that this composite is again a characteristic map (basically, because characteristic maps compose), and so we have just defined a stratification $(X, f^\dagger)$ on $X = \left|T_n^{\mathrm{op}}\right|$. This stratification is important, and in [4, §2.4] we call it a *cell n-diagram* (it's the 'true dual' notion of manifold n-diagrams). Still, $(X, f^\dagger)$ won't yet look like the type of pasting diagrams that many of us are used to—for that, we still need to quotient the stratification further. To do so, framed conicality re-enters the stage: after dualization, the fact that $(\mathbb{R}^n, f)$ was framed conical now guarantees that, for closures $\overline{x}$ of cells x in $M_n^\dagger$, the substratification of $(X, f^\dagger)$ obtained by restricting to $\overline{x}$ will look like a product stratification $Z \times (C, c)$ (where Z is a closed k-ball and (C, c) is a stratified closed l-ball whose interior is its own stratum, with $k + l \leq n$). The quotient map $(X, f^\dagger) \to (\tilde{X}, \tilde{f}^\dagger)$ we seek is the map assembled from the projections $Z \times (C, c) \to (C, c)$ for each closed cell $\overline{x}$!

Phew, those last three paragraphs had a lot of symbols. It might be best to just see some worked out examples!

Mini-exercises 5.1 *Translate previous examples of manifold diagrams into cellular pasting diagrams, by retracing the following steps: (1) draw your manifold diagram (X, f); (2) find your coarsest subdividing mesh; (2b) remember the entrance path poset map of the subdivision; (3) dualize you mesh; (4) using the dual of the poset map you remembered, find a stratification $(X, f^\dagger)$ that your dualized mesh subdivides; (5) find the cell closures that look like product stratifications and quotient them out; done! Can you find two different manifold 3-diagrams that have the same (quotiented) pasting diagram? Extra point: above we glossed a bit over the fact that cells are directed... so do try to draw directions on all your cells. (Link to Solutions)*

Finally, let's briefly get back to the bigger topic at hand: dualizing globalization. As a matter of fact, the above constitutes 96 percent of what I wanted to say on the topic. Indeed, we now understand dualization of manifold diagrams to pasting diagrams. And globalization is easy: just glue together the directed cells that we just described! Okay, let's slow down a tiny bit. What have we really seen above? We've found a 'intermediate notion' of cell n-diagrams obtained by taking a closed mesh and then stratifying it in a certain way; this stratification was then interpreted as a type of quotient map. This suggests a 2-step thinking process as follows. *First*, define your building blocks to be the directed cells in closed n-meshes (these, in fact, have been introduced in [3] as 'framed regular cells'). *Then*, consider the quotient maps (cell-wise) as as attaching maps of such framed regular cells. In summary, 'framed spaces' can be built by taking framed regular cells, and attaching them according to the stratifications found in cell n-diagrams (or, dually, in manifold n-diagrams). Following *precisely* this procedure, but in combinatorial rather then topological terms, will lead us to the notion of geometric computads in story 8. But this is firmly part of the global story!

6 How computable is invertibility?

There is a central difference between the notions of manifold diagrams and tangles diagrams, that many constructively inclined mathematicians will care about. Given a labeled n-truss, can you actually decide whether it combinatorially represents a manifold n-diagram resp. an m-tangle n-diagram? And an equivalent question: can there be an algorithm that lists all framed stratified homeomorphism classes of manifold n-diagrams resp. of m-tangle n-diagrams? (Of course, such an algorithm would be allowed to run indefinitely.) For both questions, the answer is YES in the case of manifold diagrams, but it is NO in the case of tangle diagrams as we will now discuss.

Classical results by Markov and Novikov establish that there can be no algorithm that decides the question of whether a given triangulated space K is an m-manifold, or even the simpler question of whether K is an m-sphere ($m > 4$). The proof proceeds, of course, like any undecidability proof, by encoding another undecidable problem: namely, if it would be possible to recognizes spheres, then it would be possible to decide whether certain homology spheres are spheres, and in the fundamental groups of these homology spheres we can encode undecidable word problems of finitely presented groups (see [7, §6.2 and §7.2]). This obstruction to recognition and enumeration of manifolds persists if we work with embedded manifolds, even if we restrict their codimension. Indeed, all homology m-spheres embed in S^{m+1} (in the smooth case, this holds up to changing smooth structures around a point) and this embedding partitions S^{m+1} into two identical contractible components (higher Mazur manifolds): in general, for $m > 4$, no algorithm can distinguish such embeddings from the standard embedding $S^m \hookrightarrow S^{m+1}$ as it would entail a comparison of fundamental groups. Moreover, nothing about these observations is pathological (Mazur manifolds arise in dimension ≥ 4 since non-trivial knots do in dimension ≥ 3), and thus they cannot be easily circumvented by putting additional conditions. The undecidability must therefore also transfer to our definition of tangle diagrams: recall, this required local neighborhoods of the form $\mathbb{R}^k \times \mathrm{Cone}(S^{m-k-1} \hookrightarrow S^{n-k-1})$, but, as we have just learned, we cannot recognize whether neighborhoods (no matter their combinatorial presentation) are actually of this form.

Maybe you don't find the above upsetting. After all, manifolds are just very complex objects (just like program runtimes can be). But from the perspective of geometric higher categories we should be a bit upset: recall from Table 1, we want to use tangles to understand morphisms in ∞-groupoids—but what good is this approach if it doesn't let us easily work with the structure associated to an invertible morphism? Is it really impossible to have a computer usefully enumerate the higher geometric structure of an invertible morphism? Alas, in a turn of fates, I will now tell you that maybe there is some hope after all. This hope relies on two ideas: first, the passage to smooth tangles, and, second, the hypothesis that generic smooth tangles may be classified analogous to results from classical manifold singularity theory—importantly, both points appear to work best (or, better said, *at least*) in codimension-1.

A priori, nothing in our framed topological definition of tangle diagrams mentions smooth structures—nonetheless, we start our journey into the smooth realm by describing a conjecture which would firmly link the two set-ups. For simplicity assume all manifolds to be compact (using the right flavor of 'compactly triangulable' would also work). A 'smooth tangle diagram' is a tangle diagram $W \hookrightarrow \mathbb{R}^n$ in which the embedding $W \hookrightarrow \mathbb{R}^n$ is a smooth embedding (in particular, W itself is smooth). Given any tangle diagram $W \hookrightarrow \mathbb{R}^n$, a 'framed smooth structure' of the diagram is a framed stratified homeomorphism between $W \hookrightarrow \mathbb{R}^n$ and a smooth tangle diagram $W' \hookrightarrow \mathbb{R}^n$. A framed smooth structure on $W \hookrightarrow \mathbb{R}^n$ in particular induces a smooth structure on the tangle manifold W. We may now state the 'FR-DIFF' conjecture as follows: given tangle diagrams, any two framed smooth structures on the diagram induce diffeomorphic smooth structures on the tangle manifold (or, equivalently, if two smooth tangle diagams are framed homeomorphic, then their tangle manifolds are diffeomorphic). The conjecture was stated in [3]. Note that 'FR-PL', i.e. the PL manifold version of the conjecture, is true, and follows from what is called the 'framed Haupvermutung' in [3, §5]. The smooth version would be quite remarkable if true: it would imply that we would have found a finite combinatorial representation for compact smooth structures (a feat that, I believe, so far hasn't yet been achieved!).

The FR-DIFF conjecture is motivated by the idea that 'higher Morse-like decompositions leave no room for exotic differential behaviour locally'. To illustrate this idea, I will outline a proof attempt of the conjecture (which also highlights its tricky part). First, we observe that slices of an m-tangle n-diagram $W \hookrightarrow \mathbb{R}^n$, i.e. restrictions to the fibers of $\mathbb{R}^n \to \mathbb{R}$, are (as a consequence of our definition) themselves $(m-1)$-tangle $(n-1)$-diagrams $W_x \hookrightarrow \mathbb{R}^{n-1}_x$ except at finitely many critical points $x \in \mathbb{R}$. This allows us to argue inductively, since framed homeomorphism in particular implies framed homeomorphism of slices (and unwinding this induction all the way down to 0-tangles yields the sort of 'higher Morse-like' reasoning alluded to before). In the inductive step, we want to extend diffeomorphisms of non-critical slices across critical points. In order to construct this extension, recall links around critical points are $(m-1)$-spheres and these are themselves described by (appropriate gluings of) $(m-1)$-tangle $(n-1)$-diagrams; inductively, we may thus construct diffeomorphism between these link spheres; all that remains to be done is to extend this diffeomorphism to a diffeomorphism of the m-disk neighborhood of the critical point. This last part is tricky when $m > 6$: not every sphere diffeomorphism extends to the disk—namely, those which are not pseudoisotopic to the identity do not. Thus, somehow implicit in the conjecture is the claim that the *inductive way* in which we constructed the link sphere diffeomorphism prevents these 'exotic' behaviours from appearing. (While I have some vague ideas on how to make this more precise, let me also say that any expert help would certainly be greatly appreciated.)

If the conjecture turns out to be false (which is definitely a possibility), there is an important case to fall back to: this is the case of tangle diagrams of codimension 1. Codimension 1 is special. It is the only codimension in which framings are naturally a *combinatorial* structure: indeed, for any codimension-1 tangle diagram $W \hookrightarrow \mathbb{R}^n$ there is always a $\mathbb{Z}_2$-worth of choices of 'compatible' normal framings, corresponding to labeling the connected components of $\mathbb{R}^n \setminus W$ with signs ± such that no two regions of the same sign share a boundary. In the presence of normal framings, classical smoothing theory once more tells us that PL and smooth structures are essentially the same thing. Therefore, since the FR-PL conjecture holds one can derive the FR-DIFF conjecture as well in the case of framed codimension-1 tangle diagrams. It will turn out that the framed codimension-1 case is important for many reasons (in particular, for the story of invertibility, as we will get back to in a moment), so now is the time to draw some examples.

Mini-exercises 6.1 *Draw smooth m-tangles in dimension $m + 1 = 1, 2, 3$. Draw the two possible choices of normal framings. In each case, pass to subdividing manifold diagrams. Observe that m-strata in your manifold diagrams come with two types of framings. (Can you formally distinguish them? Consider the projection $\mathbb{R}^{m+1} = \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$.) Label these strata with f and f^{-1} accordingly: you have written down structural equations for an invertible 1-morphism f. (Link to Solutions)*

Here is another reason to care about codimension-1 tangles: they arise naturally from a comparison with classical singularity theory. In classical singularity theory, in order to study the local behavior of $\mathbb{R}$-valued functions on an m-manifold W, one considers function germs $f : \mathbb{R}^m \to \mathbb{R}$ with critical point at 0 and critical value 0. Then origin-preserving diffeomorphism of $\mathbb{R}^m$ act on the space of such germs which yields a stratification by orbits. The codimension-0 strata recover the well-known quadratic normal forms of Morse functions. The locally finite part of the stratification is Arnold's famous ADE classification (with the usual quadratic function germs from Morse theory being the 'A_1 strata'): the fundamental poset of this stratification is depicted in [8, Cor. 8.7] (up to some signs that differ in the $\mathbb{R}/\mathbb{C}$ cases, and, depending on convention, up to arrow direction). But what do I mean by the 'locally finite part'? It turns out the stratification isn't nice looking everywhere: Arnold also finds a 'locally uncountably-infinite' part of the stratification, which appears starting in codimension 5 because of jet space dimensions outgrowing general linear group dimensions—*to some extent*, the differential machinery breaks down at this point (several attempts for recovery were made, for instance, by considering actions by homeomorphisms in place of diffeomorphisms).

In the locally finite part, however, the world is whole and the stratification of the germ space is even conical: for codimension-k strata S, open neighborhoods look like $Z \times (\mathrm{Cone}(L), \mathrm{cone}(l))$. Arnold describes these cones $\{f\} \times (\mathrm{Cone}(L), \mathrm{cone}(l))$ $(f \in Z \subset S)$ by parametrizing them with so-called *versal unfoldings* $f + F_\lambda$ (with k-dimensional parameter space $\lambda \in \mathbb{R}^k$). And these now relate back to our tangle diagrams: namely, given one of Arnold's unfoldings one can try to produce an $(m + k)$-tangle $(m + k + 1)$-diagram simply by considering the parametrized graph $\Gamma_{f+F_\lambda} \subset \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}$. Note that this is necessarily of codimension-1.

