From zero to manifold-diagrammatic higher categories

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Abstract

We write out a self-contained definition of manifold-diagrammatic higher categories with no fluff provided.

Contents

Introduction

This document contains the 'bare minimum' of self-contained mathematics needed for defining manifold-diagrammatic higher categories. More gentle introductions to manifold diagrams, combinatorial manifold diagrams, and their related models of higher categories exist, cf. in particular:

1. this paper on manifold diagrams,
2. this blog post on trusses and this blog post on geometric higher categories,
3. the nLab articles on manifold diagrams, trusses and manifold-diagrammatic categories.

*Please report mistakes and typos to my current email.

1 Trusses

1.1 1-Trusses

Definition 1.1. A *1-truss* $T \equiv (T, \leq, \preceq, \dim)$ is a set T with two partial orders (the 'face' order $\leq$ and the 'frame' order $\preceq$) as well as a 'dimension' map $\dim : (T, \leq) \to [1]^{\mathrm{op}}$ such that

1. $(T, \preceq) \cong [n] = (0 \to 1 \to \ldots \to n)$
2. either $i < i+1$ or $i+1 < i$ for all $i \prec n$
3. dim is conservative. ▲

Definition 1.2. A *1-truss map* $F : T \to S$ is a function of sets that preserves both face and frame order. Further,

- F is called *regular* if $\dim \circ F \Rightarrow \dim$,
- F is called *singular* if $\dim \circ F \Leftarrow \dim$,
- F is called *dimension-preserving* if $\dim \circ F = \dim$,

where $\Rightarrow$ and $\Leftarrow$ denote natural transformations of functors $(T, \leq) \to [1]^{\mathrm{op}} = (0 \leftarrow 1)$. ▲

Notation 1.1. Given a truss T, denote by $T_{(i)}$ the subset of objects x with $\dim(x) = i$. ▲

1.2 1-Truss bundles

To define bundles of 1-trusses, we first define what are the valid fiber transitions. We dub these '1-truss bordisms'.

Remark 1.1. Below, a *Boolean profunctor* is an ordinary profunctor $H : C \nrightarrow D$ whose values are either the initial set $\emptyset \equiv \bot$ or the terminal set $* \equiv \top$. If C and D are discrete, then such a profunctor H is simply a relation of sets. In this case, we call the profunctor H a *function* if it is a functional relation or a *cofunction* if the dual profunctor H^{op} is a function. ▲

Remark 1.2. For any map of posets $F : P \to Q$, the fiber $F^{-1}(x \to y)$ over an arrow $x \to y$ of Q defines a Boolean profunctor $F^{-1}(x) \nrightarrow F^{-1}(y)$ by mapping (a, b) to $\top$ iff $a \to b$ is an arrow in P. ▲

Definition 1.3. Given 1-trusses T and S, a *1-truss bordism* $R : T \nrightarrow S$ is a Boolean profunctor $T \nrightarrow S$ satisfying the following:

1. R restricts to a function $R_{(0)} : T_{(0)} \nrightarrow S_{(0)}$ and a cofunction $R_{(1)} : T_{(1)} \nrightarrow S_{(1)}$.
2. Whenever $R(t, s) = \top = R(t', s')$, then either $t \prec t'$ or $s' \prec s$ but not both. ▲

Importantly, 1-truss bordisms are morphisms of a category $\mathfrak{T}^1$ that embeds into the category of profunctors **Prof** (unlike general Boolean profunctors). Cf. the discussion of classifying categories in Appendix A.

