Framed combinatorial topology

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Dedicated to our grandparents:

Dietlinde Schmidt and Winfried Wegener,
Johanna Sternberg and Fritz Dorn;
Laia Perlmutter and Richard Hanau,
Julia Huntington and Randall Douglas.

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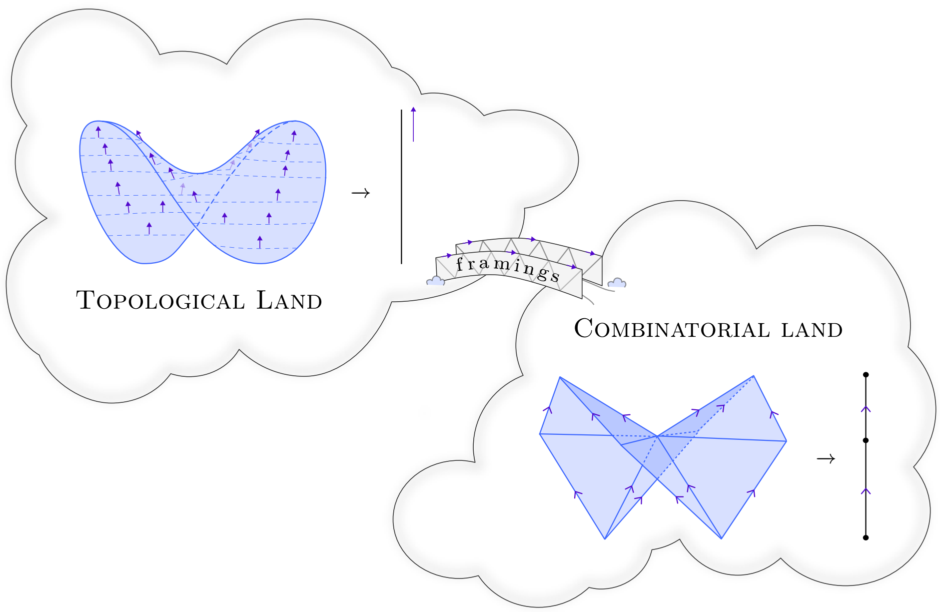
Introduction



Framed combinatorial topology is a novel theory providing a combinatorial description of phenomena that arise at the intersection of *stratified topology*, *Morse* and *singularity theory*, and *higher algebra*. The theory synthesizes several elements of classical combinatorial topology with a new approach to *framings* that locally endows combinatorial structures with a rigid frame of spatial directions. Adding framings as a foundational ingredient leads to a new and greatly refined perspective on many topological and geometric phenomena as well as their combinatorial counterparts. This often results in unexpectedly good behavior when compared to classical, nonframed combinatorial notions of space.

In discussing this behavior and its contrast with that of classical structures, we emphasize two broad themes, 'computability in combinatorial topology' and 'combinatorializability of topological phenomena'. The first theme of computability concerns whether certain combinatorial structures (such as simplicial complexes homeomorphic to spheres) can be algorithmically recognized and classified. The second theme of combinatorializability concerns whether certain constructions in the study of continuous (or smooth) spaces can be faithfully described through discrete combinatorial means.

Combining these themes, we will find that in the context of framed combinatorial topology we can overcome a set of fundamental classical obstructions to the computable combinatorial representation of topological phenomena. Building on these themes, as of yet unexplored connections between smooth manifold structures, singularity theory, and higher algebra, linked through the language of framed combinatorial topology, will emerge towards the end of the book.

We begin this introduction by elaborating the themes of computability and of combinatorializability in, respectively, Section I.1 and Section I.2. We then give a more formal overview of our results in Section I.3, a chapter-by-chapter outline in Section I.4, and an outlook on the larger program and aims of the subject in Section I.6.

I.1. ◊Computability in combinatorial topology

Computability is the ability to solve a *general problem* by a *general method*, that is, the ability to write a step-by-step procedure which for each specific instance of a problem computes a solution. Combinatorial topology provides, in a sense, a computation-oriented foundation for the study of spaces, by encoding space in discrete structures [**RS72, Bry02**]. However, many fundamental problems in combinatorial topology turn out to be computably intractable; such problems include the following [**Mar58, VKF74, CL06, Wei04, JLLT22**]:

Consider this list of references [Reduced to two originals, one survey. In 'Quantum Geometry' book, sec. 6.3, same first two reference are picked]

(1) *Disk recognition.* The statement 'The simplicial complex K is homeomorphic to the n-disk' cannot be computably verified for general finite complexes K. This uncomputability issue remains in the piecewise linear setting: the statement 'The simplicial complex K piecewise linearly subdivides the n-simplex' cannot be verified. In particular, one cannot classify all topological subdivisions of the n-disk, nor all piecewise linear subdivisions of the n-simplex.
(2) *Homeomorphism problem.* More generally, it is impossible to algorithmically decide whether two simplicial complexes K and L have homeomorphic, or piecewise linearly homeomorphic, geometric realizations. Similarly, given two embedded, or piecewise linearly embedded, simplicial complexes $K \hookrightarrow \mathbb{R}^N$ and $L \hookrightarrow \mathbb{R}^N$, one cannot in general determine whether the embeddings are ambient homeomorphic, respectively ambient piecewise linearly homeomorphic. Yet more generally, it is undecidable whether stratified simplicial complexes are stratified homeomorphic or stratified piecewise linearly homeomorphic; similarly

for stratified embedded complexes and stratified ambient homeomorphism, or stratified piecewise linearly embedded complexes and stratified ambient piecewise linear homeomorphism.

(3) *Manifold classification.* Manifolds are comparably intractable: the statement 'The simplicial complex K is homeomorphic to a manifold' cannot be computably verified in general, and neither can the statement 'The simplicial complex K is a piecewise linear manifold'. In particular, one cannot classify all simplicial complexes homeomorphic or piecewise linearly homeomorphic to manifolds, nor constructively enumerate homeomorphism or piecewise linear homeomorphism types of manifolds.

One could view these failures of computability as unavoidable imperfections of mathematics as we know it, or one can see them as failures of the interplay of standard simplicial methods and traditional topological notions. Adopting the latter viewpoint, one may hope for a form of combinatorial topology with better computability properties, for instance in which one can recognize combinatorial disks, decide combinatorial homeomorphism, and classify combinatorial manifolds.

The first central theme of this book is that, though classical simplicial methods often do not provide an entirely computable foundation for combinatorial spaces, there is a different approach, using *framed* combinatorial spaces, that may provide a more suitable foundation for *computable* combinatorial topology. Our theory of 'framed combinatorial topology' differs in two fundamental respects from classical piecewise linear topology: first, we endow simplices and simplicial complexes with a combinatorial framing structure akin to local directions [**Gra03**], and second, we generalize the resulting class of 'framed *simplicial* complexes' to a broader class of 'framed *regular* cell complexes'. Though classical regular cells are much less tractable even than simplices—indeed even the list of cell shapes is uncomputable—it will turn out that framed regular cells arise as iterated constructible combinatorial bundles and therefore both these cells and their complexes are, remarkably, algorithmically classifiable.

A classical frame of an m-dimensional vector space is an ordered choice of m linearly independent vectors. We will define a combinatorial frame of an m-simplex to be an ordered choice of m vectors in the spine of the simplex. To make sense of a frame on a simplicial complex, we need a notion of the compatibility of frames along faces shared between simplices. The restriction of a frame of a simplex to a face gives not only information about a frame of the face but also about how that restricted frame embeds in the ambient frame of the simplex; we will be primarily concerned with the resulting notion of *embedded framed* simplex, and a framed simplicial complex will be a simplicial complex with compatible embedded frames on all its simplices.

Regular cell complexes, that is those complexes whose attaching maps are injective, generalize simplicial complexes by allowing cells of 'polytopic' shapes instead of merely 'triangular' shapes [**Zie12**]. Regular cells can be identified with the geometric realizations of their face posets [**Bjö84, LW69**], and via that identification they obtain piecewise linear simplicial subdivisions. We use that simplicial structure, together with our notion of framed simplicial complexes, to define framings of regular cells and identify a tractable class of such cells: namely, cells whose framing structure is 'contractible' as guaranteed by a combinatorial notion of framed collapse which closely mirrors the classical notion of combinatorial-topological collapse [**Whi50**] but respects frame orders.

A framed regular cell complex, finally, will be a regular cell complex with compatible choices of framings on each of its cells.

A space with a homeomorphism to a regular cell complex is 'cellulated', as a space with a homeomorphism to a simplicial complex is 'triangulated'. The fact that cellulated spaces have played a less prominent role than triangulated spaces in classical combinatorial topology is partially due to the aforementioned fundamental computability obstruction: it is impossible to classify all the possible shapes of regular cells, in the sense that one cannot produce a list of all regular cells with a given number of faces, in general; said another way, there is no general algorithm for deciding whether a given poset is the face poset of a regular cell, even though there are only finitely many posets of a given size. Even when constraining the class of cells further, say, to the class of *convex* polytopic shapes, classifications of these cells in elementary combinatorial terms are often not known.

Endowing regular cells with a framing overcomes these fundamental issues: framed regular cells, in contrast to their nonframed counterparts, are classifiable. More precisely, we will discover that collapsibly framed regular cell complexes are classified by a novel elementary combinatorial structure, called 'trusses', which are iterated constructible bundles of oriented fence posets. As a result, given a poset together with a framing of its underlying simplicial complex, we can algorithmically recognize whether the poset is the face poset of a framed regular cell.

Framed regular cells strike an unlikely and delicate balance, being simultaneously a class of shapes that is tractable (i.e., algorithmically recognizable and classifiable in elementary combinatorial terms) and also a class of shapes that is quite general and has a rich category of maps. In combination, this provides a unique set of properties, many of which are unthinkable with simplicial structures and unknown with any other class of shapes: cell subdivisions obtain elementary combinatorial representation as certain maps, the class of cells is closed under geometric dualization, and, most importantly, framed realizable spaces admit *minimal* (i.e., coarsest) cell structures. In particular, any n-framed regular cell complex that can be (not necessarily injectively) framed realized in $\mathbb{R}^n$ has a computable unique minimal cell structure. Having a computably unique representation of these complexes makes algorithmically decidable almost any question about them; for instance, it follows that framed homeomorphism of these complexes is decidable, in stark contrast to the classical (nonframed simplicial) situation. In this and other related respects, working with framed regular cells and their complexes provides, finally, a computable framework for combinatorial models of spaces.

I.2. ◊Combinatorializability of topological phenomena

Combinatorics is primarily concerned with discrete, and often finite, structures whose constituents can be counted. Topology, by contrast, is primarily concerned with the continuous structure of spaces. The 'combinatorializability' of topological phenomena refers to the ability to faithfully encode continuous objects (spaces, manifolds, continuous maps, bordisms, et cetera) in discrete or finite data structures. This faithful encoding depends both on having a combinatorial representation of the object in question, and on knowing that representation is unique up to some specified combinatorial equivalence relation.

There are by now various known instances of topological phenomena that cannot be faithfully combinatorialized, or even combinatorialized at all, giving an impression of a mysterious and insurmountable divide between topological spaces and any discrete representations of those spaces. A headline instance of this divide is the disproven 'Hauptvermutung', a conjecture that, roughly speaking, claimed that *topological* homeomorphism coincides with *piecewise linear* homeomorphism [**RCS⁺96**]. This conjecture would in particular imply that combinatorial spaces (that is, geometric realizations of simplicial complexes) that are homeomorphic are also piecewise linear homeomorphic. This intuitive, presumptive claim was eventually disproven [**Mil61**] by the explicit construction of homeomorphic finite simplicial complexes that are not piecewise linear homeomorphic.

A flurry of results followed in subsequent decades, quantifying the divide not only between the 'continuous' and the 'combinatorial', but also between the 'combinatorial' and the 'smooth' conceptions of space [**KS69, HM74, KS77, Fre82, Don83**]. Recently, a disproof of the triangulation conjecture [**Man16**] established an especially stark gap, that in every dimension greater than 4 there exist compact topological manifolds that do not even admit a triangulation. (It will be pertinent later that most instances of the classical topological–combinatorial gap rely on certain infinitary or 'wild' topological constructions.) By contrast, smooth manifolds always admit triangulations and all triangulations of a smooth manifold are combinatorially equivalent [**Whi40**]. However, smooth manifolds that are not smoothly isomorphic may nevertheless be combinatorially isomorphic [**Mil56, Mil10**], and combinatorial manifolds need not admit any smooth structure [**Ker60**].

Consider this list of references. [I moved refs to end of sentence. Ethan Akin ref removed. Selection is a bit biased? (Could cover more topics?) But also we go into more details below.]

One might dream of a topological foundation or combinatorial framework in which the mismatch between the continuous, combinatorial, and smooth conceptions of space would, at least to some extent, be lessened. One could imagine, firstly, a discrete, perhaps infinitary, combinatorial theory that faithfully represents a delineated class of relevant continuous phenomena, and secondly, a discrete, perhaps finitary, combinatorial theory that suitably encodes smooth behavior. Each of these two comparative visions has been pursued, to some but not complete satisfaction: for instance, an 'o-minimal' approach to tame topology provides a method for excluding certain wild topological structures [**VdD98, Shi14**], while a 'matroid' perspective aims for a direct combinatorial description of smooth structures [**Mac91, GM92, Bjö99**].

The second central theme of this book is that, in contrast to the classical gap between topological and combinatorial phenomena, in our framed combinatorial setting there is a faithful comparison between framed topological and framed combinatorial phenomena. Moreover, addressing the second vision above, we expect framed combinatorial structures also faithfully encode all framed smooth phenomena, and therefore will provide an unexpected unification of the continuous, combinatorial, and smooth perspectives on space. While the resulting 'topologie modérée' [**Gro97**] shares some of the common characteristics of existing approaches like o-minimal topology, it is ultimately a fundamentally different approach, distinguished by its systematic foundational use of *bundle constructibility* conditions.

The concrete framed topological structures used in the above comparisons will not be mere tame *spaces*, but rather tame *stratifications*. These stratifications are *framed* by an embedding in standard euclidean space. Their tameness is captured by the condition that the stratification admit a refining 'mesh', which is a refinement by framed regular cells; this refinement requirement is analogous to working with triangulable spaces and therefore excluding, a priori, certain wild behavior. These meshes are iterated constructible bundles of stratified 1-manifolds, and will be a precise topological counterpart of the iterated constructible combinatorial structure of trusses mentioned earlier.

The chain of associations, from a tame stratification to its mesh cellulation to the corresponding combinatorial truss, does not by itself necessarily ensure a faithful combinatorialization of tame stratified topology. As a space can have various inequivalent triangulations, a tame stratification could in theory have various inequivalent meshes (and therefore corresponding trusses)—however, we will prove, crucially, that such a stratification always has a unique coarsest compatible mesh. This uniqueness is an unexpected and stark counterpoint to the classical situation: given two triangulations of a space, traditionally one aims (and fails) to construct a mutual *refinement* and thereby verify their combinatorial equivalence; now instead, given two mesh cellulations of a stratification, we construct a canonical mutual *coarsening* and thus establish the desired combinatorial equivalence. The proof of this canonical coarsening will rely, of course, again on the generality of the shapes of framed regular cells. The canonical coarsest cell refinement provides the desired faithful combinatorialization of topological phenomena in standard framed euclidean space. In this context, we will also establish the 'tame Hauptvermutung', that for tame stratifications, framed homeomorphism classes coincide with framed piecewise linear homeomorphism classes.

Regarding the combinatorialization of smooth phenomena, we will conjecture that any smooth manifold can be represented as a tame stratification (via a generic embedding in euclidean space) and that the resulting combinatorial representation as a truss faithfully encodes the smooth structure. The conjectures are based on a connection of so-called elementary tame singularities and classical differential singularities, and broader parallels to higher-dimensional Morse theory [**Cer70, Arn75, Arn81, MNB22**]. We will revisit the context and plausibility of this smooth combinatorialization conjecture in the outlook, Section I.6 below.

I.3. Overview of core concepts

We collect and summarize our main theorems, and along the way further describe and illustrate our key definitions. Recall that a framed simplex is an ordinary simplex together with a frame, that is, a choice of order of its spine vectors. More generally, a framed regular cell is an ordinary regular cell together with a suitably compatible choice of frames on each simplex in the cell's face poset. Though it is impossible to classify regular cells, by contrast framed regular cells are classifiable. The classifying combinatorial structure will be a special case of the notion of *trusses*, which are iterated constructible poset bundles defined as follows.

Consider whether to add pointer to overview diagram somewhere in here. [not needed imo]

DEFINITION 1 (Trusses). *A '1-truss' is a fence poset equipped with a total 'frame' order on its elements. An 'n-truss' is a length-n tower of constructible bundles of 1-trusses.*

The notions of '1-trusses', their 'constructible bundles', and 'n-trusses' are given more precisely in, respectively, Definition 2.1.6, Definition 2.1.74, and Definition 2.3.1. Elements of a 1-truss that are targets of poset arrows are called 'singular' or 'dimension 0', while elements that are sources of poset arrows are called 'regular' or 'dimension 1'. A 1-truss is 'closed' if both its endpoints are singular, and 'open' if both are regular; an n-truss is 'closed' or 'open' if all its fiber 1-trusses are closed or open respectively. In Figure I.1 we illustrate a closed 2-truss and an open 3-truss; singular elements are red, regular elements are blue, and the frame orders are indicated by green arrows.

A closed truss is called a 'truss block' if its total poset has an initial element; these truss blocks provide the combinatorial correlate of framed regular cells.

THEOREM 2 (Classification of framed regular cells). *The category of framed regular cells is equivalent to the category of truss blocks.*

This result will appear as Theorem 3.1.1. It will immediately imply the algorithmic enumerability of framed regular cells (Corollary 3.3.29). Moreover, this enumeration is efficient and cells are generated 'bottom up' in an elementary fashion, which sets the class of framed regular cells apart from

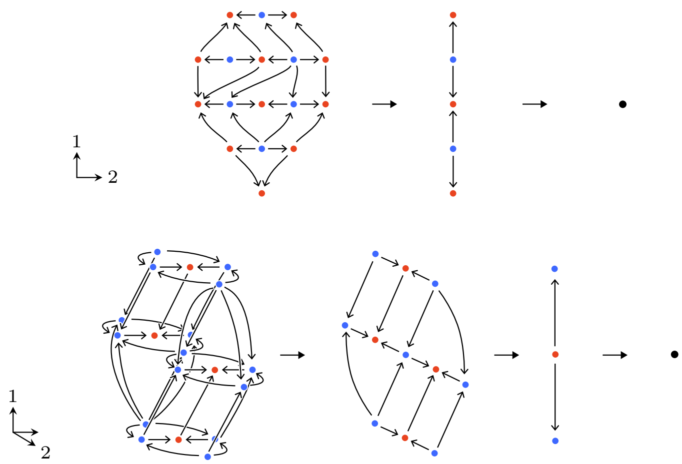


FIGURE I.1. A closed 2-truss and an open 3-truss.

other, at least theoretically computationally tractable classes of shapes such as convex polytopes: a brief comparative discussion for the latter is included in Section 1.3.2.5.

The classification of framed regular cells builds on the classification of regular cell complexes by cellular posets. Namely, given a framed regular cell, the total poset of the classifying truss block is the face poset of the cell; sequentially projecting out frame vectors from this poset determines a tower of 1-truss bundles. In Figure I.2 and Figure I.3 we illustrate a few framed regular cells and their corresponding truss blocks. A more extensive menagerie of framed regular cells is kept in Chapter C.

There is an arrow missing in the last row [addressed]

The classification of framed regular cells by truss blocks implies a corresponding classification for framed regular cell complexes. As a regular cell complex is a presheaf on the category of regular cells and their injective maps [**EZ50, Dug99**] whose cells injectively include into the complex, a 'regular block complex' is a presheaf on the category of truss blocks and their injective maps whose blocks injectively include into the complex.

THEOREM 3 (Classification of framed regular cell complexes). *The category of framed regular cell complexes is equivalent to the category of truss block complexes.*

This result appears in the main text as Theorem 3.1.3.

Trusses have a geometric counterpart, called *meshes*, which rephrases key elements of their definition in stratified topological terms.

DEFINITION 4 (Meshes). *A '1-mesh' is a contractible 1- or 0-manifold, stratified by open intervals and points, and equipped with a framing. An 'n-mesh' is a length-n tower of constructible bundles of 1-meshes.*

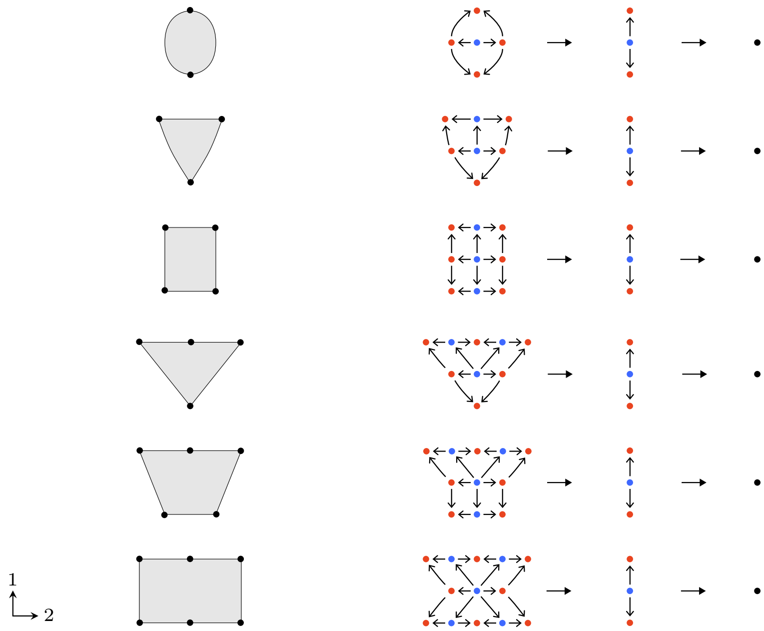


FIGURE I.2. Framed regular 2-cells and their classifying truss blocks.

The notions of '1-mesh', their 'constructible bundles', and 'n-mesh' are given more precisely in, respectively, Definition 4.1.9, Definition 4.1.28, and Definition 4.1.69. An n-mesh is 'closed' if its total space is compact, and is 'open' if its total space is an open disc. In Figure I.4 we illustrate a closed 2-mesh and an open 3-mesh.

The correspondence of meshes and trusses is at the core of our combinatorialization of topological phenomena.

THEOREM 5 (Equivalence of meshes and trusses). *The topological category of closed, respectively open, meshes is weakly equivalent to the discrete category of closed, respectively open, trusses.*

This result appears in a more precise form as Theorem 4.2.1. Recall that the fundamental poset of a stratified space has an element for each stratum and an arrow indicating when a stratum intersects the closure of another stratum. The above equivalence takes a mesh, a tower of stratified spaces, to the truss given by the tower of corresponding fundamental posets. As an illustration, note that the meshes in Figure I.4 yield, on application of fundamental posets, the trusses in Figure I.1.

Recall that a basic unsolvable problem of classical combinatorial topology is to classify subdivisions of the n-simplex. By contrast, leveraging the connection between framed regular cell complexes and block complexes, we

Consider whether we need more illustrations of basic players, eg mesh block, etc [considering the level of rigor .. i'd say no. in any case, definitely not for mesh blocks.]

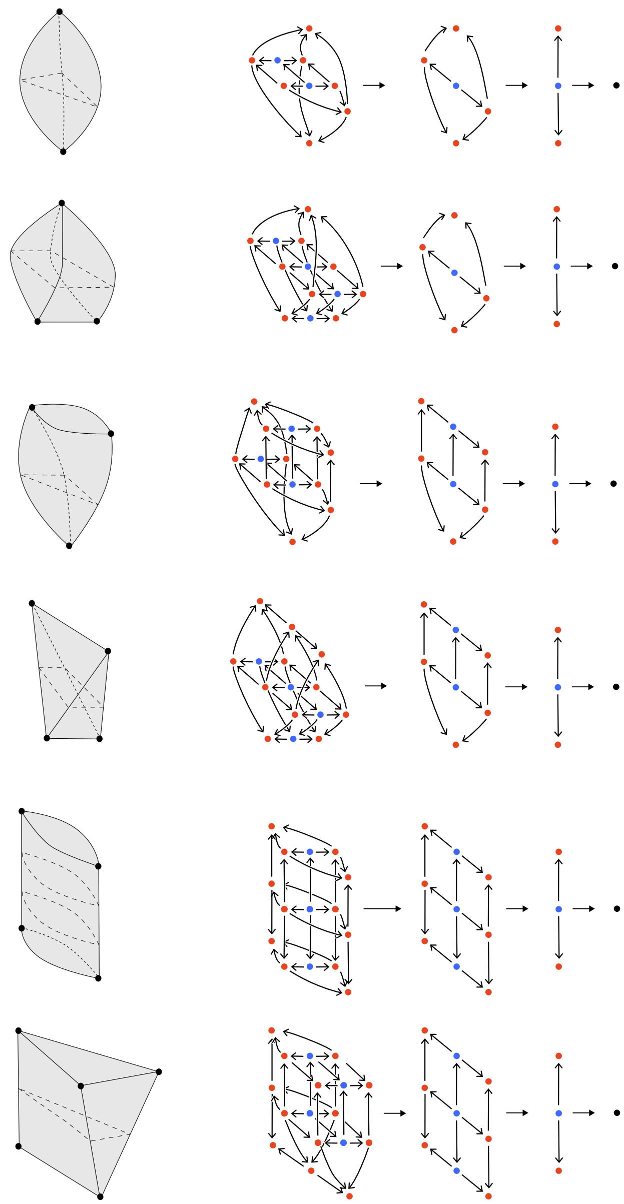


FIGURE I.3. Framed regular 3-cells and their classifying truss blocks.

can classify framed subdivisions of framed regular cells in terms of combinatorially defined 'subdivision' maps in the category of trusses. A simple such subdivision, realized by framed regular cell complexes, is illustrated in Figure I.5.

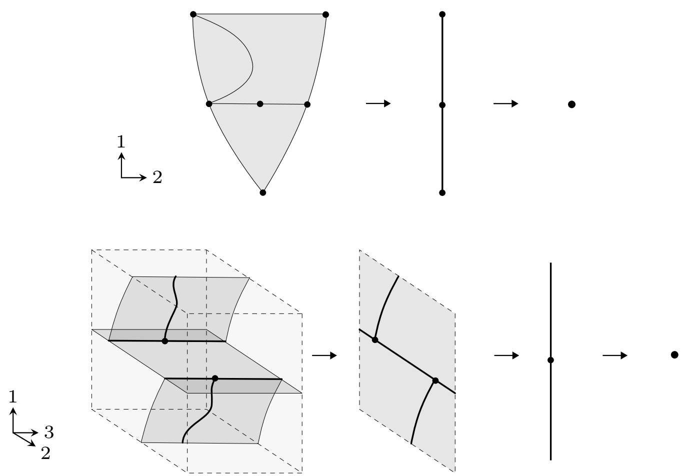


FIGURE I.4. A closed 2-mesh and an open 3-mesh.

THEOREM 6 (Combinatorial classification of framed cell subdivisions). *A framed regular cell complex framed subdivides a framed regular cell if and only if it corresponds to a truss that combinatorially subdivides a truss block.*

This result appears in more precise form as Theorem 4.2.8.

Couldn't we omit the words 'as a framed stratified space' from the theorem? [Rephrased]

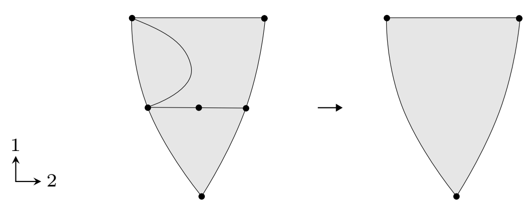


FIGURE I.5. A subdivision of a framed regular cell.

There is no reference to Fig I5. Will go through the commit history to unwind when I'm next on the intro [was meant to tacitly illustrate the theorem, added trivial reference]

From here to the break: not so clear why, really, this is here. [rewritten]

Meshes naturally support a construction of geometric dualization, which inverts dimension of each individual stratum. The construction provides an endofunctor on the category of meshes, turning closed meshes into open meshes, and mesh *blocks* (the geometric analog of truss blocks) into so called mesh *braces*.

THEOREM 7 (Dualization of meshes). *The topological category of closed meshes is weakly equivalent to the topological category of open meshes.*

This self-duality appears later as Corollary 4.2.9, and is based on the preceding combinatorialization result of Theorem 5: the fundamental poset of the dual

n-mesh is dual to the fundamental poset of the original mesh, and dimensions of all strata in the mesh are dualized—the collection of geometric shapes and their incidences is fully inverted. Figure I.6 illustrates the duality of meshes in two cases: the first dualizes a (closed) 3-mesh block to an (open) 3-mesh brace, the second dualizes a 3-mesh consisting of two 3-cells to a corresponding open 3-mesh. Note that open meshes themselves have (open) framed regular cells as their strata.

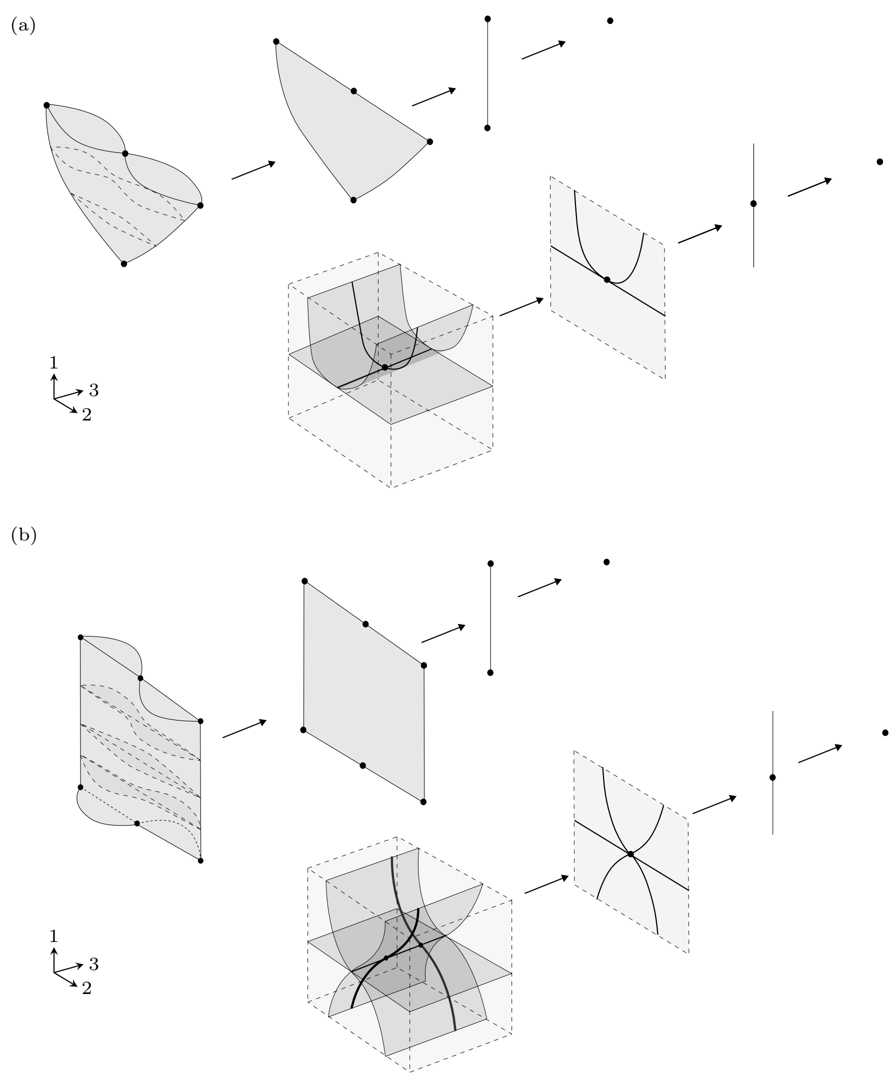


FIGURE I.6. Dualization of meshes.

Meshes are a flexible, computationally tractable class of highly structured stratifications; they furthermore provide access to a much broader, almost

Removed ', built up from mesh blocks and their duals, '

If seeking smooth connection to previous section, reconsider how completely general class of stratifications by considering those stratifications that admit a refinement by a mesh, as follows.

Definition 8 (Tame stratifications). *An ‘n-tame stratification’ is a stratification of a subspace of* $\mathbb{R}^n$ *that admits a refinement by an* n*-mesh.*

This definition will appear in a more precise form in Definition 5.1.1. In Figure I.7 we illustrate two tame stratifications of an open 4-cube, by depicting three pertinent slices. The first stratification is the classical third Reidemeister move [**Rol03**], and the second is the classical swallowtail singularity [**TF18**]; that these indeed admit mesh refinements is illustrated in a moment in Figure I.8 and Figure I.9.

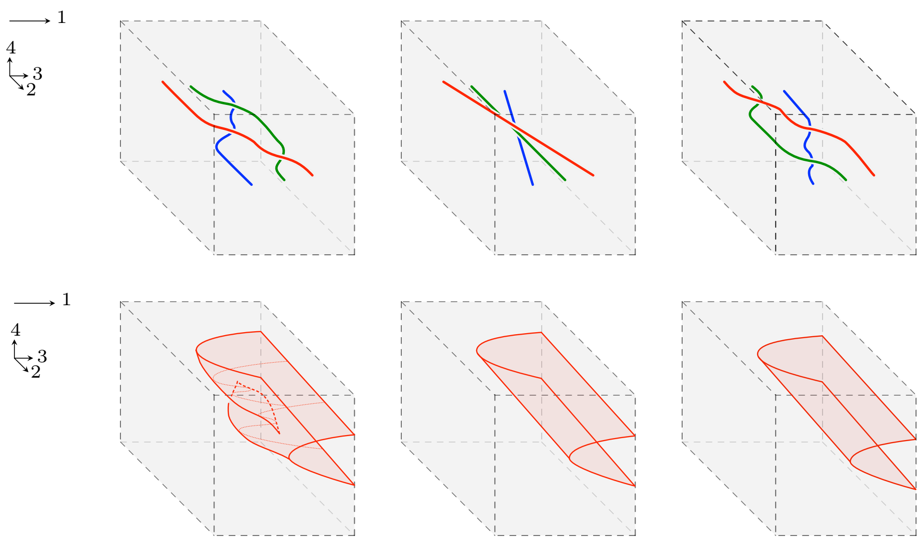


Figure I.7. Two tame stratifications of the 4-cube.

At the heart of the computability of our combinatorial model of stratified framed topology is the rather unexpected fact that among the set of all refining meshes of a tame stratification, there is always a canonical *coarsest* choice. Heuristically, the coarsest refining mesh of a stratification is built up just from the indispensable critical loci of certain projections of the strata. Needless to say, this situation is in stark contrast to any simplicial model of (stratified) topology, in which two triangulations almost never have a mutual coarsening and typically do not even admit a mutual refinement, preventing any canonical or computable comparison.

Theorem 9 (Coarsest meshes of tame stratifications). *Any tame stratification has a unique refining mesh that is coarser than any other refining mesh.*

This will be established as Theorem 5.2.23. In Figure I.8 and Figure I.9 we depict the unique coarsest mesh for the third Reidemeister move and for the swallowtail singularity.

All the RIII pictures I think need some editing to make the crossings jump out more — they really appear to just be in the central plane atm. [done]

This is an actual question: is it essential to the correct-

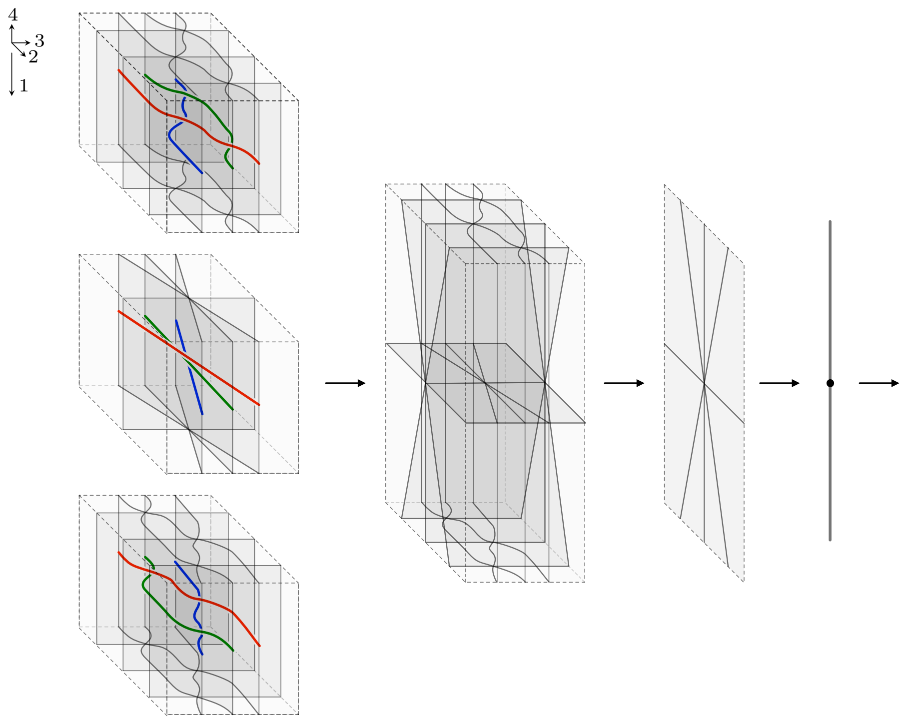


FIGURE I.8. The coarsest mesh refining the third Reidemeister move.

A tame stratification is refined by a mesh, and a mesh is combinatorialized by a truss; to complete the combinatorialization of tame stratifications, we translate the initial stratification into a stratified structure on the truss. A 'stratified poset' is a poset together with a 'stratification map' to another poset (encoding the set of strata and the combinatorial entrance paths between them); a 'stratified truss' is a truss together with a stratification of its total poset. Furthermore, a stratified truss is 'normalized' if it cannot be simplified while preserving the stratification; this property of being normalized corresponds to a mesh being maximally coarsened while still refining a given stratification. CLD HERE

THEOREM 10 (Classification of tame stratifications). *Framed stratified homeomorphism classes of tame stratifications are in bijective correspondence with isomorphism classes of normalized stratified trusses.*

This will be established as Theorem 5.1.23. Figure I.10 illustrates a tame stratification, its coarsest refining mesh, and the corresponding normalized stratified truss; the stratification on the truss records which strata of the mesh assemble into each stratum of the initial tame stratification.

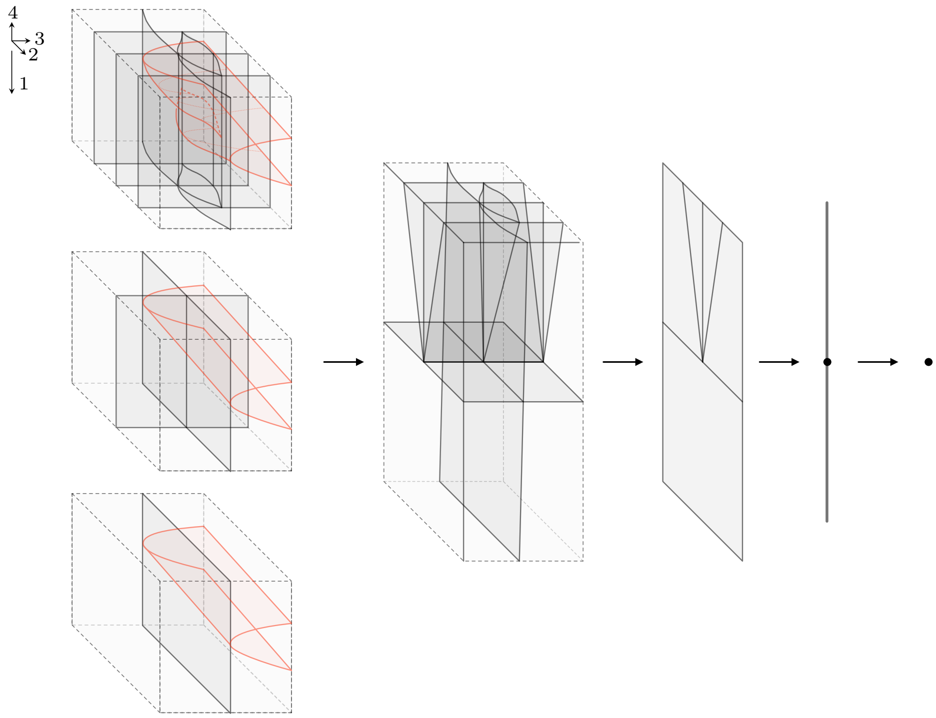


Figure I.9. The coarsest mesh refining the swallowtail singularity.

Tame stratifications are intrinsically topological structures, considered up to homeomorphism, while stratified trusses are intrinsically combinatorial or piecewise linear structures, considered up to combinatorial or piecewise linear equivalence—the classification of framed stratifications by stratified trusses thus provides a faithful bridge between the topological and piecewise linear contexts. Recall the classical, false Hauptvermutung, that homeomorphic simplicial complexes are piecewise linear homeomorphic. The failure of correspondence between the topological and the piecewise linear remains even for subspaces (or substratifications) of euclidean space: given two piecewise linear embedded triangulated spaces in euclidean space that are ambient homeomorphic, they need not be ambient piecewise linear homeomorphic. By contrast, in the framed setting we will have a tight correspondence between the topological and piecewise linear, as follows.

Theorem 11 (Tame Hauptvermutung). *Tame piecewise linear stratifications that are framed stratified homeomorphic are also piecewise linear framed stratified homeomorphic.*

This result will be established later as Theorem 5.1.27. A brief comparison to the o-minimal topological setting, in which a version of the Hauptvermutung has also been established [**Shi14, Shi13**], is provided in Section 5.3.2.3.

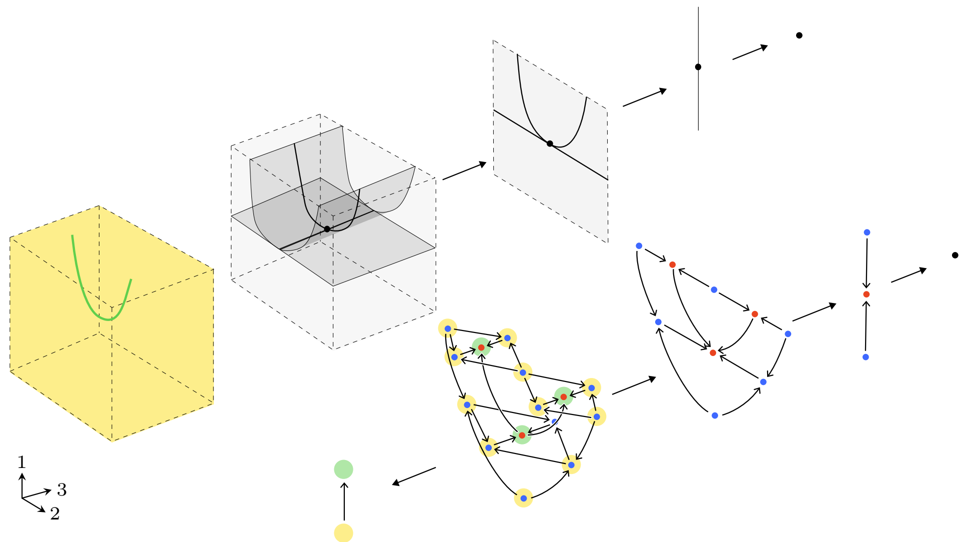


FIGURE I.10. A tame stratification, its coarsest mesh, and its normalized stratified truss.

Having a combinatorial or piecewise linear handle on tame stratifications, via stratified trusses, resolves several fundamental decidability problems for framed homeomorphism and framed stratified homeomorphisms. As a first result we will show that if an n-framed regular cell complex admits a framed piecewise-linear realization in standard framed $\mathbb{R}^n$ (meaning, simply, a map that restricts to framed embeddings on each cell), then the problem of framed homeomorphism becomes decidable.

THEOREM 12 (Decidability of framed homeomorphism). *Given two finite realizable n-framed regular cell complexes, one can algorithmically decide whether they are framed homeomorphic.*

This is recorded later as Theorem 5.1.31.

A second central problem concerns the question of framed *stratified* homeomorphism which, for the class of all tame stratifications, is resolved by the following result.

THEOREM 13 (Decidability of framed stratified homeomorphism). *Given two tame stratifications, one can algorithmically decide whether they are framed stratified homeomorphic.*

This is recorded later as Theorem 5.1.30, and will rely on the combinatorialization of such stratifications using (normalized) stratified trusses.

Finally, we mention two key conjectures. Tame stratifications and their framed maps exhibit conceptual analogies with stratified Morse theory, which understands stratified spaces through studying differentiable functions on them: combinatorial frames provide a notion of direction or 'flow' akin to that induced by a Morse function. In contrast to the classical theory, by iterating this idea in each dimension, 'attaching maps' in tame stratification become determined inductively by 1-dimensional data. We expect that this reduction of dimensionality will allow us to extend the above computable combinatorialization of framed topological phenomena to framed *smooth* phenomena.

CONJECTURE 14 (Framed homeomorphism implies diffeomorphism). *Given two smooth compact manifolds smoothly embedded in euclidean space, and defining tame stratifications there, if they are ambient framed homeomorphic then they are diffeomorphic.*

CONJECTURE 15 (Framed stratifications are dense in smooth embeddings). *Any smooth embedding of a smooth compact manifold into euclidean space has an arbitrarily small perturbation that is a tame stratification.*

These conjectures reappear later as Conjecture 5.4.22 and Conjecture 5.4.23. Because tame stratifications can be faithfully combinatorialized as stratified trusses, these conjectures together would imply a first-of-its-kind faithful combinatorial representation of smooth structures on manifolds.

Fig edits: specify chapter references to sections as appropriate.

I.4. Chapter outlines

We briefly outline how core notions and results are organized across chapters. A summary of this organization is illustrated in Figure I.11.

Chapter 1 introduces framed combinatorial structures. The first such structure, 'framed simplices', is a combinatorial analog of classical framed vector spaces. A 'framed simplicial complex' will be a collection of compatibly framed simplices. After recalling the classical combinatorial-topological definition of regular cell complexes, we then further generalize framed simplicial complexes to 'framed regular cell complexes'.

In Chapter 2, we develop our fundamental combinatorial notion of 'trusses', as certain iterated constructible bundles of posets. This development begins with '1-trusses', which are framed fence posets, morphisms between them called '1-truss bordisms', and families of them called '1-truss bundles'. 1-truss bundles over simplices turn out to have an unexpected total order on the top-dimensional simplices in their total posets, and this leads to a crucial method of 'truss induction'. Finally, we describe 'n-trusses', as iterated 1-truss bundles, their corresponding 'n-truss bordisms' and 'n-truss bundles', and their elementary constituents 'n-truss blocks'.

Chapter 3 proves the equivalence of the category of truss blocks and the category of framed regular cells, and more generally the equivalence of the category of regular presheaves on truss blocks and the category of framed regular cell complexes, as stated in Theorem 2 and Theorem 3 above. Truss blocks are translated into framed regular cells by an appropriate

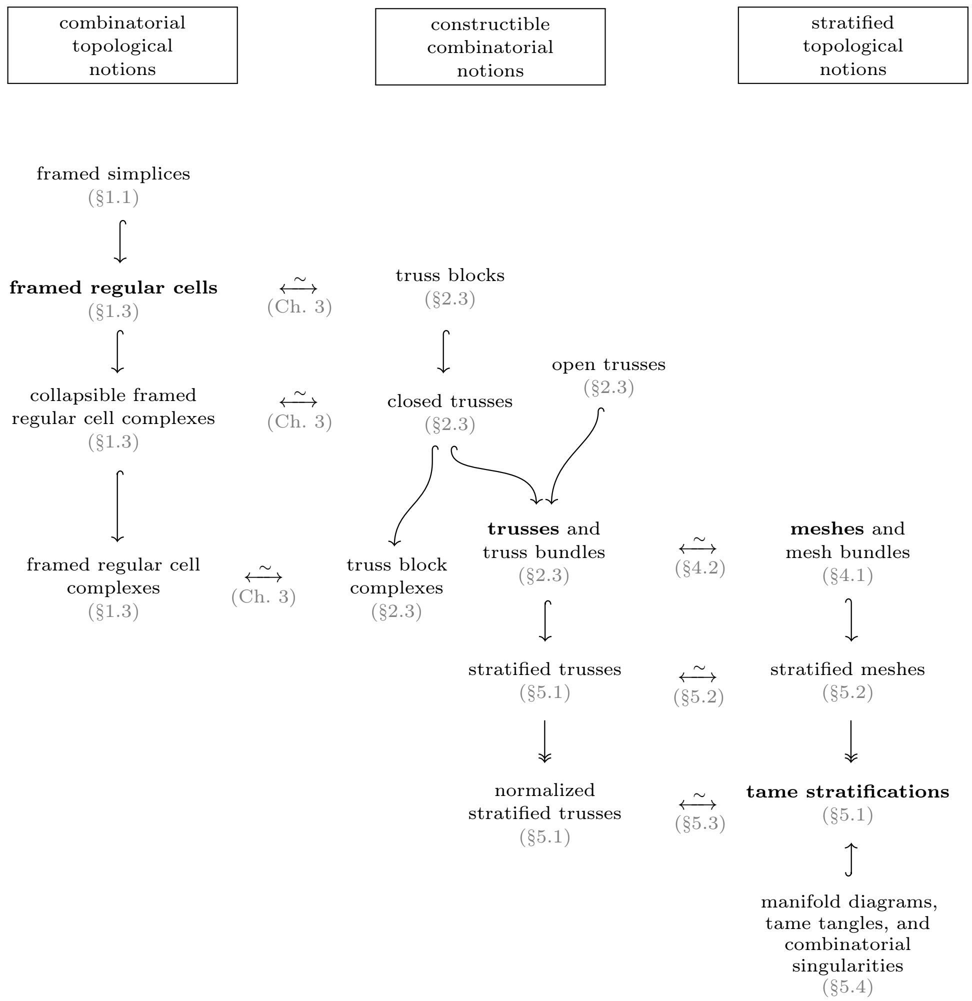


FIGURE I.11. Overview of notions.

combinatorial 'gradient' functor; the converse functor, termed 'integration', entails the more technical construction of a tower of 1-truss bundles from the framing information on the regular cell.

In Chapter 4, we introduce our fundamental stratified topological notion of 'meshes', as certain iterated constructible bundles of stratified manifolds. From the outset, meshes appear as a topological analog of the combinatorial structure of trusses, and the notions of '1-mesh', '1-mesh bundle', and 'n-mesh' parallel the corresponding truss notions. Indeed, we prove, as claimed in Theorem 5, that the topological category of meshes is weakly equivalent to the discrete category of trusses; one direction constructs a 'fundamental truss', which combinatorially encodes a mesh, and the other direction produces a 'mesh realization', which geometrically realizes a truss. This fundamental translation allows us to establish the classification of subdivisions of framed regular cells (Theorem 6) and the self-duality of meshes (Theorem 7).

Finally, Chapter 5 radically broadens the class of stratifications under investigation, introducing 'tame stratifications' as those stratifications that admit a mesh refinement. The core work of the chapter is the proof that every tame stratification has a coarsest refining mesh, as claimed in Theorem 9. We leverage that result to establish the combinatorial classification of tame stratifications in terms of normalized stratified trusses (Theorem 10). We then bridge the topological to piecewise linear chasm, proving the tame Hauptvermutung (Theorem 11) that for tame stratifications, framed stratified homeomorphism implies piecewise linear framed stratified homeomorphism. As a final application, we establish the decidability of framed stratified homeomorphism (Theorem 13). In the final portion of the chapter, we will describe a future outlook for framed combinatorial topology, including theories of manifold diagrams, tangles [**Con70**], and the combinatorial representation of smooth structures and singularities (Conjecture 14 and Conjecture 15).

Chapter A provides a detailed discussion of classical linear frames, corresponding notions of indframes and proframes (which conceptually inspire the constructions in Chapter 3), as well as their generalizations to the cases of partial and embedded frames, and the affine analogs of these structures. Based on the affine notions of framings and proframings, this appendix also describes the geometric realization of framed regular cell complexes into 'framed spaces'. Chapter B reviews and elaborates various elementary notions from stratified topology, including the construction of fundamental posets and higher categories of stratifications. Chapter C provides a menagerie of framed regular cells in dimensions 2, 3, and 4, illustrating the range of familiar, curious, novel, and exotic shapes that arise.

I.5. Reader's guide

Contents of this research book are organized in a largely linear fashion. However, this linearity hides that there are, in fact, two intertwining conceptual tracks: one is the '*classical* combinatorial structures' track starting in Chapter 1, the other the '*constructible* combinatorial structures' track starting in Chapter 2. The former builds directly on familiar classical notions, such as simplices and regular cells, introducing framings as an additional structure on those. This perspective will be especially helpful to the reader who has had some previous exposure to classical combinatorial topology. The latter chapter makes no direct contact with any classical notions; instead, it defines and explores a new notion of constructible combinatorial stratified line bundles. The equivalence of the two perspectives is proven in Chapter 3, though the (at times technical) proofs should not distract the reader from the useful core intuitions relating the perspectives developed earlier.

Both Chapter 1 and Chapter 2 introduce new combinatorial notions and are, therefore, definition and example driven chapters by design that are relatively light on theorems and proofs. This is distinctly different from the flavor of Chapter 4 and Chapter 5, which focus on putting the earlier definitional frameworks to use. Together, these chapters contain most of this book's 'headline results'. In both, the constructible perspective will be more important than the classical perspective on framed combinatorial structures. Therefore, to get to these chapters, a reading of Chapter 1 and Chapter 3 is not strictly necessary (at least for the majority of the discussed results). A summary of chapter dependencies, with a rough delineation of 'classical' and 'constructible' tracks, is given in Figure I.12.

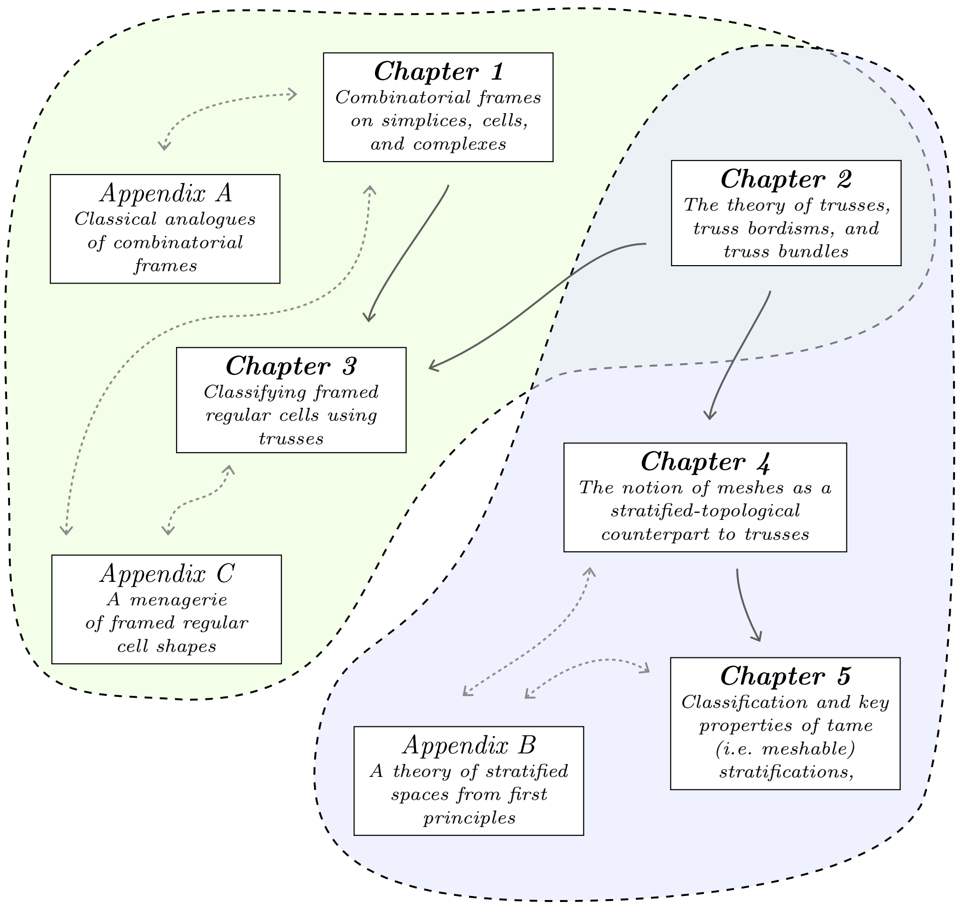


FIGURE I.12. Chapter dependencies.