Mini-exercises 6.2 *For $m = 1$, consider $f(x) = x^3$ (this is a germ in the stratum A_2, which is of codimension 1). Arnold gives us the unfolding $f(x) + F_\lambda(x) = x^3 + \lambda x$ (so $k = 1$). Passing to the parametrized graph, which 2-tangle in $\mathbb{R}^{k+m+1} = \mathbb{R}^3$ will be produced? Write down a few other 2-tangles which are framed stratified homeomorphic to the one you just produced. (Link to Solutions)*

The procedure provides a heuristic translation from Arnold's ADE singularities into tangle diagrams (tested up to the D_4 singularity in [4, §3.4])—but, of course, we would want to see a formal explanation for what is going on here! While such an explanation is missing as of yet, let me mention one direction of investigation: this direction tries to first understand singularities from a purely framed-combinatorial perspective, without reference to the differential machinery (which, as we saw, has its issues anyways). Here's a very brief outline. First, we need to know that notions of meshes and trusses naturally extend to definitions of *bundles* of meshes and trusses (this is unsurprising since, after all, both notions are themselves defined in terms of bundles). Studying mesh and truss bundles then also enables us to study tangle diagrams in bundles. This, in particular, allows us to describe how tangle diagrams can be *perturbed* into one another. With sufficient care (and directly leveraging the combinatorial set-up, e.g. by replacing polynomial degrees by some framed combinatorial invariant, such as truss element count), we can define a notion of 'perturbation-stable tangle germs', which are those local neighborhoods of critical points in tangle diagrams that cannot be perturbed into simpler such germs. The fact that, in low dimensions, this combinatorial approach *reproduces* singularities from Arnold's ADE list is remarkable.

Certainly, a full understanding of the resulting 'framed combinatorial theory of singularities' is still lacking, and [4] merely tries to put down a few basic strokes of how the subject could look like, while leaving much room for further exploration and theory-building. The ultimate goal would be to achieve a full classification of perturbation-stable codimension-1 tangle germs, analogous to the classification given by Arnold, and to understand the relation between the two approaches (the hypothesis is, of course, that tangle germs also have an ADE-like classification!). In doing so, at least one interesting thing needs to happen: the 'locally uncountably-infinite' stratification encountered in classical differential singularity theory will have to disappear—indeed, everything is countable to begin with in framed combinatorial singularity theory! This would be a central advantage of the framed combinatorial approach.

Yet more interesting to us, as a result of such a hypothetical classification of stable tangle germs, we would obtain a presentation of codimension-1 tangle diagrams: indeed, any codimension-1 tangle diagram would be, up to perturbation, a composite of stable tangle germs. This would address the questions that this story started with at least in codimension-1: indeed, we now find ourselves in the position to algorithmically enumerate all 'stable' tangle diagrams, simply by composing the tangle germs from our classification. (Of course, we cannot make anything that was previously incomputable computable, but it is interesting to learn that, while we cannot enumerate manifolds up to diffeomorphism (or homeomorphism), we might be able to enumerate stable tangles up to framed homeomorphism!) Moreover, for the combinatorial purposes of geometric higher categories, a classification of stable tangle germs would yield a workable structure for describing invertible 1-morphisms.

We will end with a cliff-hanger. We saw that codimension-1 tangle diagrams had combinatorial (namely, $\mathbb{Z}_2$-classified) normal framings, *and* that they behaved nicely with respect to smooth structure, *and* that they had an important hypothetical relation to classical differential ADE singularities... but what about the other codimensions? Can we come up with a finite list of 'elementary' singularities in higher codimensions as well? Well, it gets certainly more difficult. You can still define perturbation-stability of tangle germs in higher codimensions but you easily encounter infinities (for instance, in codimension-2, this is because there are infinitely many non-equivalent knots, see [4, Rmk. 3.3.8]). There might, however, be a different natural approach to the question. This starts with the observation that, for the purposes of studying invertible morphisms in geometric higher categories, codimension-1 appears to be all we need: an invertible n-morphism is really just an invertible 1-morphism between $(n-1)$-morphisms. Geometrically, this corresponds to a stabilization of normal 1-framings to n-framings: in fact, there's a natural combinatorial construction that takes a conical manifold n-diagrams (the 'n-morphism') and a framed codimension-1 k-tangle diagram (the 'invertibility datum') and produces an manifold $(n+k)$-diagram by taking the 'product' of the tangle and the conical diagram, which is a sort of 'stratified stabilization' construction—we will return to this construction in story 8 in the context of describing higher invertible morphisms in geometric computads. Still, we haven't addressed the original question: where are the elementary higher-codimension (non-stabilized) singularities hanging out? It turns out this question is actually related to (minimal) invertible cell structures of Thom spectra... but this, finally, brings us into the realm of *global models*.

II Global models

In this part of the series we will be concerned with the application of the geometric ideas from the first part to the description of global phenomena in higher structures. While the preceding part was mostly based on ideas that have already appeared in written form, in this part, in many places we will try to look beyond the theory that has been formally developed (though still, most of the ideas aren't conceptually new). In particular, not only will we encounter many more open questions, but the expositional style will shift from 'definitions and theorems' to 'examples, illustrations and ideas'. This will somewhat culminate in the last story of the series, story 9, which discusses a higher type theory of geometric computads. Secretly, it will not be me *telling* you about type theory, it will be me *asking* you about it! Indeed, one hope of putting out this (unfinished) material was to allow for experts to chime in and answer some of the open question about the ideas presented here.

7 Coherences as isotopies

Easing our way into the global study of higher structures, we will first recall a few basic but important intuitions. The following question is our starting point: why, actually, are n-categories difficult? Well, the difficulty is their weakness (pun intended). Indeed, there is, of course, the simpler notion of *strict* n-categories which is more or less straight-forward to define in all dimensions $n \in \mathbb{N}$. While strict n-categories treat 'deformations' of one pasting diagram into another as a strict equality between these pasting diagrams, (weak) n-categories aim to keep track of all of these deformations (and the deformations between deformations, and so on) by higher morphisms that are referred to as *coherences*. In the groupoidal case the resulting distinction can be quantified by a concrete comparison: while the theory of n-groupoids (i.e. of n-categories in which all morphisms are invertible) is equivalent to the homotopy theory of homotopy n-types, this fails to be the case for the theory of strict n-groupoids. Of course, to many this is very old news. But, at some point in time the news weren't old: indeed, there was a paper in the 1990s that proved that strict higher groupoids could model homotopy types, and only several years later a mistake in that proof was pointed out. Legend has it that this mistake even played a role in the later inception of homotopy type theory.

The mistake can be traced back to one of the simplest pasting diagram coherences, namely, the *exchange* of two 2-morphisms: the exchange coherence is, in the case of degenerate 1-morphisms, exactly one half of the usual Eckmann-Hilton argument. Indeed, by now it is folklore knowledge that 'the exchange (or Eckmann-Hilton argument) cannot be strictified without ending up with a less general theory of higher structures'. In contrast, many coherences *can* be strictified. For instance, the theories of (weak) 2-categories and strict 2-categories are equivalent, and so all the coherences you could possibly write down for 2-categories may, in fact, be replaced by strict equalities. In 3-categories still many coherences can be strictified: one approach to 'maximal strictification' yields the notion of Gray categories. In essence, Gray categories are a notion of 3-categories in which the *only* non-strictified coherence *is* the exchange. As it turns out, the theory of Gray categories can be shown to be equivalent to that of other inceptions of (weak) 3-categories; in other words, complementing our earlier slogan, 'the exchange is the only coherence you need for a fully general theory of 3-structures'.

At the same time that the above story became part of the community's collective understanding of higher structures, an idea emerged that there could be an easy organizational principle for the question of which coherences could be strictified and which could not: this idea rooted in the relation between coherences and the isotopy deformations of the 'topological duals of pasting diagrams' (the latter, of course, should be synonymous with manifold diagrams!). But while the idea was there, definitions were lacking. I believe Todd Trimble was one of the first to act on this by trying to develop a definition of surface diagrams that should precisely provide the 'geometric semantics' for Gray categories (generalizing the case of string diagrams which provide geometric semantics for strict 2-categories as described by Hotz and Joyal & Street). Under this interpretation, the exchange (with degenerate 1-morphisms) would become the braid diagram which, indeed, is an isotopy of an embedding of two points into $\mathbb{R}^2$. Similarly, higher 'essential' coherences (i.e. those that should not be strictified) would correspond to some sort of higher-dimensional isotopies—that, at least, was the vague idea.

Today, with the help of the tools that we previously introduced, we can be much less vague. Namely, using the framed-combinatorial-topological machinery of manifold diagrams from story 3 and story 4 as well as the dualization procedure described in story 5, we can not only make precise what we mean by 'higher isotopies', but we can also rigorously translate such isotopies into laws for pasting diagrams. (Here and henceforth, 'pasting diagram' means cell diagram in the sense of story 5; as we mentioned there, cell diagrams are the 'better pasting diagrams' as they do not forget information in the way traditional pasting diagrams do.) We have already seen one example of this in exercise 5.1, but let's recall it here: in Figure 1 we depict, in the indicated order, (1) the braid as a manifold 3-diagram, (2) its coarsest subdividing mesh, (3) the dual cell complex of that mesh, and (4) the resulting exchange coherence as an equivalence of pasting diagrams.

Generalizing the braid example, the general notion of isotopy can be easily defined in framed-topological terms. A manifold n-diagram $(\mathbb{R}^n, f)$ is a *k-isotopy* if all strata of f are of dimension greater or equal to k. In particular, the braid is a 1-isotopy while, say, the Reidemeister III move is a 2-isotopy (see exercise 3.3). So a k-isotopy should be thought of as an isotopy of $(k-1)$-isotopies. Let's think about a few more examples in the contexts of braidings (in a moment, we will see some other examples as well).

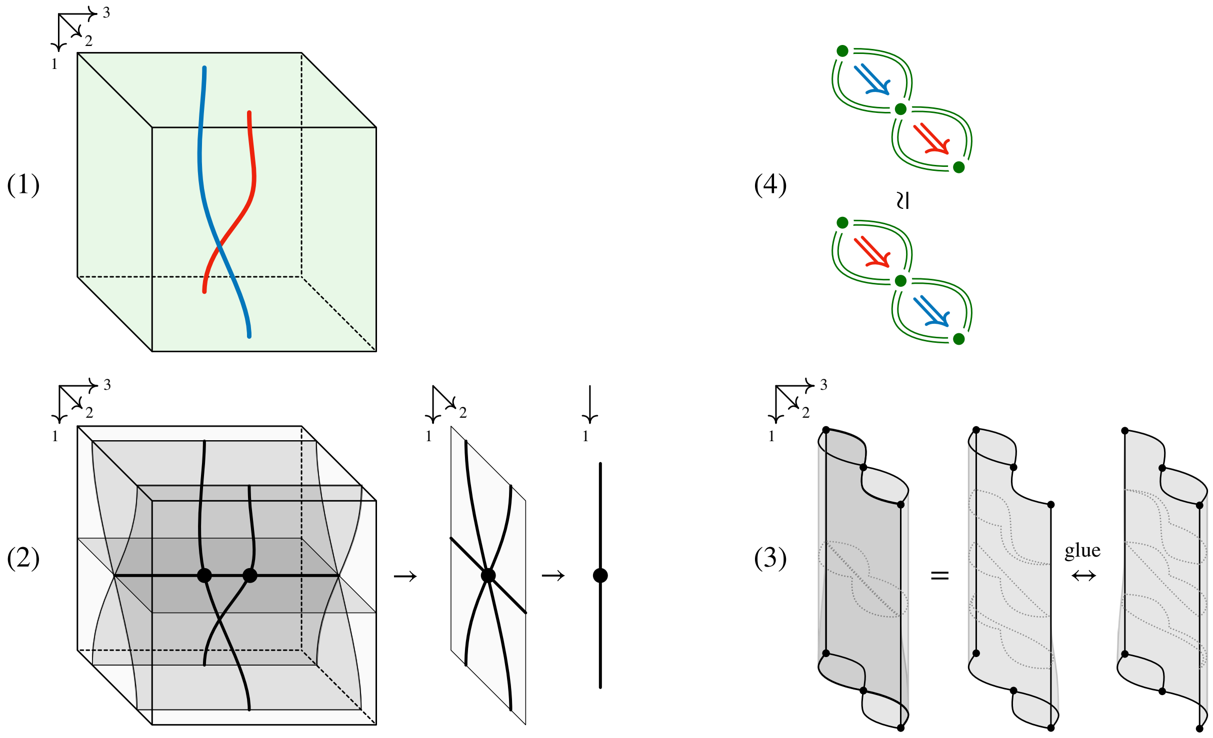

Figure 1: Steps in the formal translation of a braid isotopy into an exchange law of pasting diagrams.

