

From zero to manifold diagrams

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Tallinn, 1 June 2023

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Part 2

(we likely skip this)

Part 1

*Please report mistakes and typos to my [current email](#).

Part 0: References + Prerequisites

These notes the 'bare minimum' of self-contained mathematics needed for defining manifold-diagrammatic higher categories. More gentle introductions to manifold diagrams, combinatorial manifold diagrams, and their related models of higher categories exist, cf. in particular:

1. [this paper](#) on manifold diagrams,
2. [this blog post](#) on trusses and [this blog post](#) on geometric higher categories,
3. the *n*Lab articles on [manifold diagrams](#), [trusses](#) and [manifold-diagrammatic categories](#).

Some prerequisites to read these notes include:

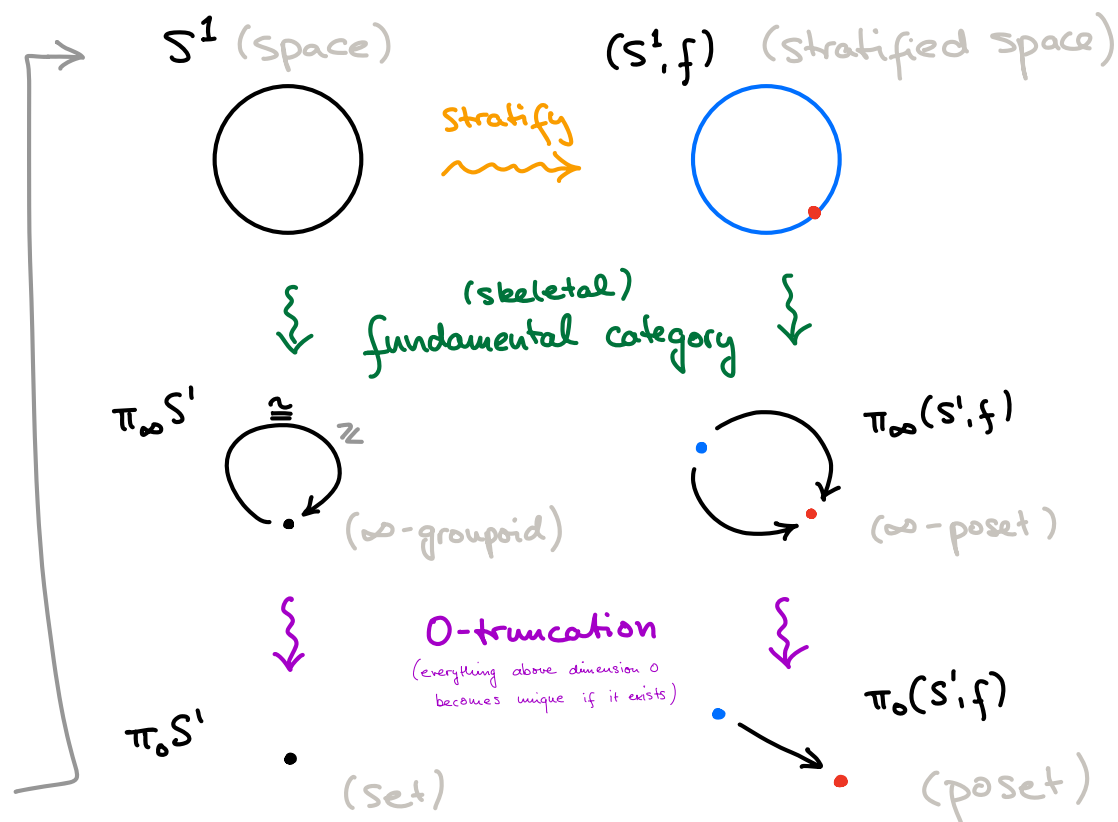
1. Basic familiarity with [category theory](#) (in particular, notions of [hom functors](#), [profunctors](#) and [partial orders as categories](#)).
2. Some familiarity with [higher categories](#) (at least the basic idea of higher morphisms being morphisms between morphisms between ..., and the conceptual [relation of that idea to homotopy theory](#)).
3. Basic intuition about [stratified spaces](#)

1 Trusses \equiv fundament categories of strat. 0-types

1.1 1-Trusses in framed-directed space

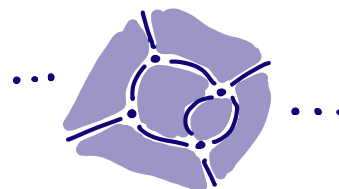
Definition 1.1. A 1-truss $T \equiv (T, \leq, \leq, \dim)$ is a set T with two partial orders (the 'face' order \leq and the 'frame' order \leq) as well as a 'dimension' map $\dim : (T, \leq) \rightarrow [1]^{\text{op}}$ such that

1. $(T, \leq) \cong [n] = (0 \rightarrow 1 \rightarrow \dots \rightarrow n)$
2. either $i < i+1$ or $i+1 < i$ for all $i < n$
3. \dim is conservative.



important e.g. :

regular
cell complexes



Stratified
homotopy
0-types

$(\pi_\infty(x, f) \simeq \text{poset})$

Definition 1.2. A 1-truss map $F : T \rightarrow S$ is a function of sets that preserves both face and frame order. Further,

- F is called *regular* if $\dim \circ F \Rightarrow \dim$,
- F is called *singular* if $\dim \circ F \Leftarrow \dim$,
- F is called *dimension-preserving* if $\dim \circ F = \dim$,

where \Rightarrow and \Leftarrow denote natural transformations of functors $(T, \leq) \rightarrow [1]^{\text{op}} = (0 \leftarrow 1)$. ▲