An attempt is made, despite this book notably developing a *new area* of research rather than acting as a reference to or expansion of an existing one, to keep contents as self-contained as possible. In particular, the book should be readable by any reader with basic knowledge of simplicial topology and category theory. Additional topics are reviewed in the appendices (in particular Chapter B, which provides an introduction to stratified topology):

note that all three appendices are designed mostly independently of the main text and can thus be read separately or in parallel as the reader sees fit. Moreover, key terms and symbols are indexed in the backmatter with references to the relevant environments where they are defined. Finally, we remark that sections containing technical proofs and constructions that can be safely skipped are marked with the symbol ✠ .

Fig edit: the arrow from C5 to AppB suggests a dependency there, which doesn't exist. Rethink where AppB goes and how. Maybe the arrows to appendices are of a different type, and maybe they go 'up', suggesting that it's useful but not essential. That better reflects the role of AppA and AppC too. Maybe with that modification the placement of AppB works, but also warrants an arrow to C4. But if there's an arrow from AppB to both C4 and C5, then why not from AppC to C1?

Mention other backmatter eg glossary, symbol list, list of figures, etc [not needed here imo]

I.6. Outlook

We briefly summarize our immediate outlook for framed combinatorial topology beyond the present book; a more detailed discussion of these future directions appears in the final Section 5.4.

Tame stratifications are already highly, if implicitly, structured by their canonical mesh refinements and the relation of that refinement to the ambient frame on euclidean space. However, the stratification itself need not be in any sort of generic relation to the ambient frame. Moreover, a priori strata in the stratification may have 'irregular' attachment maps. For certain theoretical and computational purposes, it is essential to restrict attention to appropriately generic and regular stratifications. We can define and detect such stratifications in purely topological, as opposed to smooth, terms, by insisting that each point in the stratification has a neighborhood with a trivialization whose 'trivial direction' is transversal to the ambient framing. We dub this condition framed conicality, in analogy with the classical notion of conical stratification (see Chapter B), and call the resulting framed conical stratifications *manifold diagrams*: indeed our definition provides a solution to the long-standing search for a formal generalization of 'string diagrams' to all higher dimensions [**BD98, Bur93**]. This notion of manifold diagrams is as powerful as one could hope: first, because of the combinatorializability of tame stratifications, manifold diagrams are also completely combinatorializable, into an appropriate notion of transverse trusses, thus providing a computable and combinatorial theory of 'diagrammatic reasoning'; second, combining the framed conicality and the dualization of meshes, manifold diagrams naturally dualize to a notion of higher cell pasting diagrams that formalizes arbitrary composability structures in higher categories.

Framed conicality appears here, but may be suppressed in C5. Probably suppress here too. [outdated after discussion]

Building on the conceptual link between stratified Morse theory and framed stratifications, and drawing from the established connection between embedded manifold bordisms and higher categories via the tangle hypothesis, we can leverage the theory of manifold diagrams to identify a combinatorially tractable class of *tame tangles*: specifically, these are embeddings of manifolds in euclidean space that allow refinements through manifold diagrams. Of course we expect any smooth embedding of a manifold has an arbitrarily small deformation to a tame tangle, and so nothing is lost by excluding more general embeddings. Our combinatorial encoding of (tame) tangles immediately provides a novel computational toolkit: we can stratify the space of tangles by algorithmically computable local or global complexity measures, and formalize computable notions of tangle perturbation, simplification, and stability. Having a robust algorithmic approach to tangles is already novel in dimension 4 [**Kam17**], but indeed our definitions and tools apply in all dimensions and all codimensions. The divergence of our theory from the classical view of tangles becomes more interesting in higher dimensions [**Lur08**]. A sufficiently small open neighborhood around a critical point in a tangle is called a 'tangle singularity', or just a 'singularity' for brevity. The traditional view has been that singularity classification becomes profoundly unmanageable as the dimension increases [**MY82, Fun11, BGMP24, GZA85**]: first arise uncountable continuous moduli of distinct singularity types, then the moduli space of singularities itself becomes infinite dimensional, and generally demons abound (except for the so-called *simple* singularities [**AWT86**]). By contrast, in the framed combinatorial setting, singularities are countable in all dimensions.

That context of manifold diagrams and tangle singularities considered, there arise various open problems and directions for investigation; for instance, the classification of perturbation-stable singularities. A 'perturbation-stable singularity' is one that cannot be simplified by small deformations, and is therefore in a sense an 'elementary singularity'; this stability condition is straightforward to formalize using the combinatorial complexity measures at our disposal. We expect that the set of isomorphism classes of perturbation-stable singularities in any fixed dimension is *finite*, at least when considering tangles in codimension 1. While this classification exhibits intriguing parallels to the classification of classical *differential* singularities (especially, the simple ones), a full description remains elusive as the dimension grows.

Complementary to singularities, which are the most local sort of tangles, are 'tangle isotopies', or 'isotopies' for brief, which are singularity-free tangles that encode the ways manifolds can pass by one another at a distance in euclidean space. As there are distinguished elementary singularities, namely those that are perturbation-stable, similarly there are 'elementary isotopies', namely those that cannot be deformed into a composite of simpler isotopies. Naturally we may then pose the problem: classify elementary isotopies. Again, we expect that the set of isomorphism classes of elementary isotopies in any fixed dimension is *finite*, but a precise classification remains unknown even in relatively low dimensions.

Given a sufficiently perturbation-stable k-dimensional tangle M^k in euclidean space $\mathbb{R}^n$, the composite map $M^k \hookrightarrow \mathbb{R}^n \to \mathbb{R}^m$ (where the last map is the standard projection in the canonical proframe of euclidean space) should be a prototypical 'm-Morse function', in the sense that all its local singularities and global isotopies would be, in an appropriate sense, elementary. Less precisely than the previous problems, we may ask for the development of a direct definition of m-Morse functions (without reference to tangle embeddings), cf. [**B$^+$67, Sae93, GK11**], which retains the combinatorial and computational flavor of our tangles and manifold diagrams, and therefore admits a tractable classification and attendant application to smooth manifold topology. We expect not only that such a combinatorial higher Morse theory exists, but that the resulting combinatorial invariants detect, for instance, all smooth structures on manifolds. The realization of such an expectation depends, most likely, on the validity of our aforementioned conjectures about framed homeomorphism and framed stratifications—indeed they would imply that every combinatorial tame tangle built from (combinatorially represented) smooth singularities has a canonical smooth structure and that every perturbation-stable smooth tangle has such a combinatorially tame representation.

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1 : Add Sergei and AMS folks

2 : Add MPIM

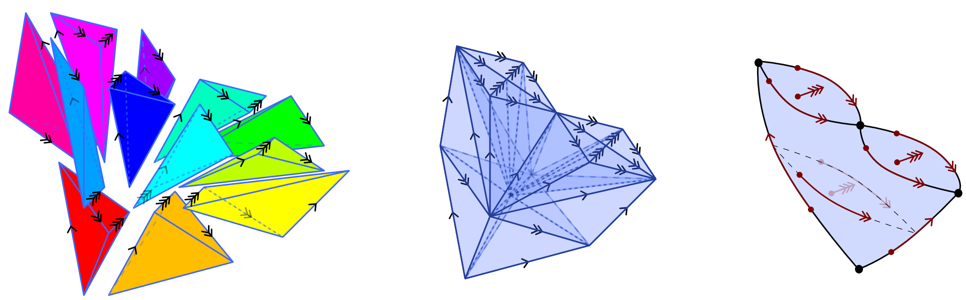
Add TypeDB and Symbolica

3 : Add Filippos and anyone else who proofread – Yuhang. Glen? Manuel?

1 : Add Lukas Heidemann

CHAPTER 1

Framed combinatorial structures



In this chapter, we introduce notions of framings on classical combinatorial structures. We begin by defining frames on simplices, in Section 1.1, along with the related concepts of partial and embedded frames on simplices. We then define framings on simplicial complexes, in Section 1.2, and explain the condition of collapsibility for framed simplicial complexes. Finally, we define framings on regular cells and regular cell complexes, in Section 1.3, in terms of locally collapsible framings on the associated barycentric subdivision simplicial complexes.

1.1. Framed simplices

The notion of a frame on a simplex, and later on other combinatorial objects, is of course inspired by and modeled on the classical notion of frames. Classically, a coordinate trivialization of an m-dimensional vector space V is specified by a linear isomorphism $V \xrightarrow{\simeq} \mathbb{R}^m$. Preimages of standard unit vectors $e_i \in \mathbb{R}^m$ under this isomorphism define an ordered list of vectors $(v_1, v_2, \ldots, v_m)$ in V called a 'frame' of V.

The guiding intuition in the translation of frames in linear algebra into the combinatorics of simplices, is that directed edges of simplices play the role of vectors. However, simplices are combinatorially specified by sets of vertices and thus they do not have a distinguished origin. Moreover, their vectors (i.e., directed edges) are 'affine' in that different vectors may start at different points in the simplex. This fact complicates the translation of the classical intuition of frames in linear algebra into the combinatorics of simplices, and we will emphasize this difference by referring to the 'affine combinatorics' of framed simplices.

The basic analogy of vectors in a vector space with 'vectors' in a simplex will lead to a notion of 'frames' on a simplex as follows. We say two vectors in a simplex are 'composable' if the endpoint of the first vector is the starting point of the second vector; in this case, their 'composite' is the unique directed edge starting at the starting point of the first vector and ending in the endpoint of the second vector. A 'basis' of an m-simplex is a set of m vectors such that all other vectors (up to reversing their direction) can be written as composites of vectors in the basis. A 'frame' of an m-simplex is an ordered basis.

To emphasize a different aspect of the analogy to classical linear frames, we can rephrase the notion of frames on a simplex as follows. Observe that the elements of any basis of an m-simplex S must be the elements of a chain of m composable vectors in the simplex; we call such a chain a 'spine' of S [**Lan21**, Def. 1.1.37]. A choice of basis therefore determines an identification $S \cong [m]$ of S with the *ordered standard simplex* $[m] = (0 < 1 < \cdots < m)$, by mapping the spine of S to the standard spine of $[m]$. Conversely, any identification $S \cong [m]$ determines a basis of the simplex S by transporting the standard spine. A frame of an m-simplex S is then an identification $S \cong [m]$ *together* with a choice of order on the set of standard spine vectors $\mathsf{spine}[m]$ of $[m]$. The standard simplex $[m]$ has of course the canonical identity identification with itself, and so we refer to the simplex $[m]$ with an order $\mathcal{F}$ of its spine vector set $\mathsf{spine}[m]$ as a 'framed standard simplex' $([m], \mathcal{F})$.

The collection of all framed standard simplices $([m], \mathcal{F})$ *with any choice of frame* $\mathcal{F}$ will together play the role of the solitary euclidean space $\mathbb{R}^m$ *with its standard frame* $\{e_1, e_2, \ldots, e_m\}$. The fact that there is only one 'framed standard euclidean space $\mathbb{R}^m$' but many 'framed standard simplices $([m], \mathcal{F})$' reflects the fact that several affine constellations of standard basis vectors can arise. For instance, considering the standard basis vectors e_1 and e_2

in $\mathbb{R}^2$, the affine concatenation of e_1 with e_2 forms the spine (e_1, e_2) of a 'framed standard simplex', while the affine concatenation of e_2 with e_1 forms the spine (e_2, e_1) of a *different* 'framed standard simplex'. Similarly, the three standard basis vectors in $\mathbb{R}^3$ admit six distinct affine concatenations, and these correspond to the six types of framed standard 3-simplices. See Figure 1.1 for an illustration. The affine combinatorics of framed simplices allows and accounts for all of these configurations.

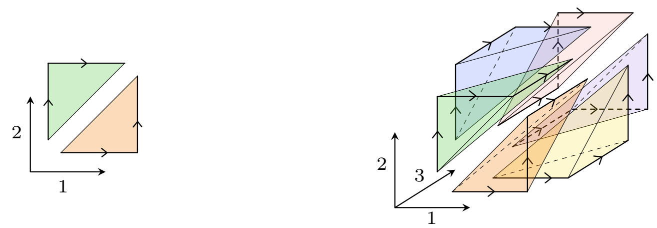


FIGURE 1.1. Distinct framed standard simplices spanned by the same standard frame vectors.

OUTLINE. In Section 1.1.1 we describe the affine combinatorial counterparts of various classical linear algebraic notions, and then define frames on simplices along with their generalization to 'partial' and 'embedded' frames. Then in Section 1.1.2 we introduce the basic notion of 'kinship' of simplicial vectors and use it to define restriction of frames to simplicial faces, with which we can characterize framed maps of framed simplices.

1.1.1. Frames, partial frames, and embedded frames.

SYNOPSIS. We begin with a description of the correspondence between classical linear algebraic notions and simplicial notions, which will also include a generalization of the classical notion of frames to partial and embedded frames. We then formally define and illustrate all three notions of frames for simplices.

It will be useful to first fix some basic terminology and notation concerning simplices [**Mun18**, §1-§3] [**Fri08**].

TERMINOLOGY 1.1.1 (Combinatorial simplices). An 'ordered m-simplex' is a totally ordered set with $m+1$ elements. An 'unordered m-simplex', also called simply an 'm-simplex', is a set with $m+1$ elements. ⌟

NOTATION 1.1.2 (Category of ordered simplices). We will denote the category of ordered simplices by Δ; its objects are the ordered simplices, and its morphisms are the order-preserving maps. ⌟

NOTATION 1.1.3 (Category of unordered simplices). We will denote the category of unordered simplices by $\underline{\Delta}$; its objects are the unordered simplices and its morphisms are all functions. ⌟

TERMINOLOGY 1.1.4 (Unordering ordered simplices). The 'unordering' functor $(-)^{\mathsf{un}} : \Delta \to \underline{\Delta}$ forgets the order of ordered simplices. ⌟

TERMINOLOGY 1.1.5 (Face maps and degeneracy maps). An injective map of (ordered or unordered) simplices is called a 'face map', and a surjective map of (ordered or unordered) simplices is called a 'degeneracy map'. ⌟

NOTATION 1.1.6 (Maps between ordered and unordered simplices). Maps $S \to T$ resp. $T \to S$ between an unordered simplex S and an ordered simplex T will be parsed as maps of unordered simplices $S \to T^{\mathsf{un}}$ resp. $T^{\mathsf{un}} \to S$. ⌟

TERMINOLOGY 1.1.7 (Ordered standard simplex). The 'ordered standard m-simplex' $[m]$ is the poset $(0 < 1 < \cdots < m)$; considering posets as categories, this simplex is, equivalently, the category $(0 \to 1 \to \cdots \to m)$. ⌟

Since it will be used so frequently, we usually refer to the ordered standard m-simplex simply as 'the m-simplex $[m]$'. Every ordered m-simplex S is canonically isomorphic to the standard m-simplex $[m]$. We may therefore work with a skeleton of Δ as follows.

NOTATION 1.1.8 (The skeleton of standard simplices). Abusing notation, we denote the skeleton of Δ containing only standard simplices $[m]$ for $m \in \mathbb{N}$ again by Δ. ⌟

TERMINOLOGY 1.1.9 (Unordered standard simplex). The unordering $[m]^{\mathsf{un}}$ of the ordered standard simplex $[m]$ is the 'unordered standard m-simplex' $\{0, 1, ..., m\}$. ⌟

TERMINOLOGY 1.1.10 (Sets of numerals). The 'set of numerals' or 'numeral set' $\underline{m}$ is the ordered set $\{1 < 2 < \cdots < m\}$. A morphism of numeral sets is an order-preserving function. ⌟

TERMINOLOGY 1.1.11 (Geometric simplices). Given an m-simplex S, the 'geometric simplex' $|S|$ is the convex hull of the set S in the free vector space $\mathbb{R}\langle S\rangle$. Similarly given a map of simplices $f : S \to T$, the map of geometric simplices $|f| : |S| \to |T|$ is the restriction of the linear extension $\mathbb{R}\langle f\rangle : \mathbb{R}\langle S\rangle \to \mathbb{R}\langle T\rangle$. ⌟

1.1.1.1. The fundamental analogy. We now develop a fundamental though loose analogy between some notions in linear algebra and corresponding notions in affine simplicial combinatorics, which will motivate and guide our definitions of frames on simplices. See Chapter A for recollections of the relevant linear and affine algebra background context.

As the fundamental starting point, for the notion of an m-dimensional vector space V, we take as an analog the notion of an unordered m-simplex S. The standard vector space $\mathbb{R}^m$ will have its analog being the standard simplex $[m]$. Next, for the notion of a vector in a vector space, we take as an analog the notion of 1-simplex in a simplex, as follows.

TERMINOLOGY 1.1.12 (Simplicial vectors). A 'vector' v in an unordered simplex S is a map $v : [1] \to S$. We say the vector is 'nondegenerate' (or 'nonzero') if the map v is injective, and otherwise say that the vector is a 'degenerate' (or 'zero') vector. We typically assume vectors are nondegenerate unless specified to the contrary. ⌟

Given two vectors $v_0, v_1 : [1] \to S$ with $v_0(1) = v_1(0)$ their composite is, of course, the vector $z : [1] \to S$ defined by $z(i) = v_i(i)$.

An ordered basis of a vector space is an ordered collection of vectors that are suitably independent. Analogously, a *basis* is a set of vectors in a simplex S such that proper composites of basis vectors are exactly the remaining non-basis vectors of S. An *ordered* basis of S is a basis set with an order such that basis vectors are composable in that order, leading to the following notion of 'directed spine'.

TERMINOLOGY 1.1.13 (Directed spine of an unordered simplex). A 'directed spine' of an unordered m-simplex is an ordered set of m nondegenerate vectors, such that the starting point of each vector is the ending point of the preceding vector (if one exists) and such that the vectors together cover every vertex of the simplex. ⌟

Note that a choice of directed spine on a given unordered m-simplex S is the same as an isomorphism of unordered simplices from the standard simplex $[m]$ to the given simplex S, and thus is also the same as a choice of ordering of the simplex S. A 'spine vector' of a simplex with a directed spine is, of course, one of the vectors in the ordered collection that is the directed spine. We may reexpress this notion of spine vector in terms of the corresponding ordered simplex as follows.

TERMINOLOGY 1.1.14 (Spine vector of an ordered simplex). A 'spine vector' in an ordered simplex S is an (order-preserving) nondegenerate vector $[1] \to S$ that cannot be written as the composite of more than one (order-preserving) nondegenerate vector. The ordered collection of spine vectors of the ordered simplex S is denoted $\mathsf{spine}\, S$. ⌟

EXAMPLE 1.1.15 (The spine of the standard simplex). The directed spine $\mathsf{spine}[m]$ of the standard simplex $[m]$ is the ordered set $((0 \to 1), (1 \to 2), \ldots, (m-1 \to m))$. It may be canonically identified with the numeral set $\underline{m}$, by identifying the spine vector $(i-1 \to i) \in \mathsf{spine}[m]$ with the numeral element $i \in \underline{m}$. ⌟

In classical linear algebra, an ordered basis of a vector space [**Axl24**, Def. 2.26] is, of course, synonymous with a (linearly independent) frame of that vector space.[1] However, a crucial *disanalogy* between linear algebra and affine combinatorics is that, though for a vector space, an ordered basis

[1]At the outset we will take the word 'frame' to implicitly include a sense of linear independence, though later generalizations of the notion of frame, both linear algebraic and affine combinatorial, will allow certain dependencies.

simply is a frame, by contrast for a simplex, a directed spine will not by itself provide a frame. Recall from Figure 1.1 that there are multiple distinct 'framed simplices' built out of the same ordered basis vectors; indeed there are as many such simplices as there are orders on the already ordered set of basis vectors. As that figure adumbrates and as defined precisely later, the further information required for a frame on a simplex with a directed spine will be *an order on the spine*.

Classical linearly-independent frames can be generalized by allowing the frame to be partial, that is, not entirely spanning the vector space, or allowing the frame to be redundant, that is, spanning but with dependencies, or indeed allowing both partiality and redundancy at the same time. A convenient way of encoding a 'partial frame' on a vector space V is via an injection $V \hookleftarrow \mathbb{R}^k$ (which, in particular, selects k vectors in V as the images of the standard basis in $\mathbb{R}^k$). Dually, a convenient way of encoding a redundant frame on a vector space V is via a projection $V \twoheadleftarrow \mathbb{R}^n$. Importantly, if V has euclidean structure, we may equivalently consider projections $V \twoheadrightarrow \mathbb{R}^k$ that *split* the injections $V \hookleftarrow \mathbb{R}^k$ to define partial frames, and similarly, injections $V \hookrightarrow \mathbb{R}^n$ that split the projections $V \twoheadleftarrow \mathbb{R}^n$ to define redundant frames [**ML98**, §I.5]. This *splitting* perspective not only strongly informs the formulation of simplicial frames, but is also the reason that we refer to the above redundant frames instead as 'embedded frames'. Finally, note that the pushout of these two notions, that of an 'embedded partial frame', comes from a general map $V \hookleftarrow W \twoheadleftarrow \mathbb{R}^n$ or alternatively from the corresponding splitting $V \twoheadrightarrow W \hookrightarrow \mathbb{R}^n$.

To motivate and prepare for the corresponding notions of generalized frames in the simplicial combinatorial case, we will need simplicial analogs of the notions of projection and injection of vector spaces. Projections pose no difficulty and correspond to ordinary simplicial degeneracy maps.

TERMINOLOGY 1.1.16 (Simplicial degeneracies). A 'degeneracy' map $S \twoheadrightarrow T$ between unordered simplices S and T is any surjective map of sets from S to T. ⌟

TERMINOLOGY 1.1.17 (Kernel). The 'kernel' $\ker(S \twoheadrightarrow T)$ of a degeneracy map $S \to T$ is the subset of the simplicial vectors in S that are mapped to zero vectors by the degeneracy. ⌟

Note that not every subset of simplicial vectors is the kernel of a degeneracy, which is dissimilar to the case of vector spaces (where every subspace is the kernel of some projection). Moreover, there is, of course, the usual companion notion of simplicial 'face' map $S \hookrightarrow T$, that is an injective map of unordered simplices. However, that notion too is decidedly insufficient as a combinatorial analog of vector space inclusions. Instead, we must generalize notions of faces, kernels, and images to appropriate 'affine' notions as follows. These are easiest to phrase in the case of ordered simplices.

TERMINOLOGY 1.1.18 (Affine kernel). For a degeneracy $S \twoheadrightarrow T$ of ordered simplices, the 'affine kernel' $\ker^{\mathrm{aff}}(S \twoheadrightarrow T)$ is the subset of the spine vectors in $\mathsf{spine}\, S$ that are mapped to zero vectors by the degeneracy. ⌟

Note well that the affine kernel $\ker^{\mathrm{aff}}(S \twoheadrightarrow T)$ need not form the spine vectors of any subsimplex of the simplex S, and so need not be the image of a simplicial face map.

The kernel of any linear projection is a subspace, and any subspace is the kernel of the quotient by that subspace. Similarly, the affine kernel of any simplicial degeneracy is a subset of the spine, and any subset $U \subset \mathsf{spine}\, S$ is the affine kernel of the degeneracy $S \twoheadrightarrow T$ that degenerates exactly those spine vectors; that correspondence suggests the following notions.

TERMINOLOGY 1.1.19 (Affine face). An 'affine face' map $f : S \hookrightarrow T$ of ordered simplices is an ordered map $f : \mathsf{spine}\, S \hookrightarrow \mathsf{spine}\, T$. ⌟

TERMINOLOGY 1.1.20 (Affine image). The 'affine image' of an affine face map $f : S \hookrightarrow T$ of ordered simplices is the image $\mathrm{im}^{\mathrm{aff}}(f) := \mathrm{im}(f : \mathsf{spine}\, S \hookrightarrow \mathsf{spine}\, T) \subset \mathsf{spine}\, T$ in the spine of the target. ⌟

TERMINOLOGY 1.1.21 (Affine cokernel). The 'affine cokernel' of an affine face map $f : S \hookrightarrow T$ of ordered simplices is the complement $\mathrm{coker}^{\mathrm{aff}}(f) = \mathsf{spine}\, T \setminus \mathrm{im}^{\mathrm{aff}}(f) \subset \mathsf{spine}\, T$ of the affine image in the spine of the target. ⌟

OBSERVATION 1.1.22 (Canonical simplicial splittings). Observe that any affine face map $f : S \hookrightarrow T$ has a canonical *splitting* simplicial degeneracy $S \twoheadleftarrow T : g$ whose affine kernel $\ker^{\mathrm{aff}}(g)$ is the affine cokernel $\mathrm{coker}^{\mathrm{aff}}(f)$; that is the splitting map degenerates exactly those spine vectors that are not in the affine image of the affine face map. ⌟

The notions of simplicial degeneracy and affine faces combine to a general notion of affine simplicial maps.

TERMINOLOGY 1.1.23 (Affine maps). An 'affine' map $S \to R$ from an unordered simplex S to an ordered simplex R is a sequence $S \twoheadrightarrow T \hookrightarrow R$ consisting of a degeneracy $S \twoheadrightarrow T$ from S to an ordered simplex T, and an affine face $T \hookrightarrow R$. ⌟

In particular, an affine map $S \to R$ is an affine face, written again $S \hookrightarrow R$, if it decomposes into maps $S \cong T \hookrightarrow R$ for an ordered simplex T.

Recall from the discussion above that a frame on an unordered m-simplex S can be described as an isomorphism $S \xrightarrow{\sim} [m]$ to the standard simplex (providing a directed spine of S) together with a choice of order on the spine of the standard simplex (providing by the isomorphism an order on the spine of S). Equipped with the affine simplicial combinatorial analogs of vector space projections, inclusions, and maps, namely degeneracies, affine faces, and affine maps, we can provide a compact preview of the simplicial notions of generalized frames, namely partial frames, embedded frames, and embedded partial frames on simplices.

A *partial* frame of an unordered simplex S will be a degeneracy $S \twoheadrightarrow [k]$ together with an order on the spine of the target simplex. An *embedded* frame of an unordered simplex S will be an affine face $S \hookrightarrow [n]$ (which determines and is determined by its canonical splitting $S \twoheadleftarrow [n]$) and an order on the spine of S. An *embedded partial* frame of an unordered simplex S will be an affine map $S \twoheadrightarrow T \hookrightarrow [n]$ (whose affine face again determines and is determined by the corresponding splitting $T \twoheadleftarrow [n]$) and an order on the spine of T.

The prelude analogy described so far, between elementary structures in linear algebra (including the earlier description of splitting maps to define frames on euclidean vector spaces V) and affine simplicial combinatorics, is displayed in Figure 1.2. The crucial notions of frames and generalized frames on simplices are detailed, with further explanation, examples, and illustration, over the whole of this first section, and a yet more detailed discussion of their classical linear algebraic counterparts can be found in Chapter A. Frames, partial frames, and embedded frames on vector spaces and simplices are illustrated in Figure 1.3, along with a preview of their generalizations to simplicial complexes and regular cells.

1.1.1.2. The definition of framed simplices. We introduce frames on simplices, as a combinatorial analog of frames of euclidean vector spaces. The role of frame vectors will be played by the spine vectors of a simplex. As a linear frame is an ordered list of its frame vectors, a simplicial frame will be an ordered collection of spine vectors.

DEFINITION 1.1.24 (Frame on the standard simplex). A **frame $\mathcal{F}$ of the standard m-simplex** $[m]$ is a bijection $\mathcal{F} : \mathsf{spine}[m] \to \underline{m}$ from the set $\mathsf{spine}[m]$ of spine vectors of the simplex to the set of numerals $\underline{m} = \{1, 2, \ldots, m\}$. ⌟

We may of course equivalently think of a frame $\mathcal{F}$ in terms of the inverse function $\mathcal{F}^{-1} : \underline{m} \to \mathsf{spine}[m]$ from the set of numerals to the spine, or more concretely as an ordered list $(v_1, v_2, \ldots, v_m)$ of spine vectors $v_i = \mathcal{F}^{-1}(i) \in \mathsf{spine}[m]$ of the simplex. That is, the frame is an order on the spine, as suggested in Figure 1.2. Frames of unordered simplices may then be defined as follows.

DEFINITION 1.1.25 (Frame on a simplex). A **frame** of an unordered m-simplex S is an isomorphism $S \cong [m]$ together with a frame $\mathcal{F}$ on $[m]$. ⌟

We usually denote framed simplices S by pairs $(S \cong [m], \mathcal{F})$. We may also keep the isomorphism $S \cong [m]$ implicit, and simply say that $\mathcal{F}$ is a frame on S. Of course, the choice of isomorphism $S \cong [m]$ is the same as a choice of spine for S, and the order $\mathcal{F}$ on the standard spine gives an order on the chosen spine for S; we therefore often think of the frame of S simply as a choice of spine and order on that spine.

Linear algebra	**Affine combinatorics**
an m-dimensional vector space V	an unordered m-simplex S
a nonzero vector v in V	a directed edge v in S
the zero vector 0 in V	a vertex x in S
an ordered basis of V	a directed spine of S $\equiv$ an isomorphism $S \xrightarrow{\sim} [m]$
a projection $V \twoheadrightarrow W$	a degeneracy $S \twoheadrightarrow T$
an injection $V \hookrightarrow W$	an affine face $S \hookrightarrow T$: a choice of directed spines of S and T and an inclusion of $\mathsf{spine}\, S$ into $\mathsf{spine}\, T$
a map $V \twoheadrightarrow W \hookrightarrow U$	an affine map $S \twoheadrightarrow T \hookrightarrow R$
the standard vector space $\mathbb{R}^m$	the standard simplex $[m]$
the standard ordered basis of $\mathbb{R}^m$	the standard directed spine of $[m]$
$\Vert$	$\nparallel$
the standard frame of $\mathbb{R}^m$	the standard directed spine of $[m]$ *with an order on that spine*
a frame of $\mathbb{R}^m$	a directed spine of $[m]^{\mathsf{un}}$ with an order on that spine
a frame of V: an isomorphism $V \xrightarrow{\sim} \mathbb{R}^m$	a frame of S: an isomorphism $S \xrightarrow{\sim} [m]$ with an order on the spine of $[m]$
a k-partial frame of V: an injection $V \hookleftarrow \mathbb{R}^k$ $\equiv$ a splitting *projection* $V \twoheadrightarrow \mathbb{R}^k$	a k-partial frame of S: a *degeneracy* $S \twoheadrightarrow [k]$ with an order on the spine of $[k]$
an n-embedded frame of V: a projection $V \twoheadleftarrow \mathbb{R}^n$ $\equiv$ a splitting *injection* $V \hookrightarrow \mathbb{R}^n$	an n-embedded frame of S: an *affine face* $S \hookrightarrow [n]$ and an order on the spine of S
an n-embedded partial frame of V: a map $V \hookleftarrow W \twoheadleftarrow \mathbb{R}^n$ $\equiv$ a splitting map $V \twoheadrightarrow W \hookrightarrow \mathbb{R}^n$	an n-embedded partial frame of S: an affine map $S \twoheadrightarrow T \hookrightarrow [n]$ and an order on the spine of T

FIGURE 1.2. The analogy between notions in linear algebra and notions in affine combinatorics.

EXAMPLE 1.1.26 (Frames on simplices). In Figure 1.4 we illustrate a few framed m-simplices $(S \cong [m], \mathcal{F})$. The frame $\mathcal{F} : \mathsf{spine}[m] \to \underline{m}$ is indicated in three different ways, as follows.

(1) *Numeral labels*: the spine vector $v \in \mathsf{spine}[m]$ is labeled by its numeral value $\mathcal{F}(v) \in \underline{m}$.

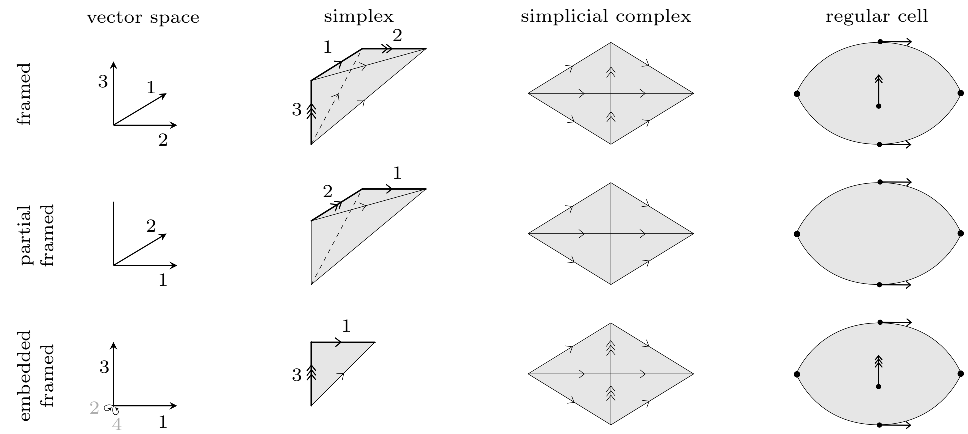


FIGURE 1.3. Frames in linear algebra and affine combinatorics.

(2) *Arrowheads*: the numeral label of the spine vector is specified by the number of arrowheads along the simplicial vector.
(3) *Coordinate frame*: the labeled spine vectors, thought of as vectors in the linear space spanned by the picture of the simplex, are translated so their sources are coincident, and the resulting labeled 'coordinate frame' is drawn in or near the simplex. ⌟

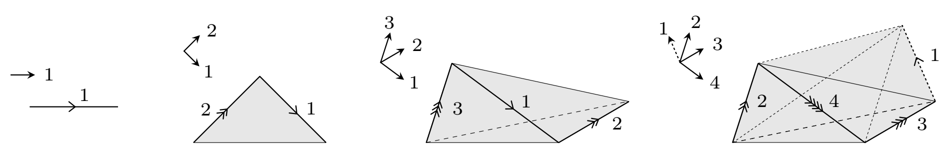


FIGURE 1.4. Framed simplices.

NOTATION 1.1.27 (Frame labels). In subsequent figures, we will primarily indicate the frame labels by the multi-arrowhead notation, though we sometimes also retain the coordinate frame and occasionally the numeral labels themselves for emphasis or clarity. (Note that in Figure 1.1 we depicted the framed 2-simplices and framed 3-simplices using only the coordinate frame notation.) ⌟

A linear embedding of a framed simplex into euclidean space may preserve the frame structure in the following sense.

TERMINOLOGY 1.1.28 (The standard oriented components of euclidean space). For all $1 \leq i \leq n$, the complement of the linear subspace $\{0\}^i \times \mathbb{R}^{n-i}$ in $\{0\}^{i-1} \times \mathbb{R}^{n-i+1}$ (both being subspaces of $\mathbb{R}^n$) has two components ϵ_i^- and ϵ_i^+ given by $\{0\}^{i-1} \times \mathbb{R}_{<0} \times \mathbb{R}^{n-i}$ resp. $\{0\}^{i-1} \times \mathbb{R}_{>0} \times \mathbb{R}^{n-i}$. We call ϵ_i^- and ϵ_i^+ the 'ith negative' resp. 'ith positive standard component' of $\mathbb{R}^n$. ⌟

Definition 1.1.29 (Framed realization of a framed simplex). A **framed realization** of a framed m-simplex $(S \cong [m], \mathcal{F})$ with frame vectors $v_i = \mathcal{F}^{-1}(i)$ is a linear embedding $r_{\mathcal{F}} : |S| \hookrightarrow \mathbb{R}^m$ of the geometric simplex $|S|$ into $\mathbb{R}^m$ such that, on associated vector spaces, the translation vectors $\vec{v}_i = v_i(1) - v_i(0)$ are mapped into the ith positive standard component $\epsilon_i^+ \subset \mathbb{R}^m$, for all $i \in \underline{m}$.[2] ⌟

Example 1.1.30 (Framed realization of a framed simplex). In Figure 1.5 we illustrate framed realizations of the two framed 2-simplices and the corresponding images vectors of $\vec{v}_i$ based at the origin. See also again Figure 1.1, where one sees the images of one framed realization of each of the six framed 3-simplices. ⌟

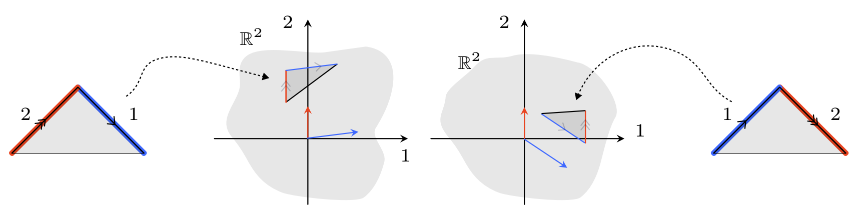


Figure 1.5. Framed realization of framed simplices.

1.1.1.3. Partial frames. We now introduce a simplicial combinatorial analog of the notion of linear partial frames. Recall that a partial frame of a vector space may be seen as an injection $V \hookleftarrow \mathbb{R}^k$ [**Ste51**, §7.7], or considered instead as a projection $V \twoheadrightarrow \mathbb{R}^k$ that is split by that injection. Analogously, instead of having a frame defined on the whole of the spine of a simplex S, we may consider a frame defined only on part of the spine; that portion of the spine will be the complement of the affine kernel of a corresponding degeneracy, and we formulate the definition in terms of that degeneracy, as follows.

There are problems with the ambiguity over whether simplices are ordered or unordered ... it's unclear what class of maps is used ...

Definition 1.1.31 (Partial frame on a simplex). A **k-partial frame** on an unordered m-simplex S is a degeneracy $S \twoheadrightarrow [k]$ together with a frame $\mathcal{F}$ on $[k]$. ⌟

Note that in an m-partial frame of an m-simplex $(S \twoheadrightarrow [m], \mathcal{F})$ the degeneracy $S \twoheadrightarrow [m]$ must be an isomorphism, and thus m-partial frames of m-simplices are frames on m-simplices.

Terminology 1.1.32 (Unframed subspace of a partially framed simplex). The 'unframed subspace' of a k-partially framed simplex $(S \twoheadrightarrow [k], \mathcal{F})$ is the kernel $\ker(S \twoheadrightarrow [k])$. Note that this 'subspace' is a subset of the nondegenerate vectors of the simplex S. ⌟

This terminology does not make sense / contact at present. Affine kernels were only defined for maps of ordered simplices, but partially framed simplices are defined on unordered simplices.

[2] Technically, by 'linear map' $|S| \hookrightarrow \mathbb{R}^m$ we mean an 'affine map' [**Mun18**, §1], as discussed in more detail in Terminology A.2.3.

Example 1.1.33 (Partial frames on simplices). In Figure 1.6 we illustrate the two distinct 1-partially framed 2-simplices, and the three distinct 1-partially framed 3-simplices. Similarly, in Figure 1.7 we illustrate three of the six distinct 2-partial framings of the 3-simplex; the other three are obtained by exchanging all the 1 and 2 frame labels. We depict the degeneracies $S \twoheadrightarrow [k]$ by highlighting their unframed subspace (in green) and illustrate the target framed simplices $([k], \mathcal{F})$ as in Example 1.1.26. Note that the partial frame may also be recorded by labeling vectors v in S with $i \in \underline{k}$ (or with that many arrowheads) whenever $w = (S \twoheadrightarrow [k])(v) \in \mathsf{spine}[k]$ with $\mathcal{F}(w) = i$. ⌟

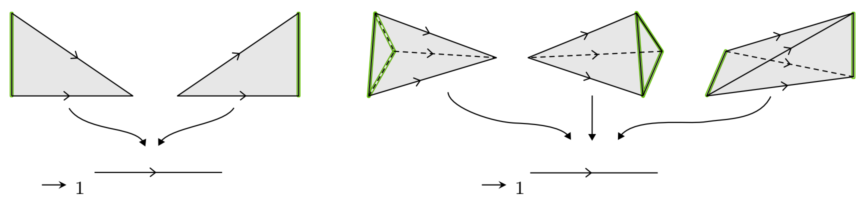


Figure 1.6. The 1-partially framed 2-simplices and 3-simplices.

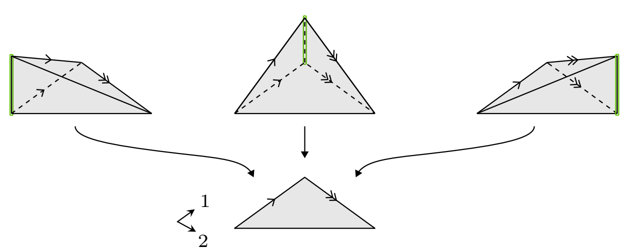


Figure 1.7. Half of the 2-partially framed 3-simplices.

A linear embedding of a partially framed simplex into euclidean space may preserve the frame structure in the following sense.

Definition 1.1.34 (Framed realization of a partially framed simplex). Consider a k-partially framed m-simplex $(S \twoheadrightarrow [k], \mathcal{F})$ with unframed subspace $U := \ker(S \twoheadrightarrow [k])$. A **framed realization** of $(S, \mathcal{F})$ is a linear map $r_{\mathcal{F}} : |S| \to \mathbb{R}^k$ such that, on associated vector spaces, the translation vector $\vec{v} = v(1) - v(0)$ is mapped into the ith positive standard component $\epsilon_i^+ \subset \mathbb{R}^k$ whenever $w = (S \twoheadrightarrow [k])(v) \in \mathsf{spine}[k]$ and $\mathcal{F}(w) = i$, and to $0 \in \mathbb{R}^k$ when $v \in U$. ⌟

This definition had issues. Old version commented out. Partially fixed. Still gap between S and $[m]$. Rewrite. (This def is still tentative because it depends on the affine kernel that isn't well defined.)

Example 1.1.35 (Framed realization of a partially framed simplex). In Figure 1.8 we illustrate framed realizations of a 1-partially framed 2-simplex and of a 2-partially framed 3-simplex. ⌟

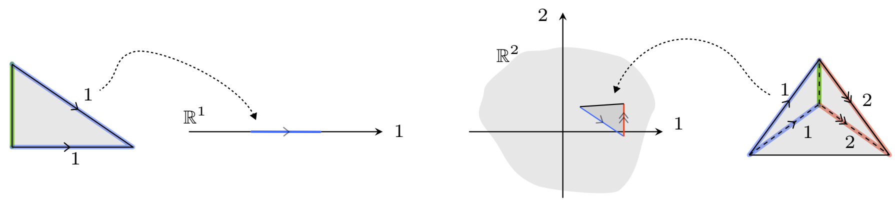


FIGURE 1.8. Framed realization of partially framed simplices.

1.1.1.4. Embedded frames. We introduce a simplicial combinatorial analog of the notion of linear embedded frames. Recall, a linear embedded frame is a vector space embedding $V \hookrightarrow \mathbb{R}^n$. Such frames may alternatively be considered in terms of a corresponding splitting map $V \twoheadleftarrow \mathbb{R}^n$; the image of the standard basis vectors under this splitting provides a classical redundant (or 'overcomplete') frame in the vector space V [**Chr03**, Cor. 1.1.3].

DEFINITION 1.1.36 (Embedded frame on the standard simplex). An **n-embedded frame $\mathcal{F}$ of the standard m-simplex** $[m]$ is an injective function $\mathcal{F} : \mathsf{spine}[m] \hookrightarrow \underline{n}$ from the spine of the simplex into the set of numerals $\{1, 2, \ldots, n\}$. ⌟

We may of course equivalently think of an n-embedded frame $\mathcal{F}$ of $[m]$ in terms of the partial inverse function $\mathcal{F}^{-1} : \underline{n} \to \mathsf{spine}[m]_+$ from the set of numerals to the 'augmented spine' $\mathsf{spine}[m]_+ := \mathsf{spine}[m] \sqcup \{0\}$; more concretely, the n-embedded frame is seen as an ordered list $(v_1, v_2, ..., v_n)$ where $v_i = \mathcal{F}^{-1}(i) \in \mathsf{spine}[m]$ if $i \in \underline{n}$ is in the image of the frame $\mathcal{F} : \mathsf{spine}[m] \hookrightarrow \underline{n}$, and $v_i := 0 \in \mathsf{spine}[m]_+$ otherwise. Note that an m-embedded framed standard m-simplex is the same as a framed standard m-simplex as previously defined. For not-necessarily-standard simplices, we have the following corresponding notion.

DEFINITION 1.1.37 (Embedded frame on a simplex). An **n-embedded frame** of an unordered m-simplex S is an isomorphism $S \cong [m]$ together with an n-embedded frame $\mathcal{F}$ on $[m]$. ⌟

We usually denote n-embedded framed m-simplices S by pairs $(S \cong [m], \mathcal{F})$, though may also leave the isomorphism $S \cong [m]$ implicit. As in the case of non-embedded frames, the choice of isomorphism $S \cong [m]$ is the same as a choice of spine for S (and thus correspondence of the chosen spine with the standard spine of the standard simplex), so we may think of the embedded frame of S as a choice of spine together with an injective assignment of numerals to the spine vectors.

REMARK 1.1.38 (Embedded frames via affine subspaces). To make contact with the analogy displayed earlier in Figure 1.2, note that an n-embedded frame $\mathcal{F}$ of an unordered m-simplex S may alternatively be described by an affine face $S \hookleftarrow\!\!\!\rightarrow [n]$ (i.e., an isomorphism $S \cong [m]$ together with an affine face $[m] \hookleftarrow\!\!\!\rightarrow [n]$) and an ordering of the affine image of that affine face. ⌟

Check whether this remark does actually correspond to the table

Once the defs are actually sorted, check throughout the correspondence with the table

EXAMPLE 1.1.39 (Embedded frames on simplices). In Figure 1.9 we illustrate a few n-embedded framed m-simplices $(S \cong [m], \mathcal{F})$. As before, the frame $\mathcal{F} : \mathsf{spine}[m] \hookrightarrow \underline{n}$ of $[m]$ is indicated in three ways: the spine vector $v \in \mathsf{spine}[m]$ is labeled by its numeral value $\mathcal{F}(v) \in \underline{n}$, that numeral value is the number of arrowheads along the vector, and the m labeled spine vectors are translated into a labeled coordinate frame. In that coordinate frame, we also depict the numerals that are not in the image of the frame $\mathcal{F} : \mathsf{spine}[m] \hookrightarrow \underline{n}$ as infinitesimal 'curled-up' dimensions; this evokes the partial inverse function $\mathcal{F}^{-1} : \underline{n} \to \mathsf{spine}[m] \sqcup \{0\}$ which sends those numerals to zero in the augmented spine. ⌟

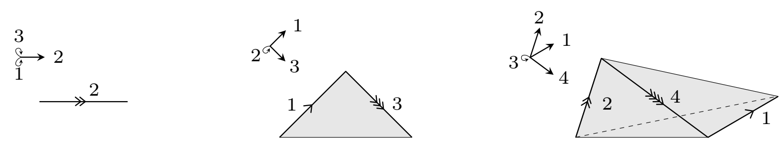


FIGURE 1.9. Embedded framed simplices.

A linear embedding of a framed simplex into euclidean space may preserve the frame structure in the following sense.

DEFINITION 1.1.40 (Framed realization of an embedded framed simplex). A **framed realization** of an n-embedded framed m-simplex $(S \cong [m], \mathcal{F})$ with nonzero frame vectors $v_i = \mathcal{F}^{-1}(i)$, $i \in \mathrm{im}(\mathcal{F})$, is a linear embedding $r_{\mathcal{F}} : |S| \hookrightarrow \mathbb{R}^n$ of the geometric simplex $|S|$ into $\mathbb{R}^n$ such that, on associated vector spaces, the translation vectors $\vec{v}_i = v_i(1) - v_i(0)$ are mapped into $\epsilon_i^+ \subset \mathbb{R}^n$, for all $i \in \mathrm{im}(\mathcal{F})$. ⌟

EXAMPLE 1.1.41 (Framed realization of embedded framed simplices). In Figure 1.10 we illustrate framed realizations of the three 3-embedded framed 1-simplices. The framed realization of the vector with frame label 3 must be an affine vector whose associated linear vector (i.e., after translating its basepoint to the origin) is in ϵ_3^+, which is to say a positive multiple of the basis vector e_3. The framed realization of the vector with frame label 2 must be an affine vector whose associated linear vector is in ϵ_2^+, that is the open half of the plane $\langle e_2, e_3 \rangle$ with positive e_2 coordinate. Similarly the framed realization of the frame label 1 vector has associated linear vector in ϵ_1^+, the open half of 3-space $\langle e_1, e_2, e_3 \rangle$ with positive e_1 coordinate.

Similarly, in Figure 1.11 we illustrate framed realizations of the six 3-embedded framed 2-simplices. We abuse notation slightly by only depicting the images of the simplices, and labeling the image vectors by the frame labels of the implicit corresponding source 2-simplex. As for the 3-embedded framed 1-simplices in the previous figure, all the vectors with frame label 3 point strictly in the positive e_3 direction, all the vectors with frame label 2 point in a positive e_2 direction inside the $\langle e_2, e_3 \rangle$ plane, and all the vectors with frame label 1 point in some positive e_1 direction in 3-space. ⌟

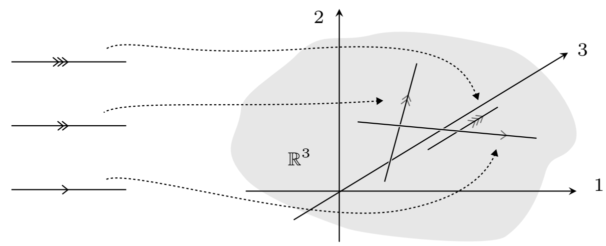


FIGURE 1.10. Framed realization of the 3-embedded framed 1-simplices.

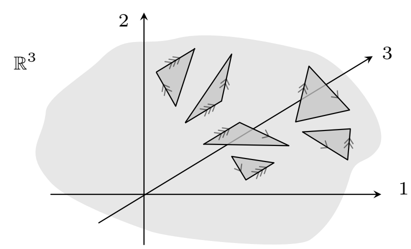


FIGURE 1.11. Framed realization of the 3-embedded framed 2-simplices.

While we will be interested ultimately only in non-partial embedded frames, we also mention the following generalization of embedded frames to the partial case. Instead of an embedded frame on the 'complete' simplex S, we may consider an embedded frame on S that is defined everywhere but on a subset of the vectors of S, namely as before those vectors in a corresponding affine kernel. Following Figure 1.2 embedded partial frames of unordered simplices can be defined as follows.

DEFINITION 1.1.42 (Embedded partial frame on a simplex). An **n-embedded k-partial frame** on an unordered m-simplex S is a degeneracy $S \twoheadrightarrow T$ together with an n-embedded frame $(T \cong [k], \mathcal{F})$ on the target k-simplex T. ⌟

Since the intermediate simplex T has, as part of the data of its embedded frame, a given isomorphism to the standard simplex, we usually omit the simplex T entirely and consider n-embedded k-partially framed simplices S to be pairs $(S \twoheadrightarrow [k], \mathcal{F})$, where $\mathcal{F}$ is an n-embedded frame of the standard simplex $[k]$.

REMARK 1.1.43 (Embedded partial frames via affine maps). To make contact with the earlier analogy of Figure 1.2, note that we may think of the structure of an embedded partially framed simplex as an affine map $S \twoheadrightarrow T \hookrightarrow [n]$, together with an ordering of the spine of T. ⌟

Check that last remark, and the earlier one referenced (CXD)

Terminology 1.1.44 (Unframed subspace of an embedded partially framed simplex). The 'unframed subspace' of an n-embedded partially framed simplex $(S \twoheadrightarrow [k], \mathcal{F})$ is the kernel $U = \ker(S \twoheadrightarrow [k])$. As before this 'subspace' is actually a subset of the nondegenerate vectors. ⌟

Example 1.1.45 (Embedded partial frames on simplices). In Figure 1.12 we illustrate a few n-embedded k-partially framed m-simplices $(S \twoheadrightarrow [k], \mathcal{F})$: we depict degeneracies $S \twoheadrightarrow [k]$ by highlighting their unframed subspace (in green) and illustrate the framed simplices $([k], \mathcal{F})$ as in Example 1.1.39. Note that the embedded partial frame $\mathcal{F}$ may also be recorded by labeling vectors v in S with $i \in \underline{n}$ (or with that many arrowheads) whenever $w = (S \twoheadrightarrow [k])(v) \in \mathsf{spine}[k]$ and $\mathcal{F}(w) = i$. ⌟

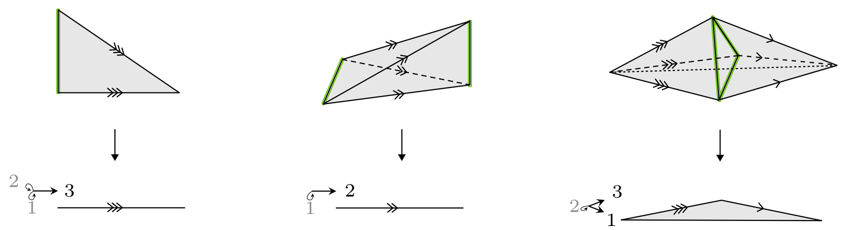


Figure 1.12. Embedded partially framed simplices.

A linear embedding of an embedded partially framed simplex into euclidean space may preserve the frame structure in the following sense.

Definition 1.1.46 (Framed realization of an embedded partially framed simplex). Consider an n-embedded partially framed m-simplex $(S \twoheadrightarrow [k], \mathcal{F})$ with unframed subspace $U := \ker(S \twoheadrightarrow [k])$. A **framed realization** of $(S \twoheadrightarrow [k], \mathcal{F})$ is a linear map $r_{\mathcal{F}} : |S| \hookrightarrow \mathbb{R}^n$ such that $\vec{v} = v(1) - v(0)$ maps to $0 \in \mathbb{R}^n$ when $v \in U$, and into $\epsilon_i^+ \subset \mathbb{R}^n$ when $w = (S \twoheadrightarrow [k])(v)$ and $\mathcal{F}(w) = i$. ⌟

Whether this is final still depends on fixing the the fact that affine kernel, thus unframed subspace, isn't yet defined where it's being used. [keraff->ker]

Example 1.1.47 (Framed realization of an embedded partially framed simplex). In Figure 1.13 we illustrate framed realizations of a 3-embedded 1-partial frame of the 3-simplex and of a 3-embedded 2-partial frame of the 4-simplex. ⌟

1.1.2. Mapping frames on simplices. Having defined frame structures on simplices, we next consider simplicial maps that preserve these frame structures. The characterization of such maps will require us to consider their action on frames of individual simplicial vectors.

Synopsis. We first discuss how frames on simplices restrict to their vectors and, more generally, to their subsimplices. We then define framed maps of framed simplices as those simplicial maps that preserve embedded frames on each vector (and equivalently, on each subsimplex) of their domain simplex. We also define the slightly more general notion of subframed maps.

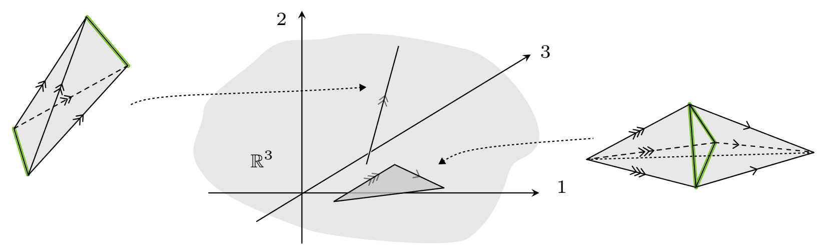


FIGURE 1.13. Framed realization of embedded partially framed simplices.

1.1.2.1. Restricting frames. A linear isomorphism $V \xrightarrow{\simeq} \mathbb{R}^n$ (providing an ordinary frame) or more generally a linear embedding $V \hookrightarrow \mathbb{R}^n$ (conceived of as an embedded frame) can be restricted to any linear subspace $W \hookrightarrow V$ to obtain another linear embedding $W \hookrightarrow \mathbb{R}^n$ (that can again be seen as an embedded frame). We discuss the combinatorial analog of this process, that is, how frames and embedded frames of simplices restrict to embedded frames on simplicial faces. In geometric terms, our definition can be conveniently expressed as follows.