Mini-exercises 7.1 *Check that you understand how directed cells are attached in the cell* 3*-diagram dual to the braid (see Figure 1 (3)). Note, that there are two braids (the 'over-' and 'under-braid'), and no* 2*-isotopy relates them (if boundaries remain fixed). This changes in codimension-3. Write down the two generic braidings of two points in* $\mathbb{R}^3$*, and construct an 2-isotopy between them (this is a manifold* 5*-diagram, so you may want to use 'movies of movies' depicted frame-by-frame). You have arrived at a coherence called the* syllepsis*! For fun: can you imagine the attaching maps in the dual cell* 5*-diagram? (Link to Solutions)*

My favorite example of the 'coherences as isotopies' principle are actually the categorical laws governing the sequence: categories, functors, natural transformations, modifications, ... Sometimes, the kth element in this sequence is also referred to as an $(n-2)$-transfor, but let's instead be verbose and call them 'k-morphism in n**Cat**', the $(n+1)$-category of n-categories, for $k \leq n+1$. For concreteness, we start with the familiar case $n = 1$, i.e. we consider the (strict!) 2-category of 1-categories, their functors, and their natural transformations. The usual definitional laws of these objects, such as 'functoriality' and 'naturality', can now be recovered purely from manifold diagrammatic composition and isotopy. Naturality is the more interesting case (functoriality is merely a consequence of associativity of manifold diagram composition, can you see how?). Consider functors $F_i : A \to B$, $G_i : B \to C$, and natural transformational $\alpha : F_1 \to F_2$, $\beta : F_2 \to F_3$. Then the *naturality law* is precisely the exchange coherence as shown in Figure 2. Note, we recover the usual form of the law (which, in turn, implies the more general one) if we set $A = \mathbf{1}$ to be the terminal category, which then makes F_i objects and α a morphism in the category B.

With the warm-up case done, let us continue onwards and upwards. Consider $n = 2$. Now, natural transformations (or, more precisely, pseudo-natural transformations) not only gain additional coherence data, but they themselves can be related by 3-morphisms, also called modifications. Their laws are described, e.g., here. But importantly, as long as we remember the definition of isotopies, we don't really need to remember these laws individually—we can simply derive them! Instances of modification laws are derived in this way in Figure 3. The same figure also shows an example of a yet higher law (as an isotopy) which will have to be satisfied by a 4-morphism in 3**Cat**. Readers familiar with the protoypical example of Gray categories (namely, the category 2**Cat** of strict 2-cats, strict 2-functors, pseudo-natutral transformations, modifications) will not be surprised by the pattern that we are illustrating here. In effect, we are exactly following the aforementioned long-known geometric intuition for describing higher-dimensional versions of Gray categories!

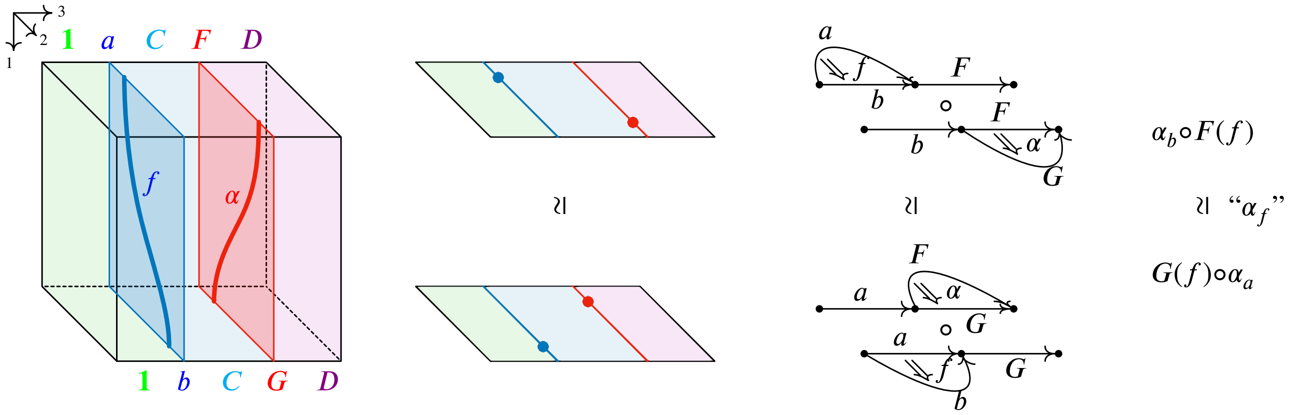

Figure 2: The naturality law of natural transformations as an isotopy.

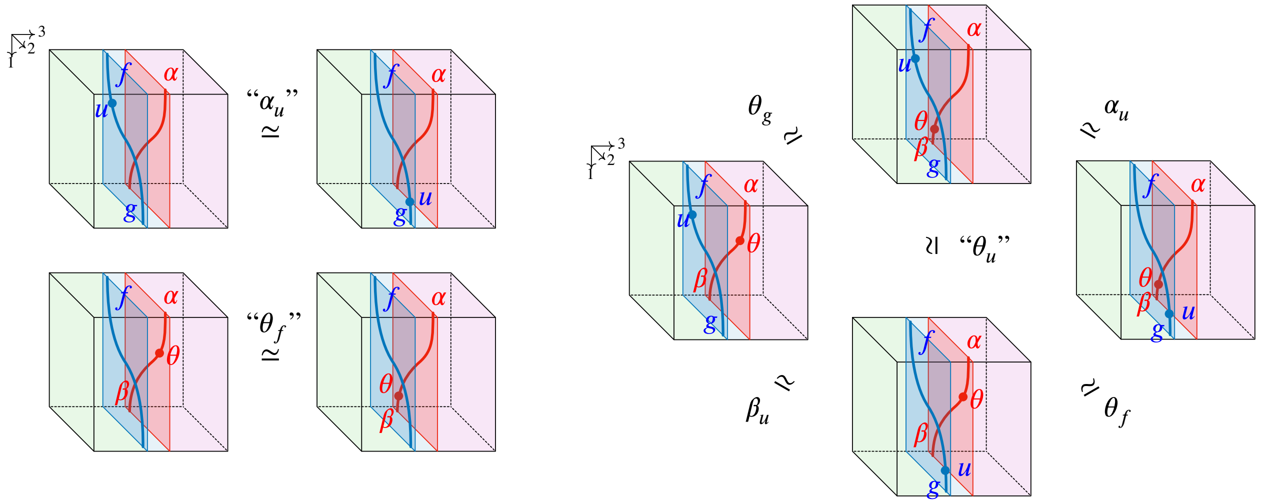

Figure 3: Laws of higher morphisms in n**Cat**.

Manifold diagram isotopies, like manifold diagrams, are nicely computable in all dimensions: that is, without much effort we can write a computer programme that enumerates all manifold isotopies (ordered, say, by the size of their combinatorial representation). At this point you may say: great, we can just take manifold isotopies and use them as pasting diagram laws in order to define our notion of n-categories. But this, of course, is not as easy as I made to sound: there are infinitely many isotopies in all dimensions $n \geq 3$. And while you can list them all, it is generally rather tedious to have to verify whether infinitely many laws are satisfied. A much better approach would be to find a nice generating set for isotopies, that is, some dimension-wise finite set of *elementary* isotopies such that any other isotopy can, up to higher isotopy, be written as a composite of only elementary ones (at least up to some 'shape-equivalence' relation, which accounts for changing conical neighborhoods of the strata that are being isotoped... but that's a minor technicality which I will not expand upon). For instance, in the case of tangle isotopies in dimension 3, the braid is the *only* elementary isotopy: this is illustrated in Figure 4 where another isotopy, namely, the 'triple braid', is isotoped into a composite of three ordinary braids. Much work has been done that indirectly relates to this question of finding elementary isotopies (e.g. on higher braid groups, or the combinatorics of E_n algebras), and maybe our framed-combinatorial perturbation theory (see story 6) can be leveraged to address the question to some extent... but no one has puzzled these ideas together yet!

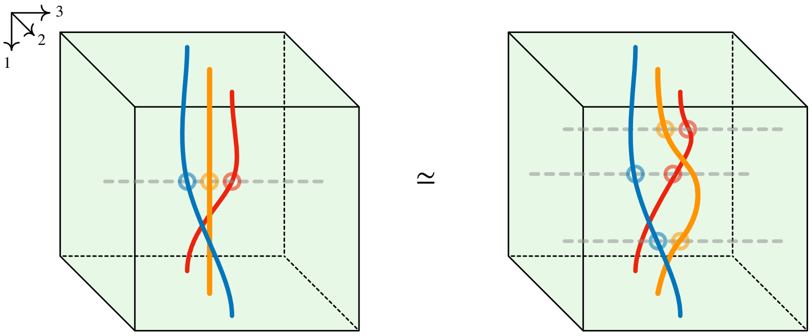

Figure 4: The triple braid is isotopic to a composite of three braids

8 Geometric computads

We ended the last story with the observation that distilling notions of n-categories from the geometry of manifold diagrams may be not as straight-forward as one could hope for. In this section, we will consider a class of higher structures for which all geometric coherence laws are always automatically satisified: namely, we will consider *free* higher categories, also known as computads. The fact that defining free geometric higher categories is exceedingly simple will be one of the fundamental observations on which story 9 will build.

We can jump straight to the definition. We will say a manifold k-diagram $(\mathbb{R}^k, f)$ is a *stratum k-type* if the diagram is of the form $(\mathrm{Cone}(S^{k-1}), \mathrm{cone}(\partial f))$, and we call $(S^{k-1}, \partial f)$ the *type boundary* of f. (Yes, we've seen stratum types before! As a consequence of the framed conicality condition introduced in story 3, any k-stratum in a manifold diagram locally looks like a product of $\mathbb{R}^k$ and a stratum $(n-k)$-type, and we refer to this conical diagram as the stratum's type.) We can now define: a *geometric n-computad* is an $[n]$-graded set C together with, for each $c \in C_i$, a stratum type $(\mathbb{R}^i, f_c)$ and a 'labeling' function $l_c : \mathscr{E}_0 f_c \to C_{\leq i}$ with $l_c\{0\} = c$, such that k-strata s in f_c have stratum type $f_{l_c(s)}$ and $l_{l_c(s)}$ factors through l_c by the induced map $\mathscr{E}_0 f_{l_c(s)} \to \mathscr{E}_0 f_c$. So, in words, a computad C is a bunch of 'i-morphisms' (indexed by the sets C_i) for $i \leq n$, and each morphism has a stratum type whose boundary is consistently labeled in lower-dimensional morphisms.

Comparing to the classical notion of computad, we make an immediate and important observation about the above definition. Classically, building computads is an inductive *two-step* process: namely, in the first step, we attach a set of new generating morphisms whose boundaries (or 'types') are described by existing morphisms; in the second step, we *freely generate* new morphisms from the newly attached generating morphisms (and this prepares us to then attached yet more generating morphisms in the next inductive step). The process of free generation is usually straight-forward for strict higher categories, but it can get harder for models of weak categories: often, these weak models have existential conditions that guarantee the existence of coherences (sometimes also referred to as 'fillers' or 'contractions'), which means we must add all these coherences as new morphisms while also adding new morphisms that compose existing morphisms, leading to an indefinite process of generating new morphisms. In contrast, for geometric computads there is just one simple step that does it all—boundaries of stratum types can expres all the needed composites and coherences without the need to generate any new extra terms!