Definition 1.4. A *1-truss bundle* over a 'base' poset $(P, \leq)$ is a poset map $q : (T, \leq) \to (P, \leq)$ in which, for each $x \in P$, the fiber $(T^x, \leq) = q^{-1}(x)$ is equipped with the additional structures $(\preceq, \dim)$ of a 1-truss, and, for each arrow $x \to y$ in P, the fiber $q^{-1}(x \to y)$ is a 1-truss bordisms $T^x \nrightarrow T^y$ (cf. Rmk. 1.2). ▲

Definition 1.5. A *1-truss bundle map* $F : q_1 \to q_2$ of 1-truss bundles $q_i : T_i \to P_i$ is a map $F : T_1 \to T_2$ that factors through q_i by a 'base' map $F_0 : P_1 \to P_2$, such that F is fiberwise a 1-truss map (cf. Def. 1.2). We further say F is regular resp. singular resp. dimension-preserving if it is fiberwise so. ▲

1.3 n-Truss bundles and n-trusses

Definition 1.6. An *n-truss bundle T* over a poset P is a tower of 1-truss bundles (see Def. 1.4)

$$T_n \xrightarrow{q_n} T_{n-1} \xrightarrow{q_{n-1}} \cdots \xrightarrow{q_2} T_1 \xrightarrow{q_1} T_0 = P \qquad \blacktriangle$$

Definition 1.7. An *n-truss bundle map* $F : T \to T'$ is a tower of 1-truss bundle maps $F_i : q_i \to q_i'$ where F_{i-1} is the base map of F_i and $F_n \equiv F : T_n \to T_n'$. The adjectives 'regular' resp. 'singular' resp. 'dimension-preserving' apply to F if they apply to all F_i. If T and T' have the same base P, then F is called *base-preserving* if $F_0 = \mathrm{id}_P$. ▲

Terminology 1.1. An n-truss bundle over the terminal poset $*$ is called an *n-truss*. ▲

2 Labels

2.1 Truss bundles with labels

Definition 2.1. Given a category C, a *C-labeled n-truss bundle* $T = (\underline{T}, \mathsf{lbl}_T)$ over P consists of an 'underlying' n-truss bundle $\underline{T} = (T_n \to \ldots \to T_1 \to P)$ together with a 'labeling' functor $\mathsf{lbl}_T : T_n \to C$. In other words, T is of the form

$$C \xleftarrow{\mathsf{lbl}_C} T_n \xrightarrow{q_n} T_{n-1} \xrightarrow{q_{n-1}} \cdots \xrightarrow{q_2} T_1 \xrightarrow{q_1} T_0 = P \qquad \blacktriangle$$

Remark 2.1. If $C = *$ is the terminal category in the previous definition, then we recover ordinary n-truss bundles. ▲

Definition 2.2. A *labeled n-truss bundle map* $F = (\underline{F}, \mathsf{lbl}_F) : T \to T'$ from a C-labeled n-truss bundle T to a C'-labeled n-truss bundle T' consists of an n-truss bundle map $\underline{F} : \underline{T} \to \underline{T'}$ and a functor $\mathsf{lbl}_F : C \to C'$ such that $\mathsf{lbl}_{T'} \circ \underline{F} \cong \mathsf{lbl}_F \circ \mathsf{lbl}_T$ commutes up to natural isomorphism. Adjectives 'regular', 'singular', 'dimension-preserving', 'base-preserving' apply if they apply to $\underline{F}$. Further, we say F is *label-preserving* if $\mathsf{lbl}_F = \mathrm{id}_C$. ▲

Labeled truss bundles are a central ingredient in truss theory (see Appendix A).

2.2 Normalization theorem

Definition 2.3. Given C-labeled n-truss bundles T and T' over P, a *normalizing map* $F : T \to T'$ is a labeled n-truss bundle map which is:

1. regular;
2. endpoint-dimension-preserving, meaning it is dimension-preserving on the *endpoints* of all 1-truss fibers;
3. base-preserving;
4. label-preserving-on-the-nose, meaning that $\mathsf{lbl}_F = \mathrm{id}$ and $\mathsf{lbl}_{T'} \circ F = \mathsf{lbl}_T$ commutes strictly. ▲

(There's a dual version of the definition that replaces 'regular' by 'singular'.)