REMARK 1.1.48 (Restricting frames via framed realization). Consider an n-embedded framed m-simplex $(S \cong [m], \mathcal{F})$, and a simplicial j-face $f : T \hookrightarrow S$. Pick any framed realization $r : |S| \hookrightarrow \mathbb{R}^n$. The 'frame restriction' of the n-embedded frame of S to the face T is the unique n-embedded frame of the simplex T for which the linear embedding $r \circ |f| : |T| \hookrightarrow |S| \hookrightarrow \mathbb{R}^n$ is a framed realization. ⌟

Describing this geometric process of frame restriction in purely combinatorial terms, without reference to the affine framed structure of $\mathbb{R}^n$, is more subtle. To properly account for the combinatorial situation, we introduce the notion of simplicial vectors being 'akin', which provides a combinatorial analog of vectors being non-orthogonal. (We will refer to this relationship of combinatorial non-orthogonality generally as 'kinship' of simplicial vectors.)

DEFINITION 1.1.49 (Akin simplicial vectors). The vectors $v = (a \to b)$ and $w = (c \to d)$ in the simplex $[m] \equiv (0 \to 1 \to \cdots \to m)$ are **akin**, denoted $v \pm w$, if there is a vector u that is a factor of both, i.e., such that $v = \tilde{v} \circ u \circ \tilde{\tilde{v}}$ and $w = \tilde{w} \circ u \circ \tilde{\tilde{w}}$ for some possibly degenerate vectors $\tilde{v}, \tilde{\tilde{v}}, \tilde{w}, \tilde{\tilde{w}}$. ⌟

Note that, like the relation of non-orthogonality of linear vectors in a euclidean space, the kinship relation between simplicial vectors is reflexive and symmetric but not transitive.

EXAMPLE 1.1.50 (Kinship of simplicial vectors). In Figure 1.14 we illustrate the kinship of vectors in the 3-simplex. To emphasize the informal conceptual relationship of this notion with the geometry of non-orthogonality, the simplex is drawn with its three spine vectors (highlighted in red, green, and blue) being orthogonal in the ambient euclidean 3-space. The three red vectors are akin, the four green vectors are akin, and the three blue vectors are akin; no other vectors are akin. Note that indeed, vectors are akin precisely when they are non-orthogonal in this geometric 3-simplex. ⌟

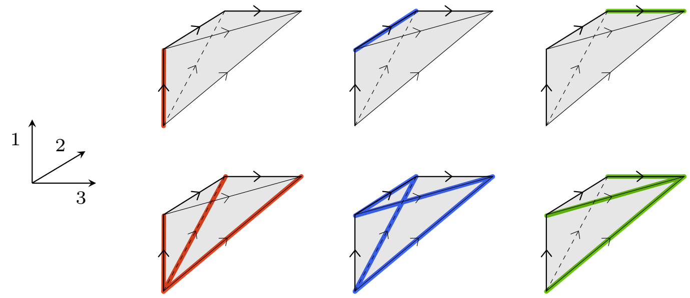


FIGURE 1.14. Kinship of simplicial vectors.

Using the notion of kinship, we can describe the restriction of a frame to any vector of the standard simplex, and subsequently to any face of the standard simplex. The frame order on the spine of the simplex plays a paramount role in this process: the frame label of a general vector will be the lowest numeral among the frame labels of the spine vectors akin to the given vector.

CONSTRUCTION 1.1.51 (Frame restriction to simplicial vectors of the standard simplex). Given an n-embedded frame $\mathcal{F} : \mathsf{spine}[m] \hookrightarrow \underline{n}$ of the simplex $[m]$, and a vector $v : [1] \to [m]$ of that simplex, the 'restriction' $\mathcal{F}|_v : \mathsf{spine}[1] \hookrightarrow \underline{n}$ of the frame to the vector is the n-embedded frame of the simplex $[1]$ whose single label is the minimal frame label of the spine vectors akin to the vector v, i.e., $\mathcal{F}|_v(0 \to 1) = \min\{\mathcal{F}(w) \,|\, w \pitchfork v\}$. ⌟

This restriction procedure produces a plethora of combinatorial arrangements quite distinct from any permutation of its application to the standard frame on the simplex.

EXAMPLE 1.1.52 (Frame restriction to simplicial vectors). In Figure 1.15 we illustrate various embedded framed 3-simplices, along with the corresponding embedded framed restrictions to their 1-faces. In Figure 1.16 we similarly illustrate restriction to the vectors of embedded framed 4-simplices.

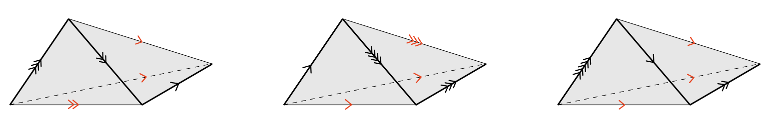


FIGURE 1.15. Restriction of embedded frames to vectors of a 3-simplex.

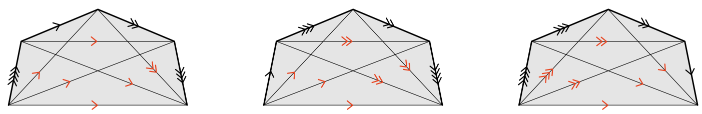


FIGURE 1.16. Restriction of embedded frames to vectors of a 4-simplex.

⌟

The restriction of an embedded frame to any j-face of the simplex is determined directly by the restrictions to the 1-faces, as follows.

CONSTRUCTION 1.1.53 (Frame restriction to simplicial faces of the standard simplex). Let $\mathcal{F} : \mathsf{spine}[m] \hookrightarrow \underline{n}$ be an n-embedded frame of the simplex $[m]$, and let $f : [j] \hookrightarrow [m]$ be a j-face of that simplex. The 'frame restriction' $\mathcal{F}|_f : \mathsf{spine}[j] \hookrightarrow \underline{n}$ of the frame to the j-face is the n-embedded frame whose label value on the spine vector $v : [1] \to [j]$ is the numeral $\mathcal{F}|_{f \circ v} \in \underline{n}$. ⌟

The restriction of an embedded frame of a (not standard) simplex is obtained by translating the frame restriction for the standard simplex across the given (by the frame) isomorphism to the standard simplex, as follows.

NOTATION 1.1.54 (Images of faces under simplicial maps). Given a map $\alpha : S \to [m]$ from an m-simplex S to a standard simplex, and a j-face $f : T \hookrightarrow S$, we can construct the commuting diagram

$$\begin{array}{ccc} T & \xhookrightarrow{f} & S \\ {\scriptstyle \alpha|_f}\big\downarrow\!\!\downarrow & & \big\downarrow{\scriptstyle \alpha} \\ [l] & \xhookrightarrow{f} & [m] \end{array}$$

where $\alpha|_f : T \twoheadrightarrow [l]$ is surjective, and (abusing notation) $f : [l] \hookrightarrow [m]$ is a face, called the 'image face' of f under α. The construction is, of course, simply an application of image factorizations, i.e., $f \equiv \mathrm{im}(\alpha \circ f)$. ⌟

DEFINITION 1.1.55 (Frame restriction to faces of simplices). The **frame restriction** of an n-embedded framed m-simplex $(S \cong [m], \mathcal{F})$ to a simplicial j-face $f : T \hookrightarrow S$ is the n-embedded frame $(T \cong [j], \mathcal{F}|_f)$ of T, where $T \cong [j]$ is the restricted isomorphism $(S \cong [m])|_f$ and the n-embedded frame $\mathcal{F}|_f$ is obtained by restricting $\mathcal{F}$ to the image face $f : [j] \hookrightarrow [m]$. ⌟

EXAMPLE 1.1.56 (Frame restriction to faces). In Figure 1.17 we depict a 4-embedded framed 3-simplex along with the restriction of its frame to various faces. ⌟

One may of course similarly define frame restrictions of embedded *partial* frames of simplices, as follows.

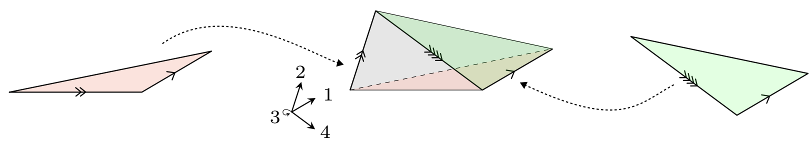


FIGURE 1.17. Restriction of an embedded frame to faces of a simplex.

DEFINITION 1.1.57 (Partial frame restrictions to simplicial faces of simplices). The **frame restriction** of an n-embedded k-partially framed m-simplex $(S \twoheadrightarrow [k], \mathcal{F})$ to a j-face $f : T \hookrightarrow S$ is the n-embedded partial frame $(T \twoheadrightarrow [l], \mathcal{F}|_f)$ of T where $T \twoheadrightarrow [l]$ equals the restricted degeneracy $(S \twoheadrightarrow [k])|_f$ and the n-embedded frame $\mathcal{F}|_f$ is obtained by restricting $\mathcal{F}$ to the image face $f : [l] \hookrightarrow [k]$. ⌟

REMARK 1.1.58 (Restricting partial frames via framed realization). As for embedded frame restrictions in Remark 1.1.48, the notion of embedded partial frame restriction is characterized geometrically as follows. Consider an n-embedded partial frame $(S \twoheadrightarrow [k], \mathcal{F})$ and a simplicial face $f : T \hookrightarrow S$; pick any framed realization $r : |S| \to \mathbb{R}^n$. The restriction of the frame to the face is the unique n-embedded partial frame of the simplex T which is framed realized by the composite linear map $r \circ |f| : |T| \hookrightarrow |S| \to \mathbb{R}^n$. ⌟

1.1.2.2. Framed maps. Given two embeddings $W \hookrightarrow \mathbb{R}^n$ and $V \hookrightarrow \mathbb{R}^n$, thought of as embedded frame structures on the vector spaces W and V, there are of course no interesting non-injective maps $V \to W$ that commute with the embeddings, but we may nevertheless consider a map $V \to W$ to respect the frame structure when for every vector $v \in V$, either the vector is sent to zero by the map $V \to W$ or the map $V \to W$ commutes with the embeddings at that vector.

We now introduce the analogous combinatorial notion of framed map, wherein simplicial vectors are either degenerated or have their embedded frame preserved.

DEFINITION 1.1.59 (Framed map of framed simplices). Given n-embedded framed simplices $(S \cong [l], \mathcal{F})$ and $(T \cong [m], \mathcal{G})$, a **framed map** $F : (S \cong [l], \mathcal{F}) \to (T \cong [m], \mathcal{G})$ is a simplicial map $F : [l] \to [m]$ such that for every vector $v : [1] \to [l]$ in the simplex $[l]$, either its frame label is preserved, i.e., $\mathcal{F}|_v = \mathcal{G}|_{F \circ v}$, or the vector is degenerated, i.e., $F \circ v : [1] \to [m]$ is constant. ⌟

There are two natural subclasses of framed maps worth distinguishing, as follows, namely those for which the simplicial map is injective or surjective.

TERMINOLOGY 1.1.60 (Framed faces). A framed map $F : (S \cong [l], \mathcal{F}) \hookrightarrow (T \cong [m], \mathcal{G})$ such that $F : [l] \hookrightarrow [m]$ is a simplicial face is called a **framed face**. Note that this implies $\mathcal{G}|_F = \mathcal{F}$. ⌟

Terminology 1.1.61 (Framed degeneracies). A framed map $F : (S \cong [l], \mathcal{F}) \twoheadrightarrow (T \cong [m], \mathcal{G})$ such that $F : [l] \twoheadrightarrow [m]$ is a simplicial degeneracy is a **framed degeneracy**.

Notation 1.1.62 (Category of embedded framed simplices). Denote by FrSimp_n the category of n-embedded framed simplices and their framed maps. (Note that the objects of this category are simplices of dimension necessarily at most n.)

Observation 1.1.63 (Epi–mono factorization of framed maps). As a map of simplices factors as a degeneracy map (epimorphism) followed by a face map (monomorphism), similarly any framed map of framed simplices factors as a framed degeneracy map followed by a framed face map.

Example 1.1.64 (Framed and non-framed maps). In Figure 1.18 we illustrate three framed maps between 2-embedded framed simplices; the left one is a simplicial face map (with highlighted image), the middle one is a simplicial degeneracy (with highlighted affine kernel), the right one is a general simplicial map (with indicated image and affine kernel). These are framed as all the frame labels on nondegenerated vectors are preserved.

In Figure 1.19, by contrast, we illustrate three non-framed maps. The first is again a simplicial face, the second a simplicial degeneracy, and now the third is an unordered simplicial isomorphism. These are not framed maps, as some frame label of a nondegenerated vector is not preserved. Note that in the second case, the map is not framed even though all spine vectors are either degenerated or have their frame label preserved, because the non-spine vector frame label is not preserved; framed maps cannot be naively detected by their frame behavior on spines.

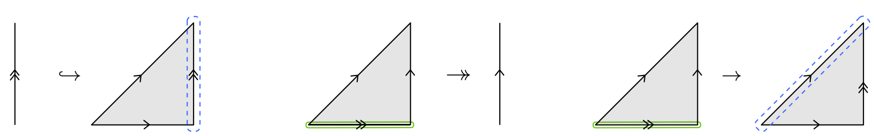


Figure 1.18. Framed maps of framed simplices.

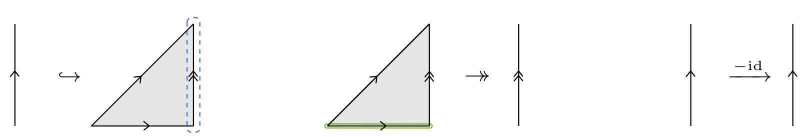


Figure 1.19. Non-framed maps of framed simplices.

Framed maps of n-embedded simplices can also be understood in geometric terms.

Remark 1.1.65 (Framed maps in linear algebraic terms). Consider n-embedded framed simplices $(S \cong [l], \mathcal{F})$ and $(T \cong [m], \mathcal{G})$ and a simplicial map $F : S \to T$. For framed realizations $r : |S| \hookrightarrow \mathbb{R}^n$ and $q : |T| \hookrightarrow \mathbb{R}^n$, the map F is framed if for any vector $\vec{v}$ mapped into ϵ_i^+ by r, its image $w = F(v)$ is mapped into $\epsilon_i^+ \cup \{0\}$ by q. ⌟

Example 1.1.66 (Framed maps via framed realization). For each of the three framed maps from Figure 1.18, we illustrate, in Figure 1.20, a framed realization of the source and target, together with an indication of the associated geometric map of subspaces of euclidean space, showing that the type of each of the frame vectors is preserved (or degenerated). ⌟

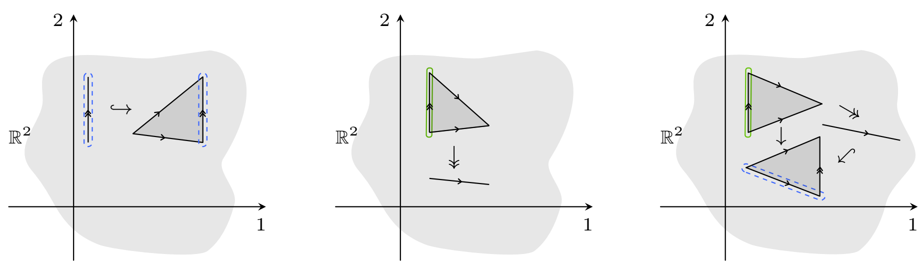


Figure 1.20. Framed maps via framed realization.

Note that the geometric description of framed maps of embedded framed simplices immediately generalizes to the case of embedded partially framed simplices. In combinatorial terms, this may be phrased as follows.

Definition 1.1.67 (Framed maps of embedded partially framed simplices). Given n-embedded partially framed simplices $(S \twoheadrightarrow [j], \mathcal{F})$ and $(T \twoheadrightarrow [k], \mathcal{G})$, a **framed map** $F : (S \twoheadrightarrow [j], \mathcal{F}) \to (T \twoheadrightarrow [k], \mathcal{G})$ is a simplicial map $F : S \to T$ that descends to a framed map of n-embedded framed simplices $F_n : ([j], \mathcal{F}) \to ([k], \mathcal{G})$, that is, $F_n : [j] \to [k]$ commutes with $F : S \to T$ and the degeneracies $S \twoheadrightarrow [j]$ and $T \twoheadrightarrow [k]$. ⌟

Notation 1.1.68 (Category of embedded partially framed simplices). Denote by $\mathsf{PartFrSimp}_n$ the category of n-embedded partially framed simplices and their framed maps. ⌟

Framed maps either preserve the frame label of a vector or degenerate that vector to zero; there is a more general notion of 'subframed map' in which vectors may degenerate not just to the zero vector but to any vector with more specialized frame label. We first describe this geometric viewpoint, and then give a purely combinatorial definition of subframed maps.

Remark 1.1.69 (Subframed maps in linear algebraic terms). Consider n-embedded framed simplices $(S \cong [l], \mathcal{F})$ and $(T \cong [m], \mathcal{G})$ and an unordered simplicial map $F : S \to T$. For framed realizations $r : |S| \hookrightarrow \mathbb{R}^n$ and $q : |T| \hookrightarrow \mathbb{R}^n$, we say F is 'subframed' if for any vector v in S mapped into ϵ_i^+ by r, its image $w = F(v)$ is mapped into the closure $\overline{\epsilon}_i^+$ by q. ⌟

Note that subframed maps may in particular send vectors from a positive component ϵ_i^+ into a negative component $\epsilon_j^- \subset \overline{\epsilon}_i^+$ (where $j > i$).

Definition 1.1.70 (Subframed map). Given n-embedded framed simplices $(S \cong [l], \mathcal{F})$ and $(T \cong [m], \mathcal{G})$, a **subframed map** $F : (S \cong [l], \mathcal{F}) \to (T \cong [m], \mathcal{G})$ is an unordered simplicial map $F : S \to T$ such that for every ordered vector $v : [1] \hookrightarrow S \cong [l]$, *either* the frame label of v is preserved in the sense that $F \circ v : [1] \to T \cong [m]$ is an ordered vector with $\mathcal{F}|_v = \mathcal{G}|_{F \circ v}$, *or* the frame label of v is specialized in the sense that $F \circ v : [1] \to T \cong [m]$ is a possibly unordered vector with $\mathcal{F}|_v < \mathcal{G}|_{F \circ v}$, *or* the vector v is degenerated in the sense that $F \circ v : [1] \to T \cong [m]$ is constant. ⌟

The definition of subframed maps extends, as with that of framed maps, to the embedded partially framed case, by insisting that a vector without a frame label is either mapped to zero or again to a vector without a frame label.

Example 1.1.71 (Subframed maps). The three maps in Figure 1.19 are, though not framed, subframed maps. In the first, the vector with frame label 1 is mapped to the vector with the (more specialized) frame label 2. Similarly in the second, the non-spine vector with frame label 1 is mapped to the target vector with frame label 2. In the third, the frame-label-1 vector again specializes to a frame-label-2 vector, but now with reversed orientation. In Figure 1.21 we illustrate the framed realizations of these three maps, conveying that a vector with a positive coordinate in the 1-axial direction may specialize to a 2-axial vector, with either a positive or negative coordinate, since the closure of the halfplane ϵ_1^+ contains both the halflines ϵ_2^+ and ϵ_2^-. ⌟

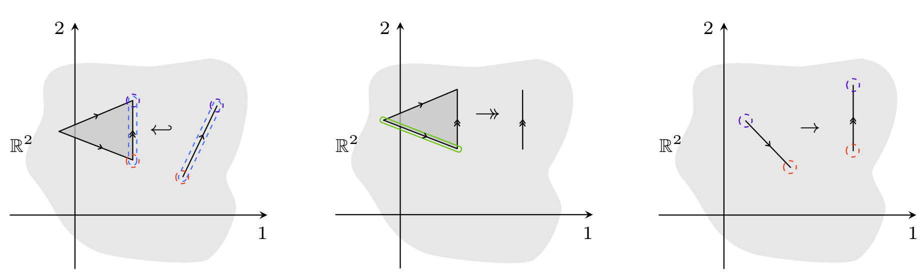


Figure 1.21. Subframed maps via their framed realizations.

1.2. Framed simplicial complexes

Our goal in this section will be to introduce a combinatorial notion of framings on simplicial complexes. This is a globalization of the notion of frames on simplices introduced in the previous section: just as manifolds are spaces that are locally modeled on euclidean space, simplicial complexes are modeled on simplices; and just as framed manifolds are locally modeled on framed euclidean space (i.e., endowed with a continuous choice of frames in each tangent space), framed simplicial complexes will be modeled on framed simplices. The notion of framed simplicial complexes has two important features which distinguish the combinatorial approach to framings from the classical geometric approach to framings of manifolds.

Firstly, framings of simplicial complexes are not local linear structures but *piecewise affine* structures in the following sense. As previously discussed, simplices are extended affine spaces rather than infinitesimal geometric objects; frames of simplices are, correspondingly, affine frames, i.e., frames not based at any specific point of the simplex but defined up to translation. We will define framings on simplicial complexes by piecing together affine frames of each of their simplices. This stands in contrast to the classical tangential notion of framings which defines framings locally, i.e., by picking a frame in the tangent space of each point.

Secondly, simplicial complexes are naturally *singular* spaces and generally not manifolds. As a consequence, framed simplicial complexes will in fact provide a combinatorial model of framed singular spaces and not just of classical framed manifolds. Classically, singular spaces are gluings of manifold strata, and singular spaces themselves need not be manifolds. The question of framing singular spaces is subtle since the usual machinery of tangent spaces relies on local euclidean trivializations; these need not exist everywhere in singular spaces. We will not attempt to geometrically define framed singular spaces in this chapter, but instead focus on leveraging the tools of affine combinatorics: namely, in combinatorial terms, individual open simplices will play the role of manifold strata, and the question of how framings can transition between strata of different dimension will find an answer using the notion of 'n-embedded' frames developed in the preceding section.

Note that framed singular spaces arise naturally in familiar situations. Consider for example the map $\mathbb{D}^2 \to \mathbb{R}^2$ that folds a 2-disk onto itself as shown in the bottom row of Figure 1.22. The standard 2-frame of $\mathbb{R}^2$ may be pulled back along this map, but of course this only yields a 2-frame of $\mathbb{D}^2$ at points where the differential is non-singular. If we regard $\mathbb{D}^2$ as a stratified space as indicated on the bottom left, then this yields an example of a framed singular space with 2-dimensional strata carrying 2-framings, 1-dimensional strata carrying (2-embedded) 1-framings, and the central 0-dimensional stratum carrying a (2-embedded) 0-frame. Combinatorially, such framed singular spaces can be naturally modeled as framed simplicial complexes, a notion that will globalize the definition of frames on individual simplices from the previous section. The framed simplicial complex corresponding to the framed singular space of the bottom row is illustrated in the top row of the figure. This analogy between framed topological spaces and framed combinatorial complexes can be extended to the case of partial frames as well. Indeed, the figure at the beginning of the Introduction shows how the singular 1-framing induced by a Morse function $\mathbb{D}^2 \to \mathbb{R}$ can be represented combinatorially using a partial 1-framed simplicial complex.

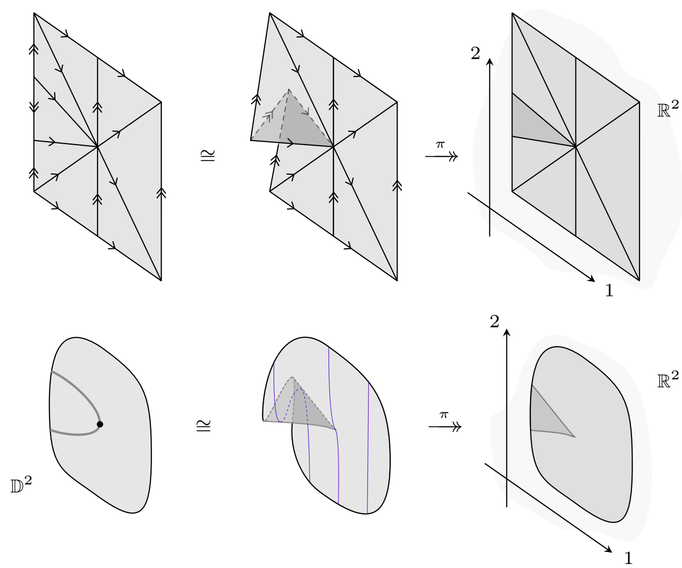


FIGURE 1.22. A framed simplicial complex representing the cusp singularity.

OUTLINE. We introduce a combinatorial notion of framed simplicial complexes in Section 1.2.1, which will be a straightforward generalization of our earlier notions of framed simplices. In Section 1.2.2 we will then introduce the important class of collapsible framings which are contractible by a frame-order respecting sequence of simplicial collapses. Such collapsible framings provide a natural 'trivial' local model for global framings and will be a key ingredient in the later definition of framed regular cells. Moreover, local collapsibility yields a condition for combinatorial framings to be at least locally trivializable, recovering a combinatorial notion of 'progressive' framings that is more analogous to the classical singularity-free tangential framings of manifolds mentioned above.

1.2.1. Framings on simplicial complexes.

SYNOPSIS. We first establish the specific terminology for unordered and ordered simplicial complexes used throughout this work. We then define framings on these complexes simply as simplex-wise framings for each simplex in the complex that consistently restrict to framings on subsimplices.

1.2.1.1. Unordered and ordered simplicial complexes. Just as orderings $S \cong [m]$ of unordered simplices played a crucial role in our discussion of framed simplices, they will be relevant for our definition of framed simplicial complexes. While the transition from unordered to ordered complexes is intuitively straightforward, a precise treatment requires a degree of technical bookkeeping. To keep this manageable, we adopt the language of simplicial sets [**Fri08, Rie11**], which provides the necessary combinatorial machinery to handle these orderings consistently.

TERMINOLOGY 1.2.1 (Simplicial sets). A 'simplicial set X' is a presheaf $X : \Delta^{\mathrm{op}} \to \mathsf{Set}$ on the simplex category Δ, mapping simplices $[m]$ to sets $X[m]$. The 'category of simplicial sets' is the category of presheaves on Δ, and is usually denoted by SSet. ⌟

We will tacitly Yoneda embed $\Delta \to \mathsf{SSet}$ and, abusing notation, use the simplex $[k] \in \Delta$ to also denote the representable simplicial set that it defines under this embedding (this is sometimes denoted by $\Delta[k]$ in the literature). By the Yoneda lemma, the set of maps $x : [k] \to X$ is naturally identified with the set $X[k]$ of k-simplices in a simplicial set X.

In piecewise linear topology one commonly uses the weaker notion of 'simplicial complexes' in place of simplicial sets, which is better suited to the construction of piecewise linear realizations. Commonly, simplicial complexes and their maps are introduced as follows.

TERMINOLOGY 1.2.2 (Simplicial complexes and their maps). A 'simplicial complex' consists of a set of vertices $K(0)$ together with a list of subsets $K(1), K(2), \ldots$ of the vertex powerset $\wp K(0)$ with the property that 'i-simplices' $x \in K(i)$ are sets of cardinality $(i+1)$ and any subset of an i-simplex is itself a j-simplex, $j \leq i$, of K.

Maps of simplicial complexes $K \to L$, called 'simplicial maps', are maps of vertex sets $f : K(0) \to L(0)$ whose image on each i-simplex $x \in K(i)$ yields a j-simplex $f(x) \in L(j)$ (for some j). The category of simplicial complexes and their simplicial maps will be denoted by $\mathsf{SimpCplx}$. ⌟

Like simplicial sets, simplicial complexes may also be characterized as (certain) presheaves. Recall the category $\underline{\Delta}$ of unordered simplices (see Notation 1.1.3).

REMARK 1.2.3 (Simplicial complexes as presheaves). Every simplicial complex K gives rise to a presheaf $K : \underline{\Delta}^{\mathrm{op}} \to \mathsf{Set}$ by defining $K(S)$ to be the set of functions $x : S \to K(0)$ whose image $\mathrm{im}(x)$ lies in some $K(j)$, and defining $K(f : S' \to S)$ to act by precomposition with f. This construction gives rise to a full and faithful embedding of $\mathsf{SimpCplx}$ into the category $\mathrm{PSh}(\underline{\Delta})$ of presheaves on $\underline{\Delta}$. ⌟

Again, we will tacitly Yoneda embed simplices in $\underline{\Delta}$ into presheaves $\mathrm{PSh}(\underline{\Delta})$.

REMARK 1.2.4 (Simplices are simplicial). Note that the Yoneda embedding $\underline{\Delta} \to \mathrm{PSh}(\underline{\Delta})$ lands in the subcategory $\mathsf{SimpCplx} \hookrightarrow \mathrm{PSh}(\underline{\Delta})$, making simplices in particular simplicial complexes. ⌟

Equipped with these notions, we are ready to describe what it means to 'order', and, conversely, 'unorder' simplicial complexes. Recall the 'unordering' functor $(-)^{\mathsf{un}} : \Delta \to \underline{\Delta}$ which forgets orders (see Terminology 1.1.4). This first extends to simplicial sets as follows.

Terminology 1.2.5 (Unordering functor). The 'unordering functor' $(-)^{\mathsf{un}} : \mathsf{SSet} \to \mathrm{PSh}(\underline{\Delta})$ forgets the order of vertices in each simplex: formally, this can be defined as the left adjoint to precomposing presheaves with $(-)^{\mathsf{un}} : \Delta \to \underline{\Delta}$. ⌟

Rourke and Sanderson introduce 'ordered simplicial complexes' as simplicial complexes with an order on their vertices [**RS71**, §1]; we will use the same term for the following more general notion, which requires each simplex in a simplicial complex to be (consistently) ordered, but this *local* order need not induce a *global* order on the set of vertices.

Terminology 1.2.6 (Ordered simplicial complexes). A simplicial set X is called an 'ordered simplicial complex' if its unordering X^{un} is an ordinary simplicial complex. ⌟

Terminology 1.2.7 (The category of ordered simplicial complexes). The 'category of ordered simplicial complexes' $\mathsf{SimpCplx}^{\mathrm{ord}}$ is the full subcategory of SSet consisting of ordered simplicial complexes and their simplicial maps. ⌟

The phrase 'a presheaf map of ordered simplicial complexes' is potentially confusing. Did this just mean … . Also in the next environment the phrase 'their simplicial maps' is used.

The unordering functor restricts to a functor $(-)^{\mathsf{un}} : \mathsf{SimpCplx}^{\mathrm{ord}} \to \mathsf{SimpCplx}$.

Terminology 1.2.8 (Ordering complexes and their maps). An 'ordering' of a simplicial complex K (resp. of a simplicial map $F : K \to L$) is a choice of a preimage K^{ord} of K (resp. a preimage $F^{\mathsf{ord}} : K^{\mathsf{ord}} \to L^{\mathsf{ord}}$ of F) under the unordering functor. For fixed orderings K^{ord} and L^{ord} of simplicial complexes K and L, a simplicial map $F : K \to L$ is said to be 'order-preserving' if there is a (necessarily unique) preimage $F^{\mathsf{ord}} : K^{\mathsf{ord}} \to L^{\mathsf{ord}}$. ⌟

As a trivial example, an ordering of the unordered standard simplex $[m]^{\mathsf{un}}$ is, of course, the ordered standard simplex $[m]$, and so is any other total order of the set $\{0, 1, ..., m\}$.

Remark 1.2.9 (Nondegenerate simplices in simplicial sets and complexes). A simplex $x : [k] \to X$ in a simplicial set X is 'nondegenerate' if there is no non-identity degeneracy map $d : [k] \to [j]$ through which x factors; otherwise, x is 'degenerate'. We write $X(k)$ for the subset of $X[k]$ containing only the nondegenerate k-simplices of X. ⌟

Note that a k-simplex x in an ordered simplicial complex K is nondegenerate if and only if the presheaf map $x : [k] \to K$ is a (componentwise) injection, in which case we write $x : [k] \hookrightarrow K$.

Remark 1.2.10 (Characterizing ordered simplicial complexes). A simplicial set X is an ordered simplicial complex if $x : [k] \to X$ is (componentwise) injective for all $x \in X(k)$, *and*, no two $x, y \in X(k)$ have the same sets of 0-simplices in $X[0]$. ⌟

1.2.1.2. The definition of framed simplicial complexes. We now introduce framings on ordinary simplicial complexes. Recall the definition of embedded frames $\mathcal{F}$ of m-simplices S from Definition 1.1.36 which endows S with an isomorphism $\alpha : S \cong [m]$ that orders S, together with an n-embedded frame $\mathcal{F}$ of $[m]$ (given by an injection $\mathcal{F} : \mathsf{spine}[m] \hookrightarrow \underline{n}$). Recall also, for a j-face $f : T \hookrightarrow S$, such an n-embedded frame of S restricts to an n-embedded frame of T (defined by the restricted isomorphism $\alpha|_f : T \cong [j]$ and the restricted n-embedded frame $\mathcal{F}|_f$, see Notation 1.1.54 and Definition 1.1.55).

Definition 1.2.11 (Framings of simplicial complexes). An **n-framing** $(\alpha, \mathcal{F})$ of a simplicial complex K endows each m-simplex $x : S \hookrightarrow K$ with an n-embedded frame $(\alpha_x : S \cong [m], \mathcal{F}_x)$ such that, for any j-face $f : T \hookrightarrow S$, the restriction of the chosen frame of x to the face f coincides with the chosen frame of $x \circ f$; that is, $\alpha_{x\circ f} = \alpha_x|_f$ and $\mathcal{F}_{x\circ f} = \mathcal{F}_x|_f$. ⌟

Note that a simplicial complex K cannot contain simplices of dimension greater than n in order for it to admit an n-framing.

Before we give examples of framed simplicial complexes, we first discuss an important simplification of the definition, which encodes orderings as part of the simplicial complex. This hinges on the following observation.

Observation 1.2.12 (Isomorphism data of framings is an ordering). An n-framing $(\alpha, \mathcal{F})$ of a simplicial complex K gives rise to an ordering of K; indeed, for each m-simplex $x : S \hookrightarrow K$ in K, an order on the vertices of S is determined by the isomorphism $\alpha_x : S \cong [m]$ and together (since choices of α_x are compatible with faces) these orderings of vertices of simplices determine an ordering of K itself. This has an inverse: any ordering of K restricts to an ordering of each simplex $x : S \hookrightarrow K$, and thus yields (compatible) isomorphisms $S \cong [m]$. ⌟

As a consequence of the observation, we obtain the following equivalent way of phrasing the notion of n-framings.

Alternative Definition 1.2.13 (Framings of ordered complexes). An n-framing $\mathcal{F}$ of a simplicial complex K is an ordering K^{ord} of K together with an n-embedded frame $\mathcal{F}_x$ of $[m]$ for each nondegenerate m-simplex $x : [m] \hookrightarrow K^{\mathsf{ord}}$, such that, for any face $f : [k] \hookrightarrow [m]$, we have $\mathcal{F}_{x\circ f} = \mathcal{F}_x|_f$. ⌟

We will henceforth adopt this more concise reformulation of n-framings in terms of orderings, recorded by the preceding definition. We refer to the pair $(K, \mathcal{F})$, of a simplicial complex and an n-framing on it, as an 'n-framed simplicial complex', and keep the ordering of K implicit in our notation.

Convention 1.2.14 (Keeping orderings implicit). When working with a framed simplicial complex $(K, \mathcal{F})$, we often tacitly consider K as an ordered simplicial complex (for instance, when working with maps $[m] \to K$): we will always assume this ordering of K to be the ordering of K provided by the n-framing $\mathcal{F}$ via the Alternative Definition 1.2.13. ⌟

Terminology 1.2.15 (Frame labels and frame vectors). A 'frame k-vector' $v : [1] \hookrightarrow K$ in a framed simplicial complex $(K, \mathcal{F})$ is a simplicial vector in K whose 'frame label' is $\mathcal{F}|_v(0 \to 1) = k$. ⌟

Note that frame vectors fully determine the frame structure of a framed simplicial complex (since they comprise, in particular, all spine vectors of all simplices).

Remark 1.2.16 (Frame vector notation). In later examples, instead of defining frames $\mathcal{F}_x$ separately for each simplex x in K, we usually only give frame labels of the simplicial vectors in the simplicial complex. But note well that we cannot define a framing on a complex by arbitrarily labeling the simplicial vectors of the complex with frame labels in $\underline{n}$; we must also check that the labeling in fact defines a valid frame on each simplex in the complex. ⌟

Example 1.2.17 (Framings and non-framings on simplicial complexes). In Figure 1.23, we depict two 2-framed simplicial complexes. The framing is indicated by the notation wherein edges with frame label 1, that is frame 1-vectors, have a single arrow and edges with frame label 2, that is frame 2-vectors, have a double arrow. The left complex is a 2-sphere, and the right complex is a 2-torus. (Notice that a manifold need not have a tangential framing in the classical sense in order to have a combinatorial framing in our sense; the point being that our more general notion of framing allows various singularities of the framing. The notion of progressive framing described later is more closely analogous to the classical notion of (nonsingular) tangential framing.)

In Figure 1.24, we depict first an unordered simplicial complex that admits no 2-framing whatsoever. (As it happens this is the minimal triangulation of the real projective plane.) Second, we have an ordered simplicial complex that admits no 2-framing, though the underlying unordered complex certainly has other orderings for which there are 2-framings.

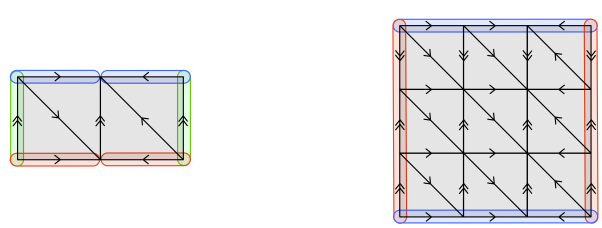


Figure 1.23. Simplicial complexes with a 2-framing.

⌟

As in the case of framed simplices, framed realizations turn out to be a convenient tool for visualizing framings of complexes. Generalizing

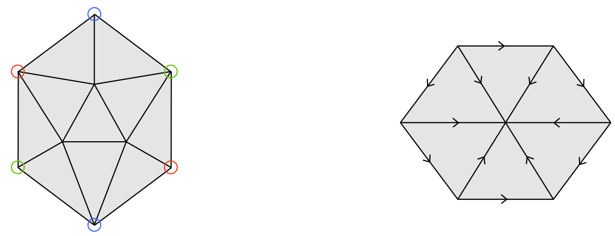


FIGURE 1.24. Simplicial complexes with no 2-framing.

the definition of framed realizations of n-embedded framed simplices (see Definition 1.1.40) to the case of framed simplicial complexes is straightforward, as follows.

DEFINITION 1.2.18 (Framed realization of a framed simplicial complex). Given an n-framed simplicial complex $(K, \mathcal{F})$, a linear map $r : |K| \to \mathbb{R}^n$ (that is, a map that is linear on each simplex) is called a **framed realization** of K if for each simplex $x : [m] \hookrightarrow K$ the restriction $r \circ |x|$ is a framed realization of $([m], \mathcal{F}_x)$. ⌟

EXAMPLE 1.2.19 (Framed realizations of framed simplicial complexes). In Figure 1.25 we illustrate various 2-framings of the 'square' simplicial complex, along with corresponding framed realizations. ⌟

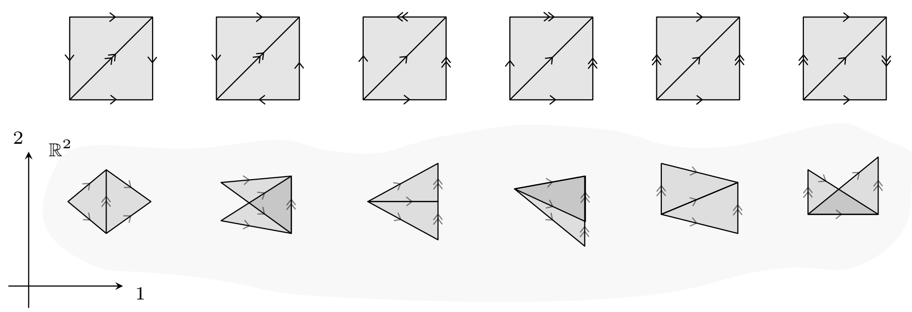


FIGURE 1.25. Simplicial complexes with 2-framings and their framed realizations.

We remark that not all framed simplicial complexes admit framed realizations in $\mathbb{R}^n$ (for example, *circularly* framing the boundary of the 2-simplex creates a non-realizable 1-framed simplicial complex).

We next define maps of framed simplicial complexes. This directly generalizes the notion of framed maps of framed simplices from Definition 1.1.59.

DEFINITION 1.2.20 (Framed maps of framed simplicial complexes). Consider n-framed simplicial complexes $(K, \mathcal{F})$ and $(L, \mathcal{G})$. A **framed simplicial**

map $F : (K, \mathcal{F}) \to (L, \mathcal{G})$ (or simply, a 'framed map') is a simplicial map $F : K \to L$ that restricts on all simplices $x : [k] \hookrightarrow K$ to a framed map $F : ([k], \mathcal{F}_x) \to ([l], \mathcal{G}_y)$ where $y = \mathrm{im}(F \circ x) : [l] \hookrightarrow L$ is the image of $F \circ x$. ⌟

Note that any framed map $F : (K, \mathcal{F}) \to (L, \mathcal{G})$ is order-preserving.

Notation 1.2.21 (The category of framed simplicial complexes). The category of n-framed simplicial complexes and framed maps will be denoted by $\mathsf{FrSimpCplx}_n$. ⌟

Definition 1.2.22 (Unframing framed simplicial complexes). The **unframing** functor $\mathsf{Unframe} : \mathsf{FrSimpCplx}_n \to \mathsf{SimpCplx}$ takes a framed simplicial complex $(K, \mathcal{F})$ to the simplicial complex K, and a framed map $F : (K, \mathcal{F}) \to (L, \mathcal{G})$ to the simplicial map $F : K \to L$. ⌟

Definition 1.2.23 (Restricted framings). Given an n-framed simplicial complex $(K, \mathcal{F})$ and a simplicial subcomplex $L \hookrightarrow K$, the **restricted framing** $\mathcal{F}|_L$ **of** $\mathcal{F}$ **to** L is the n-framing of L obtained by restricting the ordering of K to L, and setting $(\mathcal{F}|_L)_x = \mathcal{F}_x$ for simplices $x : [k] \hookrightarrow L \hookrightarrow K$. ⌟

Note that a subcomplex $L \hookrightarrow K$ of a framed simplicial complex as in the preceding definition induces a framed simplicial map $(L, \mathcal{F}|_L) \hookrightarrow (K, \mathcal{F})$.

Finally, the following generalization of *sub*framed maps of framed simplices which, later on, will naturally resurface in the context of framed cellular maps when passing to the barycentric subdivision of cells.

Remark 1.2.24 (Subframed maps of framed simplicial complexes). The notion of subframed maps of simplices described in Definition 1.1.70 generalizes to framed simplicial complexes: a 'subframed map of framed simplicial complexes' $F : (K, \mathcal{F}) \to (L, \mathcal{G})$ is a simplicial map $F : K \to L$ that restricts on all simplices $x : [k] \hookrightarrow K$ to a subframed map $F : ([k]^{\mathsf{un}} \cong [k], \mathcal{F}_x) \to ([l]^{\mathsf{un}} \cong [l], \mathcal{G}_y)$ where $y = \mathrm{im}(F \circ x) : [l] \hookrightarrow L$ is the image of $F \circ x$. Note that, unlike framed maps, subframed maps need not be order-preserving. ⌟

Our discussion of framing structures on simplicial complexes would not be complete without mentioning the following the following generalization to *partial* framings.

Definition 1.2.25 (Partial framings of simplicial complexes). A **partial n-framing** $(\alpha, \mathcal{F})$ of a simplicial complex K endows each m-simplex $x : S \hookrightarrow K$ with an n-embedded partial frame $(\alpha_x : S \twoheadrightarrow [k], \mathcal{F}_x)$ such that, for any j-face $f : T \hookrightarrow S$, the restriction of the chosen n-embedded partial frame of x to the face f coincides with the chosen n-embedded partial frame of $x \circ f$; that is, $\alpha_{x \circ f} = \alpha_x|_f$ and $\mathcal{F}_{x \circ f} = \mathcal{F}_x|_f$. ⌟

Note that a 'partial n-framed simplicial complex' $(K, \mathcal{F})$ may have m-simplices of any dimension m. Note also that Observation 1.2.12 no longer holds: a partial framing need not determine an ordering on a simplicial complex.

Going forward, we will be mostly interested in the case of (non-partial) embedded framings; nonetheless, all subsequent definitions have analogs in the partial case as well.

1.2.2. Collapsible framings. We now develop a combinatorial notion that guarantees triviality of framings on framed simplicial complexes. Our definition will be analogous to the classical notion of collapsibility of simplicial complexes [**Coh12**, §2] [**Whi39**], but will impose additional conditions to ensure collapses respect the framing structures: intuitively, these conditions recover the geometric idea that collapses should happen along frame flow lines (i.e., lines that 'integrate' the framing's k-vector fields, cf. [**For02**, §3]).

Synopsis. Recall, classically, a collapsible simplicial complex admits a sequence of elementary collapses of individual simplices, forming altogether a contraction of the complex. We will similarly define a framed simplicial complex to be framed collapsible if it admits a contracting sequence of elementary collapses, but which are required to consecutively collapse all the simplices containing a vector of highest frame label, before proceeding to collapse the simplices with next highest frame label, and so forth. Applying the condition of framed collapsibility locally in general simplicial complexes, we define a notion of progressive framings on simplicial complexes, whose role is analogous to classical tangential frameability of manifolds.

1.2.2.1. The definition of framed collapse. Classically, the notion of collapse provides a transformation of a simplicial complex into a homotopically equivalent complex by removing simplices. The notion is useful since, unlike for simplicial sets, there is no good notion of 'taking quotients' for simplicial complexes that guarantees that the resulting complex will, in fact, be another simplicial complex. For framed collapse we will impose the stronger condition that the collapse of frame k-vectors (inductively for each k) is equivalently a quotient map that quotients along the frame vector 'flow lines'.

It would still be nice to get in some idea about the combinatorial 'flow lines' of the framing, which are necessitated by framed collapsible? But I'm not sure what to say. The 'rigid geometry' viewpoint ... intuition about this ... . What's written now is a bit too technical for the intro paragraph, maybe.

The inductive nature of the collapse process hinges on the observation that framed simplices have a *unique* highest frame vector.

Terminology 1.2.26 (Highest frame vectors). Given an n-embedded framed m-simplex $([m], \mathcal{F})$, its 'highest frame vector' is the unique frame k-vector $h^{\mathcal{F}} : [1] \to [m]$ such that k is maximal among all frame vectors of the framed m-simplex. Given a framed simplicial complex $(K, \mathcal{F})$, its 'highest frame number' k is the maximal k such that $(K, \mathcal{F})$ contains a frame k-vector. ⌟

Example 1.2.27 (Highest frame vectors of framed simplices). In Figure 1.26 we depict a 4-framed 3-simplex and a 4-framed 4-simplex, highlighting in each case the highest frame vector. ⌟

Construction 1.2.28 (Elementary collapse of frame k-vectors). Given an n-framed simplicial complex $(K, \mathcal{F})$ with highest frame number k and a

FIGURE 1.26. Highest frame vectors of 4-framed simplices.

frame k-vector $v : [1] \hookrightarrow K$, the **elementary k-collapse $q_v : K \to K'$ of v** is the quotient of simplicial sets that, on each simplex $x : [m] \hookrightarrow K$ containing $v = x \circ v_x$, is induced by the degeneracy $d_{v_x(0)} : [m] \to [m-1]$.[3] ⌟

Note that the preceding construction works with simplicial sets. We are interested in the case where the quotient itself is a simplicial complex.

OBSERVATION 1.2.29 (Framed elementary collapse). Given an elementary k-collapse $q_v : K \to K'$ of $(K, \mathcal{F})$ such that K' is a simplicial complex, there is a unique framing $(K', \mathcal{F}')$ such that q_v becomes a *framed* simplicial map. We call q_v the 'framed elementary k-collapse' of v in this case. ⌟

EXAMPLE 1.2.30 (Elementary collapse of frame vectors). In Figure 1.27 we illustrate four 2-framed simplicial complexes with highest frame number 2, highlighting a frame 2-vector in each of them. In each case, we indicate the elementary k-collapse q_v. This yields a framed elementary k-collapse in all but the last case, in which the codomain of q_v fails to be a simplicial complex. ⌟

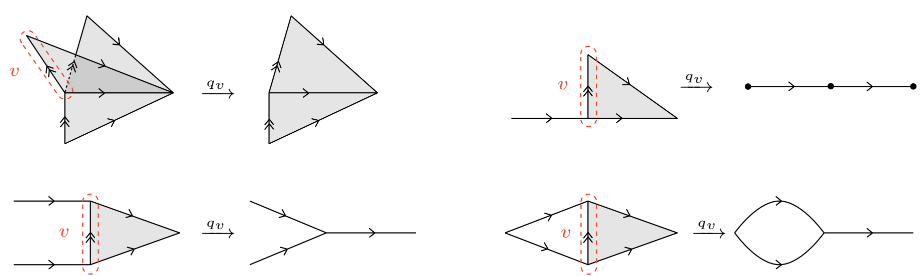


FIGURE 1.27. Elementary collapse of frame 2-vectors.

DEFINITION 1.2.31 (Framed collapsible complex). A framed simplicial complex $(K, \mathcal{F})$ with highest frame number k is **framed collapsible** if, either, $k = 0$ and K is the point, or, $k > 0$ and the following conditions hold:

[3]More formally, we can construct the elementary collapse map of a vector $v \in K(1)$ as follows. Recall, $K = \mathrm{colim}(D)$ where D is a diagram of all simplices in K: technically, this is the Yoneda embedding of the forgetful functor $\mathrm{el}(K) \to \Delta$ from the category of elements $\mathrm{el}(K)$ of K. There is a unique natural transformation $\alpha : D \to D'$ to some D' defined such that on $x : [m] \to K$, α_x is the degeneracy $d_{v_x(0)} : [m] \to [m-1]$ if x contains $v = x \circ v_x$ and $\mathrm{id} : [m] \to [m]$ otherwise. The collapse map q is the induced map of colimits $\mathrm{colim}(\alpha) : \mathrm{colim}(D) \to \mathrm{colim}(D')$.

(1) *Inductive collapsibility*. K admits a sequence of framed elementary k-collapses to a framed collapsible $(K', \mathcal{F}')$ with highest frame number $< k$. (Denote by $q_k : (K, \mathcal{F}) \to (K', \mathcal{F}')$ the canonical induced 'framed k-collapse map'.)
(2) *Flow section existence*. For every framed m-simplex $x : ([m], \mathcal{G}) \hookrightarrow (K', \mathcal{F}')$ in the collapsed complex K', and every vertex z in K whose collapse $q_k(z)$ lies in x, there exists some 'flow section lift' $l : ([m], \mathcal{G}) \hookrightarrow (K', \mathcal{F}')$ containing z and such that $q_k \circ l = x$.
(3) *Flow continuation uniqueness*. For any simplex $x : [m] \hookrightarrow K$ and any frame k-vector v whose source (resp. target) lies in x, there exists at most[4] one 'flow continuation' $(m+1)$-simplex $c : [m+1] \hookrightarrow K$ containing both l and v. ⌟

We say a framing of a simplicial complex is 'collapsible' when that framed simplicial complex is framed collapsible, and we refer to a framed collapsible framed simplicial complex simply as a 'collapsible framed simplicial complex'.

Example 1.2.32 (Collapsible framings). In Figure 1.28 we depict collapsible 1-framed simplicial complexes: these are all of 'linear' form.

Figure 1.28. Collapsible 1-framed simplicial complexes.

In Figure 1.29 we depict collapsible 2-framed simplicial complexes with highest frame number 2, together with their respective 2-collapses onto collapsible 2-framed simplicial complexes with highest frame number 1.

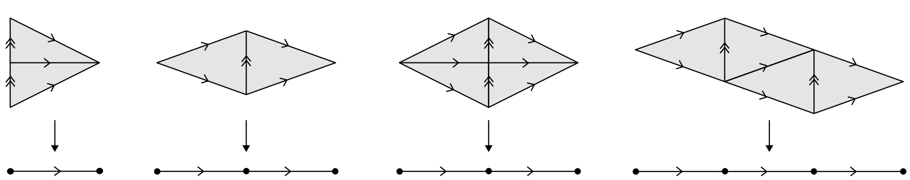


Figure 1.29. Collapsible 2-framed simplicial complexes with their 2-collapse maps.

⌟

Example 1.2.33 (Non-collapsible framings). In contrast, in Figure 1.30, we depict framed complexes that are not collapsible.

Similarly, none of the 2-framed complexes in Figure 1.27 are collapsible: the first complex fails the flow continuation uniqueness condition; the second fails the flow section existence condition; the third is not inductively collapsible; the fourth does not admit a simplicial k-collapse.

[4]The condition turns out to be equivalent to there being 'exactly one' such continuation simplex.

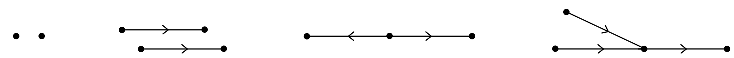


FIGURE 1.30. Non-collapsible 1-framed simplicial complexes.

Two more non-collapsible 2-complexes are shown in Figure 1.31. The first fails the 'flow continuation uniqueness condition; the circled edge has two different 2-simplices that would collapse downward onto it. The second has no elementary collapse that is simplicial. In fact, the unordered simplicial complex underlying these 2-framed complexes admits no collapsible 2-framing.

FIGURE 1.31. 2-Framed complexes which are not collapsible.

NOTATION 1.2.34 (Category of collapsible framings). We denote the full subcategory of $\mathsf{FrSimpCplx}_n$ consisting of collapsible framings by $\mathsf{CollFrSimpCplx}_n$.

REMARK 1.2.35 (Contractibility of collapsible framings). Note that framed collapsibility of framed simplicial complexes implies, in particular, their contractibility in the classical sense.

Isn't the contractibility an immediate consequence of the fact that framed collapsible implies unframed collapsible implies contractible. Ie doesn't depend on the classification. I think the content of this remark, if it exists, would need to be something about /framed/ contractibility. But what is that? cf my unease with the intro paragraph in 1.2.2. Ie I feel like a notion of framed contractibility could both form the content of a remark, and serve as the motivation / intuitive idea in that intro. [both changed]

1.2.2.2. Progressive framings. We may further impose collapsibility *locally*, and will refer to the resulting notion as 'progressivity'. The existence of a progressive framing on a triangulation of a manifold makes contact with classical notions of tangential frameability (in the sense of parallelizability, cf. [**MS74**, §2]).

NOTATION 1.2.36 (Stars). Recall, given a simplicial complex K, the 'star of a vertex' x in K, denoted $\mathsf{star}(x)$, is the minimal subcomplex of K containing all simplices that have x as a vertex.

DEFINITION 1.2.37 (Framed progressive complex). We say an n-framing $(K, \mathcal{F})$ of the simplicial complex K is **framed progressive** if for each vertex $x \in K(0)$ the restricted n-framing $(\mathsf{star}(x), \mathcal{F}|_{\mathsf{star}(x)})$ is collapsible.

REMARK 1.2.38 (Collapsibility implies progressivity). Every collapsible framing is, in particular, progressive.

EXAMPLE 1.2.39 (Progressive and non-progressive framed manifolds). In Figure 1.32, we depict two progressive framings on simplicial complexes, both

Nowhere except in the intro text is it stated that collapsible complexes have framed realizations that are framed contractible, or anything similar (ie that there's some nice contractibility property). Add remark? [YES] of which happen to be manifolds. The left image is a progressive framing on the annulus, and similarly the right image is a progressive framing on the torus. Note that the earlier framing of the torus in Figure 1.23 was not progressive, and had no progressive sub-annulus.

In Figure 1.33, we depict two non-progressive framings, one on the circle and one on the Möbius band. In fact, the Möbius band admits no triangulation that has a progressive 2-framing. Another example of a non-progressive framing was the one on the 2-sphere in Figure 1.23. Indeed that complex contains as a star subcomplex the left complex in Figure 1.31, which we observed was not collapsible. Again, in fact no triangulation of the 2-sphere will have a progressive 2-framing.

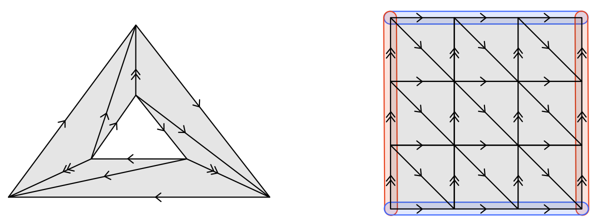


Figure 1.32. Progressive framed simplicial manifolds.