Importantly, you can also think about geometric computads topologically, or, rather, *framed topologically*. Recall from story 5 that manifold k-diagrams dualize to cell k-diagrams, which can be canonically subdivided by closed meshes (yielding a subdivision by directed cells which we called 'framed regular cells'). As a special case of this, stratum k-types dualize to 'cell k-types': these are are cell k-diagrams with a single (framed regular) k-cell and all other cells are contained in the closure of that k-cell. Thus, a geometric computad is equivalent a collection of morphisms with associated cell types. Using the quotient maps described in story 5 as attaching maps, we can glue these cell types into a 'framed cell complex' $|C|_{\mathrm{fr}}$ (i.e. a cell complex, in which each cell is the image of a framed regular cell). We refer to this framed cell complex as the *framed realization* of the geometric computad C. Let's think about some examples!

Mini-exercises 8.1 *Come up with a geometric* 2*-computad C with a single 0-morphism, and a single 2-morphism such that the underlying space $|C|$ of the framed realization $|C|_{\mathrm{fr}}$ is the 2-sphere. Similarly, come up with a minimal geometric* 2*-computad such that $|C|$ is a torus. (Link to Solutions)*

While we will not do so here in any detail, it is easy to develop the theory of geometric computads a bit further, and many basic definitions are quite straight-forward to come up with. For instance a *k-morphism* $(\mathbb{R}^k, f)$ in an n-computad C, for *any* $k \in \mathbb{N}(!)$, is of course a manifold diagram together with a labeling $l : \mathscr{E}_0 f \to C_{\leq k}$ such that k-strata s in f have stratum type $f_{l(s)}$ and $l_{l(s)}$ factors through l by the induced map $\mathscr{E}_0 f_{l(s)} \to \mathscr{E}_0 f$. A functor $F : C \to D$ of computads C and D will of course, for each $c \in C_k$, provide us with a k-morphism $F(c)$ of D which is suitably compatible with boundaries of morphisms (to make this 'compatibility' precise it is easiest to pass to the dual world of cellular diagrams and work with stratified subdivisions, see here for a more detailed treatment). One may further define a type of cylinder construction $C \times I$ on computads, which induces a notion of 'cylindrical transformations' by considering functors $\alpha : C \times I \to D$ (while we won't spell out the definition, think of these as natural transformations in which all components of the natural transformations below dimension $(n + 1)$ are isotopies). Together, these constructions amount to something like a *directed Pontryagin theorem*! Why? Well, all we need (namely, dualization and globalization) is already built-into into our theory: any functor $C \to D$ can be turned into into a dual stratification of $|C|_{\mathrm{fr}}$, simply by stratifying open cells c with the manifold diagrams $F(c)$, and any transformation $\alpha : F \Rightarrow G$ similarly induces a 'stratified cobordism' (as a stratification of $|C \times I|_{\mathrm{fr}}$) between two such stratifications!

We will illustrate the close relation of the directed Pontryagin theorem with the classical theorem with several examples in a moment. Importantly, to do so, we first need re-connect with an old friend (or foe?): invertibility. More concretely, having understood how to attach a new morphism to a computads, let's think about how to attach a new *invertible* morphism f to computad. What do we expect from an invertible morphism f? Well, it should have an 'inverse' morphism f^{-1}, and then, surely, we should also have higher morphisms $f \circ f^{-1} \leftrightarrow \mathrm{id}$ and so on. As alluded to at the end of story 6, all this additional data carried by an invertible morphism can be parametrized by framed codimension-1 tangles, or more precisely, by their 'germs': here, an *m-germ* is an m-tangle diagram whose universal manifold diagram subdivision is a stratum typ (these subdivisions were discussed in story 3). The slogan now is: given a morphism f and a framed codimension-1 germ t, there will be a morphism '$\tilde{t}(f)$' that is part of the invertibility data of f. (In fact, we could produce *finite* invertibility data by working only with the 'elementary', i.e. perturbation-stable, germs... that is, if we could classify them; but, at least in low dimensions we know how to do this, cf. [4, §3.3], and we will exploit this to keep things finite in exercises and illustrations below.)

It isn't hard to make the $\tilde{t}(f)$-construction precise, either topologically or combinatorially; we choose the former for visual intuition (the latter is e.g. in [5, Constr. 9.1.3.1]). Let f be a stratum k-type, $k > 0$, and let $t = (W \hookrightarrow \mathbb{R}^{m+1})$ be a framed codimension-1 m-germ. (More generally, the construction will work for *any* framed codimension-1 tangle t! But focussing on germs is convenient for the purpose of morphism/cell attachments.) Recall (cf. exercise 6.1), we can think of the framings combinatorially as a ±-signing of components of $\mathbb{R}^{m+1} \setminus W$. Write f_-, f_0, and f_+, for the slices of $(\mathbb{R}^n, f) \to \mathbb{R}$ over $-1, 0$ resp. $1 \in \mathbb{R}$ (all of these will be manifold $(k-1)$-diagrams). Now define $t(f)$ to be the framed compactly triangulable stratification of $\mathbb{R}^{m+k}$ for which the projection $\mathbb{R}^{m+k} = \mathbb{R}^{m+1} \times \mathbb{R}^{k-1} \to \mathbb{R}^{m+1}$ induces a stratified bundle $t(f) \to t$ with fibers $f_\pm$ over ±-components and f_0 over the tangle manifold W (technically, we also want to ensure fiber attachments $f_\pm \to f$ are those in f). While $t(f)$ is, in fact, a stratified tangle, we can also create a manifold diagram from it: consider the universal subdivision $\tilde{t} \to t$ by a manifold diagram $\tilde{t}$, and pull back the above bundle $t(f) \to t$ along this subdivision—this yields a manifold diagram $\tilde{t}(f)$ (fibered over $\tilde{t}$). The new stratum type $\tilde{t}(f)$ is exactly one of new morphism that we need to add to our computad in order to make the morphism f invertible. Let's illustrate this construction!

Mini-exercises 8.2 *For k-morphisms in the two computads built in the previous exercise add invertibility data up to and including in dimension $k + 2$. Note that by turning newly added $(k + 2)$-morphisms into strict equalities, you will recover the usual 2-categorical data of an 'adjoint equivalence' of 1-morphisms. (Link to Solutions)*

Having an idea of how to attach invertible morphisms to a geometric computad, we can now give a few examples of how the categorical Pontryagin theorem compares to the classical Pontryagin theorem (or rather, the stratified version thereof which deals with general cell complexes, see [1]). I will speak of a *groupoidal computad* to mean a geometric computad constructed by inductively attaching invertible morphisms, and I will distinguishing between the *generating morphisms* and their *invertibility data*, which are morphisms that need to be attached to ensure the generating morphisms are invertible (generating morphism together with their invertibility data will be collectively referred to as a *fat cell*). Concretely, one may be interested in two processes: (1) given a 'groupoidal' computad (i.e. a geometric computad constructed by only attaching invertible morphisms) extract a cell complex; and (2), given a cell complex we extract a computad from it. While we will make neither direction precise here, both direction can be outlined as follows.

For a given computad C, process (1) is, in fact, relatively straight-forward, but there are two approaches: either, we simply construct $|C|_{\mathrm{fr}}$ which gives a 'fattened' version of the cell complex (we realize both generating morphisms and their invertibility data), or, we realize only generating morphisms, 'squishing' invertibility data into attaching maps. The latter approach is based on the observation that, for each generating k-morphism c in a groupoidal computad, the stratum type f_c is naturally a normal framed stratification by the following '$t(f)_{\mathrm{fr}}$-construction'. First, form the unions $\mathrm{fat}(d)$ of strata in f_c belonging to the same fat cell (for other generating morphisms d). Now, starting from the earlier construction of $t(f_d)$, note, if t has normal framing η, then the stratum $W \times \{0\}$ of $t(f_d)$ has $(k-1)$-fold stabilized normal framing $\eta \oplus \epsilon^{k-1}$. These local choices of normal framings, assemble into a global choice for normal framing for the stratum $\mathrm{fat}(d)$. As a result, f_c and ∂f_c become normal framed stratifications. This can now be fed into the classical stratified Pontryagin theorem which produces the topological cell attachment maps α_c for the k-cell c we were seeking—here's a one-sentence-description of these maps: the attaching map $\alpha_c : S^{k-1} \to |C_{k-1}|$ pulls back the normal framed dual stratification of the cell complex $|C_{k-1}|$ to recover, after universal subdivision, the normal framed stratum type boundary $(S^{k-1}, \partial f_c)$ of c.

The converse process (2) is a bit more interesting—after all, we know plenty about cell complexes, but not so much about geometric (or groupoidal geometric) computads, so it would be good if we could translate examples from the former into the latter. This turns out to be more subtle, because of how normal framings work in groupoidal computads: indeed, in groupoidal computad land, normal framings for types ∂f_c of generating k-morphisms c can be produced via the $t(f)_{\mathrm{fr}}$-construction, but this only outputs very specific 'neatly' stabilized normal framed stratifications of the 'directed sphere' $S^{k-1} \hookrightarrow \mathbb{R}^k$. In contrast, if we run the classical Pontryagin construction for attaching maps $\alpha_c : S^{k-1} \to X_k$, and identify S^{k-1} with the directed sphere $S^{k-1} \hookrightarrow \mathbb{R}^{k+1}$, then we might end up with normal framed stratification ∂g_c of the directed sphere that doesn't look 'neat' at all. However, usually (and conjecturally: *always*), we can find a neatly stabilized framing *close by*! That is, up to wiggling the stratification g_c a bit, and then passing to equivalent normal framings of its strata (this amounts to a changing the framing by a continuous choice of rotations homotopic to the identity), we may turn a non-neatly-normal-framed stratification $(S^{k-1} \hookrightarrow \mathbb{R}^k, \partial g_c)$ into one with neat normal framing describing the type boundary $(S^{k-1}, \partial f_c)$ of some computad morphism c. Instead of giving dry details, let's work through this in an example of process (2)!

Consider the minimal cell complex of $\mathbb{C}P^2$ as an example. The cell complex can be build in three stages, and from this we will extract a groupoidal computad with three generating morphisms. The first stage is somewhat trivial: our $\mathbb{C}P^2$ cell complex has a single 0-cell constituting its 0-skeleton $\mathbb{C}P^2_0$ (call that cell x), and so we give our computad C a single 0-morphism (also called x). In the second stage, to build the 2-skeleton $\mathbb{C}P^2_2$, we attach a 2-cell (call it e) to x by the unique attaching map $\alpha_e : \partial D^2 \to D^0$. Since the dual stratification of D^0 is trivial, pulling it back along α_e yields the trivial stratification of $\partial D^2 = S^1$. There are several choices for making the sphere directed, i.e. identifying S^1 with the subpace $S^1 \hookrightarrow \mathbb{R}^2$, but no matter which choice we make we will obtain a neat stratum type boundary. We then run our cone construction $(\mathrm{Cone}(S^1), \mathrm{cone}(S^1))$ to obtain a valid stratum type $(\mathbb{R}^2, f_e)$. We thus attach a 2-morphism e of stratum type f_e to C. Importantly, we now *also* attach all invertibility data for e! Parts of that data (up to dimension 4) is shown in Figure 5.