Terminology 2.1. We sometimes write normalizing maps as $F : T \xrightarrow{\mathsf{norm}} T'$, and say T' is a *reduct* of T. A labeled truss whose only reduct is itself is called *normalized* (or, said to be *in normal form*). ▲

Theorem 2.1. *(Normalization ends in normal forms). For any labeled truss T, the category $\mathsf{Norm}(T)$ of reducts of T and normalizing maps between them has a unique terminal object $[\![T]\!]$ (called the* normal form *of T).*

3 Combinatorial manifold diagrams

Certain labeled trusses are the 'combinatorial' analogues of geometric manifold diagrams (see Appendix B).

3.1 Ingredients

Definition 3.1. (Stratified truss). A *stratified n-truss* T is a labeled n-truss T whose labeling lbl_T is a quotient map of posets whose preimages are connected. ▲

Definition 3.2. (Open truss). A 1-truss is called *open* if its endpoint have dimension 1. A (labeled) n-truss T is called open if all 1-truss fibers in all 1-truss bundles that comprise T are open. ▲

Definition 3.3. (Open neigborhood). Given an open n-truss T and an element $x \in T_n$, define the *neighborhood* $T^{\leq x}$ of x to be the unique open truss that comes with a dimension-preserving map $F : T^{\leq x} \hookrightarrow T$ such that $F : T_n^{\leq x} \hookrightarrow T_n$ is an inclusion of the downward closure of x into the poset $(T_n, \leq)$. ▲

Terminology 3.1. (Atomic truss). Given an open n-truss T with $x \in T_n$ such that $T^{\leq x} = T$, we say T is an *atomic* open n-truss with *cone point* x. ▲

Definition 3.4. (Stratified open neighborhood). Given a stratified open n-truss T and an element $x \in T_n$, define the *stratified neighborhood* $T^{\leq x}$ to be the unique stratified trusses that stratified embeds in T with underlying truss map being the (non-stratified) neighborhood inclusion of x. ▲

Terminology 3.2. (Cone types). A stratified open n-truss T is said to be a *combinatorial cone type* if the underlying truss of T is atomic with cone point x, and $\{x\} = \mathsf{lbl}_T^{-1} \circ \mathsf{lbl}_T(x)$ (in words: 'x is its own stratum'). ▲

Definition 3.5. (Cylinders). Given a labeled m-truss T defined the *k-cylinder* $\mathbb{I}^k \times T$ of T to be the labeled $(m + k)$ obtained from T be adding k trivial truss bundles $* \to *$ to its underlying truss. ▲

Remark 3.1. (Products). More generally, one can similarly define labeled $(k + m)$-trusses $U \times T$ as *products* between unlabeled k-trusses U and labeled m-trusses T. ▲

3.2 Definition

Putting the preceding notions together (and inspired by the geometric definition of manifold diagrams, see Appendix B), we obtain a combinatorial version of framed conicality as follows.

Definition 3.6. A *combinatorial manifold n-diagram* T is a stratifed open n-truss such that for all $x \in T$ we have

$$[\![T^{\leq x}]\!] = \mathbb{I}^k \times C_x$$

where C_x is a combinatorial cone type. ▲

4 Manifold-diagrammatic higher categories

We define two classes of maps, which together assemble into a double category over which we consider presheafs.

4.1 Embeddings

Definition 4.1. *Embeddings of combinatorial manifold diagrams* T, S are described by spans of the form

$$T \xleftarrow{\mathsf{norm}} T' \hookrightarrow S$$

where $T' \xrightarrow{\mathsf{norm}} T$ is a normalizing map and $T' \hookrightarrow S$ is map of labeled trusses whose underlying map is a regular and injective. Embeddings compose by pullback composition of spans. ▲