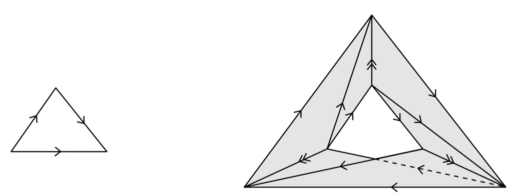


Figure 1.33. Non-progressive framed simplicial manifolds.

⌟

1.3. Framed regular cell complexes

In the final section of this chapter, we will introduce n-framings on regular cell complexes. Our goal will be the definition of a category of 'n-framed regular cell complexes' $\mathsf{FrCellCplx}_n$, fitting into the following diagram of categories (in which vertical arrows are fully faithful embeddings of categories, while horizontal arrows forget framing structures):

$$\begin{array}{ccc} \mathsf{FrSimpCplx}_n & \xrightarrow{\mathsf{Unframe}} & \mathsf{SimpCplx} \\ \big\downarrow & & \big\downarrow \\ \mathsf{FrCellCplx}_n & \xrightarrow{\mathsf{Unframe}} & \mathsf{CellCplx} \end{array} .$$

Regular cell complexes, as we recall in Section 1.3.1, are complexes whose cellular attaching maps are injective; informally, these complexes are gluings of cells of general 'polytopic shape'. Despite their generality, regular cell complexes, unlike their non-regular counterpart, are combinatorializable. The fundamental property of regular cell complexes that enables this combinatorialization is the homotopical triviality of their closed cells. This entails that one can describe a class of so-called cellular posets, which are exactly the face posets of regular cell complexes; geometric realizations of cellular posets recover the (cellular) homeomorphism type of their corresponding regular cell complexes. The resulting translation between regular cell complexes and cellular posets provides the claimed combinatorialization of regular cell complexes.

Crucially, however, this combinatorialization is computably *intractable* in that, given a poset, there can be no general algorithm to determine if that poset is cellular [**VKF74, CL06**]. In particular, it is impossible to algorithmically write down a list classifying 'all the shapes' of regular cells up to some general bound in, say, the number of boundary cells. By *framing* regular cells this intractability will find a natural resolution.

The definition of framed regular cell complexes will directly rely on our previous work on framings of simplicial complexes. Namely, an n-framed regular cell complex will be an n-framing of the simplicial complex that underlies the cellular poset of the complex, together with the additional condition that the framing is collapsible on each cell. This approach combines two ingredients: firstly, use the correspondence of regular cells and cellular posets to endow regular cells with canonical simplicial structure; secondly, require simplicial framings on cells to be collapsible. The second condition assures that framings of cells are trivial (in the sense that the framed cell is framed realizable as a subspace of $\mathbb{R}^n$) which, intuitively, reflects that framed regular cells, just like framed simplices, will play the role of small, trivializable framed pieces from which larger framed spaces will be built. An example of a framed regular cell complex, with its corresponding framed simplicial complex is given in Figure 1.34.

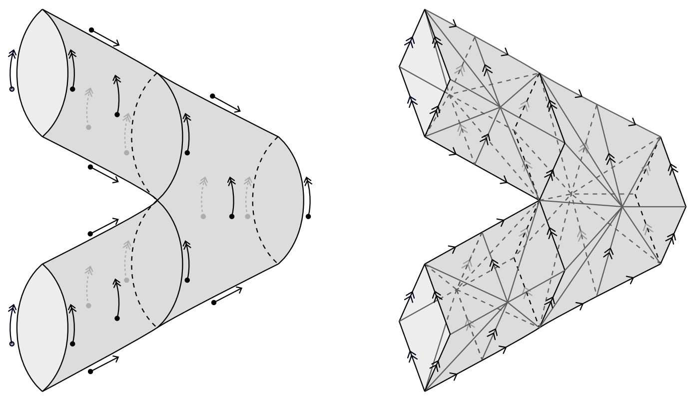


FIGURE 1.34. A framed regular cell complex and its corresponding framed simplicial complex.

In contrast to the case of nonframed regular cells, framed regular cells can now be computably recognized and classified. The classification will be constructed in Chapter 3. At the same time, the generality of framed regular cells, as opposed to mere framed simplices, will be at the very heart of fundamental results in framed combinatorial topology—for instance, working with framed cells will enable the construction of canonical (namely, *coarsest*) cellulations of tame stratifications, as we will explain in Chapter 5. This highlights that the passage from framed simplices to framed regular cells is not incidental, but of central importance in framed combinatorial topology.

OUTLINE. In Section 1.3.1, we recall various concepts from the classical theory of regular cell complexes, interpreting the latter notion both in combinatorial terms and as (cell-wise) stratified spaces. In Section 1.3.2, we then introduce the notion of framings on regular cell complexes, illustrate examples of framed regular cells, define framed maps of their complexes, and provide the aforementioned comparison functor from framed simplicial complexes.

I don't think there's enough in place to use this as an example of a canonical cellulation (as nice as that would be). It would have to be embedded, for instance. As is it sounds like the canonical cellulation applies to the manifold itself, which is confusing / not the case.

Maybe the thing to do is refer to the figure inside or at the end of the previous paragraph ('The definition of ...') — just as a picture of a framed regular cell complex as a framing of its simplicial complex.

Also later we don't generally use the words 'canonical cellulation' which makes it difficult for the reader to figure out what is meant. Maybe independently of the example, that sentence can be reconsidered or specified.

Need to rewrite the outline, and possibly flesh it out, after the necessary refactor of 1.3.2. [rewritten + added synopsis

1.3.1. Regular cell complexes.

SYNOPSIS. We recall the classical definition of regular cell complexes from the perspective of stratified spaces and discuss several related constructions, such as their fundamental posets. We introduce the category $\mathsf{CellCplx}$ of *combinatorial* regular cell complexes, which provides a combinatorial counterpart to the category of *geometric* regular cell complexes and their cellular maps, and generalizes the category of simplicial complexes $\mathsf{SimpCplx}$; in particular, we construct a fully faithful embedding $\mathsf{SimpCplx} \hookrightarrow \mathsf{CellCplx}$.

1.3.1.1. Regular cell complexes as cellular posets. Regular cell complexes generalize simplicial complexes, but are better behaved than CW complexes in that attaching maps are required to be 'regular', extending open interiors of cells to closed topological balls. Several illustrative examples of regular cells in dimension 2 and 3 are shown in Figure 1.35.

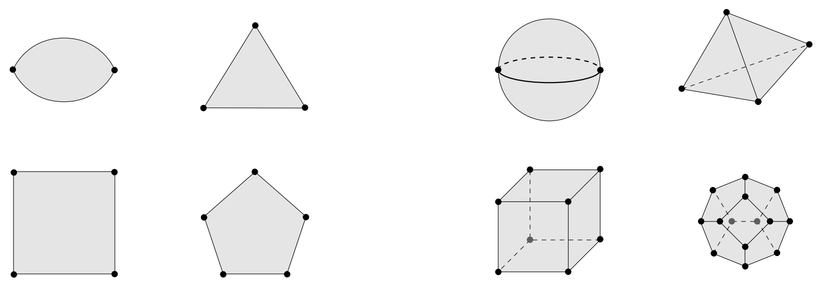


FIGURE 1.35. An illustration of regular 2-cells and 3-cells.

Regular cell complexes are commonly defined in terms of their homeomorphic attaching maps [**LW69**]. For us, it will be convenient to formulate the definition in terms of *stratified spaces*, and we begin with a brief recollection the latter notion as well as key concepts adjacent to it.

Recall, a stratified space is a space equipped with a decomposition into disjoint subspaces called *strata* (see Definition B.1.9).

TERMINOLOGY 1.3.1 (Fundamental posets of stratifications). Each stratified space has an associated 'fundamental poset' whose elements are its strata s and whose arrows are generated by the boundary relation between strata, recording an arrow $s \to t$ when s's boundary intersects t. (The definition of stratified spaces requires that no cycles appear in that relation.) ⌟

Conversely, we can turn posets into stratifications by a process of *stratified realizations* as we now describe.

TERMINOLOGY 1.3.2 (Upper and strict upper closures). Given a poset P and an element $x \in P$, then the 'strict upper closure' $P^{>x}$ of x in P is the full subposet with objects $y \in P$ with $y > x$. Similarly, the 'upper closure' $P^{\geq x}$ is the full subposet of objects $y \in P$ with $y \geq x$. ⌟

TERMINOLOGY 1.3.3 (Nerve and realizations of posets). Recall, the nerve NP of a poset P is the ordered simplicial complex whose k-simplices are the k-chains in P. The 'geometric realization' $|P|$ of a poset P is obtained by applying the standard geometric realization to NP. ⌟

CONSTRUCTION 1.3.4 (Stratified realizations of posets). Given a poset P, the **stratified realization** $\|P\|$ of P is the stratification of $|P|$ whose strata are the subspaces $\mathsf{str}(x) := \left|P^{\geq x}\right| \setminus |P^{>x}|$ for $x \in P$. ⌟

Recall, a *stratified map* of stratified spaces is a map of their underlying spaces such that strata in the domain are mapped into strata of the codomain.

Construction 1.3.5 (Stratified map realizations of poset maps). Given a poset map $F : P \to Q$, the **stratified map realization** $\|F\| : \|P\| \to \|Q\|$ is the stratified map that maps 0-simplices $p \in |P|$ to 0-simplices $F(p) \in |Q|$, and linearly extends this mapping to all other simplices in $|P|$. ⌟

Note, as a map of underlying spaces, the stratified map realization $\|F\|$ coincides with the usual geometric realization $|F|$. An alternative, explicit construction of stratified realizations of posets and their maps, based on a construction of geometric realizations in terms of convex combinations, can be found in Construction B.1.51 (for maps, see Construction B.2.14).

With a notion of stratifications at hand, we now formally introduce regular cell complexes.

Definition 1.3.6 (Regular cell complexes). A **regular cell complex** is a stratification whose strata are open disks (called the 'open cells' of the complex) while closures of strata are closed disks (also called the 'closed cells' of the complex). ⌟

Notation 1.3.7 (Fundamental posets of regular cell complexes). The **fundamental poset** $\mathbb{F}X$ of a regular cell complex X is its fundamental poset as a stratification: explicitly, that poset has objects that are the cells x of X, with arrows $x \to y$ whenever the closure $\overline{x}$ contains y. ⌟

The fundamental poset is the *opposite* category of the classical 'face poset' of a regular cell complex, cf. [**Bjö84, Bjö95**]. (See Remark B.1.5 for why we work with opposite posets here.)

Fundamental posets of regular cell complexes are graded by dimension, that is, they admit a functor $\dim : \mathbb{F}X \to \mathbb{N}^{\mathrm{op}}$ with discrete preimages, mapping each cell to its dimension. Cells of a regular cell complex X which are minimal elements in $\mathbb{F}X$ will be called 'facets'—these are exactly cells which are not contained in any other cell's boundary.

Convention 1.3.8 (Local finiteness). We assume all our regular cell complexes to be *locally finite*, that is, any cell is contained in the closure of finitely many other cells.[5] ⌟

Terminology 1.3.9 (Maps of regular cell complexes). A **map of regular cell complexes** $F : X \to Y$ is a stratified map, mapping cell strata into cell strata. ⌟

Remark 1.3.10 (Functoriality of fundamental posets). Note that the fundamental poset construction is functorial: for any map of regular cell complexes $F : X \to Y$ we obtain a poset map $\mathbb{F}(F) : \mathbb{F}(X) \to \mathbb{F}(Y)$, mapping a cell x of X to the cell in Y that contains the image $F(x)$. ⌟

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[5]As discussed in Chapter B local finiteness arises as a natural condition in the theory of stratifications: it implies continuity of a regular cell complex's 'characteristic map' $X \to \mathbb{F}X$ and ensures that regular cell complexes belong to the class of 'conical' stratification.

We now describe the class of posets that can be obtained as fundamental posets of regular cell complexes.

Definition 1.3.11 (Cellular posets). A poset $(X, \leq)$ is called **cellular** if the realization $|X^{>x}|$ of the strict upper closure of any $x \in X$ is homeomorphic to a sphere. ⌟

Example 1.3.12 (Cellular and non-cellular posets). In Figure 1.36 the three posets are cellular, while in Figure 1.37 the three posets fail to be cellular. (For simplicity, we only draw the generating arrows of the posets.) Note that even if the upper closures $P^{\geq x}$ realize to topological balls it need not be the case that the strict upper closures $P^{>x}$ realize to spheres. ⌟

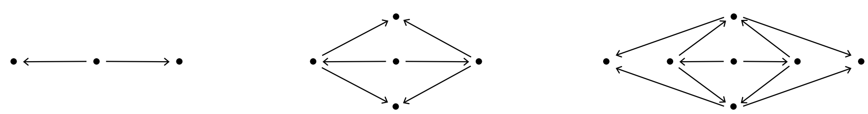


Figure 1.36. Cellular posets.

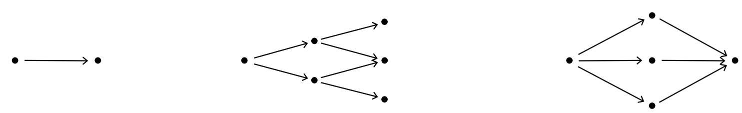


Figure 1.37. Non-cellular posets.

Centrally, stratified realizations of cellular posets are exactly regular cell complexes as recorded in the following result.

Proposition 1.3.13 (Regular cell complexes are stratified realizations of cellular posets). *Regular cell complexes are exactly stratified realizations of cellular posets, in the following sense.*

(1) The stratified realization of a cellular poset is a regular cell complex.
(2) The fundamental poset of a regular cell complex is a cellular poset.
(3) Every regular cell complex X is stratified homeomorphic to the stratified realization of its fundamental poset, that is, $X \cong \|\sqcap X\|$.[6]
(4) Every cellular poset X is canonically isomorphic to the fundamental poset of its stratified realization, that is, $X \cong \sqcap \|X\|$.

Proof. Statement (1) and (3) are discussed in [**Bjö84**, §3] (see also [**LW69**]). Statement (2) and (4) follow from these and the definitions. □

The correspondence of regular cell complexes (up to stratified homeomorphism) and cellular posets provides the *combinatorialization* of regular cell complexes by cellular posets. In light of this correspondence we introduce the following terminology.

---

[6]The isomorphism is canonical up to stratified homotopy.

TERMINOLOGY 1.3.14 (Combinatorial complexes, depth, cells, dimension, closures, boundaries). Going forward, a cellular poset X may alternatively be referred to as a 'combinatorial regular cell complex'.

Recall, the 'depth' of an object x in a poset measures the maximal length m of chains $x_0 \to x_1 \to ... \to x_m$ in the poset starting at that object, $x = x_0$. An element $x \in X$ is called a 'm-cell' if it is of depth m in X. We refer to $X^{\geq x}$ as the 'closure' of x, and to $X^{>x}$ as its 'boundary'.

Moreover, if X has an initial object which is an m-cell, then we call X itself a 'combinatorial regular m-cell' (or simply, a 'regular m-cell' if no confusion arises). We often denote such an initial element by $\perp_X$. ⌟

The term 'combinatorial regular cell complex' is used merely as a synonym for cellular posets (the choice of this terminology will be further motivated below). To contrast this with ordinary regular cell complexes, we may refer to the latter as '*geometric* regular cell complexes'.

1.3.1.2. Cellular maps of regular cell complexes. We next extend the combinatorialization of regular cell complexes to a class of maps between complexes which are 'cellular' in the following sense.

TERMINOLOGY 1.3.15 (Closure preservation for stratifications). A stratified map is said to be 'closure preserving' if it maps closures of strata *onto* closures of strata. ⌟

DEFINITION 1.3.16 (Cellular maps of regular cell complexes). A **cellular map of regular cell complexes** $F : X \to Y$ is a map of (geometric) regular cell complexes that is closure preserving.[7] ⌟

NOTATION 1.3.17 (The category of geometric regular cell complexes). Denote by $\mathsf{CellCplx}^{\mathcal{S}}$ the category whose objects are (geometric) regular cell complexes and whose morphisms are cellular maps. ⌟

Similarly, we define cellular maps for cellular posets.

TERMINOLOGY 1.3.18 (Closure preservation for posets). A map of posets $F : P \to Q$ is 'upper-closure preserving' if for each $x \in P$, the image $FP^{\geq x}$ equals $Q^{\geq Fx}$. ⌟

DEFINITION 1.3.19 (Cellular maps of cellular posets). A **cellular map of cellular posets** is a poset map that is upper-closure preserving. ⌟

Concern: check if this terminology is sufficiently standard. A priori cellular would normally mean just taking cells into cells.

[7]Cellular maps in the sense of Definition 1.3.16 have also been called 'regular cellular maps' in the literature and were introduced for general CW complexes, see [**LW69**, Def. 4.1]. We omit the qualifier 'regular' from our terminology as we only care about *regular* cell complexes, for which cellular maps in the sense of Definition 1.3.16 provide a natural definition of well-behaved maps (in particular, generalizing the case of simplicial maps). However, the definition is stricter than that of 'cellular maps of CW complexes' in the classical sense.

Notation 1.3.20 (The category of combinatorial regular cell complexes). Denote by $\mathsf{CellCplx}$ the category whose objects are cellular posets and whose morphisms are cellular maps. ⌟

Example 1.3.21 (Cellular and non-cellular maps). In Figure 1.38 we depict two cellular maps of regular cells, along with the corresponding maps of cellular posets. In each case we indicate the mapping by coloring images and preimages in the same color. In Figure 1.39, we similarly depict two maps of regular cells that are not cellular. ⌟

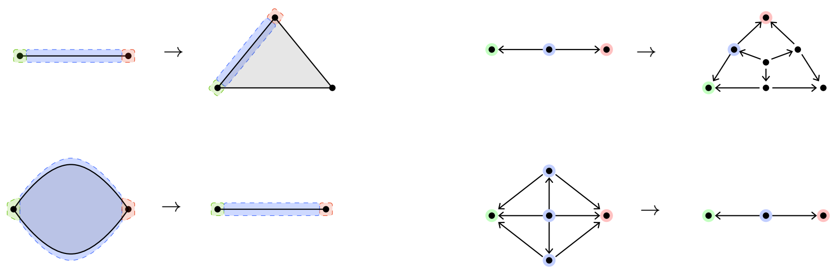


Figure 1.38. Cellular maps of regular cells.

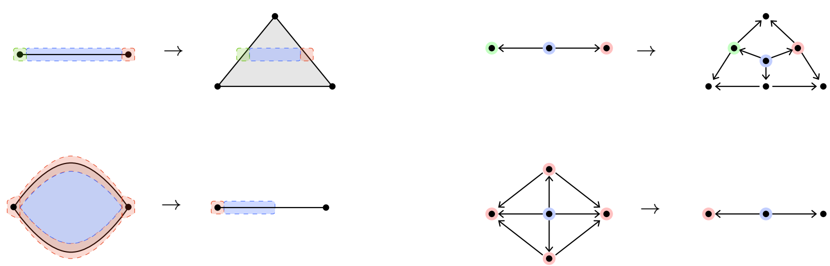


Figure 1.39. Non-cellular maps of regular cells.

The two definitions of cellular maps can be related by the following functors.

Observation 1.3.22 (Fundamental poset and stratified realization functors). The fundamental poset construction (see Remark 1.3.10), as well as the stratified realization construction (see Construction 1.3.4 and Construction 1.3.5), yield functors

$$\mathsf{CellCplx}^{\mathcal{S}} \underset{\|-\|}{\overset{\sqcap}{\rightleftarrows}} \mathsf{CellCplx} \quad .$$

⌟

Cellular posets and their cellular maps provide a combinatorial analog of geometric regular cell complexes as we now explain.

Observation 1.3.23 (The equivalence of topological and combinatorial regular cell complexes). For locally finite stratifications, the fundamental poset construction yields a functor of topologically enriched categories $\mathbb{\Pi} : \mathcal{S}\mathsf{trat}_{\mathrm{lf}} \to \mathcal{P}os_{\mathrm{lf}}$ (see Construction B.2.21); this enrichment provides the right setting to study the homotopical equivalence of our two conceptions of regular cell complexes.

Denote by $\mathcal{C}ellCplx^{\mathcal{S}}$ the topological subcategory of $\mathcal{S}\mathsf{trat}_{\mathrm{lf}}$ given by (geometric) regular cell complexes and their cellular maps.

Similarly, consider the subcategory of $\mathcal{P}os_{\mathrm{lf}}$ given by (combinatorial) regular cell complexes, i.e., cellular posets, and their cellular maps: in fact, due to the cellularity condition on maps, this subcategory has discrete hom spaces, and simply recovers the ordinary category $\mathsf{CellCplx}$.

The fundamental poset functor $\mathbb{\Pi} : \mathcal{C}ellCplx^{\mathcal{S}} \to \mathsf{CellCplx}$ yields a *weak equivalence* of topologically enriched categories; intuitively, the morphisms in the $\mathsf{CellCplx}$ capture the morphisms in $\mathcal{C}ellCplx^{\mathcal{S}}$ up to stratified homotopy and the space of such homotopies is contractible.[8] ⌟

We will not prove the preceding observation nor make direct use of it; rather it motivates that $\mathsf{CellCplx}$ is a *good* combinatorial model for geometric regular cell complexes.

As is, this is a change of notation from CellPos to CellCplx. That should be made clear, and maybe somehow avoided (by not using CellPos before?). [Just CellCplx everywhere, then motivate]

Finally, we make brief contact with the classical notions of triangulations and simplicial complexes.

Terminology 1.3.24 (Cellulations). In analogy to the notion of triangulation, we speak of 'cellulation' when decomposing a given space into the cell stratification of a regular cell complex.[9] ⌟

Observation 1.3.25 (Simplicial complexes are regular). A (geometric) simplicial complex, i.e. the geometric realization of a simplicial complex, is a simple type of (geometric) regular cell complex when stratified by its simplices. In the resulting complex, each m-dimensional cell has exactly $(m+1)$ faces of dimension $(m-1)$. ⌟

Based on this observation, we can construct the following embedding of categories.

Construction 1.3.26 (Simplicial complexes embed in combinatorial regular cell complexes). Given a simplicial complex K, its fundamental poset $\mathbb{\Pi}K$ (as a regular cell complex $|K|$) is, of course, the poset whose objects are simplices x in K with an arrow $x \to y$ whenever the simplex y is a face of the simplex x.

---

[8] A similar observation can be shown to hold in the case of PL cell complexes and PL cellular posets as introduced later in Definition 1.3.30, since Alexander's trick still holds in that case, see [**Lur09b**, §23 Lem. 2].

[9] This terminology finds a slightly more general meaning in the context of cellulable stratifications, see Section B.3.

For a simplicial map of unordered simplicial complexes $F : K \to L$, we obtain a map of fundamental posets $\sqcap\!\!\sqcap F : \sqcap\!\!\sqcap K \to \sqcap\!\!\sqcap L$ mapping a simplex x in K to the simplex Fx in L.

This yields the functor from simplicial complexes to combinatorial regular cell complexes

$$\sqcap\!\!\sqcap : \mathsf{SimpCplx} \to \mathsf{CellCplx}$$

which is a fully faithful embedding of categories. ⌟

Note that, if we were to allow non-cellular poset maps in $\mathsf{CellCplx}$ the claim of 'full faithfulness' would fail to hold in the preceding construction.

Remark 1.3.27 (Underlying simplicial complexes and barycentric subdivision). Each geometric regular cell complex X has an 'underlying simplicial complex' obtained by taking the nerve $N\sqcap\!\!\sqcap X$ (see Terminology 1.3.3) and forgetting its order. Passing to the geometric realization of that simplicial complex, we obtain a canonical 'barycentric subdivision' map $|N\sqcap\!\!\sqcap X| \to X$ (which, when stratifying the domain and codomain, respectively, by simplices and cells, is unique up to stratified homotopy); this is a consequence of Proposition 1.3.13. ⌟

Summarizing the preceding discussion, we introduce the following (abuse of) notation, which will reduce the amount of symbols needed in subsequent sections.

There's an issue that leads to confusion, which is the use of CellCplx to refer to cellular posets sometimes, and CellStrat and CellPos other times, etc. Is there a way to clean this up without messing up other things?

Notation 1.3.28 (Associated simplicial and geometric structures). A combinatorial regular cell complex $X \in \mathsf{CellCplx}$, besides being itself a cellular poset, has several *associated structures* which we organize as follows.

(1) The nerve NX of X yields an ordered simplicial complex which we refer to as the 'ordered simplicial complex representation of X'.
(2) By unordering the ordered simplicial complex representation we obtain the 'underlying simplicial complex of X' (cf. Remark 1.3.27). Abusing notation, this may itself be denoted by X.
(3) The geometric regular cell complex $\|X\|$ realizing X will, abusing notation, usually be referred to simply as the '(geometric) regular cell complex of X', and may be denoted X if no confusion arises.

The abuse of notation similarly applies to *maps* of combinatorial regular cell complexes $F : X \to Y$, which context-dependently may be used to denote maps of corresponding geometric regular cell complexes and of underlying simplicial complexes. ⌟

The next notation environment and illustration was moved from the framed section (though it involves no framing). I think it will go well here, where the combinatorial, and topological, and simplicial views are all in play. But this all can use a bit of scrubbing when I do a final pass on this section.

Example 1.3.29 (Visualizing cellular posets). In Figure 1.40 we illustrate a regular cell complex X, together with its corresponding cellular poset X and its corresponding ordered simplicial complex representation NX of X obtained by taking the nerve of X. ⌟

CLD to consider those last two headers [considered and improved]

To end this section, let us briefly address the discrepancy between 'topology' and 'piecewise linear topology' which is, in fact, also visible at the level

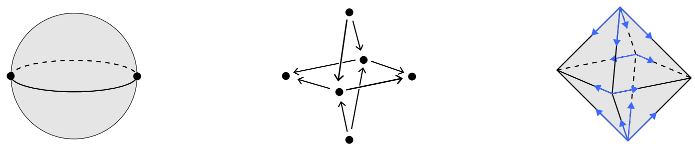


FIGURE 1.40. A geometric regular cell complex and its corresponding cellular poset and ordered simplicial complex.

of combinatorial regular cell complexes. For this, we introduce the following piecewise linear (PL) analog of the definition of cellular posets.

DEFINITION 1.3.30 (PL cellular posets). A poset $(X, \leq)$ is called **PL cellular** if the realization $|X^{>x}|$ of the strict upper closure of any $x \in X$ is PL homeomorphic to the standard PL sphere [**HS69**, §I.5].

REMARK 1.3.31 (Cellular is not always PL cellular). While 'PL cellular' trivially implies 'cellular' the converse is in general not true, i.e., you can have regular cells whose boundary is not a PL sphere. Indeed, there exist triangulations of the sphere which are not PL spheres (see [**Edw80**], [**Bry02**, Thm. 9.1]). Adjoining a new minimal element to the fundamental poset of such a triangulation yields a poset that is cellular but not PL cellular. In contrast, we will later on find that in the framed setting the adjectives 'cellular' and 'PL cellular' can be used interchangeably (see Remark 1.3.62).

The following two remarks could probably be cleaned up a bit. The first has immediate content. The second mostly repeats and forward references. Unless they are distinguished by being about cells respectively cell complexes.

Maybe this remark can be more explicit: you can have a regular cell whose boundary is not a PL sphere. [added]

TERMINOLOGY 1.3.32 (The category of combinatorial regular PL cell complexes). Denote by $\mathsf{CellCplx}^{\mathrm{PL}}$ the category of PL cellular posets and their cellular maps.

By the preceding remark, we have $\mathsf{CellCplx}^{\mathrm{PL}} \subsetneq \mathsf{CellCplx}$. We will find that 'framed' analogs $\mathsf{FrCellCplx}_n^{\mathrm{PL}}$ and $\mathsf{FrCellCplx}_n$ of these categories are in fact the same category, and we therefore need not distinguish them notationally (cf. Remark 1.3.62).

1.3.2. Framings on regular cell complexes. In this section we investigate framings on combinatorial regular cells and cell complexes. This will combine our discussion of framings on simplices and simplicial complexes, with the notion of combinatorial regular cell complexes, and their category $\mathsf{CellCplx}$, as discussed in detail in the previous section.

SYNOPSIS. We define framings on regular cell complexes as framings of their underlying simplicial complexes that are required to be collapsible on each cell, and give several examples of the resulting notion. Foreshadowing later chapters and results, we also briefly discuss the 'reasonableness' of the class of framed regular cells, including their algorithmic constructibility and piecewise linearity. We introduce maps of framed regular cell complexes as cellular maps that (in the absence of spine vectors of simplices) preserve so-called axel vectors of cells. Finally, we demonstrate how framed simplicial complexes as defined in the previous section faithfully include into our newly defined category of framed regular cell complexes.

1.3.2.1. The definition of framed regular cell complexes. We can define the notion of framings on a combinatorial regular cell complex in terms of framings on its *unordered* underlying simplicial complex. Though combinatorial regular cell complexes are gluings of their individual cells, there is no economy in defining framings cell by cell rather than all at once.

Definition 1.3.33 (Framed regular cell complex). An **n-framing $\mathcal{F}$ of a combinatorial regular cell complex** X is an n-framing $\mathcal{F}$ of its underlying simplicial complex such that, for each cell $x \in X$, the framing restricts to a collapsible framing $\mathcal{F}|_{X^{\geq x}}$ on the cell's closure $X^{\geq x}$. ⌟

We will refer to the pair $(X, \mathcal{F})$, of a combinatorial regular cell complex X together with an n-framing $\mathcal{F}$ on it, as an 'n-framed regular cell complex'. Similarly, a 'framed regular cell' is simply a combinatorial regular cell with a framing in the above sense.

Probably the definition should be rewritten so that inside it, perhaps in the notation as well, is the indication of which order is meant.

Remark 1.3.34 (The framing induced ordering of the simplicial complex X). Given an n-framed regular cell complex $(X, \mathcal{F})$, the framing $\mathcal{F}$ itself induces an order (on the underlying simplicial complex X) which will generally differ from the ordered simplicial complex NX. ⌟

This definition uses the notation X^{un}, but in the previous notation, the notation NX^{un} is used, and then it's said that abusing notation that will be just called X. Even abused, there's a mismatch here, needs fixing. Not sure if this propagates elsewhere.

This notation is now a bit out of place or redundant. This all needs smoothing out.

Example 1.3.35 (A 2-framed regular cell complex). In Figure 1.41 we depict the data of a framed regular cell complex $(X, \mathcal{F})$: on the left, we depict a regular complex X represented as an ordered simplicial complex NX (see Figure 1.40), whose order is indicated by arrows in blue with solid heads. On the right of the figure, we depict a framing on the underlying simplicial complex X using multi-headed arrow notation as introduced earlier (see Example 1.1.26). In the center of the figure, these structures are combined into a single picture.

Write text of this eg: have regular cell complex depicted as an ordered simplicial complex on the left, the framing on the right, and both at once in the middle.

A verification that the resulting data $(X, \mathcal{F})$ is indeed that of a framed regular cell complex is illustrated in Figure 1.42: we depict the framed subcomplex induced by the subcell inclusions $X^{\geq x} \hookrightarrow X$ for four different elements $x \in X$ (corresponding to two 2-cells, and two 1-cells, respectively). In each case one recursively verifies that the restricted framings induce framed regular cell structures for these subcells. ⌟

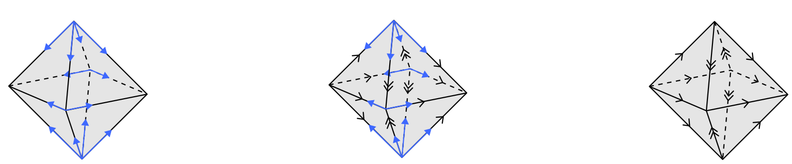


Figure 1.41. A 2-framed regular cell complex.

Note that framed realization of a framed regular cell complex has not been defined or introduced, and needs to be before it is used.

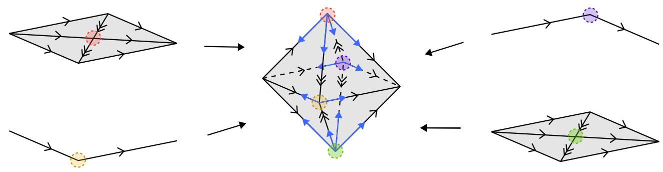


FIGURE 1.42. The collapsible subcomplexes of a 2-framed regular cell complex.

TERMINOLOGY 1.3.36 (Framed realizations of framed regular cell complexes). A 'framed realization' of an n-framed regular cell complex $(X, \mathcal{F})$ is a map $|X| \hookrightarrow \mathbb{R}^n$ which is a framed realization of the underlying framed simplicial complex of $(X, \mathcal{F})$ (in the sense of Definition 1.2.18), and which restricts to an embedding on each cell. ⌟

Framed realizations in $\mathbb{R}^n$ provide a convenient way to illustrate framings on regular cell complexes.

EXAMPLE 1.3.37 (Framed regular cell complexes via framed realization). In Figure 1.43, on the left, we re-illustrate the framed simplicial complex from Example 1.3.35 together with a map to $\mathbb{R}^2$ that provides a framed realization of the corresponding framed simplicial complex, and thus fully determines the framing. On the right of that figure we depict the corresponding map for the associated geometric regular cell complex. (While this map is not linear, one can reconstruct the left map in a 'framing-preserving' way up to contractible choice of homotopy; we forego detailing this reconstruction, but appeal to the readers intuition for now and often use regular cell complex realizations like the shown map as a convenient way of depicting framings on regular cell complexes. We will come to fully understand the relation of *topological*, *piecewise linear*, and *linear* framed realizations in later chapters.)

See description in Eg '2-framed regular cells' a bit later, referring to the cell picture as schematic notation suggestive of the other picture. The description here should be compatible with that.

The terminology here can be the simplicial picture is a 'framed simplicial realization' and the cell picture is a 'framed cell realization'.

That's only true up to contractible choice.

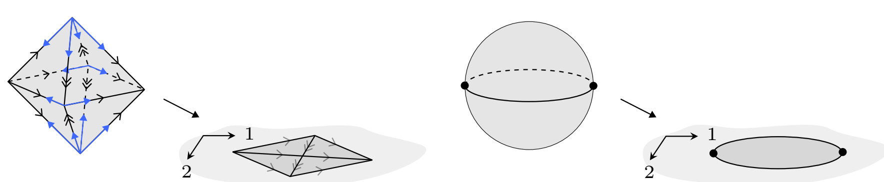


FIGURE 1.43. A 2-framed regular cell complex with a framed realization.

As another example, consider the 'front half' of the cell complex in Figure 1.34. That complex with three cells projects homeomorphically into the plane of the paper, and that projection provides a framed realization; the image of that realization, and the image of the corresponding realization of the associated simplicial complex, are shown in Figure 1.44. ⌟

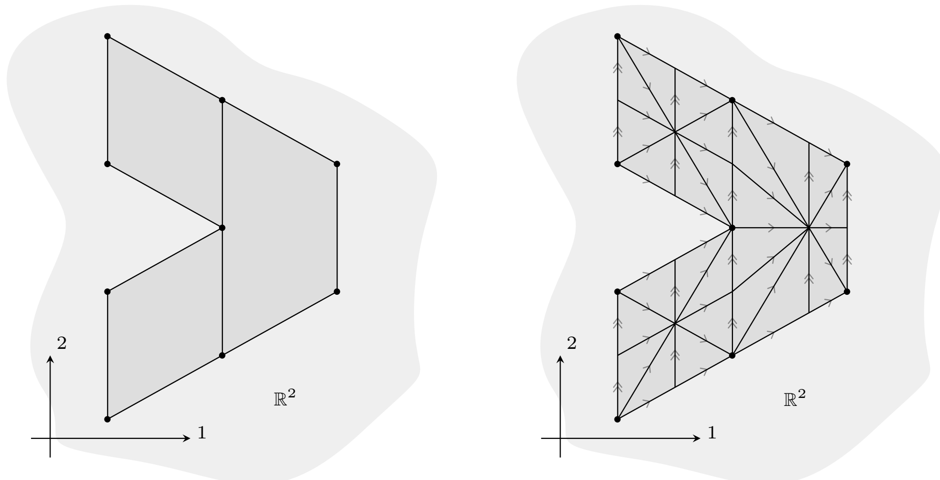


FIGURE 1.44. Framed realization of a 2-framed regular cell complex.

Framings of regular cell complexes are framings of the corresponding simplicial complex in which every cell (of the regular cell complex) corresponds to a framed collapsible subcomplex of the framed simplicial complex; we will be mostly concerned with framed regular cell complexes that are not just locally but in fact globally collapsible, as follows.

This whole discussion remains a bit sketchy because there hasn't been complete clarity on the meaning of the notations and such. But it shouldn't take too much to clean it up.

TERMINOLOGY 1.3.38 (Collapsible and progressive framed regular cell complexes). An n-framed regular cell complex $(X, \mathcal{F})$ is called 'framed collapsible' if the framing $\mathcal{F}$ of the underlying simplicial complex is framed collapsible in the sense of Definition 1.2.31. Similarly, the framed regular cell complex is called 'framed progressive' if the framing of the simplicial complex is framed progressive in the sense of Definition 1.2.37. Usually we abbreviate these terms to simply 'collapsible' and 'progressive'. ⌟

EXAMPLE 1.3.39 (Collapsible and non-collapsible framed regular cell complexes). In Figure 1.45 we illustrate two framings of the same regular cell complex. The left framing is collapsible, while the right framing is not collapsible (and indeed not even progressive). Note this latter framed regular cell complex is a coarsening of the framed simplicial complex illustrated in Figure 1.22.

Notice that in the left figure, we use the previously established framed realization notation for framed regular cell complexes, see Figure 1.44. We also denote (some) pertinent *axel vectors*, that we'll formally introduce shortly. In the right figure, we indicate the framing of the regular cells only by specifying (all) their axel vectors, but without a framed realization. ⌟

1.3.2.2. Examples of framed regular cells. We now focus attention on the nature of individual framed regular cells.

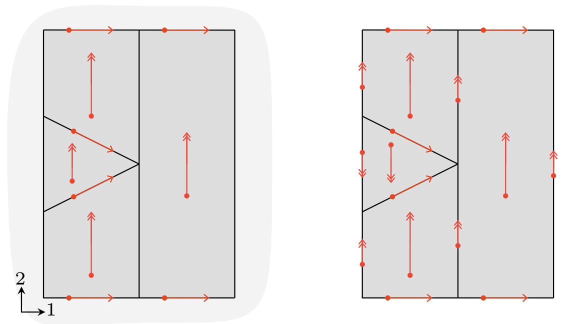


FIGURE 1.45. Collapsible and non-collapsible framed regular cell complexes.

EXAMPLE 1.3.40 (2-Framed regular cells). In Figure 1.46 and Figure 1.47 we illustrate several examples of 2-framed regular cells $(X, \mathcal{F})$. In each case we give three distinct representations. The first picture type ① is of the regular cell, seen as a simplicial complex together with its framing. The second picture type ② is of a framed realization of that framed complex. (In both pictures, we indicate the poset order $\leq$ of X by small blue arrows.) The third picture type ③ is of the regular cell together with an embedding in euclidean space; this third picture type is a schematic notation suggestive of the second picture. Henceforth we will typically depict framed regular cells simply by the last type of picture, that is of the cell embedded in euclidean space. Further examples of 2-framed regular cells can be found in Figure C.1. ⌟

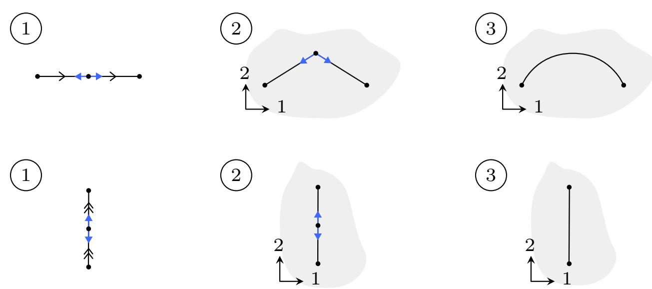


FIGURE 1.46. 2-Framed regular 1-cells.

EXAMPLE 1.3.41 (Failures of framings on regular cells). Reusing the conceptual notation introduced in the previous example, in Figure 1.48 we depict two combinatorial regular 1-cells with framings of their underlying simplicial complexes which fail to be framed regular cells. Indeed, in both cases the provided framings fail to be collapsible. In Figure 1.49 we similarly depict 2-cells with framings of their underlying simplicial complexes that fail to yield framed regular cells. In the first case, the failure arises from both of the 2-cell's 1-dimensional subcells failing to be framed regular 1-cells. A

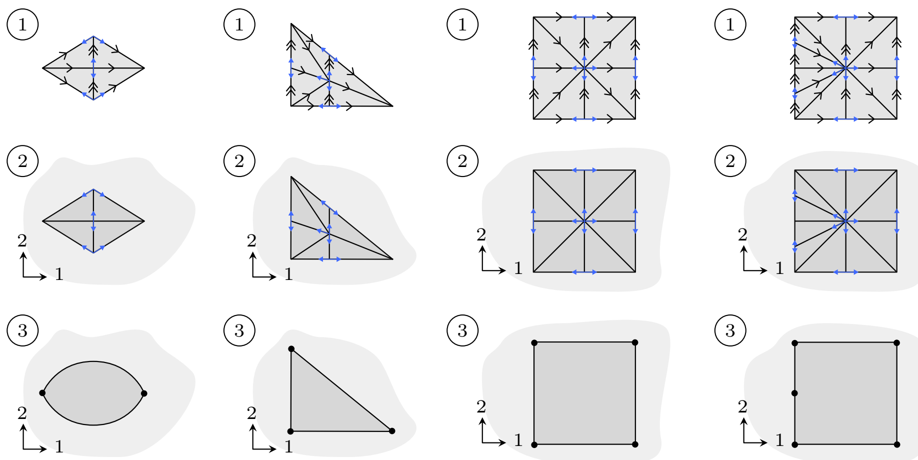


FIGURE 1.47. 2-Framed regular 2-cells.

similar failure arises in the second example. In the third case, the 2-cell itself fails to have collapsible framing. ⌟

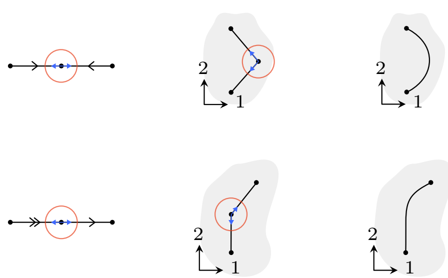


FIGURE 1.48. Framings of simplicial complexes underlying regular 1-cells that are not framed regular cells.

Edit the text of this example

EXAMPLE 1.3.42 (The simplest 3-framed regular 3-cell). In Figure 1.50, we depict the 3-globe as a framed regular cell. As before, the framed regular cell is represented in three ways: first as an ordered simplicial complex with a framing, second as an ordered simplicial complex with a framed realization, and third as a schematic realization of the regular cell itself. ⌟

EXAMPLE 1.3.43 (A failure to 3-frame the simplest regular 3-cell). In Figure 1.51, on the left we depict a framing, on the ordered simplicial complex corresponding to a 3-globe, that does not yield a framed regular 3-cell. Notice that the framed simplicial complex is identical to the framed simplicial complex from Figure 1.50, but the ordering is crucially different. That ordering controls which simplices combine into the cells of the regular cell complex, and thus how the framing interacts with those cells; though represented by tiny blue arrows, the ordering cannot be ignored. The given

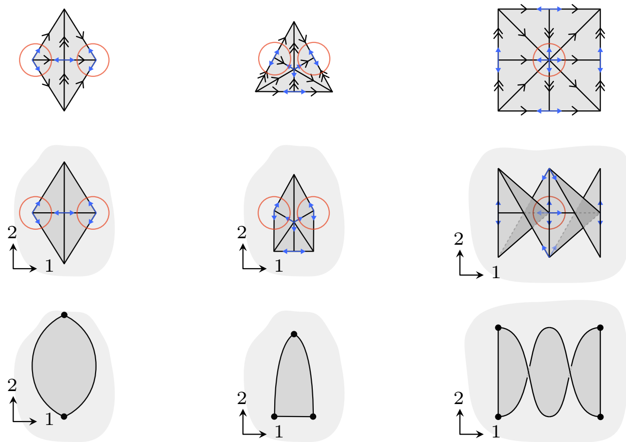


FIGURE 1.49. Framings of simplicial complexes underlying regular 2-cells that are not framed regular cells.

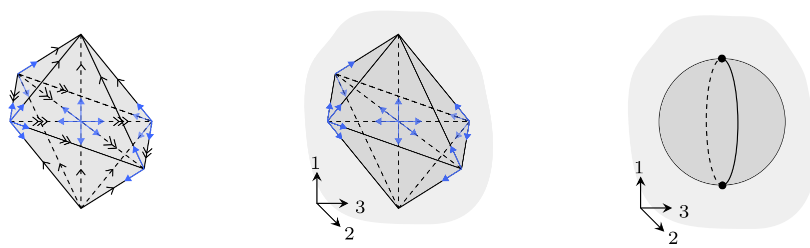


FIGURE 1.50. The simplest framed regular 3-cell.

framing fails to be collapsible on the upper closures of the four circled vertices. On the right is a corresponding schematic realization of the regular cell itself, which is again crucially distinct from the schematic realization in Figure 1.50, and does not represent a framed cell. ⌟

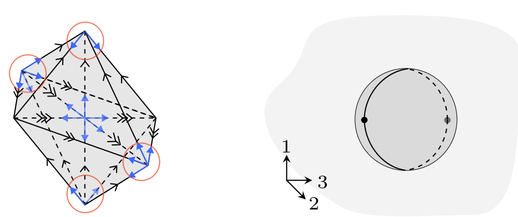


FIGURE 1.51. A framing of the simplicial complex underlying the regular 3-globe that does not yield a framed regular cell.

The previous Example 1.3.42 is the simplest example of a 3-framed regular 3-cell (indeed, its underlying regular cell complex is the simplest regular cell complex of the 3-ball). In general, 3-framed regular 3-cells can be of various shapes as the next example illustrates.

Example 1.3.44 (Framed regular 3-cells). Using framed realizations to encode framings as explained in Figure 1.46 we now illustrate, in Figure 1.52, framings on the remaining three regular cells previously shown in Figure 1.35. ⌟

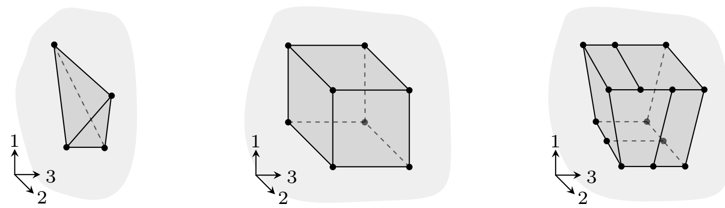


Figure 1.52. Simple framed regular 3-cells.

Example 1.3.45 (Realizations of 3-cells that are not framed regular cells). In Figure 1.53 we depict two regular 3-cells, simplices in fact, embedded in euclidean space, which, though, are not the framed realizations of any framed regular cells. Consider any linear barycentric subdivision of one of these embedded simplices. For the initial embedded simplex to be a framed regular cell, the embedded subdivision would have to be the framed realization of a framed simplicial complex. However, no matter the subdivision chosen, there will be a 2-simplex of the subdivision all of whose edges would have to have frame label 1; that is of course impossible for a framed 2-simplex. ⌟

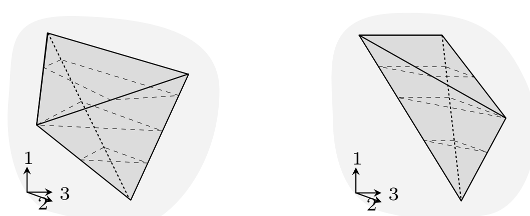


Figure 1.53. Realizations of 3-cells that are not framed regular cells.

Example 1.3.46 (More 3-framed regular 3-cells). We depict a few more 3-framed regular 3-cells in Figure 1.54 and Figure 1.55. An even larger collection of framed regular 3-cells can be found in Figures C.2, C.3, C.4, and C.5. ⌟

The first two framed regular 3-cells from Figure 1.55, together with a reflection of each, can be combined into an intriguing and fundamental framed complex, as follows.

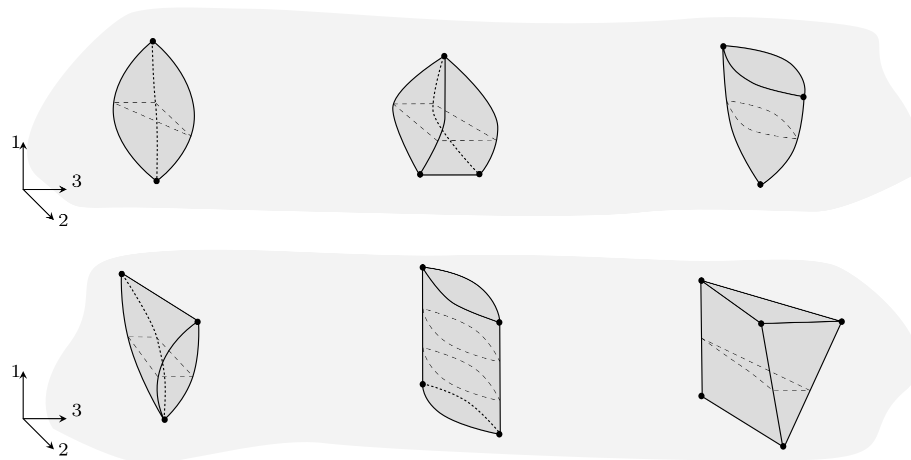


FIGURE 1.54. More framed regular 3-cells.

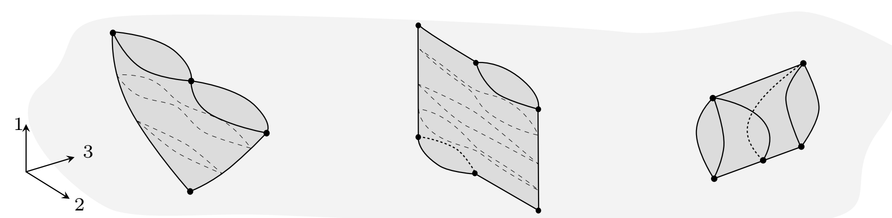


FIGURE 1.55. Yet more framed regular 3-cells.

EXAMPLE 1.3.47 (A more elaborate collapsible framed regular cell complex). In Figure 1.56 we depict a collapsible 3-framed regular cell complex made up of four framed 3-cells. This complex is geometrically dual to the Hopf circle [**GWZ86**, Fig. 1], as illustrated in the figure and as will be explained much later on in Example 5.2.36. ⌟

1.3.2.3. Framed cellular maps. We next define framed maps of framed regular cell complexes, and so in particular of framed regular cells.

Recall, in the case of framed simplices, we defined framed maps as maps that preserve frames on each vector; the setup crucially relied on vectors in m-simplices being generated by their m *spine vectors* (which provided a *basis* for the affine space of the simplex). Unfortunately, there is a priori no good analog for spine vectors in the context of regular cells in general; however, for *framed* regular cells, we may recover a notion of 'highest frame vectors' for any given framed regular cell as explained below; this turns out to be just enough to define framed maps of framed regular cells.

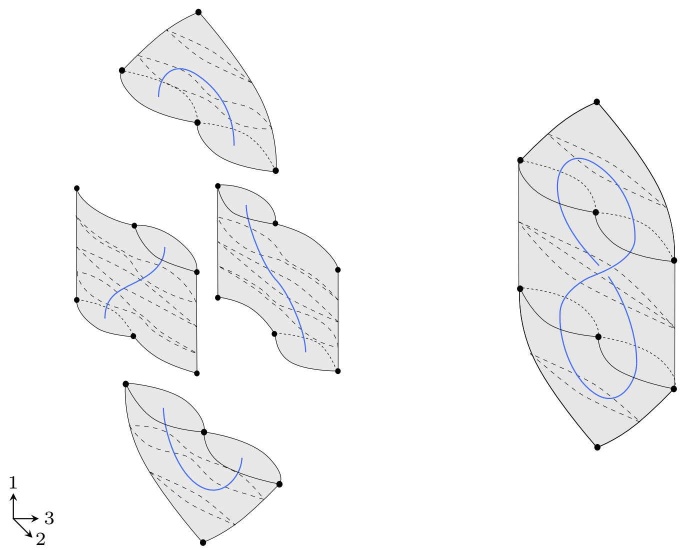


FIGURE 1.56. The cell complex dual to the Hopf circle.

Recall, the highest frame vector of a framed simplex is the unique simplicial vector whose frame label is maximal among the set of labels of frame vectors in that framed simplex.

TERMINOLOGY 1.3.48 (Highest frame vectors of framed regular cells). Given a n-framed regular cell complex $(X, \mathcal{F})$ and a cell $x \in X$, the 'highest frame subcomplex' $\mathsf{axl}\, x$ (also called simply the cell's 'axel' or 'axel vector', see below) is the simplicial subcomplex of X comprising 1-simplices that contain x and whose frame label is maximal among the labels of all such 1-simplices. ⌟

REMARK 1.3.49 (Highest frame vectors form a vector). The complex $\mathsf{axl}\, x$ always contains exactly two vectors. Moreover, it is isomorphic to the linear simplicial complex $(\bullet \!\!\rightarrow\!\! \bullet \!\!\rightarrow\!\! \bullet)$. As as full subposet of the poset X (determined on vertices) it is isomorphic to $(\bullet \leftarrow \bullet \rightarrow \bullet)$ (which matches the fundamental poset of a 1-simplex). In this sense the highest frame vector set may be regarded itself as forming a single vector, which motivates the terminology 'axel vector'. We will revisit and establish this claim in Observation 3.3.16. ⌟

EXAMPLE 1.3.50 (Highest frame vectors and axel vectors of framed regular cells). In Figure 1.57 we illustrate the highest frame vectors and axel vectors of some framed regular 1-cells and 2-cells. In the first row we show the framed ordered simplicial complex, and circle the highest frame vectors. Note that in every case, there are exactly two highest frame vectors, one that ends in the initial vertex and one that starts at it. In the second row we introduce a compact notation for this situation, by drawing a single red vector in the corresponding realized cell, now representing the single 'axel vector' of that cell. Note that the common framing label of the two highest frame vectors is again represented using multi-arrowhead notation (see Example 1.1.26).

In Figure 1.58, we similarly depict highest frame vectors for several framed regular 3-cells. Here we only use the axel vector notation. We also indicate the axel vectors for selected cells on the boundary of these 3-cells.

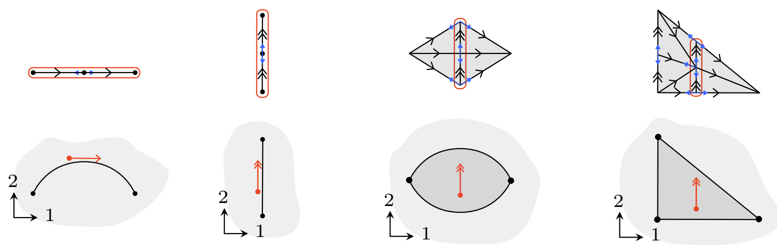


Figure 1.57. Highest frame vectors and axel vectors of framed regular 1-cells and 2-cells.

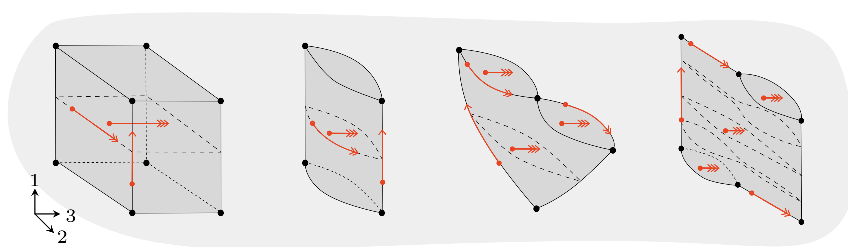


Figure 1.58. Axel vectors of framed regular 3-cells.

The definition of framed maps of framed regular cells mirrors the definition of framed maps of framed simplices (see Definition 1.1.59), and is as easily stated for all framed regular cell complexes, as follows.

Definition 1.3.51 (Framed maps of framed regular cell complexes). Given n-framed regular cell complexes $(X, \mathcal{F})$ and $(Y, \mathcal{G})$, a **framed cellular map** $F : (X, \mathcal{F}) \to (Y, \mathcal{G})$ is a cellular map of cellular posets $F : X \to Y$, such that for all $x \in X$, either F preserves $\mathsf{axl}\, x$, that is, F restricts to a framed simplicial isomorphism $(\mathsf{axl}\, x, \mathcal{F}|_{\mathsf{axl}\, x}) \cong (F(\mathsf{axl}\, x), \mathcal{G}|_{F(\mathsf{axl}\, x)})$, or degenerates it, i.e., $F(\mathsf{axl}\, x)$ is a point.