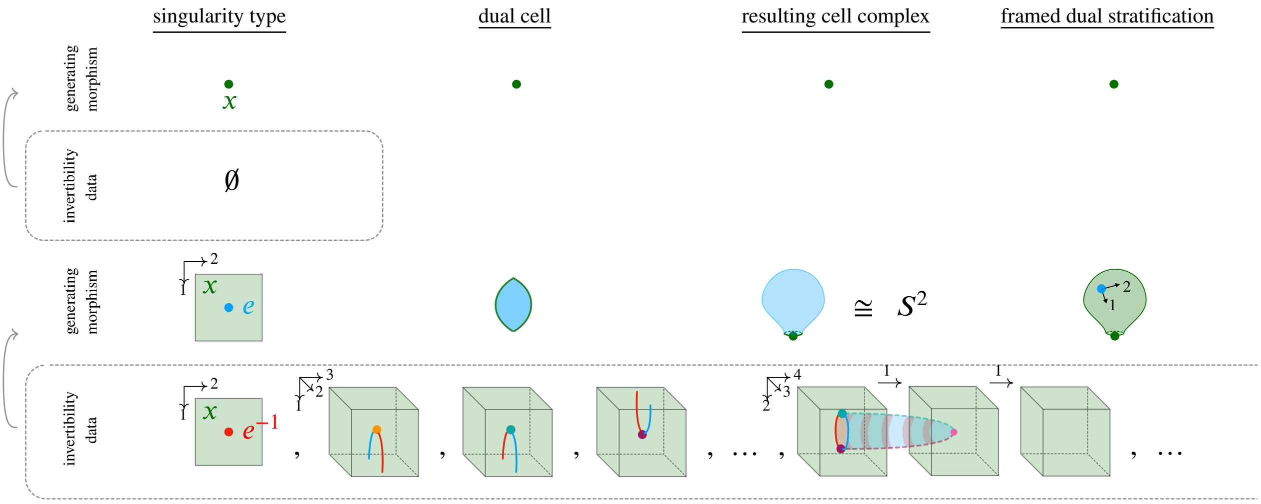

Figure 5: The first two stages of building $\mathbb{C}P^2$ as a groupoidal computad

We now enter the third and last stage. To complete the construction of $\mathbb{C}P^2 = \mathbb{C}P^2_4$, we attach a 4-cell (call it h) by an attaching map $\alpha_h : \partial D^4 = S^3 \to \mathbb{C}P^2_2 = S^2$: this attaching map is the Hopf fibration. Pulling back the normal framed dual stratification of $\mathbb{C}P^2_2$ along α_h yields a stratification ∂g_h of S^3 consisting normal framed embedded circle $e \equiv \alpha_h^{-1}(e)$ (see the normal framed stratified tangle in Figure 6 (A), and note $S^3 \cong \mathbb{R}^3 \cup \{\infty\}$). An important feature of g_h is the normal framing of the circle e rotates by 2π (when compared to the stabilized standard normal framing of $S^1 \hookrightarrow \mathbb{R}^2 \hookrightarrow \mathbb{R}^3$).

Time to translate the stratification ∂g_h into a stratum type boundary! Identify the sphere with the directed sphere $S^3 \hookrightarrow \mathbb{R}^4$. Up to some non-generic situations w.r.t. to the framing of $\mathbb{R}^4$, we can subdivide and label ∂g_h so that it becomes an 'admissible' stratum type boundary for C (i.e. a morphism with that boundary can be attached to C). Unfortunately, we may now run into the aforemention problem with normal framings! As an example, consider (B) in Figure 6 which shows $\partial \tilde{f}_h$ that *would* be a valid stratum type boundary for C; but once we reconstruct the normal framing of the fat cell e, we realize that it differs from that of e in ∂g_h! To resolve this, we need to wiggle the stratification $(S^3 \hookrightarrow \mathbb{R}^4, \partial g_h)$ a little bit, *then* subdivide the wiggled stratification to a stratum type boundary ∂f_h in C, and *then* double check that this boundary correctly represents the normal framing of the fat cell e. Finally, we attach h with stratum type $f_h = \mathrm{cone}(\partial f_h)$ to our computad C (together with invertibility data for h)! To double-check our construction, one may now verify that the realization of $C_{\leq 5}$ (the truncation of C to a 5-computad) has the same homotopy 4-type as $\mathbb{C}P^2$ (you may use either the 'fat' or 'slim' realization).

While $\mathbb{C}P^2$ is a great example to illustrate how to translate (and not to translate) attaching maps of classical cell complexes into groupoidal computads, there are already a range of simpler examples that are very fun to play around with. For instance, stopping after stage 2 in the preceding example you obtain a groupoidal computad

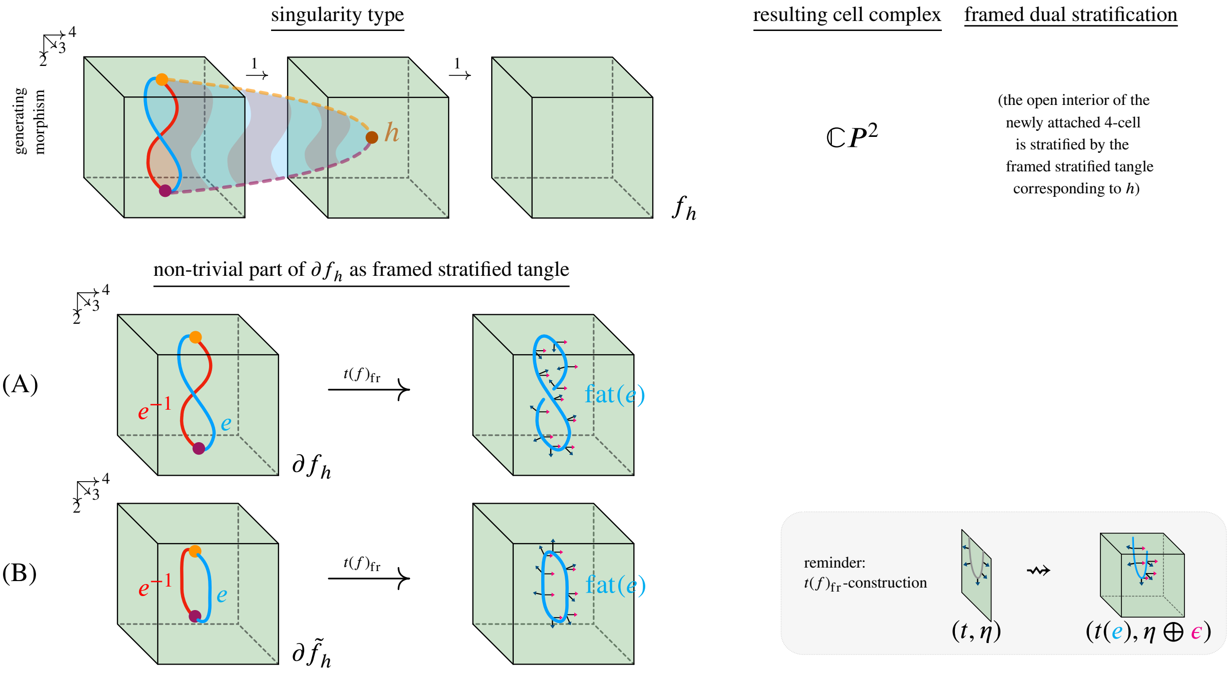

Figure 6: The last stage of building $\mathbb{C}P^2$ as a groupoidal computad

C^{S2} modelling the 2-sphere: you may then use manifold diagrammatic calculus to *prove* $\pi_3 S^2 \cong \mathbb{Z}$ (as long as you believe they model homotopy types). This, and similar fun experiments, will, however, be left to the reader!

Let's use the remaining space in this section to see one more example that relates to a deeper question posed in story 6 about singularities in higher codimension as well as tangles with different tangential structures. For this purpose we will model the Thom space $\mathbf{M}O(2)$ as a groupoidal computad. Recall, $\mathbf{B}O(2)$ can be modelled by the limit $\mathbf{Gr}(2, \infty)$ of the sequence of 2-plane Grassmannians $\mathbf{Gr}(2, n) \hookrightarrow \mathbf{Gr}(2, n+1)$. The tautological 2-plane bundle $\gamma_2(\mathbb{R}^\infty) \to \mathbf{Gr}(2, \infty)$ is the associated 2-plane bundle of an $O(2)$-principal bundle $\mathbf{E}O(2) \to \mathbf{B}O(2)$. Following the usual construction of Thom-spaces, this means $\mathbf{M}O(2)$ is the quotient of the corresponding 2-disk-bundle $p : D(\gamma_\infty(\mathbb{R}^\infty)) \to \mathbf{B}O(2)$ by the boundary $\partial D(\gamma_\infty(\mathbb{R}^\infty))$. From this, given a cell structure of $\mathbf{B}O(2)$ with cells $e : D^k \to \mathbf{B}O(2)$, one can construct a cell structure of $\mathbf{M}O(2)$ by pulling back along p to obtain cells $p^*e : D^k \times D^2 \to D(\gamma_\infty(\mathbb{R}^\infty))$ and post-composing with the quotient $D(\gamma_\infty(\mathbb{R}^\infty)) \to \mathbf{M}O(2)$. (Note, the quotiented subspace $\partial D(\gamma_\infty(\mathbb{R}^\infty))$ becomes the unique 0-cell of $\mathbf{M}O(2)$.) So all we need is a cell structure for $\mathbf{B}O(2)$!

A cell structure of $\mathbf{B}O(2)$ is described by the (beautiful) theory of Schubert cells—a concise presentation of the topic can found in Milnor's and Stasheff's book [9, §5]. This tells us that $\mathbf{Gr}(2, n)$ will have $(i + j)$-cells $e_{i,j}$ for all $i \leq j \leq n - 2$, and how to attach them. To compute the 4-skeleton $\mathbf{M}O(2)_4$, me must compute the 2-skeleton $\mathbf{B}O(2)_2$, and for the latter it suffices to compute the 2-skeleton of $\mathbf{Gr}(2, 2+2)$ (which then lifts along $p : \gamma_2(\mathbb{R}^4) \to \mathbf{Gr}(2, 4)$ to cells in $\mathbf{M}O(2)$). So, in addition to the 0-cell of $\mathbf{M}O(2)$ we fill find: a 2-cell $p^*e_{0,0}$, a 3-cell $e_{0,1}$, and two 4-cells $e_{1,1}$ and $e_{0,2}$. From here, it is not that hard to work out the $(k+1)$-cell attachment map of $p^*e_{i,j}$, *then* to produce a normal framed stratification of S^k, *then* embed $S^k \hookrightarrow \mathbb{R}^{k+1}$, *then* to wiggle that stratification a bit so that it becomes a neatly-normal-framed stratum type boundary, and *then* attach a new morphism with that stratum type boundary to our computad! Up to the homotopically-irrelevant choices involved in this process (such as the passage to directed spheres $S^k \hookrightarrow \mathbb{R}^{k+1}$), the resulting computad $C^{MO(2)}$ is shown in Figure 7 (... I claim I made the nicest possible choices).

So why is this example of an '$\mathbf{M}O(2)$ computad' interesting? Let me first highlight that, of course, nothing in the above discussion was special about dimension 2 (just that codimension-2 tangles are easier to visualize than higher codimensions), and the discussion equally applies to $\mathbf{M}O(k)$ for any k. Yet more generally, we

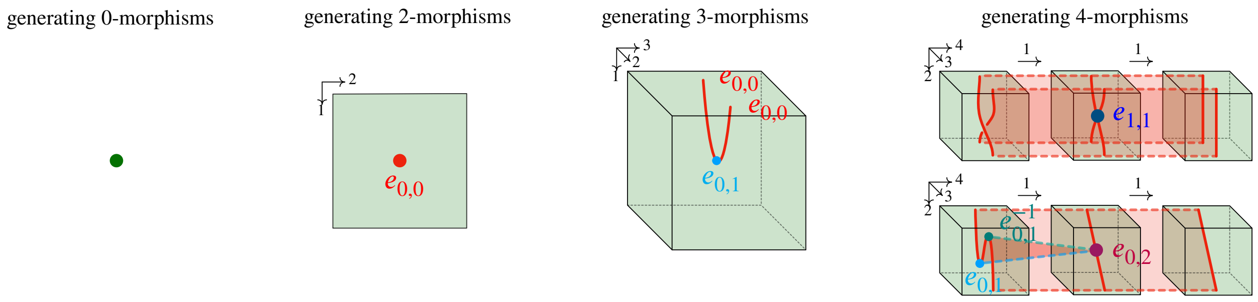

Figure 7: A groupoidal geometric computad modelling $\mathbf{M}O(2)$

could have replaced $O(k)$ by some other structure group $G(k) \to O(k)$ (for instance, $SO(k)$, or $Spin(k)$, etc.). The expectation is that by considering morphisms in the computad $C^{\mathbf{M}G(k)}$ we should obtain a presentation for working with G-structured tangles: indeed, this is precisely what we would expect by analogy with the classical Pontryagin (which states that $\Omega^G_m(\mathbb{R}^n) \cong \pi_n \mathbf{M}G(n-m)$). Crucially, in the case of Schubert cells for $O(k)$, the corresponding cells of the computad $C^{\mathbf{M}O(k)}$ look themselves like tangle germs: we saw this in our example of $C^{\mathbf{M}O(2)}$ where generating 4-morphisms are not tangle germs (after merging all non-ambient strata into a single manifold) but, in fact, they look very *elementary* (namely, they recover some of the 'perturbation-stable' codimension-2 tangle germs described in [4]). This suggests a potential link between elementary singularities in higher codimensions and Schubert calculus via the directed Thom-Pontryagin construction; but how exactly this link could work is ... a very *open question*!