Notation 1.1. Given a truss T denote by $T_{(i)}$ the subset of objects x of T with $\dim(x) = i$. ▲

1.2 1-Truss bundles

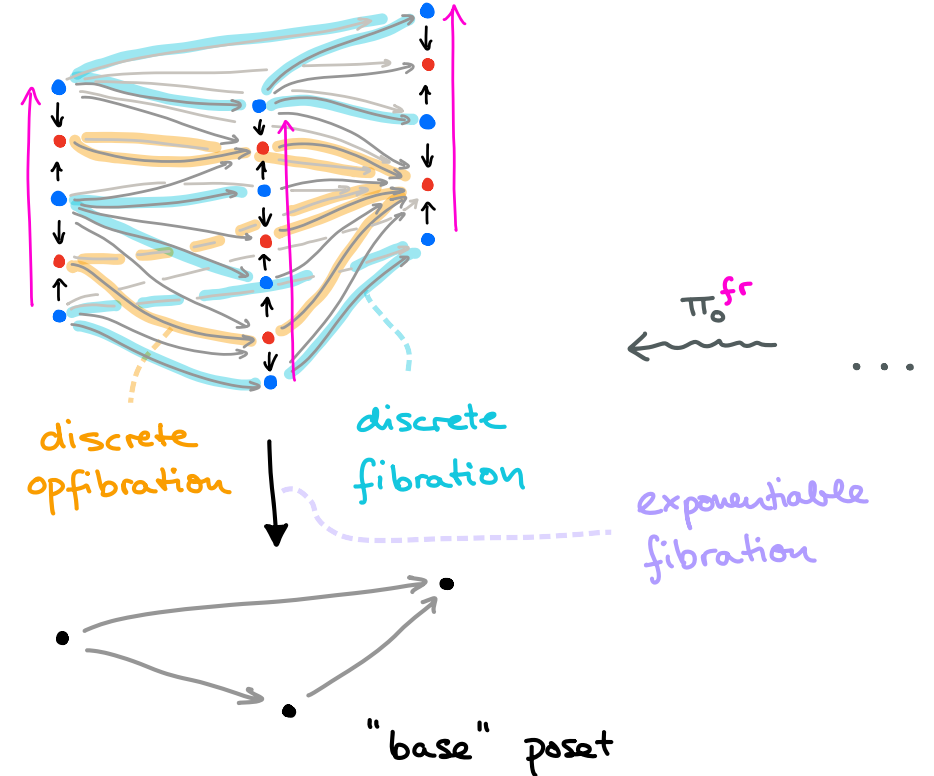
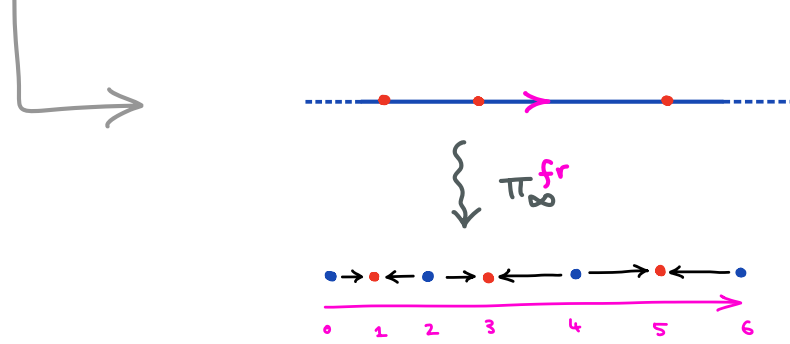
To define bundles of 1-trusses, we first define what are the valid fiber transitions. We dub these ‘1-truss bordisms’.

Remark 1.1. Below, a *Boolean profunctor* is an ordinary profunctor $H : C \nrightarrow D$ whose values are either the initial set $\emptyset \equiv \perp$ or the terminal set $*$ $\equiv \top$. If C and D are discrete, then such a profunctor H is simply a relation of sets. In this case, we call the profunctor H a *function* if it is a functional relation or a *cofunction* if the dual profunctor H^{op} is a function. ▲

Remark 1.2. For any map of posets $F : P \rightarrow Q$, the fiber $F^{-1}(x \rightarrow y)$ over an arrow $x \rightarrow y$ of Q defines a Boolean profunctor $F^{-1}(x) \nrightarrow F^{-1}(y)$ by mapping (a, b) to \top iff $a \rightarrow b$ is an arrow in P . ▲

Definition 1.3. Given 1-trusses T and S , a *1-truss bordism* $R : T \nrightarrow S$ is a Boolean profunctor $T \nrightarrow S$ satisfying the following:

1. R restricts to a function $R_{(0)} : T_{(0)} \rightarrow S_{(0)}$ and a cofunction $R_{(1)} : T_{(1)} \rightarrow S_{(1)}$.
2. Whenever $R(t, s) = \top = R(t', s')$, then either $t < t'$ or $s' < s$ but not both. ▲



Importantly, 1-truss bordisms are morphisms of a category \mathfrak{T}^1 that embeds into the category of profunctors **Prof** (unlike general Boolean profunctors). See the discussion of ‘labels’ below for details.