We often refer to 'framed cellular maps' simply as 'framed maps'.

Example 1.3.52 (Framed maps of framed regular cells). In Figure 1.59 we illustrate examples of framed maps of 2-framed regular cells; in each case we highlight image and preimage cells in the same color. On the left each map is depicted via its framed cellular realization; on the right each map is also depicted as a map of framed simplicial complexes. In each case, a key axel vector is depicted in a cell and the corresponding highest frame vectors circled in the simplicial complex; these vectors are either preserved or degenerated by the maps.

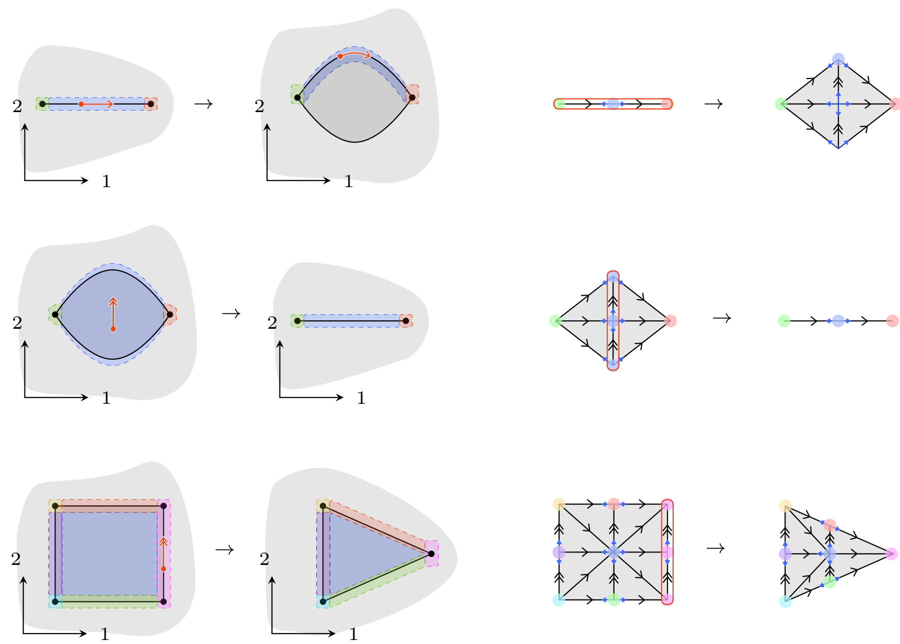


FIGURE 1.59. Framed cellular maps.

In Figure 1.60 we similarly depict cellular maps that are not framed. These maps fail to preserve the axel vector, correspondingly the highest frame vectors, in the required sense; the failures are circled. In the first case, the map sends the indicated frame 1-vectors to the inverse of frame 2-vectors; in the second case, the indicated frame 2-vectors are sent to the inverse of frame 2-vectors; in the third case, the indicated frame 2-vectors are sent to frame 1-vectors. ⌟

REMARK 1.3.53 (Framed maps of cells are subframed on simplices). Note that framed maps of framed regular cells need not descend to framed maps of their corresponding framed simplicial complexes, but they do descend to *subframed* maps of those framed simplicial complexes (see Remark 1.2.24); that is, the map need not preserve the simplicial ordering and may *specialize* frame labels of vectors. This is illustrated in Figure 1.61; the circled frame 1-vectors are mapped to frame 2-vectors or inverse frame 2-vectors. ⌟

NOTATION 1.3.54 (Categories of framed regular cell complexes). The category of n-framed regular cell complexes and their framed maps will be denoted by $\mathsf{FrCellCplx}_n$. The full subcategory of $\mathsf{FrCellCplx}_n$ on n-framed regular

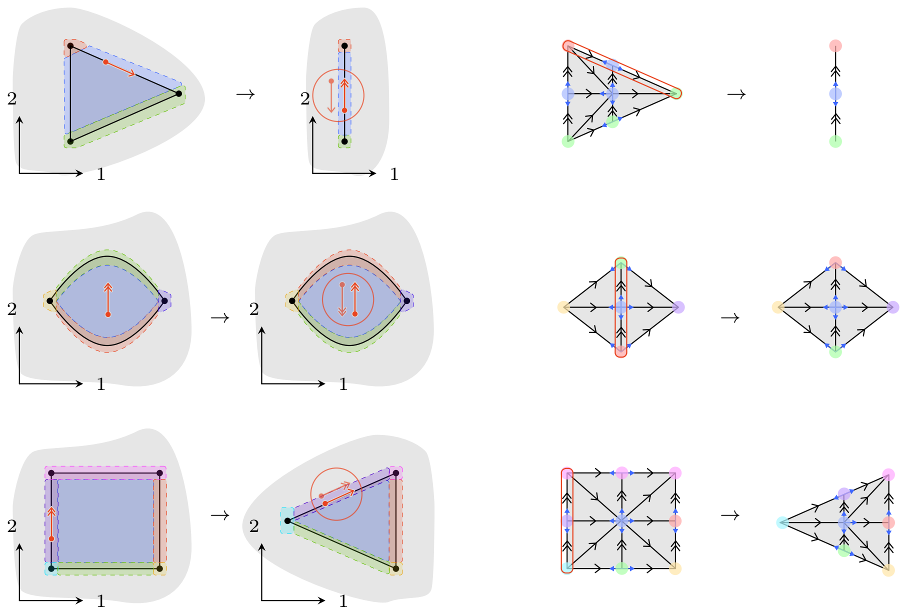


FIGURE 1.60. Cellular maps that are not framed.

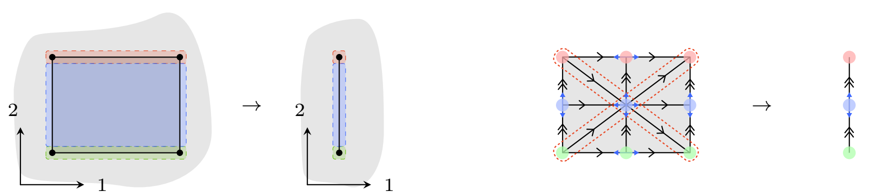


FIGURE 1.61. A framed cellular map and its corresponding subframed simplicial map.

cell complexes that are framed collapsible will be denoted by $\mathsf{CollFrCellCplx}_n$. The full subcategory of $\mathsf{FrCellCplx}_n$ on n-framed regular cells will be denoted by FrCell_n. ⌟

1.3.2.4. Framed simplicial complexes as framed regular cell complexes. We now complete our earlier quest to construct a fully faithful embedding from framed simplicial complexes into framed regular cell complexes, which descends to the ordinary embedding of (nonframed) simplicial complexes into regular cell complexes. The embedding is obtained as a framed version of the fundamental poset construction, which we outline as follows.

CONSTRUCTION 1.3.55 (Framed fundamental posets). Given an n-embedded framed simplicial complex $(K, \mathcal{F})$, we construct a framed regular cell complex $(\sqcap K, \sqcap^{\mathsf{fr}}\mathcal{F})$, referred to as the 'framed fundamental poset' of

$(K, \mathcal{F})$, where $\mathbb{\Pi} K$ is constructed via Construction 1.3.26 and $\mathbb{\Pi}^{\mathsf{fr}}\mathcal{F}$ is an n-framing constructed as follows.

First assume K is an m-simplex, $m > 0$. Given an n-embedded framed m-simplex $(K \cong [m], \mathcal{F})$, consider any framed realization $r : |K| \hookrightarrow \mathbb{R}^n$. Then $\mathbb{\Pi}^{\mathsf{fr}}\mathcal{F}$ is the unique framing of $\mathbb{\Pi}\,|K|$ such that there exists an injective framed realization $e : |\mathbb{\Pi}\,|K|| \hookrightarrow \mathbb{R}^n$ (cf. Definition 1.2.18) which factors through r by a barycentric subdivision $|\mathbb{\Pi}\,|K|| \to |K|$ (see Remark 1.3.27)

Next assume K is any simplicial complex with framing $\mathcal{F}$. The framing $\mathbb{\Pi}^{\mathsf{fr}}\mathcal{F}$ is determined by requiring that, for each simplex $x : [m] \hookrightarrow K$, $\mathbb{\Pi}^{\mathsf{fr}}\mathcal{F}$ restricts on $\mathbb{\Pi}(x) : \mathbb{\Pi}[m] \hookrightarrow \mathbb{\Pi}\,|K|$ to the framing $(\mathbb{\Pi}^{\mathsf{fr}}\mathcal{F})\big|_{\mathbb{\Pi}(x)} = \mathbb{\Pi}^{\mathsf{fr}}(\mathcal{F}|_x)$. ⌟

Construction 1.3.56 (The framed fundamental poset functor). The **framed fundamental poset functor**

$$\mathbb{\Pi}^{\mathsf{fr}} : \mathsf{FrSimpCplx}_n \to \mathsf{FrCellCplx}_n$$

takes n-embedded framed simplicial complexes $(K, \mathcal{F})$ to their framed fundamental poset $(\mathbb{\Pi} K, \mathbb{\Pi}^{\mathsf{fr}}\mathcal{F})$, and framed simplicial maps $F : (K, \mathcal{F}) \to (L, \mathcal{G})$ to the cellular poset map $\mathbb{\Pi} F : \mathbb{\Pi} K \to \mathbb{\Pi} L$. ⌟

Example 1.3.57 (The framed regular cell complex of a framed simplicial complex). In Figure 1.62, we depict two framed simplicial complexes and their associated framed cell complexes. On the far left are the two framed simplicial complexes depicted directly with their framings. In the middle left are the same two complexes depicted via their framed realizations. In the middle right are the two associated framed cell complexes depicted via their framed realizations. Note these associated complexes have the same set of cells as the set of simplices; the cells are depicted with curved boundary simply to remind us that they are conceived of as cells rather than simplices. On the far right are those framed cell complexes depicted directly via the framing on the corresponding ordered simplicial complexes. ⌟

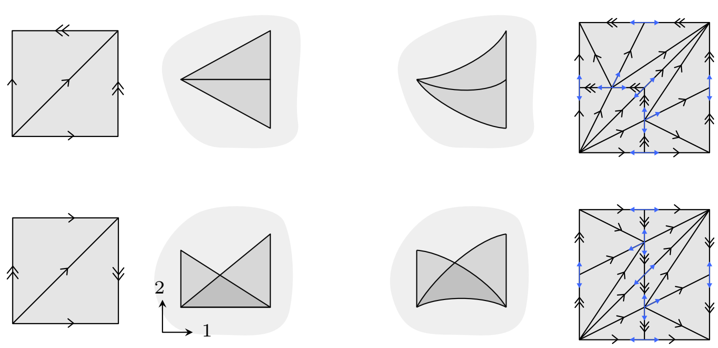


Figure 1.62. Framed simplicial complexes and their associated framed cell complexes.

We omit a detailed verification of correctness for Construction 1.3.55 and Construction 1.3.56. The punchline is recorded in the following observation.

Observation 1.3.58 (Framed simplicial complexes are framed regular cell complexes). The functor $\sqcap^{\mathsf{fr}} : \mathsf{FrSimpCplx}_n \to \mathsf{FrCellCplx}_n$ is a fully faithful embedding of categories. ⌟

In this sense, we may think of framed simplicial complexes as a special case of framed regular cell complexes, and think of the latter notion as being a generalization of the former notion.

1.3.2.5. Tractability of framed regular cells. The zoo of framed regular cells illustrated in Section 1.3.2.2, and especially the more extensive menagerie of Chapter C, raises the question of the tractability of this class as the basic shapes of a computable combinatorial theory. Forward-referencing later key results, we briefly mention several of the properties making framed regular cells 'tractable'.

Recall, a manifestation of the computational intractability of unframed regular cells and cell complexes is that there cannot be an algorithm to recognize whether a poset is cellular (thus realizes to a regular cell complex), and therefore it is impossible to *decidably* enumerate the unframed regular cells.[10] The framed situation provides a stark contrast.

Remark 1.3.59 (Recognizability of framed regular cells). There is an algorithm for recognizing framed regular cells among all framed posets (that is, posets with framings of their underlying simplicial complexes). We can therefore algorithmically (and decidably) enumerate framed regular cell complexes. This will be shown in Corollary 3.3.29. ⌟

Remark 1.3.60 (Efficient enumerability of framed regular cells). The enumeration can be made efficient by an inductive process that generates new n-framed cells by extending previous $(n-1)$-framed cells. This roots in the classification of framed regular cells in Chapter 3 characterizes their precise combinatorial structure by a 'bottom up' generation process via the notion of n-trusses introduced in Chapter 2. ⌟

As a contrasting example, the class of *convex* polytopes is well-known to be enumerable but the known enumeration requires an expensive search: namely, the enumeration must search the space of all abstract complexes testing for convex geometric realizations using the Tarski-Seidenberg theorem [**GKPS67**, §5.5].

However, not every framed regular cell need be a convex polytope. Thus, results in the convex case do not immediately carry over to the framed regular case.

Remark 1.3.61 (Framed regular cells need not be convex). Framed regular cells need not be convex. The simplest examples of non-convex cells include the n-globes, such as the 3-globe in Figure 1.50. ⌟

[10] Here, we say a set is 'decidably enumerable' relative to a larger set of combinatorial objects (such as the set of all framed posets in our specific case) if it is decidable whether a given object from the larger set will eventually appear in the enumeration.

Moved this sec to the end. I think works better as here as a big picture send off from C1, and transition to C2, where the combinatorics providing the tractability will come in.

Rephrasing in terms of 'tractability' rather than 'reasonableness' may keep things closer to the mathematics & intro narrative?

Above motivational paragraph has the logic: can't recognize therefore can't enumerate. Then below remark has the logic: can enumerate therefore can recognize. Of course they are closely related, but anyway is this mathematically/motivationally best?

What's an instance of a class of cells of the sort mentioned, ie that can be enumerated but only by an expensive search? [convex cells]

I think this remark on constructibility can be sharpened. Still feels vague even by remark standards. Of course the idea is good, and I think it belongs.

Oh, below comes the answer to the question about a contrasting class, but after it already felt missing. CLD to consider a light reorder/reformulation here.

The content of the following remark appears to be the opposite of the heading.

Finally, recall from Remark 1.3.31 that a fundamental source of unmanageability of unframed regular cells and cell complexes is the fact that regular cells or cell complexes need not be regular PL cells or PL cell complexes—that is, their links need not be piecewise linear spheres. We remark now that this failure cannot happen in the framed context.

Remark 1.3.62 (Framed cells and cell complexes are piecewise-linear). If a regular cell or cell complex admits a framing, then it is necessarily piecewise-linear—this will be shown in Corollary 3.3.31. Thus the notion of framed regular cells is identical to the notion of framed regular PL cells. ⌟

We remark that a similar result holds for convex polytopes: any convex polytope is automatically a PL cell (see [**Bjö84**, Prop. 4.5ff, Thm. 6.1]).

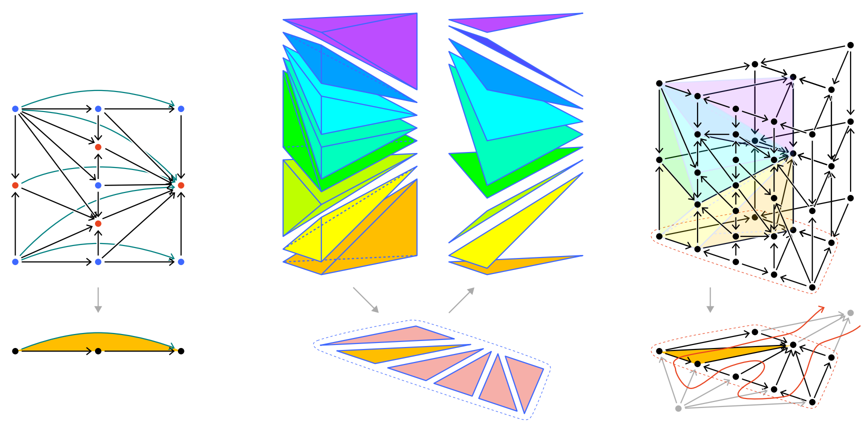
Re the next, last comment: this says again an analogy can be made, but before it was a disanalogy/contrast to the convex polytope case, and now it seems to be an alignment. (The formulation makes it seem like they'd have the same relation.)

Indeed, both framed regular cells and convex polytopes can be regarded as 'tame' classes of cells (though only framed regular cells can efficiently understood). In light of our earlier mention of the role of the Tarski-Seidenberg theorem in showing the enumerability of convex polytopes, this observation makes contact with the broader relation of framed combinatorial topology and o-minimal topology, both of which are incarnations of *tame* topology: we will revisit this relation later again in Section 5.3.2.3, by which time a wealth of other structure surrounding framed regular cells will have emerged.

In next pass, keep clear which things are about cell complexes, or cells, or both, and with what logically implication between. [headings clarify that now]

CHAPTER 2

◊Constructible framed combinatorics: trusses



In this chapter, we develop the theory of trusses.[11] Trusses are iterated constructible combinatorial bundles of framed fence posets. In Chapter 3, we will prove that trusses yield a tractable, computable combinatorial classification of the framed regular cell complexes introduced in Chapter 1. Furthermore, the stratified geometric realizations of trusses are iterated constructible bundles of framed stratified intervals, called meshes, and conversely the stratified fundamental posets of meshes are trusses. The theory of meshes will be developed in Chapter 4, and trusses will provide, via an equivalence with meshes, a concrete combinatorial model of all possible local structures of constructible framed stratified topological spaces.

We begin this chapter, in Section 2.1, by introducing 1-trusses, 1-truss bordisms and their composition, and 1-truss bundles. We then, in Section 2.2, define a scaffold order on section and spacer simplices, and thus establish the method of truss induction for reasoning about simplices in the total posets of 1-truss bundles. Finally, in Section 2.3, we define n-trusses as iterated 1-truss bundles, describe the combinatorial category of n-truss blocks, and present block sets as presheaves on truss blocks.

[11] The substantial foundational components of the theory of trusses were first formulated in [**Dor18**] under the name 'singular cubes'.

2.1. ◊1-Trusses, bordisms, and bundles

For a typical stratified space, the fundamental poset of its strata and their incidence relations is a coarse representation of the space, at best. But when the strata are contractible and arranged in a sufficiently nice way, the fundamental poset can be, in fact, a faithful encoding. The simplest such case is that of stratified 1-manifolds: a stratified 1-manifold (say with at least three strata to avoid degenerate cases) is actually homeomorphic to the geometric realization of its fundamental poset. The fundamental posets so arising are fences, that is, roughly speaking, linear or circular posets with no composable arrows. In fact, we care in the first instance about framed stratified 1-manifolds, and will restrict attention to contractible such, i.e., intervals, for simplicity; a framing of a stratified interval provides its fundamental poset fence with a total 'frame' order. A fence with a framing is the essence of our combinatorial notion of a *1-truss*. An example 1-truss is illustrated on the left in Figure 2.1, where the framing direction is indicated by a small purple arrow and the blue and red dots indicate strata with 1- and 0-dimensional realizations, respectively.

The entertainments and intricacies of framed stratified intervals, and their combinatorial encoding by 1-trusses, begin to emerge when considered in stratified families. The simplest such case is of a stratified bundle of framed stratified intervals over the standard stratified 1-simplex. In such a family there is a generic fiber (over the open bulk of the 1-simplex), and a special fiber (over the closed endpoint of the 1-simplex), and some kind of transformation from the generic to special fiber. A sufficiently nice such stratified bundle of framed stratified intervals is, like the intervals themselves, faithfully encoded by its fundamental poset bundle over the standard stratified combinatorial 1-simplex $0 \to 1$. A triple of a generic fiber 1-truss and a special fiber 1-truss and a suitable transformation between them, will be called a *1-truss bordism*; here suitable will be a collection of combinatorial conditions (namely 'bimonotone bifunctional functorial relation') ensuring, eventually, that the bordism arises as the fundamental poset of a corresponding stratified bundle. An example 1-truss bordism is illustrated in the middle of Figure 2.1, where the generic fiber 1-truss is depicted on the left, the special fiber 1-truss is depicted on the right, and the transformation relation between them is depicted by the intermediate arrows.

Two bundles over unstratified 1-simplices may be composed by concatenation to form another bundle over a 1-simplex. However, the concatenation of two stratified intervals is not another stratified interval; it is therefore not just evident how one should or can go about composing stratified bundles over 1-simplices. By contrast, a striking, transparent property of the combinatorial encoding here is that *1-truss bordisms compose* simply as their underlying relations. An example 1-truss bordism composition is illustrated on the right in Figure 2.1, where the two composable 1-truss bordisms are depicted in gray, and their composite bordism is depicted in black. As is discernible in the illustration, this situation of two composable bordisms may equivalently be considered as a *1-truss bundle* over the standard stratified combinatorial 2-simplex $0 \to 1 \to 2$. Of course, as the dimension of the base poset grows, it becomes increasingly difficult to depict or decipher the whole poset of a 1-truss bundle over the base. Precisely because the composite bordisms are combinatorially determined by their factors, it will always suffice to provide the 1-truss bordisms covering a collection of generating morphisms of the base poset; for instance, Figure 2.2 depicts (everything we need to encode) the total poset of a 4-dimensional 1-truss bundle over the 3-simplex.

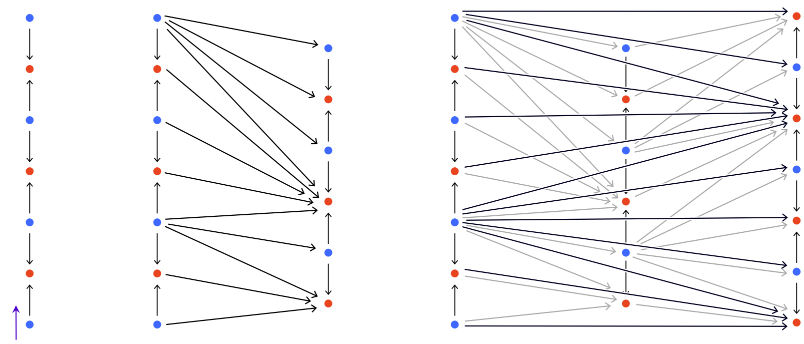


FIGURE 2.1. A 1-truss, a 1-truss bordism, and a 1-truss bordism composition.

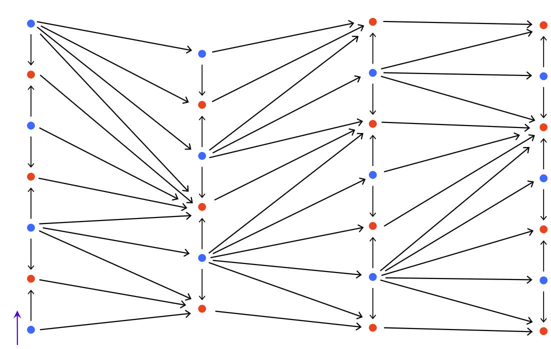


FIGURE 2.2. The total poset of a 1-truss bundle.

OUTLINE. In Section 2.1.1, we introduce 1-trusses as framed fences and maps of 1-trusses as those preserving both the framing and the fence order. In Section 2.1.2, we define 1-truss bordisms as functorial relations that are singular functional, regular cofunctional, and frame-order bimonotone, describe and illustrate assorted local behaviors of 1-truss bordisms, and observe that 1-truss bordisms compose. Finally in Section 2.1.3 we describe 1-truss bundles as collections of 1-truss bordisms, and show that a category of 1-trusses and their bordisms is a classifying category for 1-truss bundles.

2.1.1. ◊1-Trusses.

Synopsis. We introduce 1-trusses as framed fences, differentiate trivial, linear, and circular 1-trusses, reformulate the notion of linear 1-trusses in terms of a set with two interacting order relations, and distinguish singular and regular elements. We then introduce maps of 1-trusses as those preserving both orders, delineate the notions of singular, regular, and balanced maps, and classify the balanced isomorphism classes of 1-trusses according to the singularity and regularity of their endpoints. Finally we observe that there is an involutive dualization functor swapping singular and regular elements and interchanging open and closed 1-trusses.

2.1.1.1. ◊1-Trusses as framed fences. Recall the classical combinatorial notion of fences [**Sta11**]; 1-trusses will be finite fences with the additional structure of a suitable choice of dimension for objects and a consistent choice of framing for morphisms.

Definition 2.1.1 (Fence). A **fence** is a connected category with countably many objects and morphisms, such that there are no composable (non-identity) morphisms, and such that there are at most two (non-identity) morphisms with source or target any given object. ⌟

Terminology 2.1.2 (Types of fences). Fences fall neatly into three types based on the topology of their geometric realization. Recall the geometric realization of a category is the topological space associated to the simplicial set obtained by taking the nerve of the category. Notice that the geometric realization of any fence is either a connected 0-manifold or a connected 1-manifold.

› When the geometric realization is a point, we say the fence is 'trivial'.
› When the geometric realization is an interval, we say the fence is 'linear'.
› When the geometric realization is a circle, we say the fence is 'circular'.

We call a fence 'finite' if it has finitely many objects and finitely many morphisms. ⌟

Example 2.1.3 (Fences). In Figure 2.3, we illustrate fences of different types. Each fence is depicted by its geometric realization, and the direction of morphisms in the fence is indicated by directing the edges of the geometric realization. ⌟

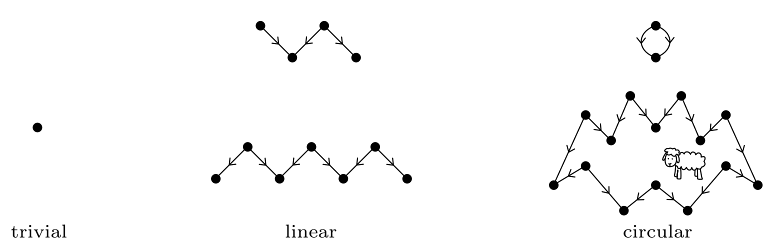


Figure 2.3. Fences of different types.

Observation 2.1.4 (Dimension functor). Every non-trivial fence T has a unique functor $T \to [1]^{\mathrm{op}}$ whose preimages are discrete categories; this functor 'folds the fence onto a single fence post'. We refer to such a functor as a 'dimension' functor, because the value of the functor on an object will be the dimension of a corresponding stratum in an associated stratified interval. Note that for a trivial fence, there are two distinct discrete-preimage functors $T \to [1]^{\mathrm{op}}$, and so an ambiguity regarding the dimension of the object. ⌟

The notion of 1-truss strengthens the notion of fence in two ways: firstly, a 1-truss includes the data of a 'progressive framing' of the edges of its geometric realization; secondly, a 1-truss comes equipped with a specified dimension functor, resolving the dimension ambiguity for trivial fences.

Terminology 2.1.5 (Progressive framing of a fence). A 'progressive framing' of a fence is a choice of direction of each edge of the geometric realization of the fence such that every vertex has at most one edge directed towards it and at most one edge directed away from it. ⌟

Definition 2.1.6 (General 1-trusses). A **1-truss** $(T, \dim, \mathcal{F})$ is a finite fence T, together with a dimension functor $\dim : T \to [1]^{\mathrm{op}}$ and a progressive framing $\mathcal{F}$. ⌟

Terminology 2.1.7 (Types of general 1-trusses). A 1-truss is 'trivial' whenever its underlying fence is. A 1-truss is 'linear' whenever its underlying fence is either linear or trivial; in the latter case we refer to it as a 'trivial linear' 1-truss. (This terminology is convenient because the two trivial linear 1-trusses will correspond to the trivially stratified linear interval and the degenerate interval.) Similarly, a 1-truss is 'circular' whenever its underlying fence is either circular or trivial; in the latter case we refer to it as a 'trivial circular' 1-truss.[12] (Again this is convenient because, in the context of circular trusses, the two trivial circular 1-trusses will correspond to the trivially stratified circle and the degenerate circle.) Note that this means that trivial 1-trusses are both linear and circular. ⌟

Example 2.1.8 (General 1-trusses). In Figure 2.4, we illustrate 1-trusses of the different types. In each case we depict the underlying fence; we indicate the dimension map by coloring preimages of 0 in red, and preimages of 1 in blue, and record the progressive framing by purple frame vectors adjacent to each edge. ⌟

Henceforth we will focus exclusively on the case of linear trusses. Much of the theory developed here, including the higher-dimensional notion of n-trusses, does generalize to the case of general (particularly circular) 1-trusses, but our main interest and applications will be in the linear case. Simplicity and brevity of exposition thus dictate the following convention.

Convention 2.1.9 (Linear 1-trusses by default). We will use the term '1-truss' to mean 'linear 1-truss' unless otherwise noted. ⌟

[12]Circular truss ⇝ **circus**.

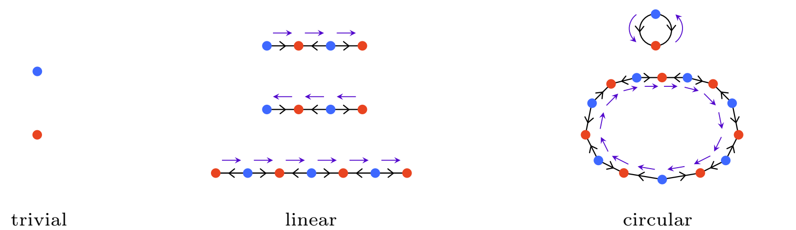


FIGURE 2.4. 1-Trusses of different types.

We may reformulate the definition of (linear) 1-trusses, without reference to fences, in terms of a set with two interacting order relations. First, note that any linear or trivial fence is a poset.[13] We usually denote the poset order of linear (or trivial) fences by $(T, \trianglelefteq)$, though in illustrations we exclusively use the arrow notation $(T, \to)$ inspired by the categorical, rather than partial order, interpretation. (The order notation $\trianglelefteq$ will be convenient for expressing strictly less or strictly greater relations with the symbols $\triangleleft$ and $\triangleright$.) Second, a progressive 1-framing $\mathcal{F}$ of a linear fence T may be equivalently encoded by a total order on the objects; we usually denote this order by $\preceq$. Together these orders provide the basis of our canonical definition, as follows.

DEFINITION 2.1.10 (1-Trusses). A (linear) **1-truss** $(T, \trianglelefteq, \dim, \preceq)$ is a finite non-empty set T together with the following structures:

(1) a partial order $\trianglelefteq$, called the 'face order' of T;
(2) a poset map $\dim : (T, \trianglelefteq) \to [1]^{\mathrm{op}}$, called the 'dimension map' of T;
(3) a total order $(T, \preceq)$, called the 'frame order' of T, for which either a succeeds b or b succeeds a if and only if either $a \triangleleft b$ or $a \triangleright b$. ⌟

NOTATION 2.1.11 (1-Trusses). We will usually keep the face orders, dimension maps, and frame orders implicit, abbreviating the 1-truss $(T, \trianglelefteq, \dim, \preceq)$ simply by T. ⌟

The terminology 'face order' for the partial order $\trianglelefteq$ reflects the relationship between 1-trusses and stratified intervals: elements a of a 1-truss will correspond to strata of dimension $\dim(a)$ in a corresponding stratified interval, and the existence of an arrow $a \triangleleft b$ of the 1-truss will correspond to the stratum b being a face of the stratum a. This motivational relationship of 1-trusses and stratified intervals is illustrated in the following example.[14]

EXAMPLE 2.1.12 (1-trusses and corresponding stratified intervals). In Figure 2.5, we depict two 1-trusses and two stratified intervals. For the

[13] Every preorder, in particular every poset, $(X, \leq)$ can be considered a category with object set X and a single morphism $x \to y$ whenever $x \leq y$. Every map of preorders can be considered as a functor of the corresponding categories.

[14] Note that the correspondence of 1-trusses and stratified intervals is fundamentally different from the earlier depiction in Example 2.1.8 of fences via their geometric realization as regular cell complexes.

1-trusses, we color blue the elements whose dimension map value is 1 and we color red the elements whose dimension map value is 0; we depict the face order by black arrows; and we indicate the overall direction of increasing frame order by a single purple vector. For the stratified intervals, the 0-strata are indicated by small black dots, and the 1-strata are indicated by black open intervals. Each 1-truss and its adjacent stratified interval correspond in the sense that the face order of the 1-truss is the fundamental poset of the stratified interval. Note that each object of the truss corresponds to a stratum of the same dimension. (The frame order of the truss corresponds to a framing of the stratified interval provided by an implicit embedding in the real line.) ⌟

FIGURE 2.5. 1-Trusses and their corresponding stratified intervals.

Inspired by much later applications to singularity theory, we adopt the following terminology.

TERMINOLOGY 2.1.13 (Singular and regular elements). An element $a \in T$ of a 1-truss T is called 'singular' if $\dim(a) = 0$, and 'regular' if $\dim(a) = 1$. We denote the subset of singular elements of T by $\mathsf{sing}(T)$, and the subset of regular elements of T by $\mathsf{reg}(T)$. ⌟

REMARK 2.1.14 (Orders on singular and regular elements). Note that, considered with the face order, the singular set $(\mathsf{sing}(T), \trianglelefteq)$ and regular set $(\mathsf{reg}(T), \trianglelefteq)$ are discrete orders, while, considered with the frame order, the singular set $(\mathsf{sing}(T), \preceq)$ and regular set $(\mathsf{reg}(T), \preceq)$ are total orders. ⌟

Since 1-trusses may be considered as fence categories or as sets with partial face orders, going forward we will refer interchangeably to either 'elements' or 'objects' of 1-trusses.

2.1.1.2. ◊Maps of 1-trusses. We next define maps of 1-trusses and distinguish certain specific classes of maps based on their behavior on singular and regular objects. As 1-trusses have two partial orders, the face and frame orders, maps thereof are conveniently and succinctly expressed in terms of diposets as follows.

DEFINITION 2.1.15 (Diposets and their maps). A **diposet** $(X, \trianglelefteq, \preceq)$ is a set X with two partial orders $\trianglelefteq$ and $\preceq$. A **diposet map** $F : (X, \trianglelefteq, \preceq) \to (Y, \trianglelefteq, \preceq)$ is a map of sets $F : X \to Y$ that independently respects both orders, i.e., induces poset maps $F : (X, \trianglelefteq) \to (Y, \trianglelefteq)$ and $F : (X, \preceq) \to (Y, \preceq)$. ⌟

Definition 2.1.16 (1-Truss maps). A **map of 1-trusses** $T \to S$ is a diposet map $(T, \trianglelefteq, \preceq) \to (S, \trianglelefteq, \preceq)$. ⌟

Note that the definition of a 1-truss map does not impose any conditions on how the map interacts with the dimension maps of the trusses. Indeed, there are several distinct such potential interactions, depending on the map's behavior on singular and regular objects.

Definition 2.1.17 (Singular, regular, and balanced maps). Let $F : T \to S$ be a map of 1-trusses.

- The map F is **singular** if it sends singular objects of T to singular objects of S. In other words, for all $a \in T$, $\dim(a) \geq \dim(Fa)$.
- The map F is **regular** if it sends regular objects of T to regular objects of S. In other words, for all $a \in T$, $\dim(a) \leq \dim(Fa)$.
- The map F is **balanced** if it is both singular and regular. In other words, for all $a \in T$, $\dim(a) = \dim(Fa)$. ⌟

Example 2.1.18 (Maps of 1-trusses). In Figure 2.6, we depict a singular, a regular, and a balanced map of 1-trusses, and one that is neither singular nor regular. ⌟

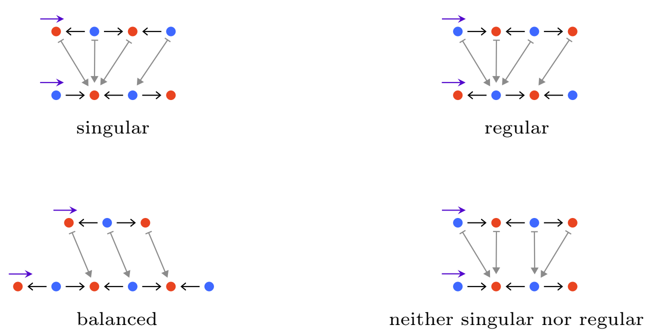


Figure 2.6. Types of maps of 1-trusses.

Notation 2.1.19 (Category of 1-trusses). The category of 1-trusses and their maps is denoted Trs_1. The wide subcategory containing only singular, respectively regular, respectively balanced maps will be denoted $\mathsf{Trs}_1^{\mathsf{s}}$, respectively $\mathsf{Trs}_1^{\mathsf{r}}$, respectively $\mathsf{Trs}_1^{\mathsf{rs}}$. ⌟

Since a map of 1-trusses that is neither singular nor regular does not abide any condition on the dimension map, a priori in the category of 1-trusses, the trivial truss of dimension 0 (i.e., with one element $a \in T$ having $\dim(a) = 0$) is isomorphic to the trivial truss of dimension 1 (i.e., with one element $a \in T$ having $\dim(a) = 1$); however as we typically want to distinguish these 1-trusses, we adopt the following convention.

Convention 2.1.20 (Balanced isomorphism by default). The term 'isomorphism of 1-trusses' will refer, unless otherwise noted, to isomorphism in the category $\mathsf{Trs}_1^{\mathsf{rs}}$, that is, to balanced bijective 1-truss maps. (Note that balanced isomorphisms preserve all the structural data of 1-trusses, namely the face order, the frame order, and the dimension map.) ⌟

Remark 2.1.21 (Balanced isomorphisms of 1-trusses are unique). If two 1-trusses are balanced isomorphic, there is a *unique* balanced isomorphism between them. There is therefore never any need to distinguish between distinct but balanced isomorphic 1-trusses. Also, in particular, there are no nontrivial balanced automorphisms of 1-trusses. (In fact, a not-necessarily-balanced isomorphism between 1-trusses is also unique, but by the previous convention we care about and concentrate on balanced isomorphism.) ⌟

As intervals are crucially distinguished by whether they are open or closed or half-open half-closed or degenerate, similarly for stratified intervals and thus correspondingly for 1-trusses. For 1-trusses, these distinctions are controlled by whether the endpoints are singular or regular.

Terminology 2.1.22 (Endpoints of 1-trusses). For a 1-truss $(T, \trianglelefteq, \dim, \preceq)$, we refer to the minimal element of the frame order $(T, \preceq)$ as the **lower endpoint** and denote it by $\mathsf{end}_- T$; similarly we refer to the maximal element as the **upper endpoint** and denote it by $\mathsf{end}_+ T$. ⌟

Terminology 2.1.23 (Endpoint types of 1-trusses). There are six 'endpoint types' of balanced isomorphism classes of 1-trusses, distinguished and referred to as follows. Let T be a 1-truss.

(1) If T has a single element and that element is regular, then T is the **trivial open** 1-truss and is denoted by $\mathring{\mathbb{T}}_0$.
(2) If T has a single element and that element is singular, then T is the **trivial closed** 1-truss and is denoted by $\bar{\mathbb{T}}_0$.

For the remaining cases, assume the 1-truss T has more than one element.

(3) If both endpoints $\mathsf{end}_\pm T$ are regular, then T is **open**. When T has $2k+1$ elements, it is denoted by $\mathring{\mathbb{T}}_k$.
(4) If both endpoints $\mathsf{end}_\pm T$ are singular, then T is **closed**. When T has $2k+1$ elements, it is denoted by $\bar{\mathbb{T}}_k$.
(5) If $\mathsf{end}_- T$ is regular and $\mathsf{end}_+ T$ singular, then T is **left-open right-closed**. When T has $2k$ elements, it is denoted by $\mathring{\bar{\mathbb{T}}}_k$.
(6) If $\mathsf{end}_- T$ is singular and $\mathsf{end}_+ T$ regular, then T is **left-closed right-open**. When T has $2k$ elements, it is denoted by $\bar{\mathring{\mathbb{T}}}_k$.

The last two cases are both referred to as 'half-open' 1-trusses. ⌟

Example 2.1.24 (Types of 1-trusses). In Figure 2.7 we depict an example of each of the six types of 1-trusses, distinguished by their endpoints. ⌟

We will mainly be concerned with the cases of entirely closed trusses and of entirely open trusses, as opposed to the half-open cases, and so we introduce notation for the following subcategories.

Notation 2.1.25 (Open and closed 1-trusses). The subcategory of Trs_1 whose objects are open trusses (including the trivial open truss) and whose

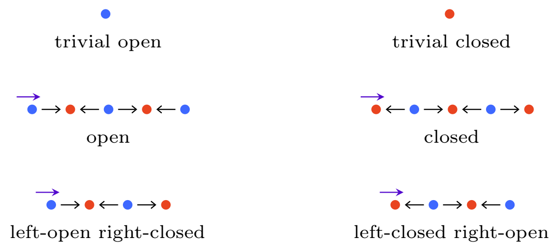


FIGURE 2.7. Endpoint types of 1-trusses.

morphisms are regular maps, will be denoted by $\mathring{\mathsf{Trs}}_1$. The subcategory of Trs_1 whose objects are closed trusses (including the trivial closed truss) and whose morphisms are singular maps, will be denoted by $\bar{\mathsf{Trs}}_1$.

2.1.1.3. ◊Dualization of 1-trusses. There is a natural dualization operation taking closed trusses to open trusses and vice versa.

CONSTRUCTION 2.1.26 (Dualization of 1-trusses). There is a covariant involutive functor, called the 'dualization functor', denoted

$$\dagger : \mathsf{Trs}_1 \cong \mathsf{Trs}_1$$

and defined as follows. Given a 1-truss $T = (T, \trianglelefteq, \dim, \preceq)$, its dual is the 1-truss $T^\dagger = (T, \trianglelefteq^{\mathrm{op}}, \dim^{\mathrm{op}}, \preceq)$. That is, the face order of $T^\dagger$ is opposite to the face order of T; the dimension map of $T^\dagger$ is the opposite of the dimension map of T (post-composed with the identification $[1] \cong [1]^{\mathrm{op}}$); and the frame order of $T^\dagger$ is the same as the frame order of T. The dual $F^\dagger : T^\dagger \to S^\dagger$ of a 1-truss map $F : T \to S$ is the map whose underlying map of sets is equal to the underlying map of sets $T \to S$ of the map F.

EXAMPLE 2.1.27 (Dualization of 1-trusses and 1-truss maps). In Figure 2.7, the 1-trusses in the left column are dual, respectively, to the 1-trusses in the right column. In Figure 2.6, the singular map dualizes to the regular map.

OBSERVATION 2.1.28 (Singular and regular are dual). Given a 1-truss T, an element $a \in T$ is singular, respectively regular, if and only if the corresponding element $a \in T^\dagger$ is regular, respectively singular. Similarly, a map of 1-trusses $F : T \to S$ is singular, respectively regular, if and only if the dual $F^\dagger : T^\dagger \to S^\dagger$ is regular, respectively singular.

OBSERVATION 2.1.29 (Closed and open are dual). Since singular and regular elements are exchanged by dualization, the endpoint types are similarly exchanged. In particular open 1-trusses dualize to closed 1-trusses and vice versa; the dualization functor restricts to an isomorphism $\dagger : \mathring{\mathsf{Trs}}_1 \cong \bar{\mathsf{Trs}}_1$.

2.1.2. ◊1-Truss bordisms. As described and illustrated in the previous section, 1-trusses provide a combinatorial model of (framed) stratified intervals. Next we introduce 1-truss bordisms, which provide a combinatorial model of certain families of stratified intervals. More specifically, 1-truss bordisms will model suitably constructible stratified bundles, of framed stratified intervals, over the standard stratified 1-simplex.[15] Two such bundles are illustrated in Figure 2.8. In the first bundle, the generic fiber is a stratified open interval with two point strata; those point strata collide into a single point stratum when entering the special fiber. In the second bundle, the generic fiber again has two point strata, but when entering the special fiber, a third point stratum spontaneously appears. In both cases, we also depict the fundamental poset of the total space of the stratified bundle, and the map of posets to the fundamental poset of the stratified interval; these 'stratified bundles of 1-truss posets' are the '1-truss bordisms' corresponding to the geometric stratified bundles. The notion of 1-truss bordism encodes a unified combinatorial description of the possible changes between the 1-truss fibers in such stratified fundamental poset bundles.

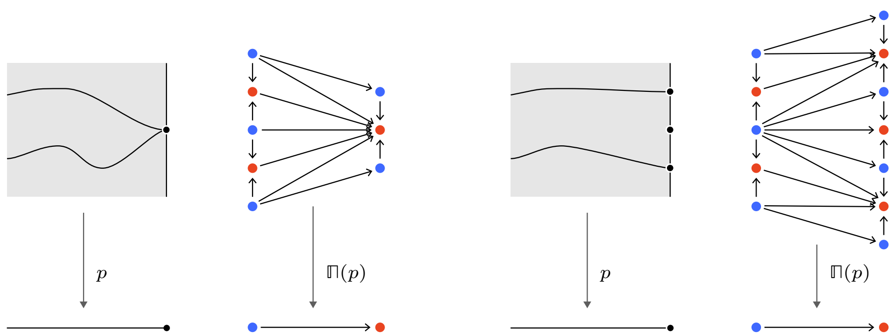


FIGURE 2.8. Stratified bundles of stratified intervals and their corresponding fundamental posets as 1-truss bordisms.

SYNOPSIS. We begin by giving the crucial definition of 1-truss bordisms as functorial relations that are functional on singular elements, cofunctional on regular elements, and bimonotone in the frame orders. We then illustrate various local phenomena that occur in 1-truss bordisms, along with an assortment of 1-truss relations that fail to be 1-truss bordisms. We observe that there is a composition operation of 1-truss bordisms, and therefore a category of 1-trusses and their bordisms. We describe a dualization operation of 1-truss bordisms, and the resulting contravariant involutive dualization functor on the category. We then show that subject to suitable boundary conditions, a 1-truss bordism is completely determined by either of its function on singular elements or its cofunction on regular elements. Finally, we explain the relationship between 1-truss maps and 1-truss bordisms, namely that singular 1-truss maps have associated mapping cylinder 1-truss bordisms and regular 1-truss maps have associated mapping cocylinder 1-truss bordisms.

[15] Recall stratified bundles generalize fiber bundles by allowing the fibers to change when passing between strata of the base; see Definition B.2.25. The condition of constructibility controls what sort of fiber changes are allowed; see Remark B.2.30.

2.1.2.1. ◊1-Truss bordisms as bimonotone bifunctional functorial relations. Inspired by the behavior of fundamental posets of stratified bundles of stratified intervals, we develop the combinatorial notion of 1-truss bordisms. In the stratified bundles illustrated above in Figure 2.8, from a given stratum of the generic fiber, there may be entrance paths (see Definition B.1.6) leading to multiple distinct strata of the special fiber. Thus in the corresponding fundamental posets, there may be an order relation between a single element of the generic fiber and multiple elements of the special fiber. The combinatorial change from generic to special fiber is therefore in no way a function of posets; rather, it is a specific sort of relation of posets.

Again in the stratified bundles of Figure 2.8, when there is an entrance path $r_0 \to s_0$ within the generic fiber and an entrance path $s_0 \to s_1$ from the generic fiber to the special fiber, then there is always a direct entrance path $r_0 \to s_1$ from the source generic stratum to the special stratum. Similarly when there is an entrance path $r_0 \to r_1$ from the generic fiber to the special fiber and an entrance path $r_1 \to s_1$ within the special fiber, then there is a direct entrance path $r_0 \to s_1$ from the generic stratum to the target special stratum. In this sense, the relation between the fundamental poset of the generic fiber and the fundamental poset of the special fiber respects composition with the poset arrows of both source and target; such a relation is called 'functorial'.

Terminology 2.1.30 (Functorial relation). For preorders X and Y, a 'functorial relation' $R : X \nrightarrow Y$ is a relation $R \subset X \times Y$ (between the object sets of the preorders) for which, if there is an arrow $r_0 \to s_0$ of X and a relation $R(s_0, s_1)$, then there is a relation $R(r_0, s_1)$, and for which, if there is a relation $R(r_0, r_1)$ and an arrow $r_1 \to s_1$ of Y, then there is a relation $R(r_0, s_1)$. ⌟

Of course, not any functorial relation can arise from the entrance paths on stratified bundles of stratified intervals, because the arrangements of generic and special, singular and regular strata are quite constrained. Notice in particular that generic singular strata converge to special singular strata, and so, on fundamental posets, there is always a relation between any singular object of the source and some singular object of the target; in this sense the relation is 'functional' on singular objects. Furthermore, notice that any special regular stratum has a nearby generic regular stratum, and so, on fundamental posets, there is always a relation between some regular object of the source and any given regular object of the target; in this sense the relation is 'cofunctional' on regular objects.

Terminology 2.1.31 (Functional and cofunctional relations). For sets (i.e., discrete preorders) X and Y, a function $f : X \to Y$ induces the associated relation $R_f := \{(x, f(x)) | x \in X\} \subset X \times Y$. A 'functional relation' is one that is associated to a function in this sense.

Similarly, a cofunction $f : X \leftarrow Y$ induces the associated relation $R^f := \{(f(y), y) | y \in Y\} \subset X \times Y$. A 'cofunctional relation' is one that is associated to a cofunction in this sense. ⌟

Finally, since a stratified bundle of stratified intervals is in particular a continuous bundle of intervals, the linear order of strata within the generic fiber is weakly preserved when entering into the special fiber, and the linear order of strata within the special fiber is weakly preserved when exiting into the generic fiber; in particular the relation on fundamental posets is weakly (frame) order preserving or 'bimonotone' in the following sense.

Terminology 2.1.32 (Bimonotone relation). For total orders X and Y, a relation $R \subset X \times Y$ (between the object sets of the orders) is 'bimonotone' if, given relations $R(x, y)$ and $R(x', y')$, there is an arrow $x \to x'$ in X if and only if there is an arrow $y \to y'$ in Y. ⌟

A pair of arrows $x \to x'$ in X and $y \to y'$ in Y is called a 'transposition' of the relation if both $R(x, y')$ and $R(x', y)$. A relation is bimonotone exactly when it has no transpositions.

Altogether, we can finally define the fundamental notion of 1-truss bordisms. Recall from Remark 2.1.14 that the singular objects, and separately the regular objects, of a truss T form discrete subposets $(\mathsf{sing}\ T, \trianglelefteq)$ and $(\mathsf{reg}\ T, \trianglelefteq)$ of the face order $(T, \trianglelefteq)$.

Definition 2.1.33 (1-truss bordisms). A **1-truss bordism** $R : T \nrightarrow S$ between 1-trusses T and S is a non-empty functorial relation $R : (T, \trianglelefteq) \nrightarrow (S, \trianglelefteq)$ of the face orders of T and S satisfying the following two conditions.

(1) *Bifunctionality*: On singular elements, the restricted relation $R : (\mathsf{sing}\ T, \trianglelefteq) \nrightarrow (\mathsf{sing}\ S, \trianglelefteq)$ is functional, and on regular elements, the restricted relation $R : (\mathsf{reg}\ T, \trianglelefteq) \nrightarrow (\mathsf{reg}\ S, \trianglelefteq)$ is cofunctional.

(2) *Bimonotonicity*: The relation $R \subset T \times S$ is bimonotone with respect to the frame orders $(T, \preceq)$ and $(S, \preceq)$. ⌟

Terminology 2.1.34 (Singular and regular functions of a truss bordism). The bifunctionality condition requires that for each singular element $a \in \mathsf{sing}\ T$ there is a unique singular element $\mathsf{sing}_R(a) \in \mathsf{sing}\ S$ such that $R(a, \mathsf{sing}_R(a))$, and for each regular element $d \in \mathsf{reg}\ S$ there is a unique regular element $\mathsf{reg}^R(d) \in \mathsf{reg}\ T$ such that $R(\mathsf{reg}^R(d), d)$. The resulting function $\mathsf{sing}_R : \mathsf{sing}\ T \to \mathsf{sing}\ S$ is called the 'singular function' of R. Similarly, the function $\mathsf{reg}^R : \mathsf{reg}\ S \to \mathsf{reg}\ T$ is called the 'regular function' of R. ⌟

Example 2.1.35 (A 1-truss bordism). In Figure 2.9, we depict a 1-truss bordism $R : T \nrightarrow S$. The domain 1-truss T is drawn on the left, the codomain 1-truss S is drawn on the right, and elements of the relation R are indicated by rightward arrows between objects of T and S. That the relation R is bimonotone is visible from there being no crossings among its edges. That the relation R is bifunctional is witnessed by the left-to-right singular function

$\mathsf{sing}_R : \mathsf{sing}\ T \to \mathsf{sing}\ S$ highlighted in red, and the right-to-left regular function $\mathsf{reg}^R : \mathsf{reg}\ S \to \mathsf{reg}\ T$ highlighted in blue. ⌟

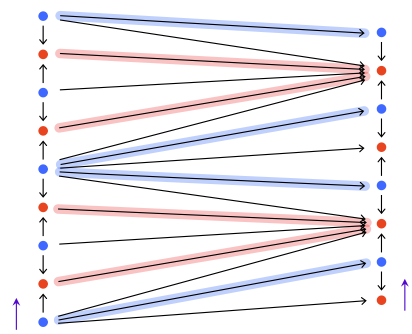


Figure 2.9. A 1-truss bordism.

Example 2.1.36 (A bifunctional functorial relation that is not a truss bordism). In Figure 2.10, we depict a bifunctional functorial relation $R : T \nrightarrow S$, between the same two 1-trusses as in the previous example, which though is not bimonotone and therefore not a truss bordism. ⌟

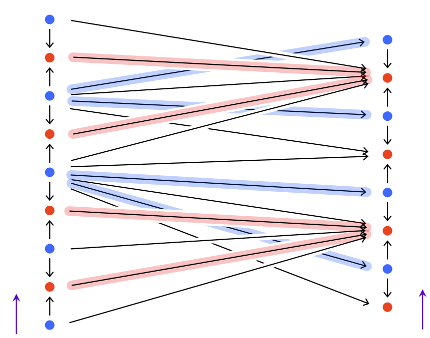


Figure 2.10. Not a 1-truss bordism.

Remark 2.1.37 (Truss bordisms are really half-bordisms). Recall that an elementary manifold bordism [**Sto15**], that is one with a single Morse critical point, can be considered as a cospan of two mapping cylinders of maps from the source and target manifolds onto a common intermediate singular manifold. See the left side of Figure 1.34 for an illustration. What we have called a 'truss bordism' is really analogous to only half of a classical geometric bordism, that is to the mapping cylinder of a single map from a manifold to an intermediate singular manifold. To obtain a truss structure more completely deserving of the name 'bordism', we would need to consider a cospan of truss bordisms. Such a cospan is illustrated in Figure 2.11. This structure indeed has the symmetric character typical of a geometric bordism, with the source and target both mapping onto an intermediate 'singular' slice. And indeed, these cospans will be the ubiquitous structure in the subsequent theory. Nevertheless, to have a concise and suggestive and familiar term for the core notion of half of such a cospan, and to avoid an interminable repetition of the prefix 'half-', we refer to what is defined above as truss bordisms simply as 'truss bordisms' and not as 'truss half-bordisms'. ⌟

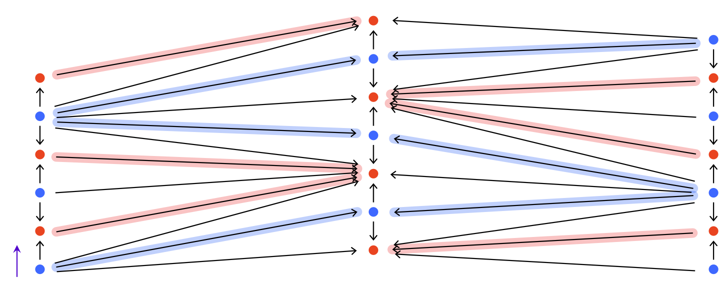


FIGURE 2.11. A 1-truss bordism cospan.

We may recast the above notions regarding 1-truss bordisms in more abstract categorical terms, as follows. (Less categorically-inclined readers may freely skip ahead to Section 2.1.2.2.) Recall that the category Bool of boolean values has two objects 'true' $\top$ (also written 1 or $*$) and 'false' $\bot$ (also written 0 or $\emptyset$), with a single non-identity morphism from false to true. Moreover, this category is monoidal under logical conjunction (i.e., multiplication on $\{0,1\}$, i.e., cartesian product on $\{\emptyset,*\}$). Considering Bool as $\{\emptyset,*\}$ provides a fully faithful monoidal functor $\mathsf{Bool} \hookrightarrow \mathsf{Set}$, which we may use to think of Bool-enriched categories as ordinary categories. Observe that Bool-enriched categories are precisely preorders. These formalities are defensible because boolean-enriched profunctors concisely and precisely encode functorial relations and their composition, as follows.