Mini-exercises 8.3 *All computads are 4-computads. Show that* $\pi_3 \mathbf{M}O(2) = 0$ *by considering hom spaces in* $C^{\mathbf{M}O(2)}$*. Produce a 4-morphisms in* $C^{\mathbf{M}O(2)}$ *that, after merging non-ambient strata, is an embedding of the Klein bottle into* $\mathbb{R}^4$*. Show that a similar 4-morphism cannot be produced in* C^{S^2} *(the groupoidal computad of the* 2*-sphere).* Extra question (dimension 5)*: Can you guess a* 5*-morphism with stratum type boundary representing the attaching map of* $e_{0,3}$ *in* $\mathbf{Gr}(2,5)$*? (Hint: look up the swallowtail singularity.) (Link to Solutions)*

9 Towards geometric type theories

Remark. *While this section has seen some improvements, it certainly remains work in progress.*

In the previous two sections we saw the following:

- In story 7 we saw that isotopies in manifold diagrams could be used to think about higher-categorical coherences. In particular, we saw that this applied to manifold diagrams in the 'higher category of higher categories' itself, observing that isotopies could naturally describe laws satisfied by natural transformations, modifications, and their higher analogs.
- In story 8, we saw how our geometric-categorical framework made it very easy to define freely generated structures: coherences and compositions were built into the notion of manifold diagram so no 'multi-step-inductive' process was needed for the construction of these structures. (We also saw examples of how free higher structures with invertible morphisms, i.e. *groupoidal* computads, could model classical cell complexes.)

One could rephrase these points as follows above: manifold diagrams naturally describe behaviour of both 'small' structures, such as finitely generated computads, and 'big' structures, such as the higher category of higher categories. In this story, I want to take this slogan one step further by asking: can we come up with a type theory of geometric computads, by describing the 'internal logic of the geometric computad of all geometric computads'? (As an immediate disclaimer: I do not aim to give a definitive answer to the question at all; at best, the answer you'll find here is a vague 'maybe yes'. Instead, the goal is to inspire people to think about the question more.)

Why could this be an interesting question in the first place? Multiple answers could be given here. To begin note that, as Jacob Lurie once said, the foundations of mathematics have been a *solved* problem for a while now (see this link for a brief discussion of what could constitute a 'universal' solution to the problem). Nonetheless, it turns out researchers keep searching for *better* solutions, and for good reasons: in many ways, existing solutions can still be improved both in their practical usability as well as their mathematical (and philosphical) asthetics. To make the case for geometric higher type theory, I want to mainly focus on the second set of reasons (which sometimes, in the long run, turns out to be not so different from the first set).

Mathematical answer**: higher universes, higher function types and higher inductive types.** Advances in mathematical foundations revealed that we can get away with only a few essential ingredient: universes, dependent (function) types, and inductive types. One example of how these pieces fit together to yield a foundational framework is the 'calculus of inductive constructions' (related, the proof assistats Coq and Lean). Importantly, in the framework of *higher directed* type theory, all of these can be seen in somewhat new and unified light.

- *Universe types can retain their (higher) directed structure.* Universe types are 'internal reflections' of the category of all types of our theory, but without sufficient higher structure in the theory, this reflection process must forget structure present in the category (for instance, the category of sets can only contain a 'set of sets' as an object; the higher structure found in the category itselfs, namely, functions of sets, needs to be modelled separately, see also philosphical discussion below).

- *Dependent function types actually become 'just' (higher) hom types.* Equipped with a universe type $\mathcal{U}$ with higher structure, it turns out that dependent function types and an ordinary function types are not that different: both internally represent hom types. Indeed, while a function types are internal representation of 1-homs $A \to B$, 'generalized' dependent function types are internal representations of 2-homs $F \to G$ where $F, G : A \to \mathcal{U}$ are 1-morphisms in our theory from a type A to the universe $\mathcal{U}$. (The word 'generalized' in the previous sentence can be dropped when F is the constant functor $\mathsf{const}_{\mathbf{1}}$ to the terminal type $\mathbf{1}$ ... we will return to this point in our later examples).

- *Higher inductive types become substantially more expressive.* Much of modern pure mathematics is the mathematics of 'higher structures', and higher inductive types can express many of the involved constructions more directly. Higher inductive have already been investigated in the context of $(\infty, 1)$-theories like homotopy type theories, but in this setting higher structure is always invertible. In full generality, we would allow higher structure that is non-invertible: a basic example (extending the standard example of $\mathbb{N}$ as an ordinary inductive type) is the poset $(\mathbb{N}, \leq)$.[2] This higher inductive type has three (in place of the usual two) constructors as follows:

$$\begin{aligned} 0 &: \mathbf{1} \to \mathbb{N} \\ \mathsf{succ} &: \mathbb{N} \to \mathbb{N} \\ \leq &: \mathrm{id}_{\mathbb{N}} \to \mathsf{succ} \end{aligned}$$

While the above three points are arguments for investigating higher directed type theories in general, the fundamental claim I'd like to make here is that geometric higher categories may provide a particularly pleasant approach to the problem: indeed, higher functoriality conditions are always automatically satisfied, higher homs are well-formed, and higher inductive types can be understood in terms of constructions geometric computads (namely, the universe itself, see later examples). Moreover, the geometric language provided by geometric higher categories yields a fun new perspective on the 'fundamental' rules of classical logic, see Table 3 below.

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[2] This poset turns out to be convenient, for instance, for representing infinite sequences of morphisms. For example, in this post this was using to model (potentially infinite) sequences of β-reductions of lambda calculus terms using higher inductive types.

Classical Logic	Category Theory	Geometry
context rules: exchange, weakening, contraction, and substitution	cartesian monoidality and pullbacks	geometry of embedded diagrams ('stabilization') and blow-up of strata.
(higher) syntax rules: composition and coherence	composition of morphisms and higher coherences	gluing of diagrams and isotopies
reflection rules	'local Grothendieck-type' constructions	geometric 'boundary' constructions

Table 3: Logic vs. Category Theory vs. Geometry

***Philosphical answer*: higher category theory as a universal meta theory.** Somewhat related to the points already made, let me also give a more philosphical perspective on why one might want to study higher directed type theory, in the form of two 'mini-stories' below.

When first learning about mathematical foundations a student may reasonably ask: why exactly *these* rules? Upon digging deeper the student finds that there are, in fact, many approaches to mathematical foundations; while all of them work with some sort of 'statements' (or 'formulas', or 'types') and 'inference rules' for these statements, their detailed workings can vary quite a lot. So maybe foundations are 'arbitrary'? The student cannot but observe that patterns seem to emerge. For instance, in predicate calculus we have quantifiers $\forall/\exists$, and in dependent type theory we have dependent types Π/Σ playing a similar role. Is this just a coincidence, or are there, in fact, some deeper mechanisms at play that make these natural choices for foundational rules? At this point, category theory comes to the student's rescue: using category theory, many of these rules can be interpreted as natural universal constructions. Phew, 'universal' sure does sound convincing! But it comes at a price: the convincing was done using the rules of category theory (which talks, if you will, about categories, functors and natural transformations, and, thus, about the 2-category of categories), and *not* by the '1-categorial' rules of the mathematical foundations the student started with. So the student thinks: it doesn't feel right that there is a non-formalized meta-theory floating around in our heads, that we made use of in the construction and/or verification of our formal foundations. Instead, shouldn't we try to formalize that meta-theory directly? It is at this point that we reach a 'chicken-and-egg' problem: even if we were to formally describe the 'theory of categories', i.e. the internal logic of the 2-category of categories, our rule choices would surely implicitly make use of our intuition about the 3-category of 2-categories (or, at least, we'd have to refer to the latter to verify we made natural choices). Thus, in our meta-theoretical endeavour we see ourselves being pushed to higher and higher dimensions n: only in the limit $n \to \infty$, the pushing stops.

This relates to another problem: the issue of type theories containing models of themselves. As mentioned earlier, one usually finds a 'type of types' $\mathcal{U}$ among the types of our type theory. However, if we start from an n-theory of types (i.e. a theory of $(n-1)$-types, which collectively organize into an n-category), then $\mathcal{U}$ will be an $(n-1)$-type itself; this necessarily involves forgetting some of the structures of n-category of $(n-1)$-types. For instance, for the 1-theory of sets (which are 0-types), the set of (small) sets will have to forget about functions between sets, which are morphisms in the category of sets. Thus, in order to fully talk about the n-theory 'internally' to the n-theory, we need to code up the structure that we just forgot. For our example of set theory, we may encode the forgotten structure by considering a *set of functions* (together with the rest of the structure of an internal category in sets). In contrast, when we let $n \to \infty$ be unbounded, we arrive at a sort of ∞-theory of ∞-types—a category of ∞-types is again an ∞-type, and so an ∞-type universe $\mathcal{U}$ need not forget any categorical structure! One more feature becomes natural when $n \to \infty$: infinite universe *hierarchies*. In n-theories, universe hierarchies must end after $(n+1)$ steps; for example, the category of sets contains a set of sets, which contains an 'element of sets' (i.e. an element corresponding to the set of sets)—but an element is a structureless (-1)-type and the hierarchy ends at this points. In contrast, the higher category of higher categories contains a small model of itself which is again a higher category; this leads to an infinite hierarchy of universes ... $: \mathcal{U}_i : \mathcal{U}_{i+1} :$ ... . In many modern type theories (despite being 1-theories, or $(\infty, 1)$-theories) this is often added as an additional feature!

Both preceding paragraphs make the case for *unbounded higher* categorical foundations, where I take 'unbounded higher' to mean type dimensions $n \to \infty$. (I don't write $n = \infty$ to highlight a small, but mainly philosophical subtlety here: while unbounded higher categorical foundations should address all dimensions, all constructions in these foundations would of course always take place in finite dimension, just that there's no bound on that dimension.) However, at first glance, constructing unbounded higher categorical foundations appears to be a horribly tedious task. But the leading thesis in this story is precisely the opposite: while n-theories seem hard for large n, maybe ∞-theories are not that bad after all; and, yet more concretely, maybe the theory of geometric computads may provide a natural approach the construction of such foundations.

Basic ingredients of geometric type theory. With definitions of manifold diagrams and geometric computads firmly establish, let us outline how a theory of 'geometric types' could potentially be made to work. We distinguish the following ingredients of the theory

1. *Primitive constituents* which include the point **1** and a (hierarchical) universe $\mathcal{U}$.
2. *Logical rules*, which govern context, syntax, and reflection rules (see Table 3).
3. *Inductive constructions* which allow the introduction of new, non-primitive terms.
4. *Inferred definitions* which record 'definitions-by-pattern-matching' (eliminating inductive types).
5. *Equality principles* dealing both with definitional and internal versions of equality.

In the below, I will attempt to use type-theoretic notation with a few important changes which seem to be necessitated by the manifold-diagrammatic set-up: (1) I will write $f : A$ to mean f is a k-morphism in the type A for *any* dimension $\dim(f) = k \in \mathbb{N}$; (2) a k-morphism's j-iterated domain $\partial^j_- f = a$ and codomain $\partial^j_+ f = b$ will be indicated by writing $\partial^j f : a \to b$ (but this is not a typing statement; the type is $f : A$!); (3) I may specify some aspects of the inference rule with natural language (written in brackets); (4) contexts may contain not only assumed variables but also boundary constraints, jointly written $(x : A \mid \partial x : a \to b)$. (In general, I will not be very careful about how contexts work: I use placeholder blobs $\bullet$ to mean the 'appropriate context'.)

B.1. Primitive constituents Within the type of types, there are two elementary type constituents: a smaller type of types and the point **1** (intuitively, the point is the basis from which all other structure is freely constructed; this is analogous to say, the category of sets being the free cocompletion of the point, or, the ∞-category of spaces being the free ∞-cocompletion of the point).

$$\frac{(\text{for } i \in \mathbb{Z})}{\vdash \mathcal{U}_i : \mathcal{U}_{i+1}} \qquad\qquad \frac{(\text{for } i \in \mathbb{Z})}{\vdash \mathbf{1} : \mathcal{U}_i} \tag{\textsc{Univ}}$$

B.2. Logical rules We briefly mention three types of rules (cf. Table 3), without going into details for any of them.