4.2 Quotients

Definition 4.2. *Quotients of combinatorial manifold diagrams* T, S are maps of labeled trusses

$$T \twoheadrightarrow S$$

whose underlying map of trusses is singular and surjective. ▲

4.3 Definition

Terminology 4.1. Together, embeddings and quotients organize into the *double category* $\mathbb{M}\mathsf{Diag}_n$ *of combinatorial manifold n-diagrams*, with horizonal morphisms being embeddings, vertical morphisms being quotients, squares being commuting diagrams of the following form:

$$\begin{array}{ccccc} T_1 & \xleftarrow{\text{norm}} & T_1' & \hookrightarrow & S_1 \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ T_2 & \xleftarrow{\text{norm}} & T_2' & \hookrightarrow & S_2 \end{array}$$

(note that the dashed arrow is unique if it exists.) The *category of manifold n-diagrams* MDiag_n will refer to just the horizontal part of this double category. ▲

Remark 4.1. (The $n = \infty$ case). Given a combinatorial manifold n-diagram f, note its cylinder $\mathbb{I} \times f$ is a combinatorial manifold $(n+1)$-diagram. This defines a chain of inclusions of (ordinary resp. double) categories, the colimit of which is a category of manifold diagrams MDiag (resp. the double category $\mathbb{M}\mathsf{Diag}$). ▲

Terminology 4.2. The category MDiag_n (and similarly, MDiag from Rmk. 4.1) contains wide subcategories MDiag^L_n resp. MDiag^R_n consisting of spans

$$T \xleftarrow{\text{norm}} S = S \qquad \text{resp.} \qquad T = T \hookrightarrow S.$$ ▲

We define a coverage for MDiag^R_n. (Note that MDiag^R_n does not have all pullbacks since, e.g., subdiagrams can intersect in non-diagrams.)

Definition 4.3. The *neighborhood coverage* J for MDiag^R_n is the coverage that assigns to $T \in \mathsf{MDiag}$ the single family $\{f_x : T^{\leq x} \to T\}_{x \in T}$ comprised of the stratified neighborhoods of T. ▲

Definition 4.4. A *manifold-diagrammatic n-category* C is a presheaf on MDiag_n such that:

1. C is a sheaf on (MDiag^R_n, J) and locally constant on (i.e. constant on the connected components of) MDiag^L_n,
2. C extends to a double-(co)presheaf $\mathbb{M}\mathsf{Diag}^{\mathrm{op,co}}_n \to \mathbb{S}\mathbf{et}$ (where $\mathbb{S}\mathbf{et}$ is the double category of squares in the category of sets, and C is covariant on vertical categories). ▲

A Classifying categories for truss bundles

A.1 Classifying 1-truss bundles

Since fiber transitions in 1-truss bundles are 1-truss bordisms, it comes as no surprise that the category of 1-truss bordisms classifies 1-truss bundles.

Definition A.1. Given 1-truss bordisms $R : T \nrightarrow S$ and $Q : S \nrightarrow U$, their composite profunctor $R \circ Q$ (composed as ordinary profunctors) is again a 1-truss bordism. (In contrast, composites of general Boolean profunctors (composed as ordinary profunctors) in general need not themselves be Boolean.) This defines the *category* $\mathfrak{T}^1$ *of 1-trusses and their bordisms*. ▲

Theorem A.1. *1-truss bundles over a base poset P up to dimension-preserving base-preserving isomorphism bijectively correspond to functors $P \to \mathfrak{T}^1$ up to natural isomorphism.*

Proof. Follows from the definition of 1-truss bundles. □

The theorem now generalizes to labelled 1-truss bundles as follows.

Definition A.2. Given a category C, the *category $\mathfrak{T}^1(C)$ of C-labeled 1-trusses and their bordisms* is defined as follows: objects of $\mathfrak{T}^1(C)$ are C-labeled 1-truss bundles over [0]; morphisms are C-labeled 1-truss bundles over [1] (with domain and codomain given by restricting to fibers over 0 resp. 1); two morphisms compose to a third iff there is a C-labeled bundle over [2] that restricts over $(0 \to 1)$, $(1 \to 2)$, and $(0 \to 2)$ to the first, second, resp. third morphism. ▲