DEFINITION 2.1.38 (Boolean profunctors). Given two preorders X and Y, a **boolean profunctor** $R : X \nrightarrow Y$ is a functor $R : X^{\mathrm{op}} \times Y \to \mathsf{Bool}$. ⌟

NOTATION 2.1.39 (Category of boolean profunctors). Preorders and their boolean profunctors form a category denoted $\mathsf{BoolProf}$. ⌟

TERMINOLOGY 2.1.40 (Underlying relations of boolean profunctors). The 'underlying relation functor' $\mathsf{rel} : \mathsf{BoolProf} \to \mathsf{Rel}$ takes preorders to their object sets, and boolean profunctors $R : X \nrightarrow Y$ to the relation $R^{-1}(\top) \subset X \times Y$. ⌟

The underlying relation functor is faithful; that is, a boolean profunctor $R : X^{\mathrm{op}} \times Y \to \mathsf{Bool}$ is completely determined by its underlying set relation

$\mathsf{rel}\, R \subset X \times Y$. However the functor is not full, but has image exactly those relations that are functorial for the preorder structure.

REMARK 2.1.41 (Boolean profunctors are functorial relations). Given a boolean profunctor $R : X^{\mathrm{op}} \times Y \to \mathsf{Bool}$ between preorders, the underlying relation $\mathsf{rel}\, R \subset X \times Y$ is functorial, and any functorial relation is the underlying relation of a boolean profunctor. ⌟

TERMINOLOGY 2.1.42 (Representability of boolean profunctors). A boolean profunctor $R : X \nrightarrow Y$ is called 'representable' if it is of the form $\mathrm{Hom}_Y(f-,-)$ for a functor $f : X \to Y$, and 'corepresentable' if it is of the form $\mathrm{Hom}_X(-, f-)$ for a functor $f : Y \to X$. ⌟

REMARK 2.1.43 (Discrete representability is functionality). If X and Y are discrete preorders, then a (co)representable boolean profunctor $X \nrightarrow Y$ is simply a (co)functional relation. ⌟

REMARK 2.1.44 (1-Truss maps and 1-truss bordisms as functors and profunctors). A 1-truss map $T \to S$ is in particular a *functor* $(T, \trianglelefteq) \to (S, \trianglelefteq)$ of the face order posets, while a 1-truss bordism $T \nrightarrow S$ is in particular a (boolean) *profunctor* $(T, \trianglelefteq) \nrightarrow (S, \trianglelefteq)$ of the face order posets. ⌟

2.1.2.2. ◊Local phenomena in 1-truss functorial relations. We describe and illustrate various local phenomena that occur in functorial relations between 1-trusses: first examples of local forms that are indeed 1-truss bordisms, then examples that violate either bifunctionality or bimonotonicity and so fail to be 1-truss bordisms.

EXAMPLE 2.1.45 (Local forms of 1-truss bordisms). In Figure 2.12 we illustrate some local behaviors in 1-truss bordisms. The top three are 'collisions' in the sense that two singular elements of the domain truss converge to the same singular element of the codomain truss. The bottom three are 'creations' in the sense that a new singular element appears in the codomain truss, with no singular element of the domain truss converging to it. The right two are also 'collapses' in the sense that the domain truss degenerates into the single singular element of the codomain truss.

The topological counterparts of each of these behaviors (which also inform the choice of terminology for these cases), in the context of stratified bundles of stratified intervals, are illustrated later in Figure 4.7. ⌟

EXAMPLE 2.1.46 (Functorial relations that are only partially bifunctional). In Figure 2.13 we illustrate three functorial relations between trusses that are not truss bordisms because they fail to be bifunctional. However, these failures are rather mild and fixable. Mild in the sense that the issue is that the singular function or regular function is only partially defined. Fixable in the sense that, by extending either the source or target truss, the relation can be completed to a truss bordism; in other words, the relation is a subrelation of an actual truss bordism.

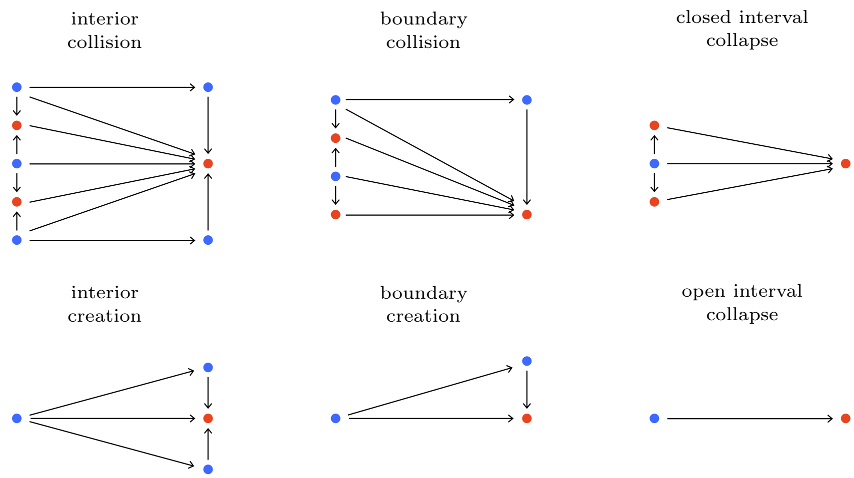


FIGURE 2.12. Local forms of 1-truss bordisms.

In the first case, the regular upper endpoint of the special fiber is not related to any regular element of the generic fiber; this is an 'upward discontinuity' of the boundary of the truss. In the second case, the singular upper endpoint of the generic fiber is not related to any singular element of the special fiber, and moreover the special fiber has a singular upper endpoint; this is a 'downward discontinuity' of the boundary of the truss. In the third case, the singular upper endpoint of the generic fiber is not related to any singular element of the special fiber, but the upper endpoint of the special fiber is regular; this is a 'boundary disappearance' of the singular endpoint of the truss.

The topological counterparts of each of these relations, in the context of stratified bundles of stratified intervals, are illustrated later in Figure 4.8. ⌟

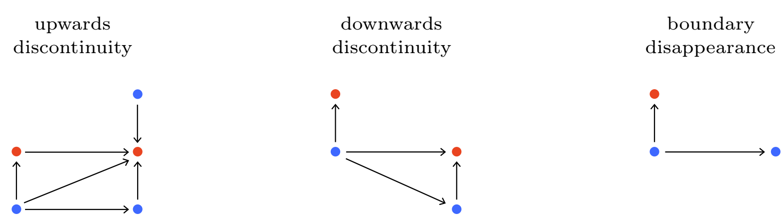


FIGURE 2.13. Fixable failures of truss bifunctionality.

EXAMPLE 2.1.47 (Functorial relations that are not bifunctional). In Figure 2.14 we illustrate three more functorial relations between trusses that are not truss bordisms, again because they fail to be bifunctional. Unlike the cases in the previous example, these failures should be considered unrecoverable, most immediately because a singular element of the generic fiber is related to a regular element of the special fiber, violating the fundamental nature of truss bordisms.

In the first case, the interior singular element of the generic fiber is not related to any singular element of the special fiber, violating functionality, and worse is related to a regular element of the special fiber; furthermore, the regular element of the special fiber is related to two regular elements of the generic fiber, violating cofunctionality; this is an 'interior evaporation' of the singular element of the truss. In the second case, now a boundary singular element of the generic fiber is not related to any singular element (violating functionality) and indeed is related to a regular element of the special fiber; this is a 'boundary evaporation' of the singular endpoint of the truss. In the third case, the singular element of the generic fiber is related to two singular elements of the special fiber, violating functionality, and the regular element of the special fiber is not related to any regular element of the generic fiber, violating cofunctionality; this is a 'point divergence' of the singular element of the truss.

The topological counterparts of these relations, in the context of stratified bundles of stratified intervals, are illustrated later—the first and second relations here correspond to the first and second images in Figure 4.9, and the third relation here corresponds to the first image in Figure 4.10. ⌟

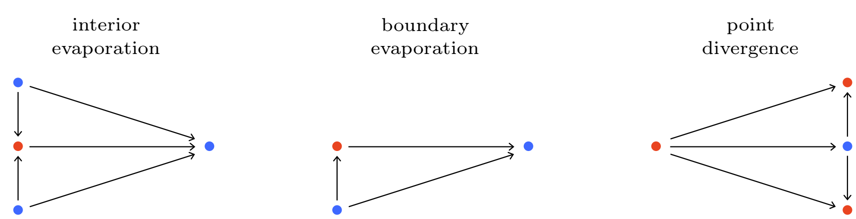


Figure 2.14. Irredeemable infractions of truss bifunctionality.

Example 2.1.48 (Bifunctional functorial relations that are not bimonotone). In Figure 2.15 we illustrate two functorial relations between trusses that are bifunctional but are still not truss bordisms because they are not bimonotone. In the first case, the relation between the singular element of the generic fiber and the regular element of the special fiber causes a violation of bimonotonicity, despite not explicitly contravening bifunctionality; we refer to this situation as a 'boundary dislocation'. (Note that by removing the relation between that singular element and that regular element, one obtains a subrelation that is in fact a truss bordism.) In the second case, the relation between the singular elements and the relation between the regular elements already violates bimonotonicity; we refer to this situation as a 'boundary divergence'. (Note that this case is especially pathological as it contains no subrelation whatsoever that is a truss bordism.)

The topological counterparts of these relations (which especially in this case clarify the terminology), in the context of stratified bundles of stratified intervals, are again illustrated later—the first relation here corresponds to the third image in Figure 4.9, and the second relation here corresponds to the second image in Figure 4.10. ⌟

FIGURE 2.15. Brutal breakdowns of truss bimonotonicity.

2.1.2.3. ◊Composition and dualization of 1-truss bordisms. 1-Trusses provide a combinatorial model of stratified intervals, and 1-truss bordisms provide a combinatorial model of suitable stratified bundles of stratified intervals, over the stratified 1-simplex. We may imagine stacking such stratified bundles end to end in an attempt to compose them, but unlike ordinary intervals, the union of two stratified intervals is not itself a stratified interval. In this case, in fact the combinatorial viewpoint provides a more evident composition than the geometric viewpoint. We need only observe that functorial relations compose and that the defining properties of 1-truss bordisms are preserved under this composition.

OBSERVATION 2.1.49 (Functorial relations compose). Given preorders X, Y, and Z, and functorial relations $R : X \nrightarrow Y$ and $S : Y \nrightarrow Z$, the composite relation $S \circ R : X \nrightarrow Z$ is given by having a relation $(S \circ R)(x \in X, z \in Z)$ if and only if there is an element $y \in Y$ for which there are both relations $R(x, y)$ and $S(y, z)$. Note that the functoriality of R and S ensures that the composite relation $S \circ R$ is also functorial. ⌟

OBSERVATION 2.1.50 (Bifunctionality and bimonotonicity compose). The properties of bifunctionality and bimonotonicity in Definition 2.1.33 are preserved under composition of functorial relations. Indeed, given 1-truss bordisms, $R : T \nrightarrow T'$ and $R' : T' \nrightarrow T''$, the functorial relation $R' \circ R : T \nrightarrow T''$ is again bifunctional: its singular function is the composite $\mathsf{sing}_{R'} \circ \mathsf{sing}_R$, and its regular function is the composite $\mathsf{reg}^R \circ \mathsf{reg}^{R'}$. The relation $R' \circ R : T \nrightarrow T''$ is also bimonotone: if there is a transposition of the composite $R' \circ R$ (with respect to the frame orders of T and T'') then there would necessarily be a transposition in at least one of the relations R or R'. ⌟

Thus altogether, the composite of two 1-truss bordism functorial relations is itself a 1-truss bordism, as desired.

Definition 2.1.51 (Composition of 1-truss bordisms). Given two 1-truss bordisms $R : T \nrightarrow T'$ and $R' : T' \nrightarrow T''$, the **composite 1-truss bordism** $R' \circ R : T \nrightarrow T''$ is the composite of R and R' as functorial relations. ⌟

Example 2.1.52 (Composition of 1-truss bordisms). In Figure 2.16 we illustrate a composition of two 1-truss bordisms. The bimonotonicity and bifunctionality of the composite relation are evident. ⌟

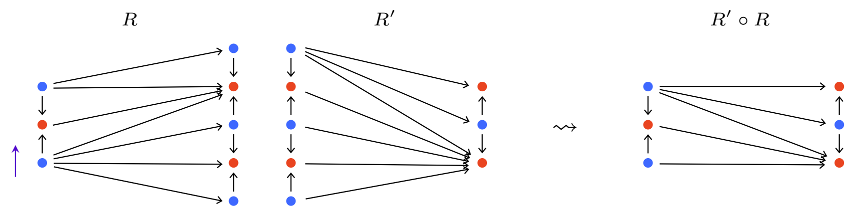


Figure 2.16. Composition of 1-truss bordisms.

There is an identity $\mathrm{id}_T : T \nrightarrow T$ for this composition of truss bordisms, namely the relation given by the hom functor $\mathrm{Hom}_{(T,\trianglelefteq)}(-,-)$. Thus 1-truss bordisms are morphisms of the following category.

Notation 2.1.53 (The category of 1-trusses and their bordisms). The 'category of 1-trusses and their bordisms', whose objects are 1-trusses and whose morphisms are 1-truss bordisms, will be denoted TBord^1. The full subcategory containing only open, respectively closed, 1-trusses will be denoted $\mathring{\mathsf{T}}\mathsf{Bord}^1$, respectively $\bar{\mathsf{T}}\mathsf{Bord}^1$. ⌟

Observation 2.1.54 (The terminal and initial 1-trusses). The terminal object of TBord^1 is the trivial closed 1-truss $\bar{\mathbb{T}}_0$. The unique bordism $R : T \nrightarrow \bar{\mathbb{T}}_0$ has a relation between every element of T and the unique (singular) element of $\bar{\mathbb{T}}_0$.

The initial object of TBord^1 is the trivial open 1-truss $\mathring{\mathbb{T}}_0$. The unique bordism $R : \mathring{\mathbb{T}}_0 \nrightarrow T$ has a relation between the unique (regular) element of $\mathring{\mathbb{T}}_0$ and every element of T. ⌟

Observation 2.1.55 (Isobordisms of 1-trusses are unique). We call a 1-truss bordism with an inverse a '1-truss isobordism'. Given two 1-trusses T and S, if there is an isobordism $R : T \nrightarrow S$, then there is a unique such isobordism. There is therefore never any need to distinguish between distinct 1-trusses that are isomorphic in the category TBord^1. Also, in particular there are no nontrivial automorphisms in TBord^1. ⌟

Remark 2.1.56 (Isobordism classes of 1-trusses). Note that the isomorphism classes of 1-trusses in TBord^1, that is the classes of 1-trusses up to invertible bordism, are the same as the balanced isomorphism classes of 1-trusses, namely $\mathring{\mathbb{T}}_k$, $\bar{\mathbb{T}}_k$, $\overset{\circ}{\overline{\mathbb{T}}}_k$, and $\overset{\circ}{\overrightarrow{\mathbb{T}}}_k$; see Terminology 2.1.23. ⌟

Recall the dual $T^\dagger$ of a 1-truss T has the same elements and frame order, but the opposite face order and dimension. As noted in Construction 2.1.26, this dual extends to a covariant involutive functor $\dagger : \mathsf{Trs}_1 \cong \mathsf{Trs}_1$ on the category of 1-trusses and their maps. The same dual on 1-trusses also extends to a *contravariant* involutive functor on the category of 1-trusses and their bordisms, as follows.

Construction 2.1.57 (Dualization of 1-truss bordisms). Given a 1-truss bordism $R : T \nrightarrow S$, the dual 1-truss bordism $R^\dagger : S^\dagger \nrightarrow T^\dagger$ is the transpose relation:

$$R^\dagger(s,t) = R(t,s).$$

This transposed relation is functorial (since the face orders of the trusses have been reversed), bifunctional (since the singular and regular elements have been switched and so the roles of functionality and cofunctionality interchanged), and bimonotone (since the transpose introduces no transpositions of the frame order). Thus dualization provides an involutive isomorphism of categories

$$\dagger : \mathsf{TBord}^1 \cong (\mathsf{TBord}^1)^{\mathrm{op}}$$

This dualization restricts to an isomorphism $\dagger : \mathring{\mathsf{T}}\mathsf{Bord}^1 \cong (\bar{\mathsf{T}}\mathsf{Bord}^1)^{\mathrm{op}}$ between the category of open 1-trusses and their bordisms and the (opposite of the) category of closed 1-trusses and their bordisms. ⌟

Example 2.1.58 (Dual 1-truss bordisms). In Figure 2.17 we depict a 1-truss bordism (the first one in the cospan of Figure 2.11) together with its dual 1-truss bordism. Notice how the transposed relation appears as a horizontal flip in this illustration, and the flipped singular function becomes the regular function, while the flipped regular function becomes the singular function. ⌟

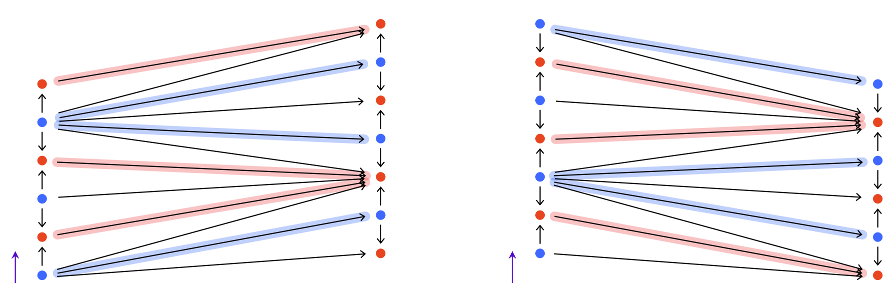


Figure 2.17. Dual 1-truss bordisms.

2.1.2.4. ◊Determination of 1-truss bordisms. Functoriality, bifunctionality, and bimonotonicity collude to rigidly constrain the structure of 1-truss bordism relations. So much so that, subject to certain boundary conditions, these relations are completely determined either by their singular function or by their regular function. We collate a few features of 1-truss bordisms, leading up to this determination property.

Observation 2.1.59 (1-truss bordism relations weakly decrease dimension). Given a 1-truss bordism $R : T \nrightarrow S$ with a relation $R(a, b)$ between elements $a \in T$ and $b \in S$, then $\dim(a) \geq \dim(b)$. In all preceding examples of 1-truss bordisms, it is visibly the case that relations are from singular to singular, regular to regular, or regular to singular elements, but never from singular to regular elements.

To prove that this is always the case, suppose by contrast that there were a relation $R(a, b)$ for singular $a \in T$ and regular $b \in S$. Consider the singular element $\mathsf{sing}_R(a)$; since b is regular and frame orders are total, either $b \prec \mathsf{sing}_R(a)$ or $\mathsf{sing}_R(a) \prec b$. Assume the former; the latter case is similar. By bimonotonicity, we have $\mathsf{reg}^R(b) \prec a$; thus there is at least one element below a in the frame order, and so there is a face order arrow $a - 1 \lhd a$ in $(T, \unlhd)$. (Here $a - 1$ denotes the predecessor of a in the total frame order $(T, \preceq)$.) By the functoriality of the relation R, this implies that $R(a - 1, \mathsf{sing}_R(a))$ holds. Since $a - 1 \prec a$, and $b \prec \mathsf{sing}_R(a)$, and we assumed $R(a, b)$, this contradicts bimonotonicity. ⌟

Recall from Terminology 2.1.22 that the minimal and maximal elements of the frame order of a truss are called the lower and upper endpoints.

Observation 2.1.60 (1-truss bordisms relate endpoints). Given a 1-truss bordism $R : T \nrightarrow S$, the lower endpoint of T is related to the lower endpoint of S, and the upper endpoint of T is related to the upper endpoint of S. In all the examples we have seen, this property is visible from the relation arrows between the bottommost and topmost elements.

To prove that this always holds, consider the lower endpoint case, as follows; the upper endpoint case is similar. Suppose the target lower endpoint $\mathsf{end}_- S$ is regular. Set $a := \mathsf{reg}^R(\mathsf{end}_- S)$. If $a = \mathsf{end}_- T$, then the endpoint relation is satisfied. Otherwise there is a predecessor of a, necessarily singular, and a relation $R(a - 1, \mathsf{sing}_R(a - 1))$; bimonotonicity, and the regularity of $\mathsf{end}_- S$, forces $\mathsf{sing}_R(a - 1) \prec \mathsf{end}_- S$, contradicting $\mathsf{end}_- S$ being a lower endpoint. If instead we suppose the source lower endpoint $\mathsf{end}_- T$ is singular, a dual argument shows that $\mathsf{sing}_R(\mathsf{end}_- T) = \mathsf{end}_- S$, ensuring the endpoint relation is satisfied.

The only remaining case is when $\mathsf{end}_- T$ is regular and $\mathsf{end}_- S$ is singular. If both trusses have just one element, they are related, and the lower endpoints preserved, since the relation is non-empty by definition. Suppose T has at least two elements; the case when S has at least two elements is similar. Since T has some element above its lower endpoint, there is a singular successor $(\mathsf{end}_- T) + 1$ and a face order $\mathsf{end}_- T \lhd (\mathsf{end}_- T) + 1$. If there is a relation

$R((\mathsf{end}_- T) + 1, \mathsf{end}_- S)$, then by functoriality $R(\mathsf{end}_- T, \mathsf{end}_- S)$, as desired. Otherwise $R((\mathsf{end}_- T) + 1, \mathsf{sing}_R((\mathsf{end}_- T) + 1))$ with $\mathsf{sing}_R((\mathsf{end}_- T) + 1) \succ \mathsf{end}_- S$. That forces there to be a regular successor $(\mathsf{end}_- S) + 1$ and a face order $(\mathsf{end}_- S) + 1 \lhd \mathsf{end}_- S$. By cofunctionality and bimonotonicity, there is a relation $R(\mathsf{end}_- T, (\mathsf{end}_- S) + 1)$ and so by functoriality a relation $R(\mathsf{end}_- T, \mathsf{end}_- S)$, as required. ⌟

The previous observation ensures that bordisms always relate endpoints. The singularity or regularity of these endpoints are interdependent, as follows.

Observation 2.1.61 (1-truss bordisms preserve singular endpoints and copreserve regular endpoints). Let $R : T \nrightarrow S$ be a 1-truss bordism. If the lower (resp. upper) endpoint of T is singular, then the lower (resp. upper) endpoint of S is singular. If the lower (resp. upper) endpoint of S is regular, then the lower (resp. upper) endpoint of T is regular.

For the lower case (the upper case is similar), it suffices of course to confirm that it cannot happen that $\mathsf{end}_- T$ is singular while $\mathsf{end}_- S$ is regular; assume by contrast that this were the situation. If either T or S has only one element, the relation necessarily violates either singular functionality or regular cofunctionality. Thus there are face orders $(\mathsf{end}_- T) + 1 \lhd \mathsf{end}_- T$ and $\mathsf{end}_- S \lhd (\mathsf{end}_- S) + 1$. By Observation 2.1.60, there is an endpoint relation $R(\mathsf{end}_- T, \mathsf{end}_- S)$. Functoriality implies $R((\mathsf{end}_- T) + 1, \mathsf{end}_- S)$ and $R(\mathsf{end}_- T, (\mathsf{end}_- S) + 1)$ and $R((\mathsf{end}_- T) + 1, (\mathsf{end}_- S) + 1)$, violating bimonotonicity and reproducing the (vertical flip of the) boundary divergence of Figure 2.15. ⌟

Combining the previous two observations, note that in particular, for a 1-truss bordism $R : T \nrightarrow S$, the singular function $\mathsf{sing}_R : (\mathsf{sing}(T), \preceq) \to (\mathsf{sing}(S), \preceq)$ preserves singular endpoints, and the regular function $\mathsf{reg}^R : (\mathsf{reg}(S), \preceq) \to (\mathsf{reg}(T), \preceq)$ preserves regular endpoints, in the following sense.

Terminology 2.1.62 (Preserving singular or regular endpoints). Given 1-trusses T and S, a function $f : (\mathsf{sing}(T), \preceq) \to (\mathsf{sing}(S), \preceq)$ on the (frame ordered) singular elements is said to 'preserve singular endpoints' if any singular lower (resp. upper) endpoint of T is sent by the function to a singular lower (resp. upper) endpoint of S; i.e., if $\mathsf{end}_-(T) \in \mathsf{sing}(T)$ then $f(\mathsf{end}_-(T)) = \mathsf{end}_-(S) \in \mathsf{sing}(S)$, and similarly with end_+ in place of end_-.

Correspondingly, a function $g : (\mathsf{reg}(S), \preceq) \to (\mathsf{reg}(T), \preceq)$ is said to 'preserve regular endpoints' if any regular lower (resp. upper) endpoint of S is sent by the function to a regular lower (resp. upper) endpoint of T; i.e., if $\mathsf{end}_-(S) \in \mathsf{reg}(S)$ then $g(\mathsf{end}_-(S)) = \mathsf{end}_-(T) \in \mathsf{reg}(T)$, and similarly with end_+ in place of end_-. ⌟

In fact, finally, we can see that a function on the (frame ordered) singular elements determines a truss bordism, provided just that it is singular-endpoint preserving, and similarly a function on the (frame ordered) regular elements determines a truss bordism, provided just that it is regular-endpoint preserving, as follows.

Lemma 2.1.63 (Bordisms determined by singular or regular functions). *Let T and S be 1-trusses.*

> Singular determined: *Given a function $f : (\mathsf{sing}(T), \preceq) \to (\mathsf{sing}(S), \preceq)$ that preserves singular endpoints, there is a unique 1-truss bordism $R : T \nrightarrow S$ with singular function $\mathsf{sing}_R = f$.*
> Regular determined: *Given a function $g : (\mathsf{reg}(S), \preceq) \to (\mathsf{reg}(T), \preceq)$ that preserves regular endpoints, there is a unique 1-truss bordism $R : T \nrightarrow S$ with regular function $\mathsf{reg}^R = g$.*

Proof. For the singular determined case, define the relation $R(a, b)$ to hold if and only if either (1) the element a is singular, and $b = f(a)$, or (2) the element a is regular, and both $f(a+1) \succeq b$ (whenever $a + 1 \in T$) and $b \succeq f(a-1)$ (whenever $a - 1 \in T$).

For the regular determined case, define the relation $R(a, b)$ to hold if and only if either (1) the element b is regular, and $a = g(b)$, or (2) the element b is singular, and both $g(b+1) \succeq a$ (whenever $b + 1 \in S$) and $a \succeq g(b-1)$ (whenever $b - 1 \in S$). □

Since 1-truss bordism singular functions preserve singular endpoints, and 1-truss bordism regular functions preserve regular endpoints, all 1-truss bordisms are determined as in this lemma, and thus constructed as in the proof. That construction has the following consequences regarding the structure of 1-truss bordisms.

Terminology 2.1.64 (Fully relating elements). We say a relation $R : T \nrightarrow S$ 'fully relates elements' if for each $a \in T$ there exists $a' \in S$ with $R(a, a')$, and for each $b \in S$ there exists $b' \in T$ with $R(b', b)$. ⌟

Observation 2.1.65 (1-Truss bordisms fully relate elements). Every 1-truss bordism fully relates elements. ⌟

Corollary 2.1.66 (Correspondence of singular functionality and regular cofunctionality). *Let T and S be 1-trusses, and let $R : (T, \trianglelefteq) \nrightarrow (S, \trianglelefteq)$ be a functorial relation, that fully relates elements, and such that the relation $R \subset (T, \preceq) \times (S, \preceq)$ is bimonotone. The relation is functional on singular elements and preserves singular endpoints, if and only if it is cofunctional on regular elements and preserves regular endpoints. In this case, it is a 1-truss bordism.*

Proof. If the relation is functional on singular elements and preserves singular endpoints, its singular function determines a 1-truss bordism by the first case of Lemma 2.1.63. Observe that the given relation agrees with the 1-truss bordism constructed in the proof of that result, using crucially the assumptions of functoriality, fully relating elements, and bimonotonicity. The other direction is entirely similar. □

2.1.2.5. ◊Mapping cylinder 1-truss bordisms. As (stratified) manifolds are the objects of two rather different looking categories, namely manifolds with maps between them, and manifolds with bordisms between them, similarly we now have 1-trusses as the objects of these two distinct categories, namely 1-trusses and their maps Trs_1, and 1-trusses and their bordisms TBord^1. However, these categories are not entirely unrelated; the mapping cylinder of a suitable map of manifolds is a (singular, stratified, half) bordism, and similarly there is a mapping cylinder construction taking certain 1-truss maps to 1-truss bordisms.

The relevant 1-truss maps are those that respect the singularity or regularity of the boundary in the following sense.

Notation 2.1.67 (Categories of endpoint-preserving truss maps). A singular 1-truss map $F : T \to S$ is said to preserve singular endpoints if the restriction of the map to singular elements $F : \mathsf{sing}(T) \to \mathsf{sing}(S)$ preserves singular endpoints (see Terminology 2.1.62). Let $\mathsf{Trs}_1^{\mathsf{s},\partial}$ denote the category of 1-trusses and their singular maps that preserve singular endpoints.

Similarly, a regular 1-truss map $G : S \to T$ is said to preserve regular endpoints if the restriction of the map to regular elements $G : \mathsf{reg}(S) \to \mathsf{reg}(T)$ preserves regular endpoints. Let $\mathsf{Trs}_1^{\mathsf{r},\partial}$ denote the category of 1-trusses and their regular maps that preserve regular endpoints. ⌟

Construction 2.1.68 (Mapping cylinders of singular and regular 1-truss maps). Given a singular map of 1-trusses $F : T \to S$, that preserves singular endpoints, Lemma 2.1.63 defines a (uniquely determined) 1-truss bordism $T \nrightarrow S$ with singular function $F : \mathsf{sing}(T) \to \mathsf{sing}(S)$. We denote that bordism $\mathrm{Cyl}(F) : T \nrightarrow S$ and refer to it informally as the 'mapping cylinder' of the 1-truss map F. This construction assembles into a functor

$$\mathrm{Cyl} : \mathsf{Trs}_1^{\mathsf{s},\partial} \to \mathsf{TBord}^1$$

Similarly, given a regular map of 1-trusses $T \leftarrow S : G$, that preserves regular endpoints, Lemma 2.1.63 defines a (uniquely determined) 1-truss bordism $T \nrightarrow S$ with regular function $\mathsf{reg}(T) \leftarrow \mathsf{reg}(S) : G$. We denote that bordism $\mathrm{coCyl}(G) : T \nrightarrow S$ and refer to it informally as the 'mapping cocylinder' of the 1-truss map G. This construction assembles into a functor

$$\mathrm{coCyl} : (\mathsf{Trs}_1^{\mathsf{r},\partial})^{\mathrm{op}} \to \mathsf{TBord}^1$$ ⌟

Example 2.1.69 (Mapping cylinders of 1-truss maps). In Figure 2.18 we illustrate 1-truss maps and their mapping (co)cylinders. The top left corner is a singular map, preserving singular endpoints. The top right corner is the mapping cylinder 1-truss bordism associated to that singular map. The lower left corner is the dual regular map (of the singular map), and it preserves regular endpoints. The lower right corner is both the mapping cocylinder 1-truss bordism associated to that regular map, and the dual 1-truss bordism of the top right bordism. ⌟

Observation 2.1.70 (Mapping cylinders commute with dualization). Given a singular map $F : T \to S$ of 1-trusses that preserves singular endpoints,

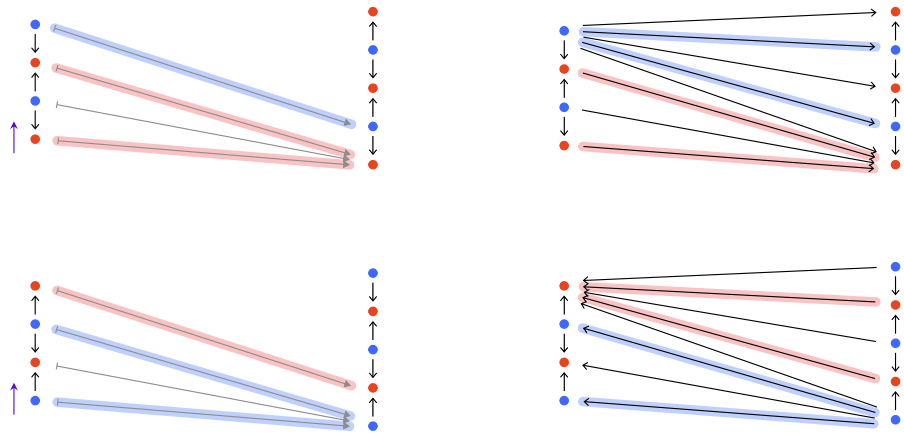


FIGURE 2.18. 1-truss maps and their mapping (co)cylinder 1-truss bordisms.

we may take its mapping cylinder 1-truss bordism $\mathrm{Cyl}(F) : T \nrightarrow S$, and then form the dual 1-truss bordism $\mathrm{Cyl}(F)^\dagger : T^\dagger \nleftarrow S^\dagger$. Or we may take the dual regular map $F^\dagger : T^\dagger \to S^\dagger$ of 1-trusses, which preserves regular endpoints, and then form the mapping cocylinder 1-truss bordism $\mathrm{coCyl}(F^\dagger) : T^\dagger \nleftarrow S^\dagger$. As illustrated in the previous example, the resulting bordisms are identical. That is, mapping cylinders respect dualization in the sense that the following diagram commutes:

$$\begin{array}{ccc} \mathsf{Trs}_1^{\mathsf{s},\partial} & \xrightarrow{\mathrm{Cyl}} & \mathsf{TBord}^1 \\ \downarrow{\scriptstyle \dagger} & & \downarrow{\scriptstyle \dagger} \\ \mathsf{Trs}_1^{\mathsf{r},\partial} & \xrightarrow[\mathrm{coCyl}]{} & (\mathsf{TBord}^1)^{\mathrm{op}} \end{array}$$

REMARK 2.1.71 (Mapping cylinders as represented functorial relations). When a singular or regular map of 1-trusses is more-or-less surjective, its mapping (co)cylinder is the relation *(co)represented* by the map, in the following sense (cf. Terminology 2.1.42).

Let $F : T \to S$ be a singular map of 1-trusses, such that the face order functor $F : (T, \trianglelefteq) \to (S, \trianglelefteq)$ is initial, i.e., for every element $s \in S$ there is an element $t \in T$ and a face order relation $F(t) \trianglelefteq s$; note that initiality implies this map preserves singular endpoints. In this case (only), the 1-truss bordism $\mathrm{Cyl}(F)$, defined by Lemma 2.1.63, is precisely the functorial relation $\mathrm{Hom}_{(S,\trianglelefteq)}(F-, -) : (T, \trianglelefteq) \nrightarrow (S, \trianglelefteq)$ represented by the 1-truss map F.

Similarly, let $T \leftarrow S : G$ be a regular map of 1-trusses, such that the face order functor $(T, \trianglelefteq) \leftarrow (S, \trianglelefteq) : G$ is final, i.e., for every element $t \in T$ there is an element $s \in S$ and a face order relation $t \trianglelefteq G(s)$; note that finality implies this map preserves regular endpoints. In this case (only), the 1-truss bordism $\mathrm{coCyl}(G)$, defined by Lemma 2.1.63, is precisely the functorial relation $\mathrm{Hom}_{(T,\trianglelefteq)}(-, G-) : (T, \trianglelefteq) \nrightarrow (S, \trianglelefteq)$ corepresented by the 1-truss map G. ⌟

2.1.3. ◊1-Truss bundles. In the previous section, we developed the notion of 1-truss bordisms, providing a combinatorial model of constructible bundles of stratified intervals over the stratified 1-simplex. Now we describe the notion of 1-truss bundle, which will provide a combinatorial model of constructible bundles of stratified intervals over not just the 1-simplex but over more general stratified spaces.

Synopsis. We begin by defining 1-truss bundles as diposets each of whose point fibers is a 1-truss and each of whose arrow fibers is a 1-truss bordism. We introduce maps of 1-truss bundles, which are just base poset maps together with total diposet maps, and so in particular are maps of 1-trusses on each fiber. We show that the category of 1-trusses and their bordisms is a classifying category for 1-truss bundles; to any 1-truss bundle there is an associated classifying functor into the classifying category, and to any functor into the classifying category there is an associated total 1-truss bundle. Finally, we mention pullbacks of 1-truss bundles, observe that the dualization functors on 1-trusses extend to 1-truss bundles, and describe suspensions of 1-truss bundles.

2.1.3.1. ◊1-Truss bundles as collections of 1-truss bordisms. A 1-truss bundle over a poset will be a compatible collection of 1-truss bordisms, one over each arrow of the base poset. To describe the compatibility, it is convenient to encode the total object of the bundle itself as a poset, and to do that we need to recast the total object of a bordism as a poset, rather than as a relation.

Terminology 2.1.72 (The associated total poset of a 1-truss bordism). By definition, a 1-truss bordism between 1-trusses T and S is a (bifunctional, bimonotone) functorial relation $R : (T, \trianglelefteq) \nrightarrow (S, \trianglelefteq)$ of the face order posets. The 'associated total poset' of the 1-truss bordism R is the partial order $(T \sqcup S, \trianglelefteq)$, whose underlying set is the disjoint union of the 1-truss elements, and whose order relation $\trianglelefteq$ has restriction to T being the face order $(T, \trianglelefteq)$, has restriction to S being the face order $(S, \trianglelefteq)$, and satisfies $(t \in T) \trianglelefteq (s \in S)$ if and only if the relation $R(t, s)$ holds. ⌟

Of course, only very special partial orders on the union of the source and target 1-trusses arise as the associated total posets of 1-truss bordisms; note that, for fixed domain and codomain 1-trusses, a partial order so arising completely determines the 1-truss bordism of which it is the associated total poset. Thus, we may and will lightly abuse terminology as follows.

Notation 2.1.73 (Total poset of a bordism). For a 1-truss bordism R (given by definition as a functorial relation), we refer without decoration to its associated total poset also simply as R. ⌟

Notice that already in our first illustration of a 1-truss bordism in Figure 2.9, we denoted each relation $R(t,s)$ by an arrow $t \to s$; as such, the collection of all arrows drawn (including those in the domain and codomain) is precisely the associated total poset of the 1-truss bordism.

We may now define 1-truss bundles, as suitably structured poset bundles, that restrict to 1-trusses over elements and to 1-truss bordisms over arrows.

Definition 2.1.74 (1-Truss bundle). Let $(B, \to)$ be a poset, and consider it to be a diposet $(B, \to, =)$ using the discrete order $=$. A **1-truss bundle** $(T, \trianglelefteq, \dim, \preceq, p)$ over $(B, \to)$ is a diposet $(T, \trianglelefteq, \preceq)$, together with a poset map $\dim : (T, \trianglelefteq) \to [1]^{\mathrm{op}}$, and a diposet map $p : (T, \trianglelefteq, \preceq) \to (B, \to, =)$, satisfying the following two conditions.

(1) *Truss point fibers*: For every element $x \in B$, the fiber $(p^{-1}(x), \trianglelefteq, \dim, \preceq) \subset (T, \trianglelefteq, \dim, \preceq)$ is a 1-truss.
(2) *Truss bordism arrow fibers*: For every arrow $x \to y$ in the base poset $(B, \to)$, the fiber $(p^{-1}(x \to y), \trianglelefteq) \subset (T, \trianglelefteq)$ is the total poset of a 1-truss bordism.

We call $(B, \to)$ the 'base poset', call $(T, \trianglelefteq, \preceq)$ the 'total diposet' and $(T, \trianglelefteq)$ the 'total poset', and as for 1-trusses, refer to $\trianglelefteq$ as the 'face order', to $\preceq$ as the 'frame order', and to dim as the 'dimension map'. ⌟

Notation 2.1.75 (1-Truss bundles). When referring to 1-truss bundles, we will usually keep the face orders, frame orders, and dimension maps, as well as the base poset order, implicit; we thus denote 1-truss bundles simply by maps $p : T \to B$. When then referring to the structures of such a 1-truss bundle, we will use the symbol '$\trianglelefteq$' for the face order, '$\preceq$' for the frame order, 'dim' for the dimension map, and '$\to$' for the base poset order. We will also freely use an arrow '$\to$', instead of $\trianglelefteq$, to indicate a face order relation, as this corresponds to our graphical illustration convention and is also nicely compatible with the order $\to$ of the base poset. ⌟

Note that in a 1-truss bundle $p : T \to B$, two elements $a, b \in T$ are related in the frame order if and only if they are in the same fiber $a, b \in p^{-1}(x)$: no elements of distinct fibers can be frame-order related since $p : (T, \preceq) \to (B, =)$ is a poset map, and frame orders are total on each fiber 1-truss.

Terminology 2.1.76 (Singular and regular elements of 1-truss bundles). Given a 1-truss bundle $p : T \to B$, we call an element $a \in T$ 'singular' if $\dim(a) = 0$ and 'regular' if $\dim(a) = 1$. We denote by $\mathsf{sing}(T)$, respectively $\mathsf{reg}(T)$, the full subposet of $(T, \trianglelefteq)$ containing all singular, respectively regular, elements. (We also freely think of $\mathsf{sing}(T)$ and $\mathsf{reg}(T)$ as the full subdiposets of $(T, \trianglelefteq, \preceq)$ on the same elements.) ⌟

Terminology 2.1.77 (Open and closed 1-truss bundles). A 1-truss bundle for which all fibers are open, respectively closed, 1-trusses, will be called an 'open', respectively 'closed', 1-truss bundle. ⌟

Example 2.1.78 (1-Truss bundle). In Figure 2.19, on the left we illustrate a 1-truss bundle $p : T \to B$. As before, singular elements are shown as red dots, and regular elements as blue dots. The face order of the total poset $(T, \trianglelefteq)$ is indicated by arrows, as is the poset order of the base. The total frame order of each fiber is indicated by a purple coordinate axis vector. (Note that we choose all such axes to point in the same, upwards direction. Flipping all these frame axes to point downward would produce a distinct 1-truss bundle.) ⌟

Terminology 2.1.79 (Generating arrows of 1-truss bundles). A 'generating arrow' of a 1-truss bundle is an arrow in the covering relation of the total poset of the bundle, i.e., a non-identity arrow that is not a composite of other non-identity arrows. ⌟

Example 2.1.80 (Generating arrows of a 1-truss bundle). As pictures of 1-truss bundles can quickly become difficult to parse, from so many arrows, we often illustrate them more sparsely by only drawing the generating arrows. On the right of Figure 2.19, we depict the same bundle as on the left, but omit all composite arrows. ⌟

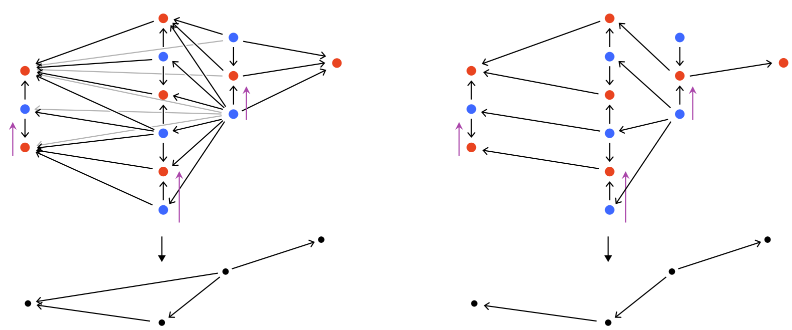


Figure 2.19. A 1-truss bundle and its generating arrows.

Construction 2.1.81 (Generating arrows in 1-truss bundles). Given a 1-truss bundle $p : T \to B$, its set of generating arrows is completely and concisely determined as follows. Let $\mathrm{cov}(B)$ denote the covering relation of the base poset. An arrow $a \to b$ of the total poset $(T, \trianglelefteq)$ is generating if and only if:

- either $p(a \to b) = \mathrm{id}$ (i.e., the arrow lies in a fiber of the projection p),
- or $p(a \to b) \in \mathrm{cov}(B)$ and either $(a \to b) \in \mathsf{sing}(T) \cup \mathsf{reg}(T)$ (i.e., the arrow has source and target singular or source and target regular) or $a \to b$ is the unique arrow of the fiber over the arrow $p(a \to b)$ of the base. ⌟

Remark 2.1.82 (Flip action on frame orders). As mentioned in Example 2.1.78, there is a $\mathbb{Z}_2$ action on the collection of 1-truss bundles by flipping the frame order of every fiber. As this flip tends to not alter the essential behavior in question, we usually depict frame orders of bundles only up to this action; specifically, we position the fibers of the bundle parallel, and assume the fiber frame orders run in the same direction, but typically do not fix which direction. ⌟

Observation 2.1.83 (Arrows lift in 1-truss bundles). Let $p : T \to B$ be a 1-truss bundle. Given an arrow $x \to y$ in the base B, and a lift of x to an element $a \in T$, the arrow lifts to some arrow $a \to a'$ of the total poset. Similarly, given a lift of y to an element $b \in T$, the arrow lifts to some arrow $b' \to b$. Both properties follow immediately from Observation 2.1.65. ⌟

Observation 2.1.84 (Unique singular or regular lifts in 1-truss bundles). The lifts in the previous observation become unique if we insist they are singular (in the first case) or regular (in the second case). Specifically, given a 1-truss bundle $p : T \to B$, an arrow $x \to y$ in the base, and a lift of x to a singular element $a \in T$, there is a unique lift to an arrow $a \to a'$ in $\mathsf{sing}(T)$. (In categorical terms, the functor $p : \mathsf{sing}(T) \to B$ is a 'discrete opfibration' [**LR20, MM12**].) Similarly, given a lift of y to a regular element $b \in T$, there is a unique lift to an arrow $b' \to b$ in $\mathsf{reg}(T)$. (The functor $p : \mathsf{reg}(T) \to B$ is a 'discrete fibration'.) These properties follow from the bifunctionality of 1-truss bordisms. ⌟

The definition of 1-truss bundles has a natural generalization allowing the base to be a category, not just a poset.

Remark 2.1.85 (Categorical 1-truss bundles). Our Definition 2.1.74 restricts attention to base *posets*, as that context will be our exclusive concern, and so gives a notion of *posetal* 1-truss bundle. However, the definition can be recast to accommodate base *categories*, yielding a notion of *categorical* 1-truss bundles. To wit, a 'categorical 1-truss bundle' is a functor $p : \mathsf{T} \to \mathsf{B}$ to a base category B, equipped with 1-truss structures on the fibers over objects, such that the fibers over morphisms are 1-truss bordisms. ⌟

Example 2.1.86 (Categorical 1-truss bundle). In Figure 2.20 we illustrate a 1-truss bundle over a category that is not a poset. Note that the 1-truss bordisms over the two parallel morphisms of the base are crucially distinct. A topological counterpart of this categorical 1-truss bundle is illustrated in Figure 4.6. ⌟

2.1.3.2. ◊Maps of 1-truss bundles. Recall that a map of 1-trusses is a diposet map, that is, a map of sets that respects both the face order and the frame order. A map of 1-truss bundles is simply a map of the total diposets, which in particular then is a map of 1-trusses on each fiber, as follows.

Definition 2.1.87 (Maps of 1-truss bundles). For 1-truss bundles $p : T \to B$ and $q : S \to C$, a **map of 1-truss bundles** $F : p \to q$ is a (base) poset map $G : (B, \to) \to (C, \to)$ and a (total) diposet map $F : (T, \trianglelefteq, \preceq) \to (S, \trianglelefteq, \preceq)$, commuting with the projections to the bases; that is, the following square

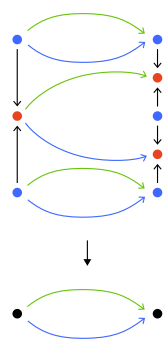


FIGURE 2.20. A 1-truss bundle over a category.

commutes:

$$\begin{array}{ccc} T & \xrightarrow{F} & S \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ B & \xrightarrow{G} & C \end{array}$$

When the base map is the identity, $G = \mathrm{id}_B$, we say that $F : p \to q$ is a map of 1-truss bundles 'over the base B' or that it is 'base preserving'. ⌟

Note that in a 1-truss bundle map, the base poset map $G : (B, \to) \to (C, \to)$ is uniquely determined by the total diposet map $F : (T, \trianglelefteq, \preceq) \to (S, \trianglelefteq, \preceq)$. Also note that a 1-truss bundle map $F : p \to q$ is a 1-truss map on each fiber; that is, for each $x \in B$, the restriction $F : p^{-1}(x) \to q^{-1}G(x)$ is a 1-truss map.

TERMINOLOGY 2.1.88 (Singular, regular, and balanced 1-truss bundle maps). Let $F : p \to q$ be a map of 1-truss bundles. If the total diposet map $F : (T, \trianglelefteq, \preceq) \to (S, \trianglelefteq, \preceq)$ sends the singular subposet $\mathsf{sing}(T)$ to $\mathsf{sing}(S)$, we call F a 'singular' bundle map; similarly if it maps $\mathsf{reg}(T)$ to $\mathsf{reg}(S)$, we call it a 'regular' bundle map; if F is both singular and regular, then we call it a 'balanced' bundle map. (Equivalently, a bundle map is singular or regular or balanced if it is so on every fiber.) ⌟

EXAMPLE 2.1.89 (1-Truss bundle maps). In Figure 2.21 we illustrate two 1-truss bundle maps. In the first case on the left, the base poset map is indicated by grey arrows; its source is the open truss with five elements and its target is the open truss with three elements. The total poset map is also indicated with grey arrows, of corresponding tonal densities. Note that this 1-truss bundle map is singular: the seven central singular elements of the source all collapse to the single central singular element of the target.

In the second case on the right, the base poset map is the identity of the open truss with five elements. The total poset map is again indicated with correspondingly grey arrows. Note that this 1-truss bundle map is regular: in the central slice all three regular elements of the source merge into the central regular element of the target, while in each of the adjacent slices, two regular elements of the source merge into a regular element of the target. ⌟

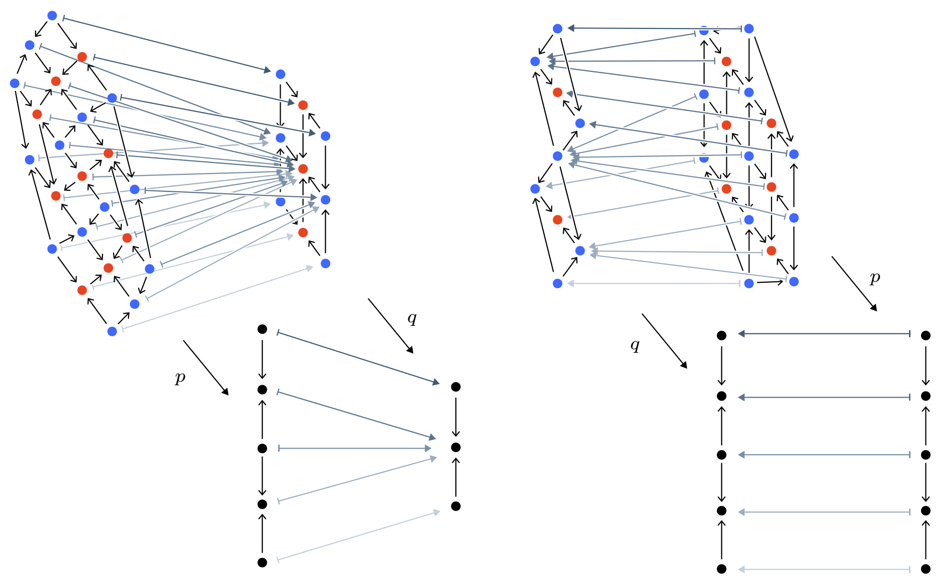


FIGURE 2.21. A singular and a regular 1-truss bundle map.

NOTATION 2.1.90 (Categories of 1-truss bundles). The category of 1-truss bundles and their maps is denoted TrsBun_1. The subcategory of bundles over a fixed base poset B and their base-preserving maps is denoted $\mathsf{Trs}_1(B)$. The subcategory of $\mathsf{Trs}_1(B)$ containing open truss bundles and their regular maps is denoted $\mathring{\mathsf{Trs}}_1(B)$; similarly the subcategory of $\mathsf{Trs}_1(B)$ containing closed truss bundles and their singular maps is denoted $\bar{\mathsf{Trs}}_1(B)$. ⌟

TERMINOLOGY 2.1.91 (Restriction of 1-truss bundles). Given a 1-truss bundle $p : T \to B$ and a subposet $A \hookrightarrow B$, the 'restriction' of the bundle to the subposet is the 1-truss bundle $p|_A : T|_A \to A$ with total set $T|_A := p^{-1}A$, and with the diposet structure, dimension map, and projection restricted accordingly. This process provides a restriction functor $-|_A : \mathsf{Trs}_1(B) \to \mathsf{Trs}_1(A)$.

2.1.3.3. ◊Classification and totalization for 1-truss bundles. Essentially by definition, 1-truss bundles over a point are 1-trusses, 1-truss bundles over the interval poset [1] are 1-truss bordisms, and 1-truss bundles over general base posets B are characterized by their behavior over points and intervals of the base. It follows, as we will describe in detail presently, that the category of 1-trusses and their bordisms TBord^1 is a *classifying category* for 1-truss bundles (see [**Bén06**]); that is, there is a correspondence between

1-truss bundles $p : T \to B$ over a base poset B and functors $\mathsf{F} : B \to \mathsf{TBord}^1$ from the base poset into the category of 1-trusses and their bordisms.

Construction 2.1.92 (Classifying functors of 1-truss bundles). We describe a map

$$(p : T \to B) \quad \mapsto \quad (\chi_p : B \to \mathsf{TBord}^1)$$

that takes a 1-truss bundle p to an associated **classifying functor** χ_p.

We construct χ_p on elements and arrows of the poset B, as follows. For each element $x \in B$, the classifying element $\chi_p(x) \in \mathsf{TBord}^1$ is the point fiber 1-truss $p^{-1}(x)$; for each arrow $x \to y$ in the base B, the classifying morphism $\chi_p(x \to y)$ of TBord^1 is the arrow fiber 1-truss bordism $p^{-1}(x \to y)$ (as given in Definition 2.1.74).

To see that the given χ_p is indeed a functor, one checks that the 1-truss bordism composite $p^{-1}(y \to z) \circ p^{-1}(x \to y)$ is equal to the 1-truss bordism $p^{-1}(x \to z)$. By the definition of composition of functorial relations, and because the total poset $(T, \trianglelefteq)$ is closed under composition of arrows, the bordism $p^{-1}(y \to z) \circ p^{-1}(x \to y)$ is a subrelation of the bordism $p^{-1}(x \to z)$; however, 1-truss subbordisms are necessarily identities. ⌟

Construction 2.1.93 (Total 1-truss bundles of classifying functors). We describe a map

$$(\mathsf{F} : B \to \mathsf{TBord}^1) \quad \mapsto \quad (\pi_{\mathsf{F}} : \mathrm{Tot}\mathsf{F} \to B)$$

that takes a functor $\mathsf{F} : B \to \mathsf{TBord}^1$ from a poset B to the category of 1-truss bordisms to an associated **total 1-truss bundle** $\pi_{\mathsf{F}} : \mathrm{Tot}\mathsf{F} \to B$.

We construct the bundle π_{F} as follows.