B.2.1. Contexts. Context keep track of assumptions (by variables), and these can be substituted for.

$$\frac{\bullet \vdash a, b : A \ (\partial a = \partial b)}{\bullet, (x : A \mid \partial x = a \to b) \vdash x : A} \qquad\qquad \frac{\bullet, x : A \vdash f(x) : A \quad \bullet \vdash c : A \ (\partial x = \partial c)}{\bullet \vdash f[c/x] : A} \tag{\textsc{Asmn}}$$

Context behave as one would classically expect: assumptions can be reordered (up to respecting the 'poset of variable dependencies'), weakened, and contracted.

B.2.2. Syntax. Terms can be composed and we record this by the following rule.

$$\frac{\bullet \vdash f_1, f_2, ..., f_k : A \ (f_1, .., f_k \text{ composable})}{\bullet \vdash \mathsf{diag}(f_1, f_2, ..., f_k) : A} \qquad (\textsc{Comp})$$

The rule is schematic: $\mathsf{diag}(...)$ represents a (labeled) manifold diagram whose constituent morphisms are those in (...). This, of course, uses that manifold diagrams are combinatorially representable and that composability is easily checked. Note that composition is automatically coherent in an appropriate higher-categorical sense since manifold diagrams can contain isotopies.

B.2.3 Reflection. $\mathcal{U}$ and $\mathbf{1}$ together fulfil an important role: they enable us to translate between type-internal and type-external perspectives on k-morphisms $f : A$ (intuitively, a k-morphism inside the type $A : \mathcal{U}_i$ corresponds to a $(k+1)$-morphism inside $\mathcal{U}_i$ with k-iterated boundary $\mathbf{1} \to A$). Modulo details, we capture this as follows.

$$\frac{\bullet \vdash f : A : \mathcal{U}_i \ (\dim(f) = k)}{\bullet \vdash \mathsf{ext}(f) : \mathcal{U}_i \ (\partial^k \mathsf{ext}(f) : \mathbf{1} \to A)} \qquad \frac{\bullet \vdash f : \mathcal{U}_i \ (\partial^k f : \mathbf{1} \to A)}{\bullet \vdash \mathsf{int}(f) : A : \mathcal{U}_i \ (\dim(\mathsf{int}(f)) = k)} \qquad (\textsc{Refl})$$

What was left implicit in these rules is that ext and int will be compatible with compositions and boundaries as well as being suitably inverse to one another (we will see examples of this). Note the rule in particular applies if $A = \mathcal{U}_{i-1}$, in which case it recovers the usual idea of 'reflection' between universes!

B.3. Inductive types Declarations of inductive types, in computad-lingo, attach a generating k-morphism to a given computad (including the universe itself).[3] Declarations are recorded in a separate memory Λ, and are valid cell attachments in a computad sense. They thus follow a similar form to how new assumptions are added to a context, but are introduced as axioms (without premises):

$$\frac{(a, b : A, \partial a = \partial b)}{\vdash c : A \ (\partial c = a \to b)} \qquad (\textsc{Decl})$$

Declarations of an inductive type A may be marked as *complete*, in which case no later attachement may modify the set of diagrams with (iterated) boundary $\mathbf{1} \to A$.

It is an important point (for the decidability of pattern matching), that at any given point there are only finitely many generators for the inductive types in our theory.

B.4. Inferred definitions Inferred definitions are terms obtained by 'eliminating' inductive types. Intuitively, such definitions describe a morphism that 'should be in $\mathcal{U}_i$ anyways'. To make definitions precise, we will use the principle of pattern matching which generalizes the principle of induction. Roughly speaking, pattern matching defines a k-morphism $f : \mathcal{U}_i$ by describing how the morphism translates a 'complete pattern' of morphisms in the domain type $\partial^k_- f = A$ into morphism of the codomain type $\partial^k_+ f = B$. Here, a 'complete pattern' $[A]$ for A is a finite subset of the rooted tree of constructions of $z : \mathcal{U}_i$ with $\partial^j z : \mathbf{1} \to A$ such that any path from the root must eventually pass through a $[A]$. Crucially, we will need a higher-dimensional version of pattern matching, as we deal not only with functors but also higher transformations. Definitions are recorded in a separate memory Δ.

$$\frac{\bullet \vdash a, b : \mathcal{U}_i \ (\partial a = \partial b)}{\bullet \vdash f : \mathcal{U}_i \ (\text{record } \mathsf{def}(f) \mid \partial f : a \to b)} \qquad \frac{\bullet \vdash f : \mathcal{U}_i \ (\mathsf{def}(f)) \quad \bullet \vdash c : \mathcal{U}_i \ (c : \mathbf{1} \to A)}{\bullet \vdash \mathsf{ev}(f, c) : \mathcal{U}_i \ (c : \mathbf{1} \to B)} \qquad (\textsc{Infr})$$

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[3] In fact, via the reflection rule, we might get away with *only* attaching new morphisms in the universe itself.

The above rules are schematic: on the left, for $k = \dim(a) = \dim(b)$ and $\partial^{k-1}(\partial a = \partial b) : A \to B$, the term $\mathsf{def}(f)$ represents the data of a mapping from a complete pattern $[A]$ of A to morphisms in B which is suitably compatible with composition and the existing actions of a and b (we will give further details in our examples and discussion below). On the right, the term $\mathsf{ev}(f, c)$ schematically represents the manifold-diagrammatic evaluation of f on c based on the definition $\mathsf{def}(f)$ assumed to be given. While the usage of pattern matching here may be rather vague, the 'ultimate' benchmark for definition rules is *canonicity*: that is, however the notion of definitions is formalized, it should not introduce any new morphisms in existing types.

B.5. Equality principles. We distinguish two types of equality: 'definitional' (a form of external equality, that is not present as a term in the theory) and 'internal' equality (called 'equivalence' here, and sometimes also called 'typal' equality).

B.5.1. Definitional equality. Definitional equality can arise in two ways: 'equality by declaration' and 'equality by inference'. The rule for the former may need no further explanation. For the latter, again, we may have to use pattern matching to determine whether two morphisms are the same when evaluated on some complete pattern.

$$\frac{(a, b : A, \partial a = \partial b)}{\vdash a = b} \quad (\text{DECL-EQ}) \qquad \frac{\bullet \vdash f, g : \mathcal{U}_i \ (\mathsf{def}(f) = \mathsf{def}(g))}{\bullet \vdash f = g} \quad (\text{INFR-EQ})$$

On the right, $\mathsf{def}(...) = \mathsf{def}(...)$ schematically represents the aforementioned procedure of comparing two morphisms by pattern matching (the rule is meant to trivially recover reflexivity $f = f$).

B.5.2. Internal equivalence. Internal equality, or 'equivalence', is more difficult to deal with. In essence, an internal equivalence should be a morphism (i.e. term with non-empty boundary) which has a inverse, meaning there must be an internal equivalence witnessing this invertibility, which leads to a co-inductive approach to equivalence. One question that this presents is how equivalence could be lifted to the nicer notion of *coherent* equivalence (see higher adjoint equivalence), which interplay better with isotopies and relates to our discussion of tangle singularities in previous sections. We omit details of how this could be made to work (since we do not know them).

Examples. Before further discussing the omissions and shortcomings of the above specification outline (in particular, the lack of details regarding reflection, definition, and equality), let us attempt to illustrate the underlying *ideas* of these rules a bit more by working through some examples.

E.1. Basic inductive types and elimination by pattern matching. Based on the rules above, here's an imaginary mathematical workflow in geometric type theory. By (UNIV) we have a type $\mathcal{U}_i : \mathcal{U}_{i+1}$ and thus by (DECL) we can declare a 0-morphism $\mathbb{N} : \mathcal{U}_i$. We further declare morphisms $0 : \mathbf{1} \to \mathbb{N}$ and $s : \mathbb{N} \to \mathbb{N}$ (note: while $0, s : \mathcal{U}_i$ by (REFL) the 1-morphisms $s \circ ... \circ s \circ 0$ in $\mathcal{U}_i$ can be thought of as 0-morphisms in $\mathbb{N}$). We have declared something, so now let's define something. Define $c_{\mathbf{1}} : \mathcal{U}_i$ with $\partial c_{\mathbf{1}} : \mathbb{N} \to \mathcal{U}_{i-1}$ by pattern matching on $z : \mathcal{U}_i$ with $\partial z : \mathbf{1} \to \mathbb{N}$. The 'root' pattern is just $[z]$ itself (i.e., up to (REFL), $[z : \mathbb{N}]$). Inspecting our previous assumptions, z can also be derived either as $z = 0$ or, using the (COMP) rule, as $z = (s \circ y)$ with $\partial y : \mathbf{1} \to \mathbb{N}$, so another complete pattern would be $[0, s \circ y]$. But let's just use the root pattern $[z]$: to z with $\partial z : \mathbf{1} \to \mathbb{N}$ we assign $\mathsf{ext}(\mathbf{1})$ with $\partial \mathsf{ext}(\mathbf{1}) : \mathbf{1} \to \mathcal{U}_{i-1}$ (the externalization of $\mathbf{1} : \mathcal{U}_{i-1}$). This defines $c_{\mathbf{1}}$ by the mapping $\mathsf{def}(c_{\mathbf{1}}) = \{z \mapsto \mathsf{ext}(\mathbf{1})\}$. That's our first definition! In the future, we will often omit writing ext and int explicitly.

E.2. Dependent function type as 2-hom type. The bread and butter of dependent type theories are dependent types. How can we model these in geometric type theory? For instance, how can we make an assumption of the form '$x : \Pi_{\mathbb{N}} F$' for an $\mathbb{N}$-indexed type family F? Well, we *easily* can. To quickly create an $\mathbb{N}$-indexed type family, use (DECL) to declare $F : \mathcal{U}_i$ with $\partial F : \mathbb{N} \to \mathcal{U}_i$. Then the traditional assumption '$x : \Pi_{\mathbb{N}} F$' is naturally modelled by the assumption $(x : \mathcal{U}_i \mid \partial x : c_{\mathbf{1}} \to F)$. Indeed, in categorical lingo, '$\Pi_{\mathbb{N}} F$' is the type of sections of the Grothendieck fibration $\int F \to \mathbb{N}$; but sections are bundle maps from the trivial bundle $\mathbb{N} \to \mathbb{N}$; so passing to classifying maps, sections become natural transformation $c_{\mathbf{1}} \to F$... note, these categorical considerations crucially rely on 2-categorical structure (indeed, we learned 2-categories abstractly describe 1-theories!). Okay, now that we understand $\Pi_{\mathbb{N}} F$ as the 'subtype' of $\mathcal{U}_i$ consisting of morphisms with boundary $c_{\mathbf{1}} \to F$, how can we work with this? Well, whenever we have $x : \mathcal{U}_i \mid \partial x : c_{\mathbf{1}} \to F$ and $n : \mathcal{U}_i \mid \partial n : \mathbf{1} \to \mathbb{N}$, then there is an obvious composite in $\mathcal{U}_i$ using (COMP) yielding $x(n) : \mathcal{U}_i \mid \partial^2 x(n) : \mathbf{1} \to \mathcal{U}_{i-1}$ (the manifold-diagrammatic composite $x(n)$ would traditionally be called a 'whiskering'). In this case, we can use (REFL) to internalize to obtain $x(n) : \mathcal{U}_{i-1} \mid \partial x(n) : \mathbf{1} \to F \circ n$ (here, the computation of $\partial x(n)$ uses that int preserves boundaries and is inverse to ext). Using (REFL) once more, we produce a 0-morphism $x(n) : F(n) \equiv F \circ n$. We recover the elimination rule of $\Pi_{\mathbb{N}} F$! And yet more can be recovered, but we leave further experimentation to the reader.