Definition 1.4. A *1-truss bundle* over a ‘base’ poset (P, \leq) is a poset map $q : (T, \leq) \rightarrow (P, \leq)$ in which, for each $x \in P$, the fiber $(T^x, \leq) = q^{-1}(x)$ is equipped with the additional structures (\leq, \dim) of a 1-truss, and, for each arrow $x \rightarrow y$ in P , the fiber $q^{-1}(x \rightarrow y)$ is a 1-truss bordism $T^x \leftrightarrow T^y$ (cf. Rmk. 1.2). ▲

Definition 1.5. A *1-truss bundle map* $F : q_1 \rightarrow q_2$ of 1-truss bundles $q_i : T_i \rightarrow P_i$ is a map $F : T_1 \rightarrow T_2$ that descends along q_i to a ‘base’ map $F_0 : P_1 \rightarrow P_2$, such that F is fiberwise a 1-truss map (cf. Def. 1.2). We further say F is regular resp. singular resp. dimension-preserving if it is fiberwise so. ▲

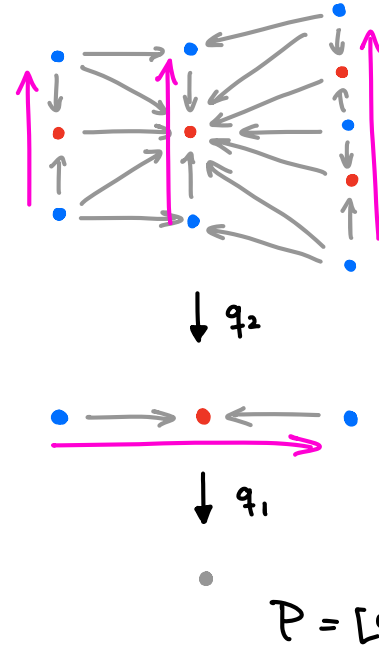
1.3 *n*-Truss bundles and *n*-trusses

Definition 1.6. An *n-truss bundle* T over a poset P is a tower of 1-truss bundles (see Def. 1.4)

$$T_n \xrightarrow{q_n} T_{n-1} \xrightarrow{q_{n-1}} \dots \xrightarrow{q_2} T_1 \xrightarrow{q_1} T_0 = P \quad \blacktriangle$$

Definition 1.7. An *n-truss bundle map* $F : T \rightarrow T'$ is a tower of 1-truss bundle maps $F_i : q_i \rightarrow q'_i$ where F_{i-1} is the base map of F_i and $F_n \equiv F : T_n \rightarrow T'_n$. The adjectives ‘regular’ resp. ‘singular’ resp. ‘dimension-preserving’ apply to F if they apply to all F_i . If T and T' have the same base P , then F is called *base-preserving* if $F_0 = \text{id}_P$. ▲

Terminology 1.1. An *n-truss bundle* over the terminal poset $*$ is called an *n-truss*. ▲



2 Labels

2.1 Truss bundles with labels

Definition 2.1. Given a category C , a *C-labeled n -truss bundle* $T = (\underline{T}, \text{lbl}_T)$ over P consists of an ‘underlying’ n -truss bundle $\underline{T} = (T_n \rightarrow \dots \rightarrow T_1 \rightarrow P)$ together with a ‘labeling’ functor $\text{lbl}_T : T_n \rightarrow C$. In other words, T is of the form

$$C \xleftarrow{\text{lbl}_T} T_n \xrightarrow{q_n} T_{n-1} \xrightarrow{q_{n-1}} \dots \xrightarrow{q_2} T_1 \xrightarrow{q_1} T_0 = P \quad \blacktriangle$$

Remark 2.1. If $C = *$ is the terminal category in the previous definition, then we recover ordinary n -truss bundles. \blacktriangle

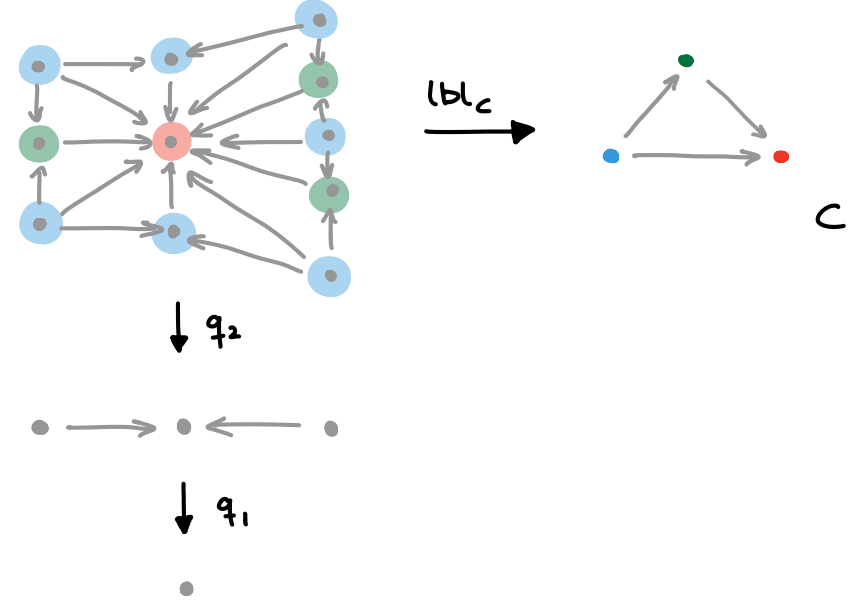
Definition 2.2. A *labeled n -truss bundle map* $F = (\underline{F}, \text{lbl}_F) : T \rightarrow T'$ from a C -labeled n -truss bundle T to a C' -labeled n -truss bundle T' consists of an n -truss bundle map $\underline{F} : \underline{T} \rightarrow \underline{T}'$ and a functor $\text{lbl}_F : C \rightarrow C'$ such that there is a natural isomorphism $\text{lbl}_{T'} \circ \underline{F} \cong \text{lbl}_F \circ \text{lbl}_T$. Adjectives ‘regular’, ‘singular’, ‘dimension-preserving’, ‘base-preserving’ apply if they apply to \underline{F} . Further, we say F is *label-preserving* if $\text{lbl}_F = \text{id}_C$. \blacktriangle

Labeled truss bundles are a central ingredient in truss theory (see [Appendix A](#)).