- The total poset $(\mathrm{Tot}\mathsf{F}, \trianglelefteq)$ has elements the pairs $(x \in B, a \in \mathsf{F}(x))$ of an element of the poset and an element of the associated 1-truss; the total poset has a morphism $(x, a) \trianglelefteq (y, b)$ exactly when the 1-truss bordism $\mathsf{F}(x \to y)$ has a relation between the element $a \in \mathsf{F}(x)$ and $b \in \mathsf{F}(y)$.
- The frame order $(\mathrm{Tot}\mathsf{F}, \preceq)$ has a relation $(x, a) \preceq (x, b)$ exactly when $a \preceq b$ in $\mathsf{F}(x)$.
- The diposet map $\pi_{\mathsf{F}} : (\mathrm{Tot}\mathsf{F}, \trianglelefteq, \preceq) \to (B, \to, =)$ is of course the projection sending (x, a) to x.
- The dimension map $\dim : (\mathrm{Tot}\mathsf{F}, \trianglelefteq) \to [1]^{\mathrm{op}}$ is given on each fiber by the dimension map of that 1-truss fiber; that this defines a poset map on $(\mathrm{Tot}\mathsf{F}, \trianglelefteq)$ follows from Observation 2.1.59 that 1-truss bordisms weakly decrease dimension. ⌟

Example 2.1.94 (Classification for a 1-truss bundle). In Figure 2.22 we illustrate on the left a 1-truss bundle $p : T \to B$ (over a 1-truss as it happens), along with on the right its associated classifying functor $\chi_p : B \to \mathsf{TBord}^1$. (The inverse association taking that functor $\mathsf{F} : B \to \mathsf{TBord}^1$ to its total bundle $\pi_{\mathsf{F}} : \mathrm{Tot}\mathsf{F} \to B$ is also indicated.) In the classifying category TBord^1, we only depict the image of this particular functor, namely the trusses $\mathring{\mathbb{T}}_1$

and $\bar{\mathbb{T}}_1$ and the morphisms between them. The functor χ_p is indicated by color matching the morphisms of the base poset B with their images. ⌟

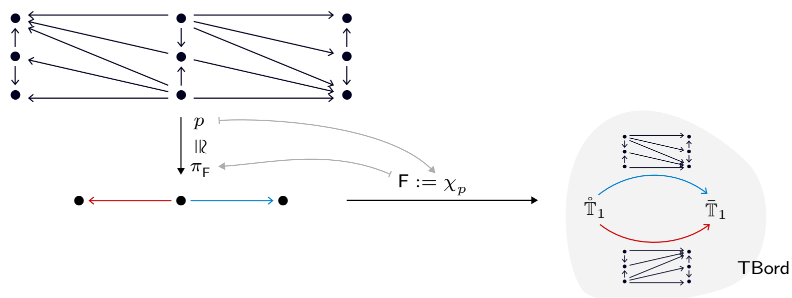


FIGURE 2.22. A 1-truss bundle and its classifying functor.

EXAMPLE 2.1.95 (The composition of 1-truss bordisms as a 1-truss bundle over the 2-simplex). Two composable 1-truss bordisms $R : T \to T'$ and $R' : T' \to T''$, together with their composite $R' \circ R : T \to T''$, define a functor $\mathsf{F} : [2] \to \mathsf{TBord}^1$ from the 2-simplex poset $[2]$ to the category of 1-trusses and their bordisms. By the previous construction, this functor has an associated total 1-truss bundle $\pi_{\mathsf{F}} : \mathrm{Tot}\mathsf{F} \to [2]$ over the 2-simplex.

In Figure 2.16 we illustrated two composable 1-truss bordisms along with their composite. In Figure 2.23 we illustrate the associated total 1-truss bundle over the 2-simplex. ⌟

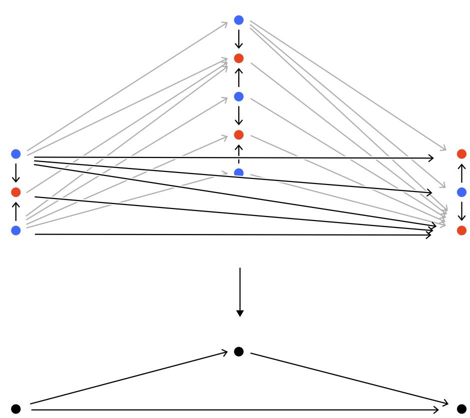


FIGURE 2.23. 1-Truss bordism composition as a bundle over the 2-simplex.

The above correspondence, between 1-truss bundles and functors into the category of 1-trusses and their bordisms, is functorial, with respect to a notion of bordism of 1-truss bundles, as follows.

Definition 2.1.96 (Bordisms of 1-truss bundles and their composition). Given 1-truss bundles $p : T \to B$ and $q : S \to B$ over a poset B, a **1-truss bundle bordism** $u : p \Rightarrow q$ is a 1-truss bundle $u : U \to B \times [1]$ such that $u|_{B\times\{0\}} = p$ and $u|_{B\times\{1\}} = q$.

The **composition** of two 1-truss bundle bordisms $u : p \Rightarrow q$ and $v : q \Rightarrow r$ is the bordism $v \circ u : p \Rightarrow r$ whose restriction $(v \circ u)|_{\{x\}\times[1]}$ is the composite bordism $v|_{\{x\}\times[1]} \circ u|_{\{x\}\times[1]}$, for all elements $x \in B$. ⌟

When the base poset B is trivial, a 1-truss bundle bordism is simply a 1-truss bordism.

Notation 2.1.97 (Category of 1-truss bundles and their bordisms). For a fixed base poset B, the 'category of 1-truss bundles and their bordisms', whose objects are 1-truss bundles over B and whose morphisms are 1-truss bundle bordisms, will be denoted $\mathsf{TBord}^1(B)$. ⌟

Observation 2.1.98 (Isobordisms of 1-truss bundles are unique). A 1-truss bundle bordism that has an inverse is called a '1-truss bundle isobordism'. As a bundle analog of Observation 2.1.55, note that, given two 1-truss bundles, if there is an isobordism between them, then there is a unique such isobordism. There is therefore no need to distinguish between distinct 1-truss bundles that are isomorphic in the category $\mathsf{TBord}^1(B)$. ⌟

Remark 2.1.99 (Isobordism classes of 1-truss bundles). As a bundle analog of Remark 2.1.56, note that the isomorphism classes of 1-truss bundles in $\mathsf{TBord}^1(B)$, that is the classes of 1-truss bundles up to invertible bordism, are the same as the classes of 1-truss bundles up to base-preserving balanced isomorphism. ⌟

Of course there is a category of functors from a base poset B to the category TBord^1 of 1-trusses and their bordisms, whose morphisms are natural transformations of functors; note that a natural transformation $\mathsf{N} : \mathsf{F} \Rightarrow \mathsf{G}$ between functors $\mathsf{F} : B \to \mathsf{TBord}^1$ and $\mathsf{G} : B \to \mathsf{TBord}^1$ is simply itself a functor $\mathsf{N} : B \times [1] \to \mathsf{TBord}^1$.

Having now a suitable category of 1-truss bundles and their bordisms, and a suitable category of classifying functors, we can describe the functorial correspondence.

Observation 2.1.100 (Classification and totalization functors for 1-truss bundles). Given a poset B, there is an equivalence of categories

$$\chi_- : \mathsf{TBord}^1(B) \rightleftarrows \mathrm{Fun}(B, \mathsf{TBord}^1) : \pi_-$$

specified as follows.

The 'classification functor' χ_- takes a 1-truss bundle $p : T \to B$ to its classifying functor $\chi_p : B \to \mathsf{TBord}^1$, and a 1-truss bundle bordism $u : p \Rightarrow q$ (by definition a 1-truss bundle over $B \times [1]$) to its classifying functor $\chi_u : B \times [1] \to \mathsf{TBord}^1$ viewed as a natural transformation $\chi_u : \chi_p \Rightarrow \chi_q$.

The 'totalization functor' $\pi_{\_}$ takes a functor $\mathsf{F} : B \to \mathsf{TBord}^1$ to its total 1-truss bundle $\pi_{\mathsf{F}} : \mathrm{Tot}\mathsf{F} \to B$, and a natural transformation $\mathsf{N} : B \times [1] \to \mathsf{TBord}^1$ to its total 1-truss bundle $\pi_{\mathsf{N}} : \mathrm{Tot}\mathsf{N} \to B \times [1]$. ⌟

Remark 2.1.101 (1-Truss bundle totalization and classification as collage and decollage). Recall the classical 'Grothendieck', i.e., 'total category' construction provides, for a category C, a correspondence between suitable opfibrations $\mathsf{D} \to \mathsf{C}$ and classifying functors $\mathsf{C} \to \mathsf{Cat}$. The above correspondence, between 1-truss bundles and classifying functors to the category of 1-trusses and their bordisms, is not a version of that Grothendieck correspondence, because truss bordisms are not functors of 1-trusses but relations between them. However, there is a 'profunctorial Grothendieck', i.e., 'collage' construction providing a correspondence between suitable exponentiable functors $\mathsf{D} \to \mathsf{C}$ and classifying (pseudo)functors $\mathsf{C} \rightsquigarrow \mathbb{P}\mathrm{rof}$ landing not in the bicategory of categories and functors and natural transformations but in the bicategory $\mathbb{P}\mathrm{rof}$ of categories and profunctors and natural transformations; see for instance [**Bén00, Str01, Bén06, Bor94**]. The above 1-truss bundle totalization and classification constructions are a combinatorial version of this collage correspondence, tailored for our purposes. ⌟

Remark 2.1.102 (Classifying categorical 1-truss bundles). Recall from Remark 2.1.85 that there is a notion of categorical 1-truss bundle $p : \mathsf{T} \to \mathsf{B}$ over a base category B. The above classification and totalization constructions carry over to the categorical case, showing that 1-truss bundles over a category B (and their bundle bordisms) correspond to functors $\mathsf{B} \to \mathsf{TBord}^1$ (and their natural transformations). ⌟

2.1.3.4. ◊Pullback, dualization, and suspension of 1-truss bundles. With the notions of classification and totalization in hand, we describe three further constructions on 1-truss bundles.

Construction 2.1.103 (Pullback of 1-truss bundles). Given a 1-truss bundle $p : T \to B$ and any poset map $G : A \to B$, the pullback of the bundle (along the map G) is the 1-truss bundle $G^*p : G^*T \to A$, together with the 1-truss bundle map $(\mathrm{Tot}G, G) : G^*p \to p$, determined as follows. The total poset $(G^*T, \trianglelefteq) := G^*(T, \trianglelefteq)$ is the pullback in the category of posets, with the resulting projection $G^*p : (G^*T, \trianglelefteq) \to (A, \to)$ and total map $\mathrm{Tot}G : (G^*T, \trianglelefteq) \to (T, \trianglelefteq)$; the frame order and dimension map on G^*T are such that the total map $\mathrm{Tot}G$ is a 1-truss isomorphism on each fiber. ⌟

Of course, when the poset map $G : A \hookrightarrow B$ is a subposet inclusion, the pullback specializes to the restriction of the 1-truss bundle, that is $G^*p = p|_A$.

Notation 2.1.104 (Pullback 1-truss bundles). Altogether, the pullback 1-truss bundle and its associated maps are concisely indicated by the usual pullback corner caret:

$$\begin{array}{ccc} G^*T & \xrightarrow{\operatorname{Tot} G} & T \\ {\scriptstyle G^*p}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle p} \\ A & \xrightarrow{G} & B \end{array}$$

⌟

Example 2.1.105 (Pullback 1-truss bundle). In Figure 2.24 we illustrate a pullback, in fact a restriction, of a 1-truss bundle. ⌟

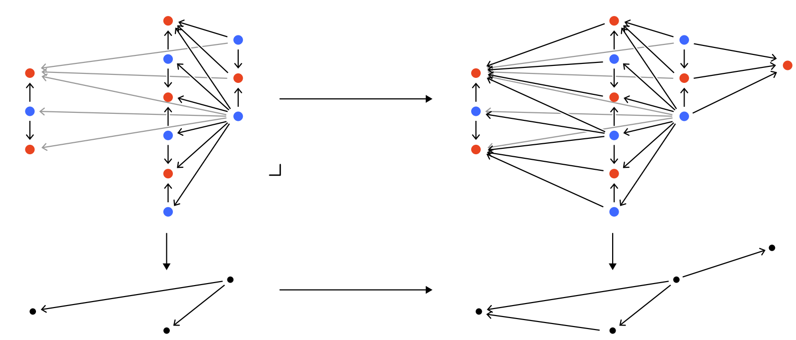


Figure 2.24. The pullback of a 1-truss bundle along a base poset inclusion.

Remark 2.1.106 (Pullback bundles via classifying functors). The pullback bundle may be reexpressed in terms of classifying functors. Given a 1-truss bundle $p : T \to B$ with classifying functor $\chi_p : B \to \mathsf{TBord}^1$, and a poset map $G : A \to B$, the pullback bundle $G^*p : G^*T \to A$ has classifying functor the composite $\chi_p \circ G : A \to \mathsf{TBord}^1$. In other words, the pullback is the total bundle of the composite classifying functor: $G^*p = \pi_{\chi_p \circ G}$. ⌟

Dualizing 1-trusses fiberwise provides a dualization of 1-truss bundles, as follows.

Construction 2.1.107 (Dualization of 1-truss bundles and their maps). Given a 1-truss bundle $p : T \to B$ with total diposet $(T, \trianglelefteq, \preceq)$, dimension map $\dim : (T, \trianglelefteq) \to [1]^{\mathrm{op}}$, and projection $p : (T, \trianglelefteq, \preceq) \to (B, \to, =)$, its 'dual 1-truss bundle' $p^\dagger : T^\dagger \to B^{\mathrm{op}}$ has total diposet $T^\dagger := (T, \trianglelefteq^{\mathrm{op}}, \preceq)$, dimension map the composite $(T, \trianglelefteq^{\mathrm{op}}) \xrightarrow{\dim^{\mathrm{op}}} [1] \cong [1]^{\mathrm{op}}$, and projection $p^\dagger : (T, \trianglelefteq^{\mathrm{op}}, \preceq) \to (B, \to^{\mathrm{op}}, =)$ elementwise equal to the original projection p. That is, as when dualizing 1-trusses, the dual bundle has opposite face order and dimension map, while the frame order is unchanged.

Given a 1-truss bundle map $F : (p : T \to B) \to (q : S \to C)$, the 'dual 1-truss bundle map' $F^\dagger : (p^\dagger : T^\dagger \to B^{\mathrm{op}}) \to (q^\dagger : S^\dagger \to C^{\mathrm{op}})$ is the map $F^\dagger : (T, \trianglelefteq^{\mathrm{op}}, \preceq) \to (S, \trianglelefteq^{\mathrm{op}}, \preceq)$ whose underlying map of sets is equal to the underlying map of sets of the bundle map F itself. We have therefore a covariant involutive functor

$$\dagger : \mathsf{TrsBun}_1 \cong \mathsf{TrsBun}_1.$$

Note that this functor does not preserve the base of the bundle. It does though restrict to the earlier dualization of 1-trusses from Construction 2.1.26, when the base is a point. ⌟

As for dualization of 1-trusses, dualization takes open 1-truss bundles to closed 1-truss bundles and vice versa, and takes singular bundle maps to regular bundle maps and vice versa.

We have not only a covariant dualization functor on bundles and their maps, but also a contravariant dualization functor on bundles and their bordisms.

Construction 2.1.108 (Dualization of 1-truss bundles and their bordisms). Given a 1-truss bundle bordism $u : p \Rightarrow q$ given by the 1-truss bundle $u : U \to B \times [1]$, the 'dual 1-truss bundle bordism' $u^\dagger : q^\dagger \Rightarrow p^\dagger$ is given by the 1-truss bundle $u^\dagger : U^\dagger \to (B \times [1])^{\mathrm{op}} \cong B^{\mathrm{op}} \times [1]$. Note that the flip of variance of the whole base poset $B \times [1]$ flips the direction of the bordism. Dualization therefore gives an involutive isomorphism

$$\dagger : \mathsf{TBord}^1(B) \cong \mathsf{TBord}^1(B^{\mathrm{op}})^{\mathrm{op}}.$$

When the base is a point, this specializes to the dualization of 1-truss bordisms $\dagger : \mathsf{TBord}^1 \to (\mathsf{TBord}^1)^{\mathrm{op}}$, given in Construction 2.1.57. ⌟

Remark 2.1.109 (Dual bundles via classifying functors). The dual bundle may be reexpressed using classifying functors. Given a 1-truss bundle $p : T \to B$, with classifying functor $\chi_p : B \to \mathsf{TBord}^1$, its dual $p^\dagger : T^\dagger \to B^{\mathrm{op}}$ has classifying functor

$$(\chi_{p^\dagger} : B^{\mathrm{op}} \to \mathsf{TBord}^1) = \big(B \xrightarrow{\chi_p} \mathsf{TBord}^1 \xrightarrow{\dagger} (\mathsf{TBord}^1)^{\mathrm{op}}\big)^{\mathrm{op}}.$$

Indeed this association of classifying functors $\chi_p \mapsto (\dagger \circ \chi_p)^{\mathrm{op}}$ is functorial and reproduces the involutive isomorphism of Construction 2.1.108. ⌟

As a final elementary construction, we describe suspensions of 1-truss bundles, obtained by adding new initial and final elements to both the base poset and the total poset of the bundle.

Construction 2.1.110 (Suspension of posets). Let X be a poset. Its 'suspension' ΣX is the poset obtained from X by adjoining two elements $\bot$ and $\top$, along with arrows $\bot \to x$ and $x \to \top$ for each $x \in X$. Note the suspension operation is functorial. ⌟

Construction 2.1.111 (Suspension of 1-truss bundles). Let $p : T \to B$ be a 1-truss bundle. The 'suspension 1-truss bundle' $\Sigma p : \Sigma T \to \Sigma B$ has base poset the suspension ΣB; total poset $(\Sigma T, \trianglelefteq)$ the suspension $\Sigma(T, \trianglelefteq)$; frame order $(\Sigma T, \preceq)$ that relates elements if and only if they are already related in $(T, \preceq)$; and dimension map restricting on $T \hookrightarrow \Sigma T$ to the dimension map of the bundle p, while mapping the initial object $\bot \in \Sigma T$ to 1 and the final object $\top \in \Sigma T$ to 0. ⌟

Remark 2.1.112 (Suspension bundles via classifying functors). Given a 1-truss bundle $p : T \to B$, its suspension bundle $\Sigma p : \Sigma T \to \Sigma B$ has classifying functor $\chi_{\Sigma p} : \Sigma B \to \mathsf{TBord}^1$ given by the unique map that restricts on $B \hookrightarrow \Sigma B$ to the classifying functor χ_p, while mapping $\bot$ to the initial truss $\mathring{\mathbb{T}}_0$ in TBord^1 and mapping $\top$ to the final truss $\bar{\mathbb{T}}_0$ in TBord^1. (See Observation 2.1.54.) ⌟

Example 2.1.113 (Suspension 1-truss bundle). In Figure 2.25 we illustrate a 1-truss bundle $p : T \to B$ on the left, together with its suspension bundle $\Sigma p : \Sigma T \to \Sigma B$ on the right. ⌟

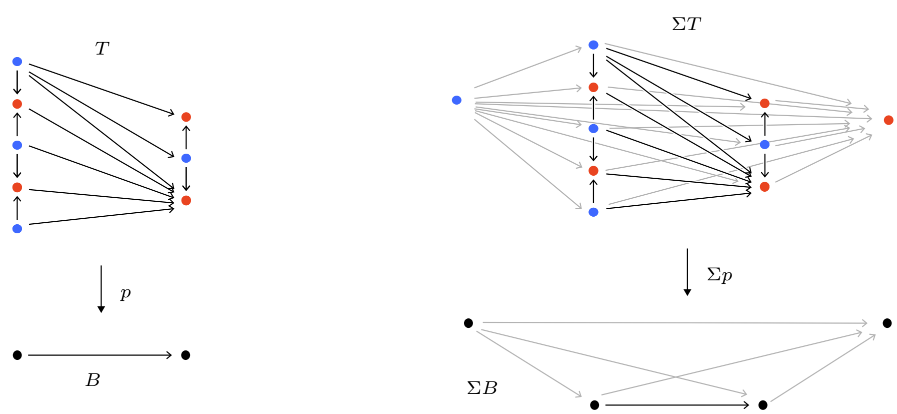


Figure 2.25. Suspension bundle of a 1-truss bundle.

2.2. ◊Truss induction and labelings

As developed in the previous Section 2.1, 1-trusses provide a combinatorial model of stratified intervals, 1-truss bordisms provide a combinatorial model of stratified bundles of stratified intervals over the stratified interval, and 1-truss bundles provide a combinatorial model of stratified bundles of stratified intervals over more general stratified spaces. In the subsequent Section 2.3, we will double down, triple down, indeed n-tuple down on this combinatorialization of stratified topology, by considering 1-truss bundles over 1-truss bundles over 1-truss bundles and so on, as a completely combinatorial description of a quite general class of suitably framed stratified spaces. In order to have and maintain a grip on the structure of this tower of iterated 1-truss bundles, we will need a handle on the simplicial structure of 1-truss bundles themselves. The primary purpose of this Section 2.2 is to develop a pair of such handles, namely the existence of a total order on the collection of sections of a 1-truss bundle over a stratified simplex, and a related total order on the collection of top-dimensional simplices in the total poset of such a bundle; we will refer to the technique of exploiting those total orders (typically by showing that a property of a section or simplex implies a corresponding property of the successor section or simplex) as *truss induction*.

Recall for instance the 1-truss bundle over the 3-simplex from Figure 2.2. The total poset of this, or indeed any, 1-truss bundle over a simplex has the quite special feature that its top-dimensional simplices are all of dimension exactly one more than the dimension of the base; such top-dimensional simplices are called *spacers*. The combinatorial structure of such a bundle is controlled, patently, by its spacer simplices and how they are glued together along their facets. The spacers of this, or indeed any such, 1-truss bundle have the further distinctive feature that a facet simplex shared between two spacers necessarily projects isomorphically to the base; such simplices are called *sections*. The remarkable property of 1-truss bundles, completely peculiar among even specialized poset bundles over posets, though manifest from a certain geometric point of view, is that there is a canonical total order on the set of sections and spacers; we call this the *scaffold order*. That scaffold order, for the 1-truss bundle previously mentioned, is illustrated cryptically in Figure 2.26; this notation for the scaffold order will be deciphered in due course.

A stratified space can be understood as a space together with a map from the space to a fundamental poset, recording the distinct strata and their relationships; similarly, our eventual combinatorial description of stratified spaces will involve a combinatorial gadget, namely an iterated 1-truss bundle, together with a suitable map from the total poset of that gadget to a fundamental poset, now recording the combinatorial strata and their relationships. The basic instances of such suitable maps will be functors, which we call *labelings*, from the total poset of a 1-truss, or 1-truss bordism, or 1-truss bundle, into a target poset or more generally target category. That labeled 1-truss bordisms have a well-defined composition, and therefore that there is a well-defined classifying category for labeled 1-truss bundles, is established by truss induction.

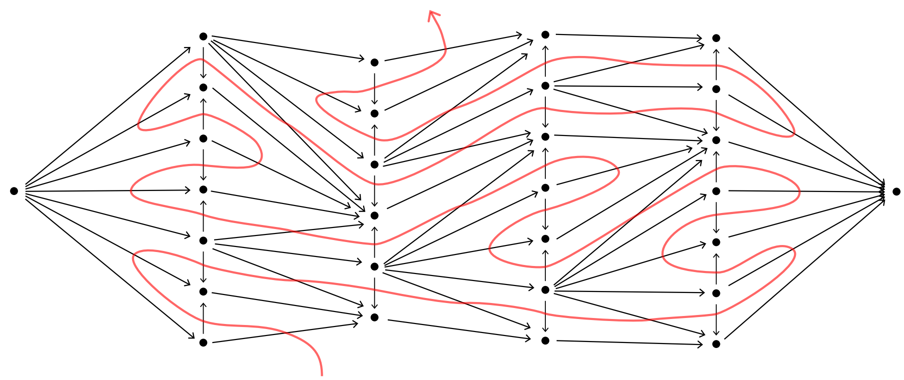


FIGURE 2.26. The scaffold order for a 1-truss bundle.

OUTLINE. In Section 2.2.1, we introduce section and spacer simplices of 1-truss bundles and classify them in terms of jump and fiber morphisms of bundle total posets. In Section 2.2.2, we define norms on the sections and spacers, and use them to construct a canonical linear order on both the set of sections and on the set of spacers, establishing the basis for our core technique of truss induction. Finally in Section 2.2.3, we define labeled 1-trusses and their bordisms, whose composition is seen to be well-defined by truss induction, and use them to provide a classifying category for labeled 1-truss bundles.

2.2.1. ◊Sections and spacers.

SYNOPSIS. We note the distinction between section simplices and spacer simplices in the total poset of a 1-truss bundle, as those simplices that project nondegenerately or degenerately to the base. We introduce jump morphisms and fiber morphisms of a 1-truss bundle, as those with regular domain and singular codomain and, respectively, projection being either a spine vector or a trivial vector of the base. We then describe the correspondence of section simplices and jump morphisms in the suspension bundle, and the correspondence of spacer simplices and fiber morphisms in the bundle itself.

2.2.1.1. ◊The definition of sections and spacers. Recall that a k-simplex of a poset P, that is a map $[k] \to P$, is called 'nondegenerate' if the map is injective on objects, and is called 'degenerate' otherwise.

DEFINITION 2.2.1 (Sections of 1-truss bundles). For a 1-truss bundle $p : T \to B$, a **k-section** is a nondegenerate k-simplex $K : [k] \hookrightarrow (T, \trianglelefteq)$ of the total poset, such that the composite map $p \circ K : [k] \to B$ is a nondegenerate k-simplex in the base poset. ⌟

Definition 2.2.2 (Spacers of 1-truss bundles). For a 1-truss bundle $p : T \to B$, a $(k+1)$**-spacer** is a nondegenerate $(k+1)$-simplex $K : [k+1] \hookrightarrow (T, \trianglelefteq)$ of the total poset, such that the composite map $p \circ K : [k+1] \to B$ is a degenerate simplex in the base poset. ⌟

Terminology 2.2.3 (Simplices in 1-truss bundles). We sometimes refer to a nondegenerate simplex $[k] \hookrightarrow (T, \trianglelefteq)$ of the total poset of a 1-truss bundle $p : T \to B$, simply as 'a simplex in the bundle'. ⌟

Note that every simplex in a 1-truss bundle is either a section or a spacer. Both k-sections and $(k+1)$-spacers of 1-truss bundles always have images that are nondegenerate k-simplices of the base poset.

Terminology 2.2.4 (Base projection of a simplex). Given an n-simplex $K : [n] \hookrightarrow T$ in a 1-truss bundle $p : T \to B$, its 'base projection' $\mathrm{im}(p \circ K) : [m] \hookrightarrow B$ is the unique nondegenerate simplex of the base poset, whose image is the image of $p \circ K : [n] \to B$. ⌟

When considering a bundle over the k-simplex, we often concentrate on sections and spacers that project to the whole base.

Terminology 2.2.5 (Base-surjective simplex). An n-simplex $K : [n] \hookrightarrow T$ in a 1-truss bundle $p : T \to [m]$ is called 'base-surjective' when its base projection $\mathrm{im}(p \circ K)$ is the whole base poset $[m]$. ⌟

Of course, every section or spacer simplex is base-surjective in the pullback bundle over the base projection of that simplex; thus it usually suffices to think about, and illustrate, only the base-surjective situation.

Example 2.2.6 (Sections and spacers in 1-truss bundles). In Figure 2.27, we highlight 2-sections (on the left) and 3-spacers (on the right) of a 1-truss bundle over the 2-simplex. All these sections and spacers are base-surjective. ⌟

As the dimension of the base poset grows, it becomes less practical to draw the section and spacer simplices in a bundle. However, there is a quite convenient notation in any dimension, by restricting attention to the spine as follows.

Notation 2.2.7 (Sections and spacers via their spines). A 1-truss bundle $T \to [k]$ over a simplex is determined by the restriction of the bundle to the spine of the base simplex. Furthermore, any base-surjective section or spacer simplex in the bundle has its entire spine living over the spine of the base. We may and will therefore think of and refer to and depict section and spacer simplices purely in terms of their spines and the projections of those spines to the spine of the base.

An example of this method is illustrated in Figure 2.28. There, on the left we show two sections (in blue and red) and two spacers (in green and

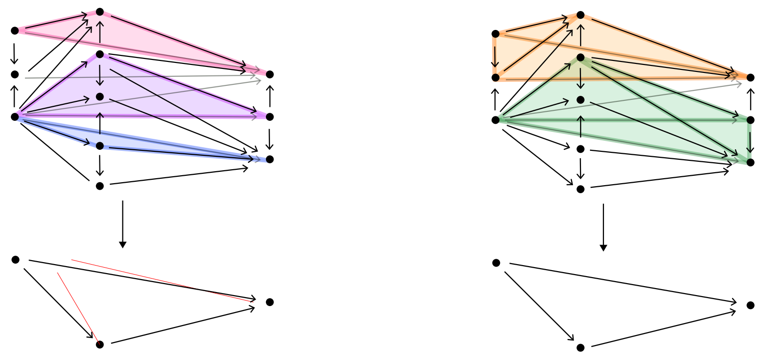


FIGURE 2.27. Sections and spacers in a 1-truss bundle.

yellow), all in a bundle over the 2-simplex. On the right, we depict the same sections and spacers just by highlighting their spines, over the spine of the base 2-simplex. ⌟

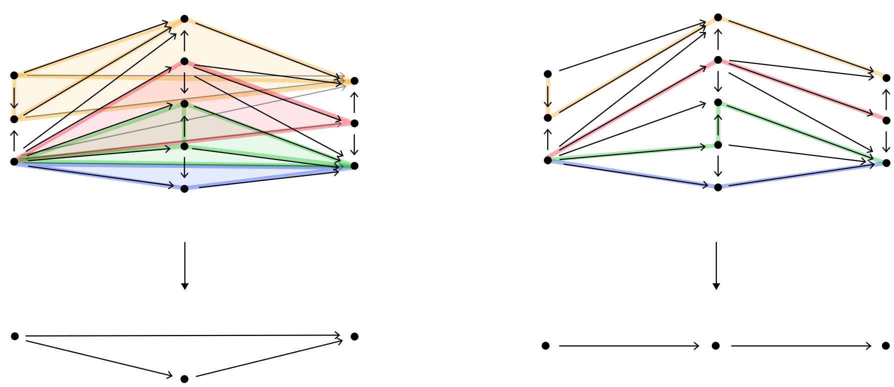


FIGURE 2.28. Spine notation for sections and spacers.

As noted, any section or spacer is base-surjective over its base projection; said another way, any simplex in a 1-truss bundle factors as a base-surjective simplex followed by a bundle inclusion, as follows.

REMARK 2.2.8 (Simplices factor through base-surjective simplices). Let $K : [n] \hookrightarrow T$ be a simplex in a 1-truss bundle $p : T \to B$, with base projection $F := \mathrm{im}(p \circ K) : [m] \hookrightarrow B$. Then $K : [n] \hookrightarrow T$ factors as a composite of the base-surjective simplex $[n] \hookrightarrow F^*T$ and the bundle pullback inclusion $F^*T \hookrightarrow T$. This factorization provides a bijection between simplices in the bundle p whose base projection is $F : [m] \hookrightarrow B$ and base-surjective simplices in the pullback bundle $F^*p : F^*T \to [m]$. ⌟

On account of this factorization, for the remainder of Section 2.2.1 and for Section 2.2.2, we will work almost exclusively with bundles over simplices, and we will implicitly assume base-surjectivity.

Convention 2.2.9 (Base-surjectivity by default). We will assume all sections and spacers are base-surjective unless otherwise noted. ⌟

Notation 2.2.10 (Set of sections and spacers). Given a 1-truss bundle $p : T \to [m]$, we denote its sets of sections and spacers as follows.

$$\Gamma_p = \{\text{sections } K : [m] \hookrightarrow T \text{ of } p : T \to [m]\ \}$$
$$\Psi_p = \{\text{spacers } L : [m+1] \hookrightarrow T \text{ of } p : T \to [m]\}$$ ⌟

2.2.1.2. ◊The spines of sections and spacers. 1-Truss bundles have such a specific combinatorial structure that both section simplices and spacer simplices (and thus all simplices) admit a manifest combinatorial classification in terms of when and how the spine of the simplex transitions from regular objects to singular objects.

Remark 2.2.11 (1-truss bundle arrows weakly decrease dimension). Recall that in a 1-truss, the nontrivial arrows $a \to b$ strictly decrease dimension, i.e., $1 = \dim(a) > \dim(b) = 0$; that is, all such arrows have regular source and singular target. Furthermore, recall from Observation 2.1.59 that in a 1-truss bordism, the relations $R(a, b)$ weakly decrease dimension, i.e., $\dim(a) \geq \dim(b)$. It follows (see Terminology 2.1.72) that all arrows in the associated poset of a 1-truss bordism, and thus all arrows in any 1-truss bundle, also weakly decrease dimension; in particular, all such arrows either have regular source and target, singular source and target, or regular source and singular target. ⌟

Observation 2.2.12 (Spines of simplices in 1-truss bundles). Consider a 1-truss bundle $p : T \to [m]$ over the m-simplex, and a k-simplex $K : [k] \hookrightarrow T$ in the bundle. The spine $\mathsf{spine}[k] = (0 \to 1 \to \cdots \to k)$ maps to the spine $\mathsf{spine}\, K([k]) = (K(0) \to K(1) \to \cdots \to K(k)) \subset T$. By the preceding remark, this spine has one of three forms:

(1) All the objects $K(i)$ are regular.
(2) All the objects $K(i)$ are singular.
(3) There is a single 'transition arrow' $K(j-1) \to K(j)$ whose source is regular and whose target is singular; all objects $K(i \leq j-1)$ are regular, and all objects $K(i \geq j)$ are singular.

We refer to the lowest number j with $K(j)$ singular as the 'transition index'. In the third case, that $K(j)$ is the target of the transition arrow; in the second case, the transition index is 0; in the first case, by convention we declare the transition index to be $k+1$.

In the first two cases, the simplex is necessarily a section, since any arrow in a fiber of the bundle would transition from regular to singular objects; we refer to such sections as 'purely regular' and 'purely singular', respectively. ⌟

Notation 2.2.13 (Base-fiber notation for 1-truss bundles). Given a 1-truss bundle $p : T \to B$, it is sometimes clarifying, if redundant, to denote an object $a \in T$ by the pair $(p(a), a) \in B \times T$, that is, by an object of the base followed by an object in the corresponding fiber. ⌟

Remark 2.2.14 (Spines of sections). Applying Observation 2.2.12, a section simplex $K : [m] \hookrightarrow T$ in a 1-truss bundle $p : T \to [m]$ necessarily has, for some transition index $0 \leq j \leq m+1$, the form

$$(0, a_0) \to (1, a_1) \to ... \to (j{-}1, a_{j-1}) \to (j, b_j) \to (j{+}1, b_{j+1}) \to ... \to (m, b_m)$$

where each object a_i is regular and each object b_i is singular. Here if $j = m+1$ then every object is regular, and if $j = 0$ then every object is singular; those are the first two cases of the previous observation. If the transition index j is strictly between 0 and $m+1$, then the section has both regular and singular objects and is an instance of the third case of the observation. ⌟

We can unify the three seemingly distinct section types (purely regular, purely singular, and mixed regular and singular) by shifting attention to the suspension of the 1-truss bundle, as follows.

Notation 2.2.15 (Suspending simplices). For convenience and compatibility with the usual representation of the standard m-simplex as $(0 \to 1 \to \cdots \to (m-1) \to m)$, we will identify the suspension $\Sigma[m]$ with the poset $(-1 \to 0 \to 1 \to \cdots \to (m-1) \to m \to m+1)$. ⌟

Construction 2.2.16 (Suspending sections). Consider a section $K : [m] \hookrightarrow T$ in a 1-truss bundle $p : T \to [m]$. By Construction 2.1.110, the suspension $\Sigma K : \Sigma[m] \hookrightarrow \Sigma T$ has $\Sigma K(\bot) = \bot$ and $\Sigma K(\top) = \top$. Equivalently, using the conventions Notation 2.2.13 and Notation 2.2.15, this may be written as $\Sigma K(-1) = (-1, r)$ and $\Sigma K(m+1) = (m+1, s)$, where r is the unique object of the initial 1-truss, and s is the unique object of the final 1-truss. Observe that ΣK is indeed a section of the bundle $\Sigma p : \Sigma T \to \Sigma[m]$, and the map $K \mapsto \Sigma K$ provides a bijective correspondence between sections in the bundle p and sections in the bundle Σp. (The inverse map is simply restricting sections $\Sigma[m] \hookrightarrow \Sigma T$ to the simplex $[m] \subset \Sigma[m]$.) ⌟

Since the suspended section $\Sigma K : \Sigma[m] \hookrightarrow \Sigma T$ begins with a regular object and ends with a singular object, it is necessarily mixed, even if the section K was purely regular or purely singular; thus considering sections in terms of their suspension unifies the section types as desired.

More than the satisfying tidiness of all suspended sections having the same structure, the shift of perspective to the suspension allows a concise classification of sections of 1-truss bundles, as follows.

Definition 2.2.17 (Jump morphisms). A **jump morphism** f in a 1-truss bundle $p : T \to [m]$ is an arrow in the total poset $(T, \trianglelefteq)$, whose domain $\mathrm{dom}(f)$ is regular, whose codomain $\mathrm{cod}(f)$ is singular, and whose base projection is a spine vector of the simplex $[m]$. ⌟

Construction 2.2.18 (Correspondence of sections of a bundle and jump morphisms of the suspended bundle). Let $p : T \to [m]$ be a 1-truss bundle over the m-simplex. To each section $K : [m] \hookrightarrow T$ of the bundle, we can associate the transition arrow $\Sigma K(j-1) \to \Sigma K(j)$ of the suspended section $\Sigma K : \Sigma[m] \hookrightarrow \Sigma T$; note that j is the transition index of the section $K : [m] \hookrightarrow T$. This transition arrow $\Sigma K(j-1) \to \Sigma K(j)$ is a jump morphism of the suspended bundle $\Sigma p : \Sigma T \to \Sigma[m]$.

Conversely, consider a jump morphism f of the suspended bundle $\Sigma p : \Sigma T \to \Sigma[m]$, with base projection the spine vector $(j-1 \to j)$ in $\Sigma[m]$ (using Notation 2.2.15 for objects of $\Sigma[m]$); we can associate a section $K : [m] \hookrightarrow T$ of the bundle $p : T \to [m]$, defined by

› for $i < j$, set $K(i) = \mathsf{reg}^{\chi_p(i \to j-1)}(\mathrm{dom}\, f)$,
› for $j \leq i$, set $K(i) = \mathsf{sing}_{\chi_p(j \to i)}(\mathrm{cod}\, f)$.

(Recall that $\chi_p(k \to l)$ is the 1-truss bordism obtained by restricting the bundle to that arrow of the base; and reg^R and sing_R are the regular function and singular function of the bordism R.)

These associations are inverse, and provide a bijective correspondence between sections of a 1-truss bundle and jump morphisms of the suspension of that bundle. ⌟

Example 2.2.19 (Correspondence of sections and jump morphisms). In Figure 2.29 we illustrate a 1-truss bundle $p : T \to [2]$, together with its suspension $\Sigma p : \Sigma T \to \Sigma[2]$ (indicated in grey). We highlight the spines of four sections (using the spine-only method from Notation 2.2.7; for each of those sections, we mark the associated jump morphism in ΣT by a big dot of the same color. ⌟

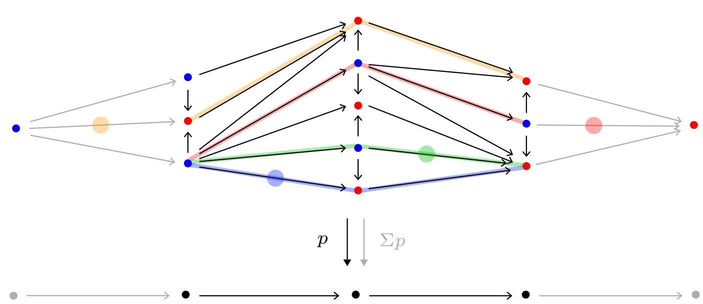


Figure 2.29. Sections and their associated jump morphisms.

We now proceed to the companion classification of spacer simplices in 1-truss bundles.

Remark 2.2.20 (Spines of spacers). Again applying Observation 2.2.12, a spacer simplex $L : [m+1] \hookrightarrow T$ in a 1-truss bundle $p : T \to [m]$ necessarily has, for some index $0 \leq j \leq m$, the form

$$(0, a_0) \to (1, a_1) \to \ldots \to (j, a_j) \to (j, b_j) \to (j+1, b_{j+1}) \to \ldots \to (m, b_m)$$

where each object a_i is regular and each object b_i is singular. Note that in this case the transition arrow is $(j, a_j) \to (j, b_j)$ and the transition index is in fact $j+1$, since $L(j+1) = (j, b_j)$. In particular, unlike for sections, every spacer has at least one regular and at least one singular vertex. ⌟

DEFINITION 2.2.21 (Fiber morphisms). A **fiber morphism** f in a 1-truss bundle $p : T \to [m]$ is an arrow in the total poset $(T, \trianglelefteq)$, whose domain $\mathrm{dom}(f)$ is regular, whose codomain $\mathrm{cod}(f)$ is singular, and whose base projection is an identity arrow in the simplex $[m]$. ⌟

CONSTRUCTION 2.2.22 (Correspondence of spacers and fiber morphisms). Let $p : T \to [m]$ be a 1-truss bundle over the m-simplex $[m]$. To each spacer $L : [m+1] \hookrightarrow T$ of the bundle, we can associate the transition arrow $(L(j) \to L(j+1)) = ((j, a_j) \to (j, b_j))$; here $j+1$ is the transition index of the spacer, and the transition arrow is a fiber morphism.

Conversely, for a fiber morphism f of the bundle $p : T \to [m]$ with base projection the identity on $j \in [m]$, we can associate a spacer $L : [m+1] \hookrightarrow T$, defined by

› for $i \leq j$, set $L(i) = \mathsf{reg}^{\chi_p(i \to j)}(\mathrm{dom}\, f)$,
› for $j+1 \leq i+1$, set $L(i+1) = \mathsf{sing}_{\chi_p(j \to i)}(\mathrm{cod}\, f)$.

These associations are inverse, and provide a bijective correspondence between spacers of a 1-truss bundle and fiber morphisms of that bundle. ⌟

EXAMPLE 2.2.23 (Correspondence of spacers and fiber morphisms). In Figure 2.30 we highlight the spines of four spacers in a 1-truss bundle $p : T \to [2]$; for each of those spacers, we mark the associated fiber morphism by a big dot of the same color. ⌟

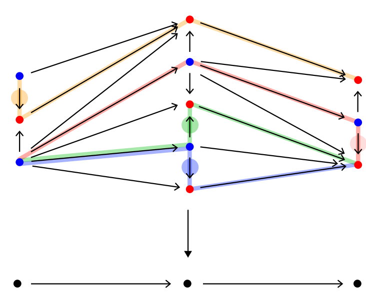


FIGURE 2.30. Spacers and their associated fiber morphisms.

2.2.2. ◊The scaffold order. We now construct a canonical linear order on the set of sections, and a related canonical linear order on the set of spacers, in a 1-truss bundle over a simplex.

SYNOPSIS. We begin by illustrating the order on the set of sections as a directed path through the jump morphisms in the spine notation for the suspended bundle. We then define a numerical norm on sections, and prove that this norm induces a total order on the set of sections. Similarly, we illustrate the order on the set of spacers as a directed path through the fiber morphisms in the spine notation for the bundle. We then define a norm on spacers in terms of the norms of boundary sections, and prove that again this norm induces a total order on the set of spacers.

2.2.2.1. ♢The case of sections. We construct a total order on the set of sections Γ_p of a 1-truss bundle $p : T \to [m]$ over the m-simplex; we will call this order the 'scaffold order of sections'. Recall from Construction 2.2.18 the correspondence of sections in a bundle and jump morphisms in the suspended bundle. The scaffold order on the sections is thus equivalent to an order on those jump morphisms, and that order on the jump morphisms has a convenient and illuminating visual representation, as shown in the next example. Moreover, the passage from each jump morphism to its successor in this order will form a core step in the subsequent formal construction of the scaffold order.

Example 2.2.24 (Scaffold order). In Figure 2.31 we illustrate *all* the sections in a 1-truss bundle over the 2-simplex (by highlighting the spines as before). We also mark the corresponding jump morphisms in the suspended bundle (by correspondingly colored dots). The scaffold order on these sections is depicted via an order on the jump morphisms; that order on the jump morphisms is indicated by the red directed path. Pragmatically, that path may be drawn (and is uniquely determined) by beginning with the jump morphism on the lower boundary, then crossing alternately and only through fiber morphisms and jump morphisms, until reaching a jump morphism on the upper boundary. ⌟

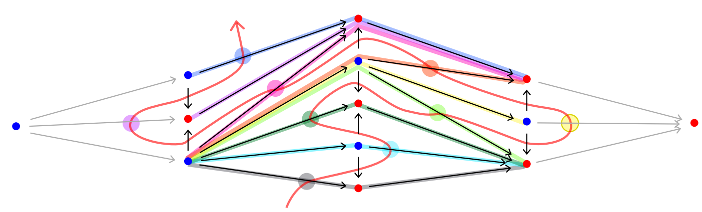


Figure 2.31. The scaffold order on sections.

We will construct the scaffold order on the sections Γ_p by defining a 'scaffold norm' $\Gamma_p \to \mathbb{N}$, and showing that the norm maps the set of sections bijectively to an interval of natural numbers; the scaffold order on sections is simply inherited from the standard order on $\mathbb{N}$.

To define the scaffold norm, it is convenient to use the notion of the frame height of an element in a 1-truss bundle, as follows. For a 1-truss bundle $p : T \to [m]$, let $\mathsf{hght} : (T, \preceq) \to (\mathbb{N}, \leq)$ be the unique map that sends every fiber $(p^{-1}(i), \preceq)$, $i \in [m]$, isomorphically to a standard framed-ordered simplex $[n] \subset (\mathbb{N}, \leq)$; that is, the frame height $\mathsf{hght}(a)$ is $j - 1 \in \mathbb{N}$ when $a \in T$ is the j-th element in the total frame order of the fiber $p^{-1}(p(a))$.

Definition 2.2.25 (Scaffold norm of sections). Consider a 1-truss bundle $p : T \to [m]$ and its set of sections Γ_p. The **scaffold norm** $\langle - \rangle$ is the function

$$\begin{aligned} \langle - \rangle : \Gamma_p &\to \mathbb{N} \\ K &\mapsto \sum_{i \in [m]} \mathsf{hght}(K(i)). \end{aligned}$$

taking a section to the sum of the frame heights of its elements. ⌟

Observation 2.2.26 (Suspension preserves scaffold norm). Recall the suspension operation on sections from Construction 2.2.16. Note that the suspension $\Sigma : \Gamma_p \to \Gamma_{\Sigma p}$ preserves the scaffold norm: $\langle K \rangle = \langle \Sigma K \rangle$ for $K \in \Gamma_p$. ⌟

In order to describe the image of the scaffold norm, we first construct the distinguished sections that minimize and maximize the norm.

Terminology 2.2.27 (Bottom and top sections). A 'bottom section' or 'top section' of a 1-truss bundle is one that minimizes or maximizes, respectively, the scaffold norm. ⌟

Construction 2.2.28 (Bottom and top sections of a 1-truss bundle). Let $p : T \to [m]$ be a 1-truss bundle over the m-simplex $[m]$. We construct a bottom section $\mathsf{K}^-_p : [m] \to T$ and a top section $\mathsf{K}^+_p : [m] \to T$, by setting the sections $\mathsf{K}^\pm_p$ on $i \in [m]$ to be the lower and upper endpoints of the corresponding fiber: $\mathsf{K}^\pm_p(i) = \mathsf{end}_\pm(p^{-1}(i))$. That this defines valid sections follows from Observation 2.1.60 that 1-truss bordisms relate endpoints.

Note that the minimal value of the scaffold norm is $\langle \mathsf{K}^-_p \rangle = 0$ and the maximal value is $\langle \mathsf{K}^+_p \rangle = \#T - \#[m]$ (where $\#T$ and $\#[m]$ are the number of elements in those posets). Denote these extremal values of the scaffold norm by $\mathsf{scaff}_\pm(p) := \langle \mathsf{K}^\pm_p \rangle$. Furthermore note that the sections $\mathsf{K}^\pm_p$ are the unique sections realizing those minimal and maximal values. ⌟

The extremal values of the scaffold norm $\mathsf{scaff}_\pm(p)$ bound an interval of natural numbers, and that interval is precisely the image of the scaffold norm on sections.

Lemma 2.2.29 (Scaffold order of sections). *For a 1-truss bundle $p : T \to [m]$ with sections Γ_p, the scaffold norm $\langle - \rangle : \Gamma_p \to \mathbb{N}$ is a bijection onto its image; that image is the set $[\mathsf{scaff}_-(p), \mathsf{scaff}_+(p)]$ of all natural numbers between the extremal values of the scaffold norm. The induced total order on the sections Γ_p is called the 'scaffold order of sections'.*

Proof. The previous Construction 2.2.28 provided unique bottom and top sections $\mathsf{K}^\pm_p$ with minimal and maximal scaffold norms $\mathsf{scaff}_\pm(p)$. We will now construct, for each non-top section $K \neq \mathsf{K}^+_p$, a successor section $\mathsf{s}(K)$

with scaffold norm $\langle \mathsf{s}(K) \rangle = \langle K \rangle + 1$, and conversely construct, for each non-bottom section $K \neq \mathsf{K}_p^-$, a predecessor section $\mathsf{p}(K)$ with scaffold norm $\langle \mathsf{p}(K) \rangle = \langle K \rangle - 1$. We will then observe that the successor and predecessor section constructions are mutually inverse; the lemma follows.

The successor section construction is briefer and more uniform in the case that all sections contain a jump morphism, i.e., when there are no purely regular or purely singular sections. Recall from Construction 2.2.16 that there is an isomorphism $\Gamma_p \cong \Gamma_{\Sigma p}$ of the sections of any bundle and the sections of its suspension, and by Observation 2.2.26 this isomorphism commutes with the scaffold norm. It therefore suffices to construct the successor in suspended bundles; we will not assume the bundle p is, per se, a suspension but we will and may assume all its sections contain jump morphisms (as is true in any suspension).

Let $K \neq \mathsf{K}_p^+$ be a non-top section of the bundle $p : T \to [m]$, with jump morphism $K(j-1) \to K(j)$ in the total poset T. We prove that in the total poset T, there is *either* an arrow $K(j-1)+1 \to K(j)$ *or* an arrow $K(j-1) \to K(j)+1$. (There cannot be both such arrows due to bimonotonicity of 1-truss bordisms.) Since K is not the top section, there is a successor $K(l)+1$ of $K(l)$ in the fiber over $l \in [m]$ for some index l.

Suppose there is a successor $K(l)+1$ for an index $l \leq j-1$. In this case, since $K(l)$ is regular, the successor $K(l)+1$ is singular. Consider the 1-truss bordism $R_l^{j-1} := p^{-1}(l \to j-1)$ and its singular function $\mathsf{sing}_{R_l^{j-1}} : \mathsf{sing}(p^{-1}(l)) \to \mathsf{sing}(p^{-1}(j-1))$. That singular function takes $K(l)+1$ to some singular element $\mathsf{sing}_{R_l^{j-1}}(K(l)+1) \in p^{-1}(j-1)$. Since $K(j-1)$ is regular, the frame order relation $K(l) \prec K(l)+1$ and bimonotonicity of the bordism R_l^{j-1} imply the frame order relation $K(j-1) \prec \mathsf{sing}_{R_l^{j-1}}(K(l)+1)$. In particular, there is a singular successor $K(j-1)+1$ in the fiber $p^{-1}(j-1)$.

Suppose instead there is a successor $K(l)+1$ for an index with $j \leq l$. Since $K(l)$ is singular, the successor $K(l)+1$ is regular. An argument dual to the previous one, using the regular function $\mathsf{reg}^{R_j^l} : \mathsf{reg}(p^{-1}(j)) \leftarrow \mathsf{reg}(p^{-1}(l))$ of the 1-truss bordism $R_j^l := p^{-1}(j \to l)$, shows there is a regular successor $K(j)+1$ in the fiber $p^{-1}(j)$.

If there is a successor $K(j-1)+1$, but no successor $K(j)+1$, then by bimonotonicity and Observation 2.1.65 that bordisms fully relate elements, there must be an arrow $K(j-1)+1 \to K(j)$. Similarly, if there is no successor $K(j-1)+1$, but there is a successor $K(j)+1$, then there must be an arrow $K(j-1) \to K(j)+1$. If there is both a (singular) successor $K(j-1)+1$ and a (regular) successor $K(j)+1$, then (by bimonotonicity) either there is an arrow $K(j-1)+1 \to K(j)$ (as desired) or there is an arrow $K(j-1)+1 \to s$ for some singular element $s \succ K(j)$; in the latter case, the structure of the singular determined 1-truss bordism in the proof of Lemma 2.1.63 implies that there is an arrow $K(j-1) \to K(j)+1$ (as desired).

We now construct the successor section $\mathtt{s}(K)$ of K, distinguishing the two cases above, where $K(j-1)+1 \to K(j)$ or $K(j-1) \to K(j)+1$. These cases are illustrated in Figure 2.32; in each case the jump morphism $K(j-1) \to K(j)$ is marked by a green dot, and the jump morphism of the successor section is marked by a purple dot. The successor is constructed in each case as follows.

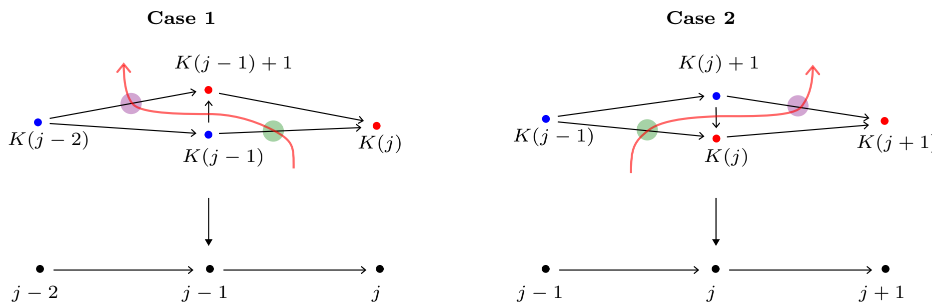


FIGURE 2.32. The construction of successor sections.

Case 1. When there is an arrow $K(j-1)+1 \to K(j)$, we construct the successor section $\mathtt{s}(K)$ by setting $\mathtt{s}(K)(j-1) = K(j-1)+1$ and $\mathtt{s}(K)(i) = K(i)$ if $i \neq j-1$. We must have $j \geq 2$, as otherwise $\mathtt{s}(K)$ would be a purely singular section, contradicting our assumption on the bundle p. Functoriality of the 1-truss bordism $p^{-1}(j-2 \to j-1)$ implies that there is an arrow $K(j-2) \to K(j-1)+1$, ensuring that $\mathtt{s}(K)$ is indeed a section.

Case 2. When there is an arrow $K(j-1) \to K(j)+1$, we construct the successor section $\mathtt{s}(K)$ by setting $\mathtt{s}(K)(j) = K(j)+1$ and $\mathtt{s}(K)(i) = K(i)$ if $i \neq j$. We must have $j \leq m-1$, as otherwise $\mathtt{s}(K)$ would be a purely regular section, contradicting our assumption on the bundle p. Functoriality of the 1-truss bordism $p^{-1}(j \to j+1)$ then implies there is an arrow $K(j)+1 \to K(j+1)$, ensuring that $\mathtt{s}(K)$ is indeed a section.

This completes the construction of successors.

The construction of predecessor sections $\mathtt{p}(K)$, for non-bottom sections $K \neq \mathsf{K}^-_p$, is given by the construction of successor sections for the bundle with opposite frame order; see Remark 2.1.82. (This opposite amounts to reading the total posets in Figure 2.32 upside down.) Observe that $\mathtt{p}(\mathtt{s}(K)) = K$ and similarly $\mathtt{s}(\mathtt{p}(K)) = K$, as required. □

2.2.2.2. ◊The case of spacers. Rather like the case of sections, we now construct a total order on the set of spacers Ψ_p in a 1-truss bundle $p : T \to [m]$ over the m-simplex; we call this order the 'scaffold order of spacers'. Recall from Construction 2.2.22 the correspondence of spacers and fiber morphisms in a bundle. The scaffold order on the spacers is therefore equivalent to an order on those fiber morphisms, and precisely as in the case of sections, that order on fiber morphisms has an efficient visual representation, as in the next example.

Example 2.2.30 (Scaffold order on spacers). Recall from Example 2.2.24 that the scaffold order of sections can be depicted by a directed path, namely the one that traverses the jump morphisms of the suspension while crossing a single fiber morphism between each two scaffold-order-adjacent jump morphisms. The order in which that path traverses the fiber morphisms is exactly the scaffold order on spacers. In Figure 2.33 we illustrate all the spacers of the same 1-truss bundle as the previous example (by highlighting the spines), mark the corresponding fiber morphisms (by correspondingly colored dots), and depict the scaffold order again by the red directed path.