Theorem A.2. *C-labelled 1-truss bundles over a base poset P up to dimension-preserving base-preserving label-preserving isomorphism bijectively correspond to functors $P \to \mathfrak{T}^1(C)$ up to natural isomorphism.*

Proof. Follows from the definition of $\mathfrak{T}^1(C)$. □

Remark A.1. *(Recovering the unlabeled case).* In particular, the preceding two definitions coincide $\mathfrak{T}^1(*) \cong \mathfrak{T}^1$ up to (essentially unique!) isomorphism of categories when $C = *$ is terminal. ▲

Remark A.2. *(Functoriality of labeled bordism).* Importantly, the construction of $\mathfrak{T}(C)$ is functorial in C. Indeed, given a functor $F : C \to D$, then $\mathfrak{T}^1(F) : \mathfrak{T}^1(C) \to \mathfrak{T}^1(D)$ acts on objects and morphisms of $\mathfrak{T}^1(C)$ by post-composing their labelings with F. This yields the *labeled bordism* functor

$$\mathfrak{T}^1 : \mathbf{Cat} \to \mathbf{Cat}.$$

A.2 Classifying n-truss bundles

For a given category $C \in \mathbf{Cat}$ we can thus apply the labeled bordism functor n times to it: the resulting category $\mathfrak{T}^n(C)$ classifies C-labeled n-truss bundless as follows.

Theorem A.3. *C-labelled n-truss bundles over a base poset P up to dimension-preserving base-preserving label-preserving isomorphism bijectively correspond to functors $P \to \mathfrak{T}^n(C)$ up to natural isomorphism.*

Proof. Inductively apply the previous theorem, starting with the highest 1-truss bundle and working your way downwards. □

Remark A.3. *(n-Truss bundles over categories).* The theorem makes it evident that nothing would have stopped us from defining n-truss bundles over *categories* B (in place of just posets): indeed, such bundles may be thought of as functors $B \to \mathfrak{T}^n(*)$ (or $B \to \mathfrak{T}^n(C)$ in the labeled case). ▲

Lukas Heidemann points out the following nice perspective on the labeled bordism functor.

Remark A.4. *(Universal construction of the labeled bordism functor)* Applying the profunctorial Grothendieck construction to the (frame-order-forgetting) functor $\mathfrak{T}^1 \to \mathbf{Prof}$, yields an exponentiable fibration $E\mathfrak{T}^1 \to \mathfrak{T}^1$. By general nonsense, the composition of the pullback $\mathbf{Cat}_{/\mathfrak{T}^1} \to \mathbf{Cat}_{/E\mathfrak{T}^1}$ and forgetful functor $\mathbf{Cat}_{/E\mathfrak{T}^1} \to \mathbf{Cat}$ has a right adjoint $\mathbf{Cat} \to \mathbf{Cat}_{/\mathfrak{T}^1}$; this adjoint is exactly the functor $C \mapsto \mathfrak{T}^1(C \to *)$. (Note: more generally, this construction applies to all normal pseudofunctors $H : D \to \mathbf{Prof}$, where it characterizes the constructions of 'vertical comma categories' $H_{//C}$ for such functors H (see [Dorn-Douglas 2021, Term. 2.3.18]) as right adjoints.) ▲

Remark A.5. *(Labels in ∞-categories)* The construction of $\mathfrak{T}^1(-)$ generalizes to an endofunctor on ∞-categories $\mathbf{Cat}_\infty$, which immediately leads to a notion of truss bundles labeled in ∞-categories. ▲

Thus there is a 'spectrum' of base/label structures on which we can reasonably consider truss bundles, ranging from posets over to categories to ∞-categories. Most of the theory works the same across the spectrum. In this article, we work with the simplest possible choice, i.e. with posets (initially as base structure, but later even as label structures for the purpose of defining 'stratifications').

B Geometric manifold diagrams and their combinatorialization

(We omit a recollection of stratified topology, see here for an introduction to stratified spaces.)