2.2 Normalization theorem

Definition 2.3. Given C -labeled n -truss bundles T and T' over P , a *reduction* $F : T \rightarrow T'$ is a labeled n -truss bundle map which is:

1. regular;
2. endpoint-dimension-preserving, meaning it is dimension-preserving on the *endpoints* of all 1-truss fibers;
3. base-preserving;
4. label-preserving-on-the-nose, meaning that $\text{lbl}_F = \text{id}$ and there is a strict equality $\text{lbl}_{T'} \circ F = \text{lbl}_T$. \blacktriangle



(framed) stratified 0-type

Terminology 2.1. We sometimes write reductions as $F : T \xrightarrow{\text{red}} T'$, and say T' is a *reduct* of T . A labeled truss whose only reduct is itself is called *normalized* (or, said to be *in normal form*). ▲

Theorem 2.1. (Normalization ends in normal forms). For any labeled truss T , the category $\text{Norm}(T)$ of reducts of T and reductions between them has a unique terminal object $\llbracket T \rrbracket$ (called the normal form of T). ▲

3 Combinatorial manifold diagrams

Certain labeled trusses are the combinatorial analogues of geometric manifold diagrams (see [Appendix B](#)).

3.1 Ingredients

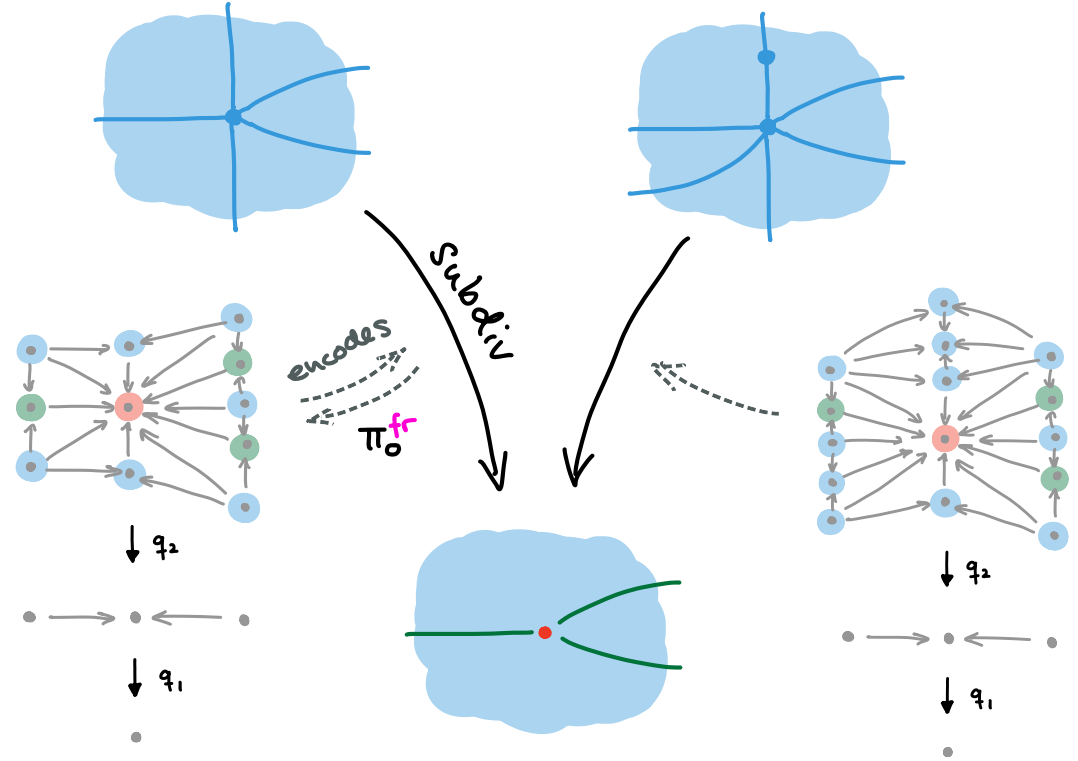
Definition 3.1. (Stratified truss). A *stratified n -truss* T is a labeled n -truss T whose labeling lbl_T is a quotient map of posets whose preimages are connected. ▲

Definition 3.2. (Open truss). A 1-truss is called *open* if its endpoints have dimension 1. A (labeled) n -truss T is called open if all 1-truss fibers in all 1-truss bundles that comprise T are open. ▲

Definition 3.3. (Open neighborhood). Given an open n -truss T and an element $x \in T_n$, define the *neighborhood $T^{\leq x}$* of x to be the unique open truss that comes with a dimension-preserving map $F : T^{\leq x} \hookrightarrow T$ such that $F : T^{\leq x} \hookrightarrow T_n$ is an inclusion of the downward closure of x into the poset (T_n, \leq) . ▲

Terminology 3.1. (Atomic truss). Given an open n -truss T with $x \in T_n$ such that $T^{\leq x} = T$, we say T is an *atomic open n -truss* with *cone point* x . ▲

Definition 3.4. (Stratified open neighborhood). Given a stratified open n -truss T and an element $x \in T_n$, define the *stratified neighborhood $T^{\leq x}$* to be the unique stratified truss that stratified embeds in T with underlying truss map being the (non-stratified) neighborhood inclusion of x . ▲



Terminology 3.2. (Cone types). A stratified open n -truss T is said to be a *combinatorial cone type* if the underlying truss of T is atomic with cone point x , and $\{x\} = \text{lbl}_T^{-1} \circ \text{lbl}_T(x)$ (in words: ‘ x is its own stratum’). ▲

Definition 3.5. (Cylinders). Given a labeled m -truss T defined the *k -cylinder* $\mathbb{I}^k \times T$ of T to be the labeled $(m+k)$ -truss obtained from T by adding k trivial truss bundles $* \rightarrow *$ to its underlying truss. ▲

Remark 3.1. (Products). More generally, one can similarly define labeled $(k+m)$ -trusses $U \times T$ as *products* between unlabeled k -trusses U and labeled m -trusses T . ▲

3.2 Definition

Putting the preceding notions together (and inspired by the geometric definition of manifold diagrams, see [Appendix B](#)), we obtain a combinatorial version of framed conicality as follows.