E.3. Dependent pair type as inductive type with one constructor. What about the traditional Σ-type? Similarly, pretty easy. As before consider an $\mathbb{N}$-indexed type family F. Instead of considering mappings into F we consider mappings from it. Using (DECL) add a new 0-morphism $\Sigma_{\mathbb{N}} F : \mathcal{U}_{i-1}$, and, analogous to the definition of $c_{\mathbf{1}}$, use (INFR) to define $c_{\Sigma_{\mathbb{N}} F} : \mathcal{U}_i$ to be the constant functor $\partial_{\Sigma_{\mathbb{N}} F} : \mathbf{1} \to \mathcal{U}_{i-1}$ to $\Sigma_{\mathbb{N}} F$. Now, using (DECL) again, declare a 2-morphism $\mathsf{in} : \mathcal{U}_i \mid \partial \mathsf{in} : F \to c_{\Sigma_{\mathbb{N}} F}$. The type $\Sigma_{\mathbb{N}} F$ thus introduced is a good way of representing the traditional Σ-type internal to geometric type theory. One straight-forward check is to verify that, given $n : \mathbb{N}$ and $w : F(n)$ we can construct $\mathsf{in}_n(w) : \Sigma_{\mathbb{N}} F$. Further experimentation is once more left to the reader. Note that neither for introducing Π nor Σ anything dependent on $\mathbb{N}$ specifically; indeed, constant functors can be also defined on free type variables $X : \mathcal{U}_i$.

E.4. Definitional equality of functors defined by pattern matching. Let's turn to the more difficult topic of equality. Assume another set of morphisms $\mathbb{N}', 0', s' : \mathcal{U}_i$ with $\partial 0' : \mathbf{1} \to \mathbb{N}'$ and $\partial s : \mathbb{N}' \to \mathbb{N}'$. How does that relate to our earlier $\mathbb{N}$? Using (INFR) we define $F : \mathcal{U}_i \mid \mathbb{N} \to \mathbb{N}'$; and this time, we use the complete pattern $[0, s \circ y]$ exhibited earlier. We define $\mathsf{def}(F) = \{0 \mapsto 0', s \circ y \mapsto s' \circ F(y)\}$. (Note the definition is recursive, but this is okay for the purposes of pattern matching, since all instances of 0-morphisms in $\mathbb{N}$ will have been constructed in finitely many steps.) We also define $G : \mathcal{U}_i \mid \mathbb{N} \to \mathbb{N}'$ in a symmetric fashion. Using (COMP) we obtain $GF : \mathcal{U}_i \mid \mathbb{N} \to \mathbb{N}$. We want to use (INFR-EQ) to show that GF and $\mathrm{id}_{\mathbb{N}}$ are equal (as an aside: the existence of j-iterated identites follows from the (COMP) rule by forming trivial manifold diagram $(\mathbb{R}^k, f) \times \mathbb{R}^j$). We use the same complete pattern $[0, s \circ y]$ as before. On 0, we evaluate $GF(0) = 0$ using the defintions of F and G, and thus $GF(0) = \mathrm{id}(0)$. On $s \circ y$ we evaluate $GF(s \circ y) = s \circ GF(y) = s \circ \mathrm{id}(y) = (s \circ y)$ (where, as part of the 'pattern matching process', we allow ourselves to use that, inductively on construction depth, we have $GF(y) = \mathrm{id}(y)$). Symmetrically, we have $FG = \mathrm{id}_{\mathbb{N}'}$. Thus together we have $F, G : \mathcal{U}_i$, with $\partial F : \mathbb{N} \leftrightarrow \mathbb{N}' : \partial G$ and $GF = \mathrm{id}$, $FG = \mathrm{id}$. Replacing $\mathbb{N}$ and $\mathbb{N}'$ by type variables X and Y, we could call the target context $(\bullet \vdash F, G : \mathcal{U}_i,\ FG = \mathrm{id}_Y,\ GF = \mathrm{id}_X)$ a '0-equivalence'.

E.5. Definitional equality of 2-morphisms. What about functors that are equivalences? Assume I, pt_0, pt_1, path_0, $\mathsf{path}_1 : \mathcal{U}_i$ with $\partial \mathsf{pt}_j : \mathbf{1} \to I$ and $\partial \mathsf{path}_j : \mathsf{pt}_j \to \mathsf{pt}_{j+1}$. Use (DECL-EQ) to set $\mathsf{path}_{j+1} \circ \mathsf{path}_j = \mathrm{id}_{\mathsf{pt}_j}$ (where indices are mod 2). Then, a high-level description of the proof that $\mathbf{1}$ and I are '1-equivalent', is the following: define functors $F, G : \mathcal{U}_i$ with boundaries $\mathbf{1} \leftrightarrow I$ by mapping (in forward direction) constantly to 0 and (in backward direction) 0-morphisms to $\mathrm{id}_{\mathbf{1}}$ (note this has boundary $\mathbf{1} \to \mathbf{1}$ as required), and 1-morphisms to $\mathrm{id}_{\mathrm{id}_{\mathbf{1}}}$. Using (INFR) again, we then define 2-morphisms $\alpha, \beta : \mathcal{U}_i$ with $\partial \alpha : GF \to \mathrm{id}_{\mathbf{1}}$ resp. $\partial \beta : FG \to \mathrm{id}_I$ by chosing the data of natural transformations (making this choice goes beyond the traditional concept of 'pattern matching' and is enforced by the requirement of canonicity... we address this central point below!!). Similarly to the previous paragraph, one shows that α and β have on-the-nose inverse α^{-1} and β^{-1}. Abstracting to type variables X and Y, we arrive at a target context $(\bullet \vdash F, G, \alpha, \alpha^{-1}, \beta, \beta : \mathcal{U}_i,\ \alpha\alpha^{-1} = \mathrm{id}_{\mathrm{id}_X},\ \alpha^{-1}\alpha = \mathrm{id}_{FG},\ \beta\beta^{-1} = \mathrm{id}_{\mathrm{id}_Y},\ \beta^{-1}\beta = \mathrm{id}_{GF})$ that could be reasonably called a '1-equivalence'.

Mini-exercises 9.1 *Given a type* $A : \mathcal{U}_i$ *and an object* $a : A$ *(i.e.* $\dim(a) = 0$*), how would you construct the hom functor* $\mathrm{Hom}(a, -)$*? (Link to Solutions)*

Discussion.

D.1. The problem of pattern matching against isotopies. If I would have to name one single obstacle to making the above specification of geometric type theory precise it would be this: the missing finite classification of elementary isotopies. Why? Because, while it is great that isotopies are automatically generated by manifold-diagrammatic composition (i.e. by (Comp)), these automatically generated morphisms also make it hard to guarantuee canonicity (as required by (Infr)).

Let's illustrate this issue, picking up from the last example above. There, we wanted to *define* a 2-morphism $\alpha : \mathcal{U}_i$. Instead, let us *declare* such α (in fact, one way to think about (Infr) is that it's just (Decl) with enough (Decl-Eq) to ensure no new terms are added anywhere else). Let's see what composites we can form. Write $\partial\alpha : F \to G$ and $\partial F = \partial G : C \to D$. Assume $a, b, f : \mathcal{U}_i$ with $\partial a = \partial b : \mathbf{1} \to C$ and $\partial f : a \to b$. Then f and α, both 2-morphisms in $\mathcal{U}_i$, can be composed in an exchange diagram $e_{f,\alpha}$, exactly like that in Figure 2! But now, by (Refl), we generated a new 2-morphism $\mathsf{int}(e_{f,\alpha}) : A$. In fact, the boundary of this morphism (by compatibility of int with boundaries and compositions) is just $\partial\mathsf{int}(e_{f,\alpha}) : \alpha_b \circ F(f) \to G(f) \circ \alpha_a$, where we abbreviated $\alpha_x := \mathsf{int}(\alpha \circ x)$ and $F(w) := \mathsf{int}(F \circ w)$ resp. $G(w) := \mathsf{int}(G \circ w)$. But that's great, our newly declared α automatically satisfies naturality! Yes, but it 'freely' does so by adding a new 2-morphism $\mathsf{int}(e_{f,\alpha})$ to A. This means, if we want to *define* α we need to equate this 2-morphism to something that already exists in A (which is precisely what we did in our above example). The problem of course doesn't end with naturality: in order to guarantee canonicity in general, we need to understand all elementary higher isotopies, of which the exchange is just the simplest example. This makes elementary isotopy classification an important question for geometric type theory: I haven't thought much about the problem myself, but I would be a bit surprised (given the combinatorial theory at our disposal) if it couldn't be solved with enough effort.

D.2. Other general peculiarities. While the canonicity issue for isotopies may be the most pressing matter in making geometric type theory precise, there are many other shortcomings in the above sketch. Obvious issues include missing restrictions on how the mentioned rules may be used, for instance, in the case of (Decl-Eq): of course, you can use the rule to trivialize your universe $\mathbf{1} = \mathcal{U}$ and such situations should likely be avoided. You may also have raised an eyebrow when I started talking about 'subtypes' $(x : A \mid \partial x : a \to b)$, ... aren't these just hom types $\mathrm{Hom}_A(a, b)$ in disguise, and shouldn't hom types be first-class types themselves? For geometric reasons (dimension shifts etc.), this might lead to complications, and thus (at least in the sketch above) hom types are not first-class types. Nonetheless, it turns out there are many ways to work with them. Moreover, it is also possible that hom types could be added to the theory with sufficient care.

D.3. Internal cartesian monoidality and internal equivalences. In addition to missing hom types, there are surely many other things the working type theorist will find lacking the story so far (Q: where is $\mathbf{0}$? A: can be added as an inductive type without constructors. Q: what about $(-)^{\mathrm{op}}$? A: could be added. etc.). Many of these points are left unaddressed here. Though two possible extensions may deserve a special mention, as they will likely be essential to a working theory. First, it could be natural to consider the universe $\mathcal{U}$ as a symmetric monoidal computad. Higher symmetric monoidality can be easily expressed in manifold diagrams (analogous to monoidal 1-categories finding semantics in string diagrams) and featured already in Table 3. But adding monoidality 'internally' adds additional interactions that need to be addressed (for instance, you'd expect the 'external' cartesian monoidality of contexts to interact appropriately with the internal one). In a different direction, you could also attempt to formalize (the aforementioned) notion of internal equivalence. As we've learned from our discussion of tangle germs and groupoidal computads, there's a geometrically natural story to be told here (...but we'd need to understand elementary singularities!). Moreover, having an *internal* notion of invertibility will be absolutely necessary in order to have any chance of 'embedding' existing lower-dimensional type theories which can talk about higher invertible morphisms (like HoTT).

Conclusion and additional remarks

This series involved a wild mixed of topics including stratifications, directed spaces, combinatorial topology, differential structures, computability questions spiced with homology spheres, singularity and higher Morse theory, transfors and computads, homotopy group calculations, enumerative geometry, and, lastly, (some sort of?) type theory. Geometric higher category theory is a place where all of these topics come together, which, I think, makes it an exciting area of research. At the same time the area is very young and not that much has been done or thought about. I hope that one thing achieved by this series of stories is that more people become aware of the existince of the area, and that maybe some people even start to think about it!

Let me also collect some **remarks** that didn't really fit anywhere in the main text. These mainly concern the comparison of the material here with existing literature.

1. In [4] it was convenient to phrase the main definition of manifold diagrams in terms of tame stratifications (the definition chosen here was given as an 'alternative', see [4, Def. 2.1.10]). These definitions are equivalently since, firstly, in [3] it was shown that 'finite triangulations' are always tame (and the proof can be adapted to the case of 'compactly-defined triangulations' considered here), and, secondly, any mesh has a compactly-defined triangulation.

2. Note also that while in [4] (for the purposes for reducing the lengths of proofs) it was technically convenient to assume links themselves to be tame, this is not necessary (see [4, Rmk. 2.1.9]).

3. While in [4] we spoke of 'tame tangles' here I called them 'tangle diagrams'. The latter choice highlights the proximity to manifold diagrams, while still distinguishing the notion from classical conceptions of tangles.

4. Another (ongoing) terminological issue: entrance path poset or fundamental poset? Entrance path truss or fundamental truss? In general, I prefer 'fundamental'. Nonetheless, I used the notion $\mathscr{E}_0$ (in other places, this is Entr or $\mathsf{Exit}^{\mathrm{op}}$) to denote fundamental posets.

5. There is also question of arrow direction conventions. When you have a point p, and attach a k-cell c to that point, then in the fundamental poset of the resulting stratification you could either have an arrow $p \to c$ ('exit' convention) or $c \to x$ ('entrance' convention). I generally prefer the entrance convention, both for representing the directions of attachments, and for the variance of classifying maps: if you have a discrete compact stratified bundle than entrance convention arrows point in the direction of functional mappings of the fibers. In the literature you mainly find the exit convention though.

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