B.1 Definition of manifold diagrams

Definition B.1. The *standard n-framing* of $\mathbb{R}^n$ is the chain of oriented $\mathbb{R}$-fiber bundles $\pi_i : \mathbb{R}^i \to \mathbb{R}^{i-1}$ ($1 \leq i \leq n$) with π_i defined to be the map that forgets the last coordinate of $\mathbb{R}^i$ (and fibers carry the standard orientation of $\mathbb{R}$ after identifying $\mathbb{R}^i = \mathbb{R}^{i-1} \times \mathbb{R}$). ▲

When considering $\mathbb{R}^n$ we tacitly always think of it as 'standard framed $\mathbb{R}^n$' and, thus, we stop mentioning the standard framing as an explicit structure all-together. Indeed, more important than defining the standard n-framing is to define the maps that preserve it.

Definition B.2. A *framed map* $F : \mathbb{R}^n \to \mathbb{R}^n$ is a map for which there exist (necessarily unique) maps $F_j : \mathbb{R}^j \to \mathbb{R}^j$ ($0 \leq j \leq n$) with $F_n = F$ such that $\pi_i \circ F_i = F_{i-1} \circ \pi_i$ with F_i preserving orientations of fibers of π_i (i.e. mapping fibers strictly monotously). ▲

A framed *stratified* map $(\mathbb{R}^n, f) \to (\mathbb{R}, g)$ is a stratified map whose underlying map $\mathbb{R}^n \to \mathbb{R}^n$ is framed. Moreover, when working with products $(\mathbb{R}^k, f) \times (\mathbb{R}^{n-k}, g)$ we will identitify $\mathbb{R}^n \cong \mathbb{R}^k \times \mathbb{R}^{n-k}$ in the standard way; and, when working with cone stratifications $(\mathrm{Cone}(S^{n-1}), \mathrm{cone}(l))$, we will standard embed $S^{n-1} \hookrightarrow \mathbb{R}^n$ and identify $\mathrm{Cone}(S^{n-1}) \cong \mathbb{R}^n$ by mapping $(x \in S^{n-1}, \lambda \in [0,1))$ to $\frac{\lambda}{1-\lambda} x \in \mathbb{R}^n$.

Definition B.3. A stratification $(\mathbb{R}^n, f)$ is *framed conical* if each point $x \in \mathbb{R}^n$ it has a framed stratified neighborhood (framed) homeomorphic to $\mathbb{R}^k \times (\mathrm{Cone}(S^{n-k-1}), \mathrm{cone}(l))$ with $x \in \mathbb{R}^k \times \{0\}$, where $0 \leq k \leq n$ and (S^{n-1}, l) is some stratification. ▲

Definition B.4. A *compactly-described triangulation* K of $\mathbb{R}^n$ is a finite stratification of $\mathbb{R}^n$ by open disks whose closures are the images of linear embeddingsnn $\Delta^k \times \mathbb{R}^l_{\geq 0} \hookrightarrow \mathbb{R}^n$ ($k + l \leq n$). This now translates to the framed stratified case as follows: a stratification $(\mathbb{R}^n, f)$ is *framed compactly triangulable* if it admits a framed stratified subdivision $(\mathbb{R}^n, K) \to (\mathbb{R}^n, f)$ of f by a compactly-described triangulation K. ▲

We can now put these concepts together to obtain the following definition of manifold diagrams.

Definition B.5. A *manifold n-diagram* is a framed conical, framed compactly triangulable stratification of $\mathbb{R}^n$.▲

B.2 Combinatorialization theorem

Theorem B.1. *Manifold diagrams, up to framed stratified homeomorphism, bijectively correspond to normalized combinatorial manifold n-diagrams.*

Proof sketch. Given a manifold n-diagram $(\mathbb{R}^n, f)$ its corresponding normalized combinatorial manifold n-diagram can be constructed by first refining f by the unique coarsest n-mesh M, and then labeling the n-truss $\mathsf{Entr}(M)$ with the labeling $\mathsf{Entr}(M \to f)$. □