Definition 3.6. A *combinatorial manifold n -diagram* T is a stratified open n -truss such that for all $x \in T$ we have

$$\llbracket T^{\leq x} \rrbracket = \mathbb{I}^k \times C_x$$

where C_x is a combinatorial cone type. ▲

4 Manifold-diagrammatic higher categories

We define two classes of maps, which together assemble into a double category over which we consider presheafs.

4.1 Embeddings

Definition 4.1. *Embeddings of combinatorial manifold diagrams* T, S are described by spans of the form

Some further highlights

- well-defined links
- dual cellular theory
- framed smooth manifold struct's

...

much remains to be discovered!

(cf. invertibility / tangles / singularities, coherences (isotopies, ...))

$$T \xleftarrow{\text{red}} T' \hookrightarrow S$$

where $T' \xrightarrow{\text{red}} T$ is a reduction and $T' \hookrightarrow S$ is map of labeled trusses whose underlying map is regular and injective. Embeddings compose by pullback composition of spans. \blacktriangle

4.2 Quotients

Definition 4.2. *Quotients of combinatorial manifold diagrams* T, S are maps of labeled trusses

$$T \twoheadrightarrow S$$

whose underlying map of trusses is singular and surjective. \blacktriangle

4.3 Definition

Terminology 4.1. Together, embeddings and quotients organize into the *double category* $\mathbb{M}\text{Diag}_n$ of *combinatorial manifold n -diagrams*, with horizontal morphisms being embeddings, vertical morphisms being quotients, squares being commuting diagrams of the following form:

$$\begin{array}{ccccc} T_1 & \xleftarrow{\text{red}} & T'_1 & \hookrightarrow & S_1 \\ \downarrow & & \downarrow & & \downarrow \\ \Downarrow & & \Downarrow & & \Downarrow \\ T_2 & \xleftarrow{\text{red}} & T'_2 & \hookrightarrow & S_2 \end{array}$$

(note that the dashed arrow is unique if it exists.) The *category of manifold n -diagrams* MDiag_n will refer to just the horizontal part of this double category. \blacktriangle

Remark 4.1. (The $n = \infty$ case). Given a combinatorial manifold n -diagram f , note its cylinder $\mathbb{I} \times f$ is a combinatorial manifold $(n + 1)$ -diagram. This defines a chain of inclusions of (ordinary resp. double) categories, the colimit of which is a category of manifold diagrams MDiag (resp. the double category $\mathbb{M}\text{Diag}$). \blacktriangle

Terminology 4.2. The category MDiag_n (and similarly, $\mathbb{M}\text{Diag}$ from Rmk. 4.1) contains wide subcategories MDiag_n^L resp. MDiag_n^R consisting of spans

$$T \xleftarrow{\text{red}} S \iff S \quad \text{resp.} \quad T \iff T \hookrightarrow S. \quad \blacktriangle$$

We define a coverage for $\mathbf{M}\mathbf{Diag}_n^R$. (Note that $\mathbf{M}\mathbf{Diag}_n^R$ does not have all pullbacks since, e.g., subdiagrams can intersect in non-diagrams.)

Definition 4.3. The *neighborhood coverage* J for $\mathbf{M}\mathbf{Diag}_n^R$ is the coverage that assigns to $T \in \mathbf{M}\mathbf{Diag}$ the single family $\{f_x : T^{\leq x} \rightarrow T\}_{x \in T}$ comprised of the stratified neighborhoods of T . \blacktriangle

Definition 4.4. A *manifold-diagrammatic n -category* C is a presheaf on $\mathbf{M}\mathbf{Diag}_n$ such that:

1. C is a sheaf on $(\mathbf{M}\mathbf{Diag}_n^R, J)$ and locally constant on (i.e. constant on the connected components of) $\mathbf{M}\mathbf{Diag}_n^L$,
2. C extends to a double-(co)presheaf $\mathbf{M}\mathbf{Diag}_n^{\text{op,co}} \rightarrow \mathbf{Set}$ (where \mathbf{Set} is the double category of squares in the category of sets, and C is covariant on vertical categories). \blacktriangle

A Classifying categories for truss bundles

A.1 Classifying 1-truss bundles

Since fiber transitions in 1-truss bundles are 1-truss bordisms, it comes as no surprise that the category of 1-truss bordisms classifies 1-truss bundles.

Definition A.1. Given 1-truss bordisms $R : T \leftrightarrow S$ and $Q : S \leftrightarrow U$, their composite profunctor $R \circ Q$ (composed as ordinary profunctors) is again a 1-truss bordism. (In contrast, composites of general Boolean profunctors (composed as ordinary profunctors) in general need not themselves be Boolean.) This defines the *category \mathfrak{T}^1 of 1-trusses and their bordisms*. \blacktriangle

Theorem A.1. 1-truss bundles over a base poset P up to dimension-preserving base-preserving isomorphism bijectively correspond to functors $P \rightarrow \mathfrak{T}^1$ up to natural isomorphism.

Proof. Follows from the definition of 1-truss bundles. \square

The theorem now generalizes to labelled 1-truss bundles as follows.

Definition A.2. Given a category C , the *category* $\mathfrak{T}^1(C)$ of C -labeled 1-trusses and their bordisms is defined as follows: objects of $\mathfrak{T}^1(C)$ are C -labeled 1-truss bundles over $[0]$; morphisms are C -labeled 1-truss bundles over $[1]$ (with domain and codomain given by restricting to fibers over 0 resp. 1); two morphisms compose to a third iff there is a C -labeled bundle over $[2]$ that restricts over $(0 \rightarrow 1)$, $(1 \rightarrow 2)$, and $(0 \rightarrow 2)$ to the first, second, resp. third morphism. \blacktriangle

Theorem A.2. C -labelled 1-truss bundles over a base poset P up to dimension-preserving base-preserving label-preserving isomorphism bijectively correspond to functors $P \rightarrow \mathfrak{T}^1(C)$ up to natural isomorphism.

Proof. Follows from the definition of $\mathfrak{T}^1(C)$. \square

Remark A.1. (*Recovering the unlabeled case*). In particular, the preceding two definitions coincide $\mathfrak{T}^1(*) \cong \mathfrak{T}^1$ up to (essentially unique!) isomorphism of categories when $C = *$ is terminal. \blacktriangle

Remark A.2. (*Functoriality of labeled bordism*). Importantly, the construction of $\mathfrak{T}(C)$ is functorial in C . Indeed, given a functor $F : C \rightarrow D$, then $\mathfrak{T}^1(F) : \mathfrak{T}^1(C) \rightarrow \mathfrak{T}^1(D)$ acts on objects and morphisms of $\mathfrak{T}^1(C)$ by post-composing their labelings with F . This yields the *labeled bordism* functor

$$\mathfrak{T}^1 : \mathbf{Cat} \rightarrow \mathbf{Cat}.$$

A.2 Classifying n -truss bundles

For a given category $C \in \mathbf{Cat}$ we can thus apply the labeled bordism functor n times to it: the resulting category $\mathfrak{T}^n(C)$ classifies C -labeled n -truss bundles as follows.

Theorem A.3. *C-labelled n -truss bundles over a base poset P up to dimension-preserving base-preserving label-preserving isomorphism bijectively correspond to functors $P \rightarrow \mathfrak{T}^n(C)$ up to natural isomorphism.*

Proof. Inductively apply the previous theorem, starting with the highest 1-truss bundle and working your way downwards. \square

Remark A.3. (*n -Truss bundles over categories*). The theorem makes it evident that nothing would have stopped us from defining n -truss bundles over *categories* B (in place of just posets): indeed, such bundles may be thought of as functors $B \rightarrow \mathfrak{T}^n(*)$ (or $B \rightarrow \mathfrak{T}^n(C)$ in the labeled case). \blacktriangle

Lukas Heidemann points out the following nice perspective on the labeled bordism functor.

Remark A.4. (*Universal construction of the labeled bordism functor*) Applying the profunctorial Grothendieck construction to the (frame-order-forgetting) functor $\mathfrak{T}^1 \rightarrow \mathbf{Prof}$, yields an exponentiable fibration $E\mathfrak{T}^1 \rightarrow \mathfrak{T}^1$. By general nonsense, the composition of the pullback $\mathbf{Cat}_{/\mathfrak{T}^1} \rightarrow \mathbf{Cat}_{/E\mathfrak{T}^1}$ and forgetful functor $\mathbf{Cat}_{/E\mathfrak{T}^1} \rightarrow \mathbf{Cat}$ has a right adjoint $\mathbf{Cat} \rightarrow \mathbf{Cat}_{/\mathfrak{T}^1}$; this adjoint is exactly the functor $C \mapsto \mathfrak{T}^1(C \rightarrow *)$. (Note: more generally, this construction applies to all normal pseudofunctors $H : D \rightarrow \mathbf{Prof}$, where it characterizes the constructions of ‘vertical comma categories’ $H_{//C}$ for such functors H (see [Dorn-Douglas 2021, Term. 2.3.18]) as right adjoints.) \blacktriangle

Remark A.5. (*Labels in ∞ -categories*) The construction of $\mathfrak{T}^1(-)$ generalizes to an endofunctor on ∞ -categories \mathbf{Cat}_∞ , which immediately leads to a notion of truss bundles labeled in ∞ -categories. \blacktriangle

Thus there is a ‘spectrum’ of base/label structures on which we can reasonably consider truss bundles, ranging from posets over to categories to ∞ -categories. Most of the theory works the same across the spectrum. In this article, we work with the simplest possible choice, i.e. with posets (initially as base structure, but later even as label structures for the purpose of defining ‘stratifications’).

B Geometric manifold diagrams and their combinatorialization

(We omit a recollection of stratified topology, see [here](#) for an introduction to stratified spaces.)

B.1 Definition of manifold diagrams

Definition B.1. The standard n -framing of \mathbb{R}^n is the chain of oriented \mathbb{R} -fiber bundles $\pi_i : \mathbb{R}^i \rightarrow \mathbb{R}^{i-1}$ ($1 \leq i \leq n$) with π_i defined to be the map that forgets the last coordinate of \mathbb{R}^i (and fibers carry the standard orientation of \mathbb{R} after identifying $\mathbb{R}^i = \mathbb{R}^{i-1} \times \mathbb{R}$). ▲

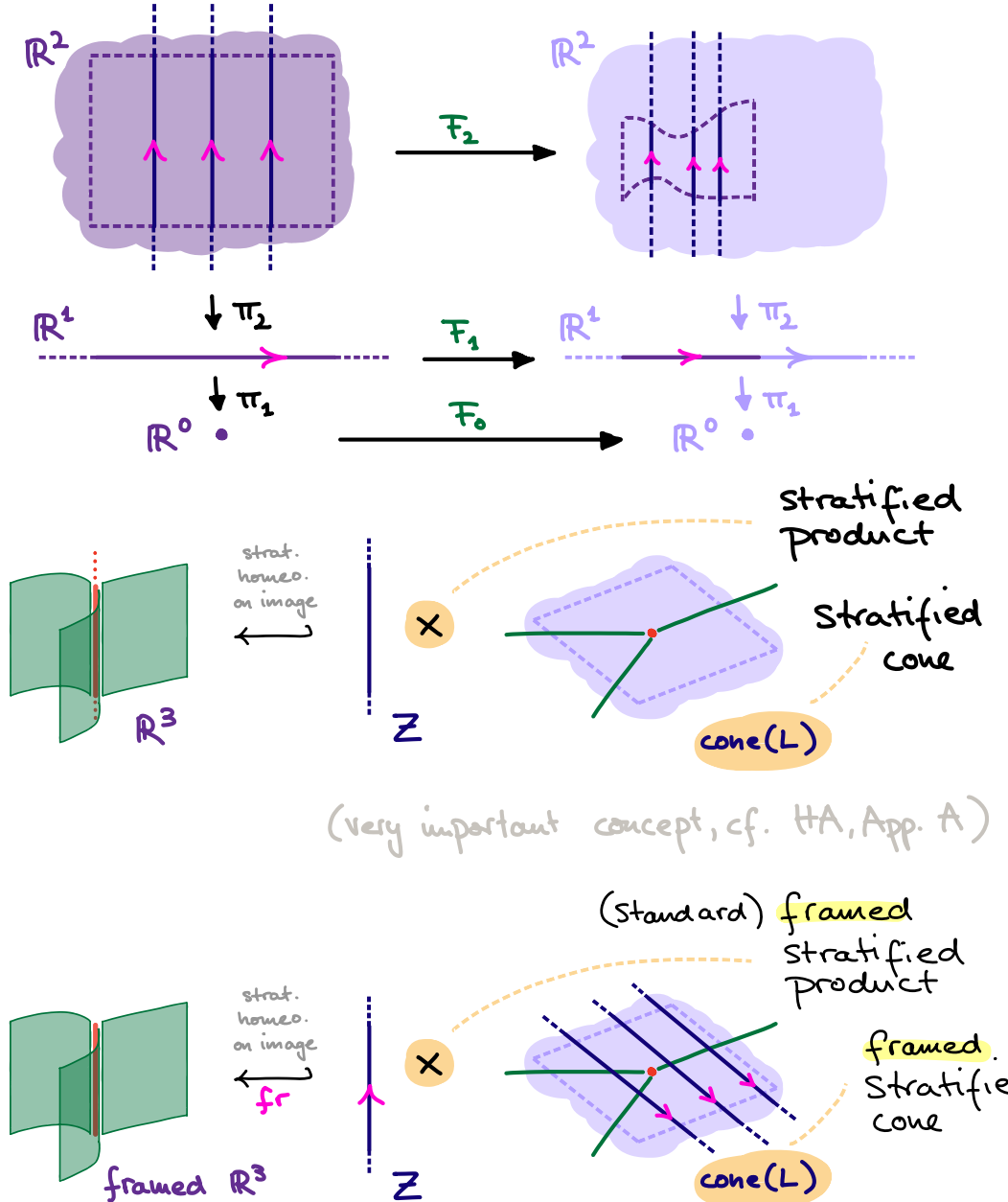
When considering \mathbb{R}^n we tacitly always think of it as ‘standard framed \mathbb{R}^n ’ and, thus, we stop mentioning the standard framing as an explicit structure all-together. Indeed, more important than defining the standard n -framing is to define the maps that preserve it.

Definition B.2. A framed map $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a map for which there exist (necessarily unique) maps $F_j : \mathbb{R}^j \rightarrow \mathbb{R}^j$ ($0 \leq j \leq n$) with $F_n = F$ such that $\pi_i \circ F_i = F_{i-1} \circ \pi_i$ with F_i preserving orientations of fibers of π_i (i.e. mapping fibers strictly monotonously). ▲

A framed stratified map $(\mathbb{R}^n, f) \rightarrow (\mathbb{R}, g)$ is a stratified map whose underlying map $\mathbb{R}^n \rightarrow \mathbb{R}$ is framed. Moreover, when working with products $(\mathbb{R}^k, f) \times (\mathbb{R}^{n-k}, g)$ we will identify $\mathbb{R}^n \cong \mathbb{R}^k \times \mathbb{R}^{n-k}$ in the standard way; and, when working with cone stratifications $(\text{Cone}(S^{n-1}), \text{cone}(l))$, we will standard embed $S^{n-1} \hookrightarrow \mathbb{R}^n$ and identify $\text{Cone}(S^{n-1}) \cong \mathbb{R}^n$ by mapping $(x \in S^{n-1}, \lambda \in [0, 1))$ to $\frac{\lambda}{1-\lambda}x \in \mathbb{R}^n$.

Definition B.3. A stratification (\mathbb{R}^n, f) is framed conical if each point $x \in \mathbb{R}^n$ it has a framed stratified neighborhood (framed) homeomorphic to $\mathbb{R}^k \times (\text{Cone}(S^{n-k-1}), \text{cone}(l))$ with $x \in \mathbb{R}^k \times \{0\}$, where $0 \leq k \leq n$ and (S^{n-k-1}, l) is some stratification. ▲

e.g. framed-directed \mathbb{R}^2



Definition B.4. A compactly-described triangulation K of \mathbb{R}^n is a finite stratification of \mathbb{R}^n by open disks whose closures are the images of linear embeddings $\Delta^k \times \mathbb{R}_{\geq 0}^l \hookrightarrow \mathbb{R}^n$ ($k+l \leq n$). This now translates to the framed stratified case as follows: a stratification (\mathbb{R}^n, f) is **framed compactly triangulable** if it admits a framed stratified subdivision $(\mathbb{R}^n, K) \rightarrow (\mathbb{R}^n, f)$ of f by a compactly-described triangulation K . ▲

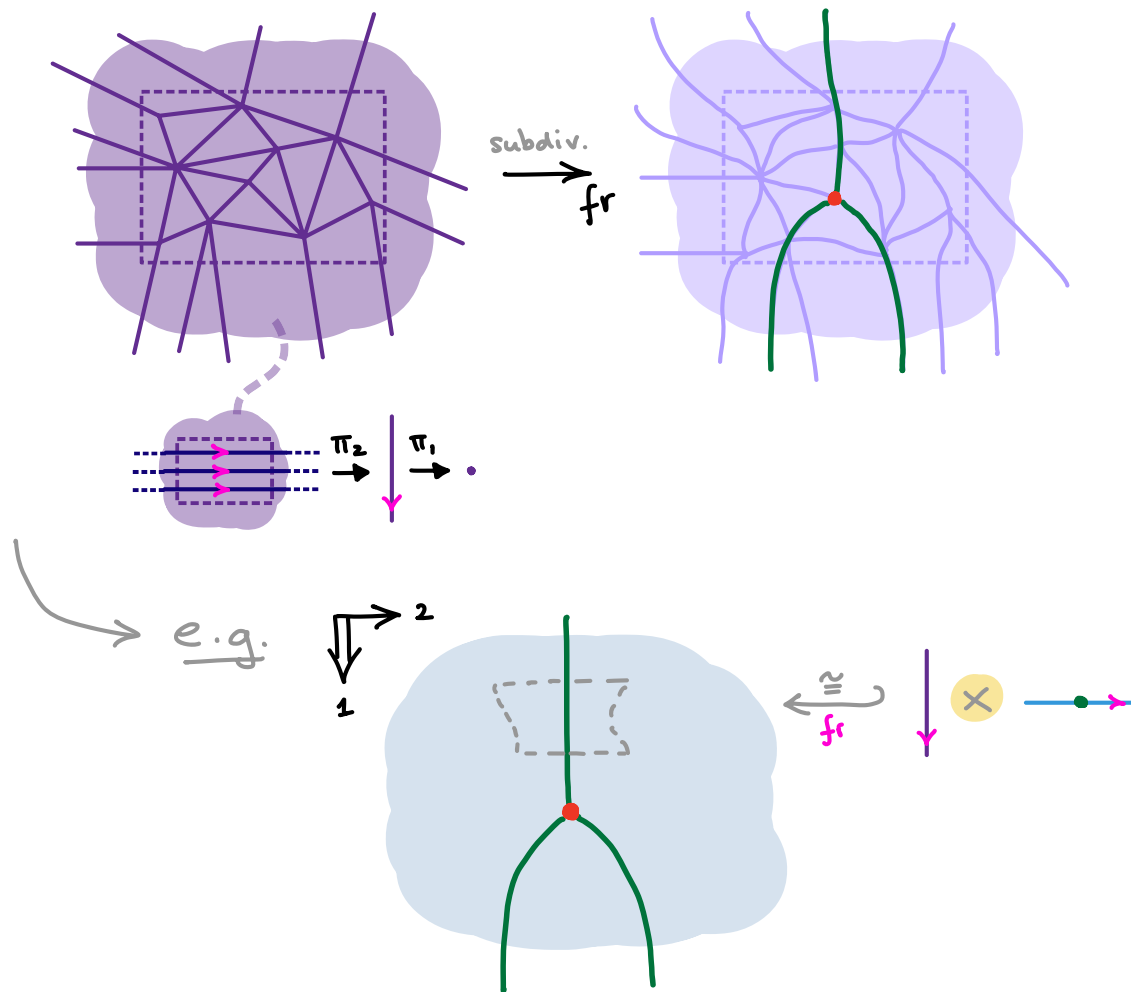
We can now put these concepts together to obtain the following definition of manifold diagrams.

Definition B.5. A **manifold n -diagram** is a framed conical, framed compactly triangulable stratification of \mathbb{R}^n . ▲

B.2 Combinatorialization theorem

Theorem B.1. **Manifold diagrams, up to framed stratified homeomorphism, bijectively correspond to normalized combinatorial manifold n -diagrams.**

Proof sketch. Given a manifold n -diagram (\mathbb{R}^n, f) its corresponding normalized combinatorial manifold n -diagram can be constructed by first refining f by the unique coarsest n -mesh M , and then labeling the n -truss $\text{Entr}(M)$ with the labeling $\text{Entr}(M \rightarrow f)$. □



Theorem B.1. provides a deep link between
stratified geometry and **combinatorics**
 ~ "directed cobordism hypothesis"

~ e.g. can naturally relate

singularities of differentiable manifolds
 vs.

combinatorics of higher morphisms

... another time!