Earlier in Figure 2.26, we illustrated a more complicated example of the scaffold order of both sections and spacers, for a 1-truss bundle over a 3-simplex, depicted again by a single directed path through the jump and fiber morphisms of the suspension. ⌟

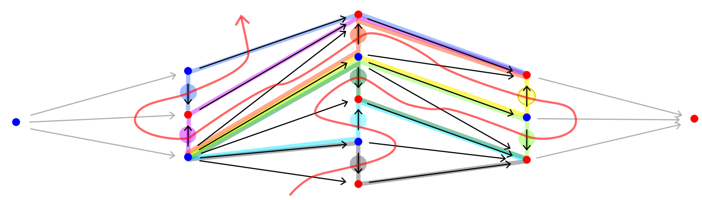


Figure 2.33. The scaffold order on spacers.

Any spacer of a bundle has exactly two facets that are sections, namely those facets obtained by omitting either the source or target of the fiber morphism. Those sections correspond to the two jump morphisms adjacent, along the illustrated directed path, to the fiber morphism of the spacer; the jump morphism preceding the fiber morphism will correspond to a 'lower boundary section' and the one subsequent to the fiber morphism will correspond to an 'upper boundary section', constructed as follows.

Construction 2.2.31 (Upper and lower boundary sections of spacers). Given a spacer $L : [m+1] \to T$ of a 1-truss bundle $p : T \to [m]$, let $L(j) \to L(j+1)$ be the fiber morphism of the spacer. When $L(j) \prec L(j+1)$, the **lower boundary section** $\partial_- L$ is the $(j+1)$th face $d_{j+1}L$ of the spacer L, and the **upper boundary section** $\partial_+ L$ is the jth face $d_j L$ of the spacer L. When by contrast $L(j+1) \prec L(j)$, then $\partial_- L$ is the jth face $d_j L$, and $\partial_+ L$ is the $(j+1)$th face $d_{j+1} L$. ⌟

Example 2.2.32 (Upper and lower boundary sections). In Figure 2.34 we highlight two spacers L and L' in a 1-truss bundle, together with their lower and upper boundary sections $\partial_\pm L$ and $\partial_\pm L'$. ⌟

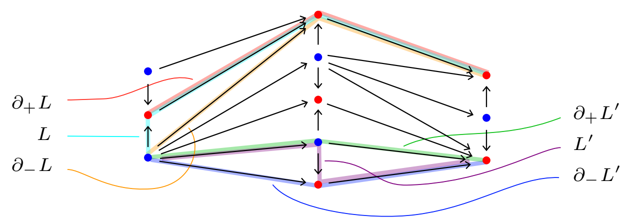


FIGURE 2.34. Upper and lower boundaries of spacers.

REMARK 2.2.33 (Upper boundaries succeed lower boundaries). The preceding construction ensures that the scaffold norms of the upper and lower boundaries of a spacer are related by $\langle \partial_+ L \rangle = \langle \partial_- L \rangle + 1$; that is, the upper boundary $\partial_+ L$ is the successor of the lower boundary $\partial_- L$ in the scaffold order $(\Gamma_p, \preceq)$ of sections. ⌟

As in the case of sections, we will construct the scaffold order on spacers Ψ_p in terms of a 'scaffold norm' $\Psi_p \to \mathbb{N} + \frac{1}{2}$.

DEFINITION 2.2.34 (Scaffold norm of spacers). Consider a 1-truss bundle $p : T \to [m]$ and its set of spacers Ψ_p. The **scaffold norm** $\langle - \rangle$ is the function

$$\begin{aligned} \langle - \rangle : \Psi_p &\to \mathbb{N} + \frac{1}{2} \\ L &\mapsto \frac{\langle \partial_- L \rangle + \langle \partial_+ L \rangle}{2} \end{aligned}$$

taking a spacer to the average of the scaffold norms of its lower and upper boundaries. ⌟

Analogously to the case of sections treated in Lemma 2.2.29, the scaffold norm of spacers will take the set of spacers bijectively to an interval of half-shifted natural numbers; the scaffold order on spacers is simply inherited from the standard order on that image.

LEMMA 2.2.35 (Scaffold order for spacers). *For a 1-truss bundle $p : T \to [m]$ with spacers Ψ_p, the scaffold norm $\langle - \rangle : \Psi_p \to \mathbb{N} + \frac{1}{2}$ is a bijection onto its image; that image is the set of all half integers between the extremal values scaff_- and scaff_+ of the scaffold norm of sections. The induced total order on the spacers Ψ_p is called the 'scaffold order of spacers'.*

PROOF. Each spacer L in the bundle p is uniquely determined by its boundary sections $\partial_\pm L$, and those boundary sections are adjacent in the scaffold norm by Remark 2.2.33. The previous Lemma 2.2.29 thus implies that the scaffold norm of spacers $\langle - \rangle : \Psi_p \to \mathbb{N} + \frac{1}{2}$ is injective.

To see that this norm surjects onto the claimed image, consider a half integer h between the extremal section scaffold norms scaff_- and scaff_+. The integers $h \pm \frac{1}{2}$ are necessarily, by Lemma 2.2.29, the scaffold norms of some section K and its successor $\mathsf{s}(K)$. Consider the cases of the construction of the successor in the proof of that Lemma. Observe that in case 1, the fiber morphism $K(j-1) \to \mathtt{s}(K)(j-1)$ determines (by Construction 2.2.22) a spacer L with $\partial_- L = K$ and $\partial_+ L = \mathtt{s}(K)$. Similarly, in case 2, the fiber morphism $\mathtt{s}(K)(j) \to K(j)$ determines a spacer L with $\partial_- L = K$ and $\partial_+ L = \mathtt{s}(K)$. Thus there is a spacer with the required scaffold norm. □

We have seen that both the set of sections and the set of spacers of a 1-truss bundle $p : T \to [m]$ have total orders, and moreover that for each section there is a spacer that increments it to the next section. We can summarize and express this situation more categorically as follows.

Terminology 2.2.36 (Fiber categories in 1-truss bundles). Let $p : T \to B$ be a 1-truss bundle, and consider a nondegenerate simplex $z : [m] \to B$. The 'fiber category' $\Phi_p(z)$ in the bundle p over the simplex z is the free category whose objects are sections $K \in \Gamma_{z^* p}$ (that is, sections over just that simplex, i.e., of the pullback bundle $z^* p$), and that has a generating morphism $L : \partial_- L \to \partial_+ L$ for each spacer $L \in \Psi_{z^* p}$. ⌟

Construction 2.2.37 (Transition functors of fiber categories). Let $p : T \to B$ be a 1-truss bundle. Consider a nondegenerate simplex $z : [m] \to B$ and let $y : [l] \to B$ be a face of the simplex z; that is, there is an injection $[l] \hookrightarrow [m]$ so that the simplex y is the composite $[l] \to [m] \xrightarrow{z} B$. There is an inclusion of pullback bundles $y^* p \hookrightarrow z^* p$. A section of the bundle $z^* p$ restricts to a section of the bundle $y^* p$, and a spacer of the bundle $z^* p$ restricts either to a spacer or to a section of the bundle $y^* p$. This restriction provides a functor $-|_{y \subset z} : \Phi_p(z) \to \Phi_p(y)$, called the 'fiber transition' in the bundle p from the simplex z to the simplex y. ⌟

Observation 2.2.38 (Structure of fiber categories and transition functors). For all 1-truss bundles $p : T \to B$, we have the following properties:

(1) All fiber categories $\Phi_p(z)$ are total orders.
(2) All transition functors $\Phi_p(z) \to \Phi_p(y)$ preserve endpoints.

The first property follows from Lemma 2.2.29 and Lemma 2.2.35, using the relation $\langle \partial_\pm L \rangle = \langle L \rangle \pm \frac{1}{2}$ between the scaffold norm of a spacer and its boundary sections. The second property follows from Construction 2.2.28 for bottom and top sections. Note that since the fiber categories are total orders, and a transition functor sends a generating morphism to either a generating morphism or an identity, and transition functors preserve endpoints, it follows that all transition functors are surjective. ⌟

2.2.3. ◊Labeled 1-trusses, bordisms, and bundles. As entertaining as 1-trusses and their bordisms and bundles themselves are, and as pretty as the inductive structure of simplices in 1-truss bundles itself may be, our eventual concern will be with stratified versions of towers of 1-truss bundles over 1-truss bundles over 1-truss bundles and so on. It will be both convenient and crucial to encode the relevant sort of stratifications and the iterated bundle structures in terms of labeled trusses. The labeling is an assignment, to each element and arrow of the truss (or truss bordism or bundle), of an object and morphism in a labeling category—that category could be a stratification poset or itself an inductively defined category of towers of truss bundles. Critically, the well-definedness of composition of labeled truss bordisms, and therefore the iterability of the labeled bordism and bundle constructions, is proven by truss induction.

SYNOPSIS. We define labeled 1-trusses and labeled 1-truss bordisms, and show that composition of labeled 1-truss bordisms is well defined; this provides a category of 1-trusses and their bordisms labeled in a given category, and therefore an iterable endofunctor on the category of categories. We then generalize these notions to labeled 1-truss bundles, and show that the category of labeled 1-trusses and their bordisms is a classifying category for labeled 1-truss bundles. Finally, we mention the labeled 1-truss bundle versions of the pullback, dualization, and suspension constructions.

2.2.3.1. ◊The definition of labeled 1-trusses and their bordisms. A labeling of a 1-truss in a category is simply an assignment of objects and morphisms in the category to the elements and face arrows of the 1-truss. Since there are no composite face arrows, there is not even a nontrivial functoriality condition on the assignment.

DEFINITION 2.2.39 (Labeled 1-trusses). Given a category C, a **C-labeled 1-truss** T is a pair $(\underline{T}, \mathrm{lbl}_T)$ consisting of a 1-truss $\underline{T}$ and a functor $\mathrm{lbl}_T : (\underline{T}, \trianglelefteq) \to \mathsf{C}$ from the face poset of the 1-truss to the category. ⌟

We refer to the 1-truss $\underline{T}$ as the 'underlying 1-truss' of the labeled 1-truss, and to the functor lbl_T as the 'labeling functor'. We can display the data of a labeled 1-truss compactly as

$$[0] \longleftarrow \underline{T} \xrightarrow{\mathrm{lbl}_T} \mathsf{C}$$

The left arrow expresses the 1-truss pedantically as a 1-truss bundle over the trivial poset; the right arrow is the labeling functor.

A labeling of a 1-truss bordism in a category is similarly an assignment of objects and morphisms to the elements and arrows of the bordism, though now of course we insist that assignment respect composition.

DEFINITION 2.2.40 (Labeled 1-truss bordisms). Given a category C, a **C-labeled 1-truss bordism** R is a pair $(\underline{R}, \mathrm{lbl}_R)$ consisting of a 1-truss bordism $\underline{R}$ and a functor $\mathrm{lbl}_R : (\underline{R}, \trianglelefteq) \to \mathsf{C}$ from the total poset of the bordism to the category. ⌟

Here and later on, we freely elide the distinction between a 1-truss bordism, its associated total poset (as in Terminology 2.1.72), and also its total poset considered as a 1-truss bundle over the interval (as in Definition 2.1.74). We can display the data of a labeled 1-truss bordism compactly as

$$[1] \longleftarrow \underline{R} \xrightarrow{\mathrm{lbl}_R} \mathsf{C}$$

The left arrow expresses the 1-truss bordism as a 1-truss bundle over the interval; the right arrow is the labeling functor. Needless to say, we consider this labeled 1-truss bordism as a morphism $R : R_0 \twoheadrightarrow R_1$ from a domain labeled 1-truss $R_0 := (\underline{R}|_0, (\mathrm{lbl}_R)|_0)$ to a codomain labeled 1-truss $R_1 := (\underline{R}|_1, (\mathrm{lbl}_R)|_1)$ (where $(\mathrm{lbl}_R)|_i$ abbreviates the restriction of lbl_R to the subposet $\underline{R}|_i \subset \underline{R}$).

Example 2.2.41 (A labeled 1-truss bordism). In Figure 2.35 we illustrate a 1-truss bordism labeled in the poset $[1] \times [1]$. The labeling is indicated by color matching the objects of the bordism and their corresponding images in the labeling poset; since this labeling category is a poset, the object map determines the labeling functor entirely. ⌟

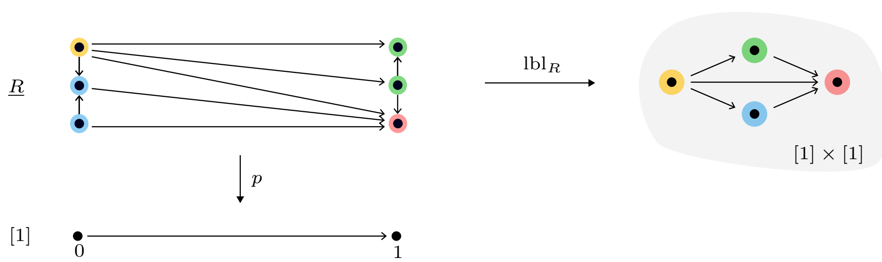


Figure 2.35. A labeled 1-truss bordism.

The natural next question is whether labeled 1-truss bordisms compose, that is whether labeling functors on two composable 1-truss bordisms suitably induce a labeling functor on the composite 1-truss bordism.

Definition 2.2.42 (Composition for labeled 1-truss bordisms). Given two C-labeled 1-truss bordisms $R_{01} : T_0 \twoheadrightarrow T_1$ and $R_{12} : T_1 \twoheadrightarrow T_2$, the **composite labeled 1-truss bordism** $R_{02} = R_{12} \circ R_{01} : T_0 \twoheadrightarrow T_2$ has underlying 1-truss bordism $\underline{R}_{02}$ being the composite $\underline{R}_{12} \circ \underline{R}_{01}$, and has labeling functor $\mathrm{lbl}_{R_{02}} : \underline{R}_{02} \to \mathsf{C}$ specified by

$$\mathrm{lbl}_{R_{02}}(x_0 \trianglelefteq x_2) = \mathrm{lbl}_{R_{12}}(x_1 \trianglelefteq x_2) \circ \mathrm{lbl}_{R_{01}}(x_0 \trianglelefteq x_1)$$

whenever $x_0 \trianglelefteq x_1$ and $x_1 \trianglelefteq x_2$ are composable arrows of the total posets $\underline{R}_{01}$ and $\underline{R}_{12}$, respectively. ⌟

Lemma 2.2.43 (Composition of labeled 1-truss bordisms is well defined). *The labeling functor in the composition of labeled 1-truss bordisms is well defined.*

Before giving the proof, we give an example of how the composition of labeled functorial relations can fail to be well defined, and an example of the well-defined composition in the 1-truss bordism case.

Example 2.2.44 (Composition of labeled functorial relations is not well defined). By contrast with the situation described in the previous lemma, composition of labeled functorial relations between 1-trusses, which one might try to specify as in Definition 2.2.42, is not well defined. We illustrate such a failure, where a label in the composite relation is overdetermined, in Figure 2.36. In the top left, there are two composable functorial relations $\underline{R}_{01}$ and $\underline{R}_{12}$ between 1-trusses. In the bottom left is their (trivial) composite functorial relation $\underline{R}_{02}$. The labeling functors $\mathrm{lbl}_{R_{01}}$ and $\mathrm{lbl}_{R_{12}}$ are indicated by color matching objects and their images, and color matching morphisms and their images. (Morphisms colored by an object color are labeled by the identity on that object.) The hypothetical composite labeling functor $\mathrm{lbl}_{R_{02}}$ would take the morphism to both the red and the blue morphisms in the labeling category, and therefore does not exist. ⌟

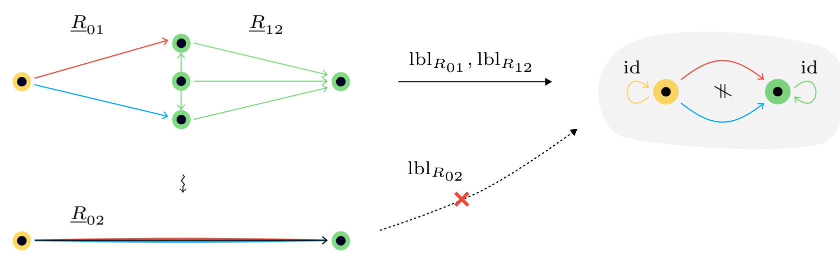


FIGURE 2.36. Failure of composition of labeled functorial relations.

EXAMPLE 2.2.45 (Composition of labeled 1-truss bordisms). In Figure 2.37, we illustrate two labeled 1-truss bordisms R_{01} and R_{12} along with their composite R_{02}. Notice that the underlying 1-truss bordisms here are those previously shown in Figure 2.23. The labeling category is a poset, so the labeling functors are determined by the color-coding of the objects.

The red arrows delineate a spacer simplex in the 1-truss bundle $\underline{W}_{012} \to [2]$ over the 2-simplex. The upper and lower sections of that spacer provide distinct factorizations of the red arrow in the composite. That spacer, along with the functoriality of the labeling of the two labeled 1-truss bordisms R_{01} and R_{12}, ensures (as explained in the proof below) that the composite red arrow has a well specified label. The blue arrows similarly delineate three spacer simplices. The boundary sections of those spacers provide four distinct factorizations of the blue arrow in the composite. Those spacers, chained together via truss induction (again as in the proof below), ensure that the composite blue arrow also has a well specified label. ⌟

PROOF OF LEMMA 2.2.43. We have C-labeled 1-truss bordisms $R_{01} : T_0 \nrightarrow T_1$ and $R_{12} : T_1 \nrightarrow T_2$; the composite $R_{02} : T_0 \nrightarrow T_2$ has underlying 1-truss bordism $\underline{R}_{02} = \underline{R}_{12} \circ \underline{R}_{01}$, and labeling supposedly defined by $\mathrm{lbl}_{R_{02}}(g_{12} \circ f_{01}) = \mathrm{lbl}_{R_{12}}(g_{12}) \circ \mathrm{lbl}_{R_{01}}(f_{01})$; we abbreviate that last composite by $\mathsf{lbl}(g, f)$. Note that any arrow e of the composite bordism $\underline{R}_{02}$, by definition of the composite relation, has some decomposition $e = g \circ f$ and therefore some label assignment $\mathsf{lbl}(g, f)$.

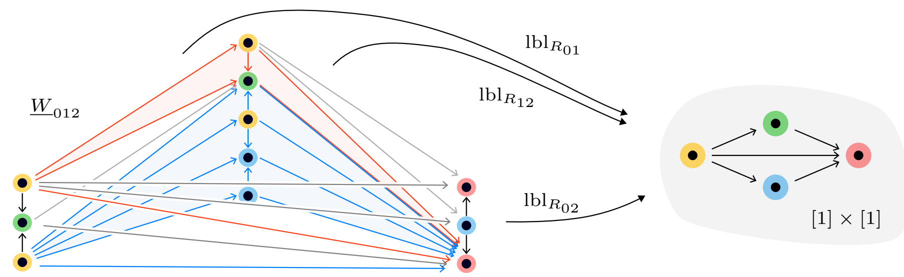


FIGURE 2.37. Composition of labeled 1-truss bordisms.

It suffices to check that whenever we have two distinct decompositions $g_{12} \circ f_{01} = g'_{12} \circ f'_{01}$, the putative labels correspond, that is

$$\mathsf{lbl}(g, f) := \mathrm{lbl}_{R_{12}}(g_{12}) \circ \mathrm{lbl}_{R_{01}}(f_{01}) = \mathrm{lbl}_{R_{12}}(g'_{12}) \circ \mathrm{lbl}_{R_{01}}(f'_{01}) =: \mathsf{lbl}(g', f').$$

The bordisms $\underline{R}_{01}$ and $\underline{R}_{12}$ and their composite $\underline{R}_{02}$ define a 1-truss bundle $\underline{W}_{012} \to [2]$, as in Example 2.1.95.

We proceed by truss induction in this bundle. The arrows f and g are the spine vectors of a section $K : [2] \to \underline{W}_{012}$, and the arrows f' and g' are the spine vectors of another section $K' : [2] \to \underline{W}_{012}$. Assume we have the scaffold order relation $K \preceq K'$ (the reverse case is the same); Lemma 2.2.29 ensures that there is a sequence of successor sections $K = K_0, K_1, \ldots, K_k = K'$, that is with $\mathsf{s}(K_i) = K_{i+1}$, starting with our section K and ending with our section K'. By induction, we may assume the sequence has length 1, that is $\mathsf{s}(K) = K'$. Now by Lemma 2.2.35 there is a spacer $L : [3] \to \underline{W}_{012}$ with lower boundary $\partial_- L = K$ and upper boundary $\partial_+ L = K'$.

Note that the spine $L(2 \to 3) \circ L(1 \to 2) \circ L(0 \to 1)$ of that spacer L composes both to the spine $K(1 \to 2) \circ K(0 \to 1) = g \circ f$ of the section K, and to the spine $K'(1 \to 2) \circ K'(0 \to 1) = g' \circ f'$ of the section K'. Since the labeling lbl_{T_1} is the restriction of both $\mathrm{lbl}_{R_{01}}$ and $\mathrm{lbl}_{R_{12}}$ to the 1-truss fiber T_1, functoriality of those labelings $\mathrm{lbl}_{R_{01}}$ and $\mathrm{lbl}_{R_{12}}$ implies

$$\begin{aligned} \mathsf{lbl}(g, f) &= \mathrm{lbl}_{R_{12}}(K(1 \to 2)) \circ \mathrm{lbl}_{R_{01}}(K(0 \to 1)) \\ &= \mathrm{lbl}_{R_{12}}(L(2 \to 3)) \circ \mathrm{lbl}_{T_1}(L(1 \to 2)) \circ \mathrm{lbl}_{R_{01}}(L(0 \to 1)) \\ &= \mathrm{lbl}_{R_{12}}(K'(1 \to 2)) \circ \mathrm{lbl}_{R_{01}}(K'(0 \to 1)) = \mathsf{lbl}(g', f') \end{aligned}$$

as required. □

NOTATION 2.2.46 (Categories of labeled 1-trusses and their bordisms). Given a category C, the 'category of C-labeled 1-trusses and their bordisms', whose objects are C-labeled 1-trusses and whose morphisms are C-labeled 1-truss bordisms, will be denoted $\mathsf{TBord}^1_{/\!/\mathsf{C}}$. ⌟

REMARK 2.2.47 (Unlabeled 1-trusses are trivially labeled). Of course, unlabeled 1-trusses and their bordisms may be considered as having 'trivial' labelings, that is labelings in the terminal category $*$. Indeed, the functor $\mathsf{TBord}^1_{/\!/*} \to \mathsf{TBord}^1$, taking an $*$-labeled 1-truss T (respectively bordism R)

to its underlying 1-truss $\underline{T}$ (respectively bordism $\underline{R}$), is an isomorphism of categories. ⌟

The construction of the category $\mathsf{TBord}^1_{/\!/\mathsf{C}}$ is functorial in the labeling category C.

Construction 2.2.48 (Relabeling by a functor). Let $\mathsf{F} : \mathsf{C} \to \mathsf{D}$ be a functor between categories. The associated **relabeling functor** between categories of labeled 1-trusses and their bordisms

$$\mathsf{TBord}^1_{/\!/\mathsf{F}} : \mathsf{TBord}^1_{/\!/\mathsf{C}} \to \mathsf{TBord}^1_{/\!/\mathsf{D}}$$

takes a C-labeled 1-truss T to the D-labeled 1-truss with underlying truss $\underline{T}$ and labeling $\mathsf{F} \circ \mathrm{lbl}_T$, and similarly takes a C-labeled bordism R to the D-labeled bordism with underlying bordism $\underline{R}$ and labeling $\mathsf{F} \circ \mathrm{lbl}_R$. ⌟

Terminology 2.2.49 (Label-forgetting functor). Note that relabeling by the terminal functor $\mathsf{C} \to *$ provides a 'label-forgetting' functor

$$(\underline{-}) : \mathsf{TBord}^1_{/\!/\mathsf{C}} \to \mathsf{TBord}^1_{/\!/*} \cong \mathsf{TBord}^1$$

which simply removes the labeling data. ⌟

The alchemical observation is that the functoriality of the construction of labeled 1-trusses and their bordisms provides, as follows, an endofunctor on the category of categories, which is therefore *iterable*—and iterate it we will.

Definition 2.2.50 (The labeled 1-truss bordism functor). The **labeled 1-truss bordism functor** is the endofunctor

$$\mathsf{TBord}^1_{/\!/-} : \mathsf{Cat} \to \mathsf{Cat}$$

that takes a category C to the category $\mathsf{TBord}^1_{/\!/\mathsf{C}}$ of C-labeled 1-trusses and their bordisms, and takes a functor $\mathsf{F} : \mathsf{C} \to \mathsf{D}$ to the relabeling functor $\mathsf{TBord}^1_{/\!/\mathsf{F}} : \mathsf{TBord}^1_{/\!/\mathsf{C}} \to \mathsf{TBord}^1_{/\!/\mathsf{D}}$. ⌟

We may recast the notion of labeled 1-truss bordisms in more abstract categorical terms, as we did for unlabeled 1-truss bordisms at the end of Section 2.1.2.1. (Readers without a categorical bent may skip ahead to Section 2.2.3.2 without consequence.)

Recall that 1-truss bordisms are in particular functorial relations between preorders, and functorial relations are the same concept as boolean profunctors. A labeling of a 1-truss bordism is of course a functor from the total face poset to a (not-necessarily posetal) category. To express the category of labeled bordisms in a concise categorical fashion, we need a context subsuming both boolean profunctors and ordinary functors; such a context is provided by the bicategory $\mathbb{P}\mathrm{rof}$ of categories, profunctors, and natural transformations.

We transport 1-trusses and their bordisms into categories and profunctors as follows.

Construction 2.2.51 (Bordisms as profunctors). The **bordism-as-profunctor pseudofunctor**

$$\iota : \mathsf{TBord}^1 \rightsquigarrow \mathbb{P}\mathrm{rof}$$

from the category of 1-trusses and their bordisms to the bicategory of profunctors, takes a 1-truss T to the face poset category $(T, \trianglelefteq)$, and takes a 1-truss bordism R to its boolean profunctor, considered as a profunctor via the inclusion $\mathsf{Bool} \hookrightarrow \mathsf{Set}$. ⌟

Note well that, as elaborated in the following remark, it is not the case that there is a sensible pseudofunctor $\mathsf{BoolProf} \rightsquigarrow \mathbb{P}\mathrm{rof}$, and so in particular not the case that the pseudofunctor $\mathsf{TBord}^1 \rightsquigarrow \mathbb{P}\mathrm{rof}$ arises as a composite $\mathsf{TBord}^1 \to \mathsf{BoolProf} \rightsquigarrow \mathbb{P}\mathrm{rof}$.

Remark 2.2.52 (1-Truss bordisms are special among boolean profunctors). Boolean profunctors compose as their underlying relations, while ordinary profunctors compose by coends (see [**Lor21**, §5]). Given a boolean profunctor R, by considering booleans as sets in the usual way, there is an associated profunctor $[R]$. For general boolean profunctors $R : X \nrightarrow Y$ and $S : Y \nrightarrow Z$ between general preorders X, Y, and Z, it need *not* be the case that the profunctor of the composite is the composite of the profunctors: $[S \circ R] \ncong [S] \circ [R]$. In particular the associated profunctor operation $[-]$ is not a pseudofunctor.

However, when the preorders X, Y, and Z are in fact 1-trusses, and the boolean profunctors R and S are in fact 1-truss bordisms, there is a unique isomorphism between the profunctor of the composite and the composite of the profunctors; that isomorphism emerges by explicitly evaluating the colimit defining the profunctor composite and following the arguments in the proof of Lemma 2.2.43. The resulting isomorphism $\iota(S \circ R) \cong \iota(S) \circ \iota(R)$ is the pseudofunctoriality data of the bordisms-as-profunctors pseudofunctor. ⌟

We have now resituated 1-trusses (and their bordisms) as having associated categories (and profunctors) in the bicategory $\mathbb{P}\mathrm{rof}$, and of course potential labeling categories also reside as objects in that bicategory. Our categorical recasting of $\mathsf{TBord}^1_{/\!/\mathsf{C}}$ will be a direct instantiation of the following abstract generalization of comma categories. (Recall that for a functor $\mathsf{F} : \mathsf{A} \to \mathsf{B}$ and an object $b \in \mathsf{B}$, the comma category F/b has as objects pairs $(a \in \mathsf{A}, f : \mathsf{F}(a) \to b)$ and morphisms $(a, f) \to (a', f')$ are those morphisms $(a \to a')$ in A such that $\mathsf{F}(a \to a')$ commutes with the given morphisms f and f' in B.)

Construction 2.2.53 (Vertical comma categories). Given a normal pseudofunctor $H : \mathsf{T} \rightsquigarrow \mathbb{P}\mathrm{rof}$ from a category T into the bicategory $\mathbb{P}\mathrm{rof}$, and a category $\mathsf{C} \in \mathbb{P}\mathrm{rof}$, the **vertical comma category** $H_{/\!/\mathsf{C}}$ is defined as follows: the objects are pairs $(t \in \mathsf{T}, \mathsf{F} : H(t) \to \mathsf{C})$, consisting of an object t of the category T, and a *functor* (not profunctor) F from the image category $H(t)$ to the category C; 1-morphisms $(t, \mathsf{F}) \to (t', \mathsf{F}')$ are pairs

$(r : t \to t', \alpha : H(r) \Rightarrow \mathrm{Hom}_{\mathsf{C}}(\mathsf{F}-, \mathsf{F}'-))$, consisting of a morphism r in the category T, and a natural transformation of profunctors from the image profunctor $H(r)$ to the Hom profunctor $\mathrm{Hom}_{\mathsf{C}}(\mathsf{F}-, \mathsf{F}'-))$. ⌟

The terminology 'vertical comma category' arises from implicitly considering $\mathbb{P}\mathrm{rof}$ not as a bicategory but as a double category with vertical functors and horizontal profunctors; the arrow of the comma category is specified to be vertical, thus a functor rather than a profunctor. Our choice of notation $H_{/\!/\mathsf{C}}$, and therefore obviously our choice of notation $\mathsf{TBord}^1_{/\!/\mathsf{C}}$ for labeled bordisms, is similarly inspired by the implicit sense that it is a sort of comma or slice category in a double categorical context.

Observation 2.2.54 (Categorical reformulation of labeled 1-trusses and their bordisms). For a category C, the category of C-labeled 1-trusses and their bordisms, as in Notation 2.2.46, is equivalent to the vertical comma category of the bordism-as-profunctor pseudofunctor over the category C:

$$\mathsf{TBord}^1_{/\!/\mathsf{C}} \simeq (\mathsf{TBord}^1 \overset{\iota}{\rightsquigarrow} \mathbb{P}\mathrm{rof})_{/\!/\mathsf{C}}$$ ⌟

There is one last yet more abstract construction of the category of labeled 1-trusses and their bordisms, using the profunctorial collage of Remark 2.1.101, as follows.

Terminology 2.2.55 (The tautological 1-truss bundle). The pseudofunctor $\iota : \mathsf{TBord}^1 \rightsquigarrow \mathbb{P}\mathrm{rof}$ from Construction 2.2.51, conceived of as a collagable classifying pseudofunctor, has a corresponding exponentiable functor $\rho : \mathsf{ETBord}^1 \to \mathsf{TBord}^1$; that functor is called the 'tautological 1-truss bundle'—indeed the fiber over each object $T \in \mathsf{TBord}^1$ is the 1-truss T as a category. (Note this bundle is a categorical 1-truss bundle in the sense of Remark 2.1.85.) ⌟

Observation 2.2.56 (Labeled 1-trusses and their bordisms via the tautological bundle). We will see a bit later that the category $\mathsf{TBord}^1_{/\!/\mathsf{C}}$ of C-labeled 1-trusses and their bordisms is a classifying category for C-labeled 1-truss bundles. Such a classifying category should be the universal category living over the classifying category TBord^1 (for unlabeled 1-trusses and their bordisms), that has a functor from its total 1-truss category (obtained as the pullback of the tautological 1-truss bundle) to the labeling category C.

That universal category can be obtained as follows. Because the tautological 1-truss bundle ρ is exponentiable, the functor $\mathsf{Cat}_{/\mathsf{TBord}^1} \to \mathsf{Cat}_{/\mathsf{ETBord}^1}$ that takes the pullback of the tautological bundle (along a classifying functor $\mathsf{F} : \mathsf{B} \to \mathsf{TBord}^1$) has a right adjoint (cf. [**Str01**]); and certainly the forgetful functor $\mathsf{Cat}_{/\mathsf{ETBord}^1} \to \mathsf{Cat}$ has a right adjoint. The composite of those adjoints provides a functor $\mathsf{Cat} \to \mathsf{Cat}_{/\mathsf{TBord}^1}$ that sends a category C to the category $\mathsf{TBord}^1_{/\!/\mathsf{C}}$ of C-labeled 1-trusses and their bordisms (with its forgetful functor to TBord^1). ⌟

2.2.3.2. ◊The definition of labeled 1-truss bundles. Given the notion of labeled 1-truss bordisms from the previous section, and considering 1-truss bordisms as 1-truss bundles over the interval, we of course have the generalization to labeled 1-truss bundles over other posets, by asking for a labeling functor from the total poset, as follows.

DEFINITION 2.2.57 (Labeled 1-truss bundles). Given a poset B and a category C, a **C-labeled 1-truss bundle** p over B is a pair $(\underline{p}, \mathrm{lbl}_p)$ consisting of a 1-truss bundle $\underline{p} : T \to B$, and a functor $\mathrm{lbl}_p : (T, \trianglelefteq) \to \mathsf{C}$ from the total poset of the bundle to the category. ⌟

We refer to the bundle $\underline{p}$ as the 'underlying 1-truss bundle', and to the functor lbl_p as the 'labeling functor'. We can display the data of a labeled 1-truss bundle $p \equiv (\underline{p}, \mathrm{lbl}_p)$ compactly as

$$B \xleftarrow{\underline{p}} T \xrightarrow{\mathrm{lbl}_p} \mathsf{C}$$

EXAMPLE 2.2.58 (1-Truss bundle labeled in a poset). In Figure 2.38 we illustrate a 1-truss bundle labeled in the poset [2]. In the previous Figure 2.35 of a ([1] × [1])-labeled 1-truss bordism, we indicated the labeling by the object mapping; though that would suffice here, given the additional complexity of this bundle, it is easier to parse the labeling by its behavior on morphisms. We therefore indicate the labeling functor also by color matching the morphisms of the total poset of the bundle and their corresponding images in the labeling poset. ⌟

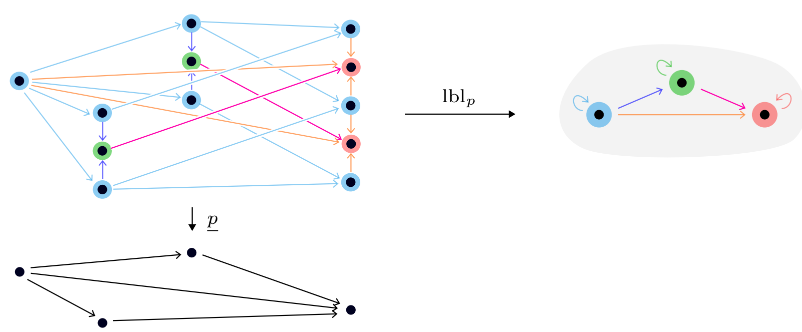


FIGURE 2.38. A 1-truss bundle labeled in a poset.

EXAMPLE 2.2.59 (1-Truss bundle labeled in a monoid). Though all of our labeled 1-truss bordism and bundle examples so far were labeled in a poset, and certainly that will be a case of core concern, still the labeling category may perfectly well be non-posetal. In Figure 2.39, we illustrate a 1-truss bundle $\underline{p} : T \to \bar{\mathbb{T}}_1$ (the one previously pictured in Example 2.1.94), together with a labeling $\mathrm{lbl}_p : T \to \mathsf{B}\mathbb{F}$ in the 'opposite flip flop monoid' $\mathbb{F}$. That monoid $\mathbb{F}$ has two non-identity, idempotent elements r and s with composition $r \circ s = s$ and $s \circ r = r$. We indicate the elements r and s by colored arrows and, as in the previous example, record the labeling functor by color matching morphisms with their images.

As it happens, this monoid is the bordism endomorphism monoid of the closed 1-truss with three singular elements: $\mathbb{F} = \mathrm{End}_{\mathsf{TBord}^1}(\bar{\mathbb{T}}_2)$. As such, this labeling may be considered as associating the closed 5-element truss to each element of the total poset T, and associating a 1-truss bordism to each arrow of that total poset. Altogether this provides a new 1-truss bundle $q : S \to T$ with base now the previous total poset T; the reader may endeavor to picture the total poset S of that bundle—we will return to such bundles in due course. ⌟

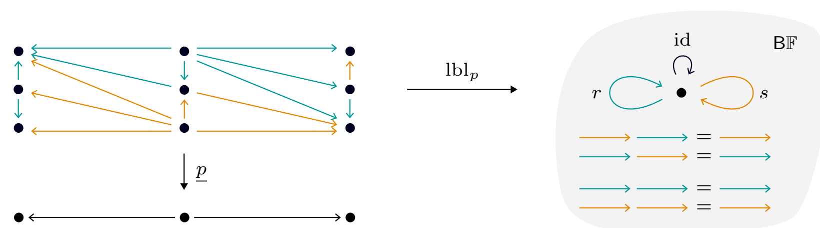


FIGURE 2.39. A 1-truss bundle labeled in a monoid.

Recall that maps of 1-truss bundles are simply maps of the total diposets; the labeled analog is immediate, as follows.

DEFINITION 2.2.60 (Maps of labeled 1-truss bundles). For categories C and D, let p be a C-labeled 1-truss bundle, and let q be a D-labeled 1-truss bundle. A **map of labeled 1-truss bundles** $F : p \to q$ is a pair $(\underline{F}, \mathrm{lbl}_F)$ consisting of a 1-truss bundle map $\underline{F} : \underline{p} \to \underline{q}$, and a functor $\mathrm{lbl}_F : \mathsf{C} \to \mathsf{D}$ such that $\mathrm{lbl}_F \circ \mathrm{lbl}_p = \mathrm{lbl}_q \circ \underline{F}$. ⌟

As in previous cases, we refer to the 1-truss bundle map $\underline{F}$ as the 'underlying bundle map'; we call the functor lbl_F the 'label category functor' or sometimes the 'relabeling functor'. We can display the data of a map $F \equiv (\underline{F}, \mathrm{lbl}_F)$ of labeled 1-truss bundles compactly as

$$\begin{array}{ccccc} B & \xleftarrow{\underline{p}} & T & \xrightarrow{\mathrm{lbl}_p} & \mathsf{C} \\ \downarrow & & \downarrow{\scriptstyle \underline{F}} & & \downarrow{\scriptstyle \mathrm{lbl}_F} \\ C & \xleftarrow[\underline{q}]{} & S & \xrightarrow[\mathrm{lbl}_q]{} & \mathsf{D} \end{array}$$

TERMINOLOGY 2.2.61 (Label-preserving and base-preserving maps). A labeled 1-truss bundle map $F \equiv (\underline{F}, \mathrm{lbl}_F)$ is called 'label preserving' if the label category functor lbl_F is the identity id_{C} of the label category, and is called 'base preserving' if the underlying bundle map $\underline{F}$ covers the identity id_B of the base poset. ⌟

Terminology 2.2.62 (Singular, regular, and balanced labeled bundle maps). A labeled 1-truss bundle map $F \equiv (\underline{F}, \mathrm{lbl}_F)$ is 'singular', 'regular', or 'balanced' if its underlying 1-truss bundle map $\underline{F}$ is, respectively. ⌟

Composition of underlying maps of bundles, along with composition of the label category functors, provides the following category.

Notation 2.2.63 (The category of labeled 1-truss bundles). The category of labeled 1-truss bundles and their maps is denoted $\mathsf{LblTrsBun}_1$. ⌟

Remark 2.2.64 (Unlabeled 1-truss bundles are trivially labeled). As in the case of bordisms in Remark 2.2.47, all 1-truss bundles have a unique labeling in the terminal category. This labeling provides a fully faithful functor $\mathsf{TrsBun}_1 \hookrightarrow \mathsf{LblTrsBun}_1$ from the category of 1-truss bundles into the category of labeled 1-truss bundles. ⌟

Terminology 2.2.65 (Restriction of labeled 1-truss bundles). Given a C-labeled 1-truss bundle $p \equiv (\underline{p} : T \to B, \mathrm{lbl}_p : T \to \mathsf{C})$ and a subposet $A \hookrightarrow B$, the 'restriction' of the labeled bundle to the subposet is the C-labeled 1-truss bundle $p|_A \equiv \big(\underline{p}\big|_A : T|_A \to A,\, (\mathrm{lbl}_p)|_A : T|_A \to \mathsf{C}\big)$. ⌟

Remark 2.2.66 (Balanced label- and base-preserving isomorphisms are unique). Recall from Convention 2.1.20 and Remark 2.1.21 that balanced isomorphisms of 1-trusses preserve all structural data and are unique when they exist. Similarly, balanced label- and base-preserving 1-truss bundle isomorphisms preserve all structural data (face order, frame order, dimension map, base projection, labeling functor) and are unique when they exist. There is therefore never any need to distinguish between distinct but balanced label- and base-preservingly isomorphic labeled 1-truss bundles. ⌟

2.2.3.3. ◊Classification and totalization for labeled 1-truss bundles. Previously in Observation 2.1.100 we saw that 1-truss bundles were classified by functors into the category TBord^1 of 1-trusses and their bordisms. As we detail presently, the labeled situation is entirely analogous: C-labeled 1-truss bundles are classified by functors into the category $\mathsf{TBord}^1_{/\!/\mathsf{C}}$ of C-labeled 1-trusses and their bordisms.

Construction 2.2.67 (Classifying functors of labeled 1-truss bundles). We describe a map

$$p \equiv (\underline{p} : T \to B, \mathrm{lbl}_p : T \to \mathsf{C}) \quad \mapsto \quad (\chi_p : B \to \mathsf{TBord}^1_{/\!/\mathsf{C}})$$

that takes a C-labeled 1-truss bundle p over a poset B to an associated **classifying functor** $\chi_p : B \to \mathsf{TBord}^1_{/\!/\mathsf{C}}$.

We construct χ_p on elements and arrows of the poset B, as follows. For each element $x : [0] \hookrightarrow B$, the classifying element $\chi_p(x) \in \mathsf{TBord}^1_{/\!/\mathsf{C}}$ is the C-labeled 1-truss $p|_x$, and for each non-identity arrow $f : [1] \hookrightarrow B$, the classifying morphism $\chi_p(f)$ of $\mathsf{TBord}^1_{/\!/\mathsf{C}}$ is the C-labeled 1-truss bordism $p|_f$.

That this construction indeed provides a functor χ_p follows directly from the Definition 2.2.57 of labeled bundles and the Definition 2.2.42 of labeled bordism composition. ⌟

Construction 2.2.68 (Total labeled 1-truss bundles of classifying functors). We describe a map

$$(\mathsf{F} : B \to \mathsf{TBord}^1_{/\!/\mathsf{C}}) \quad \mapsto \quad \pi_{\mathsf{F}} \equiv (\pi_{\underline{\mathsf{F}}} : \mathrm{Tot}(\underline{\mathsf{F}}) \to B, \mathrm{lbl}_{\mathsf{F}} : \mathrm{Tot}(\underline{\mathsf{F}}) \to \mathsf{C})$$

that takes a functor $\mathsf{F} : B \to \mathsf{TBord}^1_{/\!/\mathsf{C}}$ from a poset B to the category of C-labeled 1-trusses and their bordisms to an associated **total labeled 1-truss bundle** π_{F}.

We construct the labeled 1-truss bundle π_{F} as follows.

- The underlying 1-truss bundle $\pi_{\underline{\mathsf{F}}} : \mathrm{Tot}(\underline{\mathsf{F}}) \to B$ is the total bundle of the composite $\underline{\mathsf{F}}$ of the functor F with the label-forgetting functor $(\underline{-}) : \mathsf{TBord}^1_{/\!/\mathsf{C}} \to \mathsf{TBord}^1$ (see Terminology 2.2.49).
- The labeling functor $\mathrm{lbl}_{\mathsf{F}} : \mathrm{Tot}(\underline{\mathsf{F}}) \to \mathsf{C}$ is given on fibers over elements $x \in B$ as the labeling functor $\mathrm{lbl}_{\mathsf{F}(x)} : \underline{\mathsf{F}(x)} \to \mathsf{C}$ of the labeled 1-truss $\mathsf{F}(x) \in \mathsf{TBord}^1_{/\!/\mathsf{C}}$, and on fibers over non-identity arrows $f : [1] \to B$ as the labeling functor $\mathrm{lbl}_{\mathsf{F}(f)} : \underline{\mathsf{F}(f)} \to \mathsf{C}$ of the labeled 1-truss bordism $\mathsf{F}(f)$. ⌟

Example 2.2.69 (Classification for a labeled 1-truss bundle). Recall from Example 2.2.59 the labeled 1-truss bundle $p \equiv (\underline{p} : T \to \bar{\mathbb{T}}_1, \mathrm{lbl}_p : T \to \mathsf{B}\mathbb{F})$ with labeling in the monoid $\mathsf{B}\mathbb{F}$ described there. In Figure 2.40, on the left is that same labeled 1-truss bundle (with the labeling encoded by the colors of the arrows according to the convention for the monoid established in Figure 2.39 and recapitulated in this figure); on the right is the associated classifying functor $\chi_p : \bar{\mathbb{T}}_1 \to \mathsf{TBord}^1_{/\!/\mathsf{B}\mathbb{F}}$. (The inverse association taking that functor $\mathsf{F} : \bar{\mathbb{T}}_1 \to \mathsf{TBord}^1_{/\!/\mathsf{B}\mathbb{F}}$ to its total labeled bundle π_{F} is also indicated.) In the classifying category $\mathsf{TBord}^1_{/\!/\mathsf{B}\mathbb{F}}$, we only indicatively depict four of the eighteen objects and only four of the many morphisms among those objects; the two morphisms actually hit by this classifying functor are colored accordingly, along with their preimages in the base poset. ⌟

As in the unlabeled case, this correspondence, between labeled 1-truss bundles and functors into the category of labeled 1-trusses and their bordisms, is functorial, with respect to a notion of bordism of labeled 1-truss bundles.

Definition 2.2.70 (Bordisms of labeled 1-truss bundles and their composition). Given C-labeled 1-truss bundles p and q over a poset B, a **C-labeled 1-truss bundle bordism** $u : p \nrightarrow q$ is a C-labeled 1-truss bundle u over $B \times [1]$ such that $u|_{B \times \{0\}} = p$ and $u|_{B \times \{1\}} = q$.

The **composition** of two such labeled bordisms $u : p \nrightarrow q$ and $v : q \nrightarrow r$ is the labeled bordism $v \circ u : p \nrightarrow r$ whose restriction $(v \circ u)|_{\{x\} \times [1]}$ is the composite labeled bordism $v|_{\{x\} \times [1]} \circ u|_{\{x\} \times [1]}$, for all elements $x \in B$. ⌟

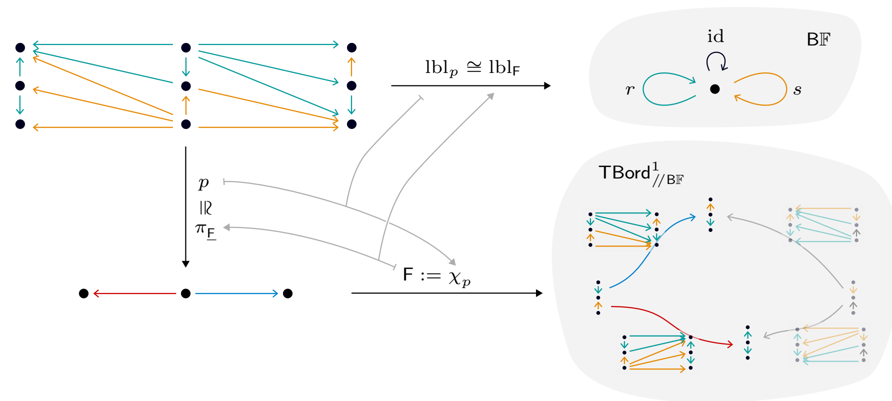


FIGURE 2.40. A labeled 1-truss bundle and its classifying functor.

NOTATION 2.2.71 (Categories of labeled 1-truss bundles and their bordisms). For a fixed base poset B and category C, the 'category of C-labeled 1-truss bundles and their bordisms', whose objects are C-labeled 1-truss bundles over B and whose morphisms are C-labeled 1-truss bundle bordisms, will be denoted $\mathsf{TBord}^1(B)_{/\!/\mathsf{C}}$. ⌟

REMARK 2.2.72 (Labeled 1-truss bundle isobordism need not be unique). By contrast with Observation 2.1.98, when there is an invertible labeled 1-truss bundle bordism (i.e. a 'labeled bundle isobordism'), that bordism need not be unique, for the simple reason that the labeling category may have non-trivial automorphisms. ⌟

There is a category of functors from a base poset B to the category $\mathsf{TBord}^1_{/\!/\mathsf{C}}$ of labeled 1-trusses and their bordisms, whose morphisms are natural transformations of functors; note that a natural transformation $\mathsf{N} : \mathsf{F} \Rightarrow \mathsf{G}$ between functors $\mathsf{F} : B \to \mathsf{TBord}^1_{/\!/\mathsf{C}}$ and $\mathsf{G} : B \to \mathsf{TBord}^1_{/\!/\mathsf{C}}$ is simply itself a functor $\mathsf{N} : B \times [1] \to \mathsf{TBord}^1_{/\!/\mathsf{C}}$. Equipped with the category of labeled 1-truss bundles and their bordisms, and with the category of classifying functors, we can describe the functorial correspondence, as follows.

OBSERVATION 2.2.73 (Classification and totalization functors for labeled 1-truss bundles). Given a poset B and a category C, there is an equivalence of categories

$$\chi_- : \mathsf{TBord}^1(B)_{/\!/\mathsf{C}} \rightleftarrows \mathrm{Fun}(B, \mathsf{TBord}^1_{/\!/\mathsf{C}}) : \pi_-$$

specified as follows.

The 'classification functor' χ_- takes a C-labeled 1-truss bundle p to its classifying functor $\chi_p : B \to \mathsf{TBord}^1_{/\!/\mathsf{C}}$, and a C-labeled 1-truss bundle bordism $u : p \Rightarrow q$ (by definition a labeled 1-truss bundle over $B \times [1]$) to its classifying functor $\chi_u : B \times [1] \to \mathsf{TBord}^1_{/\!/\mathsf{C}}$ viewed as a natural transformation $\chi_u : \chi_p \Rightarrow \chi_q$.

The 'totalization functor' π_{-} takes a functor $\mathsf{F} : B \to \mathsf{TBord}^1_{/\!/\mathsf{C}}$ to its total C-labeled 1-truss bundle π_{F}, and a natural transformation $\mathsf{N} : B \times [1] \to \mathsf{TBord}^1_{/\!/\mathsf{C}}$ to its total labeled 1-truss bundle π_{N}. ⌟

Remark 2.2.74 (Classifying categorical labeled 1-truss bundles). Recall from Remark 2.1.85 the notion of a categorical 1-truss bundle $\mathsf{T} \to \mathsf{B}$ over a base category B. A 'categorical C-labeled 1-truss bundle' is simply a categorical 1-truss bundle $\underline{p} : \mathsf{T} \to \mathsf{B}$ together with a labeling functor $\mathrm{lbl}_p : \mathsf{T} \to \mathsf{C}$. Remark 2.1.102 noted that TBord^1 provides, in fact, a classifying category for categorical, not just posetal, 1-truss bundles. Similarly, the above classification and totalization constructions carry over to the categorical case, showing that $\mathsf{TBord}^1_{/\!/\mathsf{C}}$ is a classifying category for categorical, not just posetal, C-labeled 1-truss bundles. ⌟

After having developed the machinery of truss induction in generality in Section 2.2.2, notice that we have so far used truss induction only over the 2-simplex (namely, in the proof of Lemma 2.2.43). The full power of truss induction comes to bear when we allow for truss bundles labeled in an ∞-*category* $\mathcal{C}$. For a 1-category C, we had a 1-category $\mathsf{TBord}^1_{/\!/\mathsf{C}}$ as a classifying category for C-labeled 1-truss bundles; for an ∞-category $\mathcal{C}$, we can define an analogous ∞-category $\mathcal{T}Bord^1_{/\!/\mathcal{C}}$ as a classifying category for $\mathcal{C}$-labeled 1-truss bundles, as follows. (Note though that we will not use ∞-categorical labels, and the next remark can be safely skipped.)

Remark 2.2.75 (Quasicategories of labeled 1-trusses and their bordisms). Let $\mathcal{C}$ be a quasicategory, i.e. a simplicial set that has inner horn fillers [**BV06, Joy02**]. There is a 'quasicategory of $\mathcal{C}$-labeled 1-trusses and their bordisms', denoted $\mathcal{T}Bord^1_{/\!/\mathcal{C}}$. The k-simplices of this quasicategory are the pairs $(\underline{S}, \mathrm{lbl}_S)$ consisting of a 1-truss bundle $\underline{S} \to [k]$ over the k-simplex, and a functor of quasicategories $\mathrm{lbl}_S : \underline{S} \to \mathcal{C}$.

The proof of Lemma 2.2.43, that composition of C-labeled 1-truss bordisms, and therefore the category $\mathsf{TBord}^1_{/\!/\mathsf{C}}$, is well defined, only used truss induction over the 2-simplex. That the simplicial set $\mathcal{T}Bord^1_{/\!/\mathcal{C}}$ is itself a quasicategory (i.e. has the 'composition' of inner horn fillers) follows, roughly as in that proof, but using truss induction over general k-simplices. ⌟

2.2.3.4. ◊Pullback, dualization, and suspension of labeled 1-truss bundles. The pullback, dualization, and suspension constructions carry over from the unlabeled to the labeled case, as follows.

Construction 2.2.76 (Pullbacks of labeled 1-truss bundles). Given a C-labeled 1-truss bundle $p \equiv (\underline{p} : T \to B, \mathrm{lbl}_p : T \to \mathsf{C})$ over a poset B, and a poset map $G : A \to B$, the pullback of the bundle (along the map G) is the C-labeled 1-truss bundle $G^*p \equiv (\underline{G^*p}, \mathrm{lbl}_{G^*p})$, whose underlying 1-truss bundle $\underline{G^*p}$ is the pullback $G^*\underline{p} : G^*T \to A$, and whose labeling functor lbl_{G^*p} is the composite $\mathrm{lbl}_p \circ \mathrm{Tot}G : G^*T \to \mathsf{C}$. (Recall from Construction 2.1.103 the unlabeled pullback $G^*\underline{p} : G^*T \to A$ and its total poset map $\mathrm{Tot}G : G^*T \to T$.) ⌟

We can display the labeled pullback bundle $G^*p \equiv (\underline{G^*p} : G^*T \to A, \mathrm{lbl}_{G^*p} : G^*T \to \mathsf{C}) = (G^*\underline{p}, \mathrm{lbl}_p \circ \mathrm{Tot}G)$ as

$$\begin{array}{ccccc} A & \xleftarrow{G^*\underline{p}} & G^*T & \xrightarrow{\mathrm{lbl}_p \circ \mathrm{Tot}G} & \\ {\scriptstyle G}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{Tot}G} & & \mathsf{C} \\ B & \xleftarrow{\underline{p}} & T & \xrightarrow{\mathrm{lbl}_p} & \end{array}$$

Of course, when the poset map $G : A \hookrightarrow B$ is a subposet, the pullback specializes to the restriction of labeled 1-truss bundles.

Remark 2.2.77 (Pullback of labeled bundles via classifying functors). As in the unlabeled case, the labeled pullback bundle may be expressed in terms of classifying functors. Given a labeled 1-truss bundle p over a poset B and a poset map $G : A \to B$, the classifying functor $\chi_{G^*p} : A \to \mathsf{TBord}^1_{/\!/\mathsf{C}}$ of the pullback is simply the composite $\chi_p \circ G$ of the poset map G with the classifying functor $\chi_p : B \to \mathsf{TBord}^1_{/\!/\mathsf{C}}$ of the initial labeled bundle. ⌟

Fiberwise dualization provides a dualization of labeled 1-truss bundles, as follows.

Construction 2.2.78 (Dualization of labeled 1-truss bundles and their maps). Given a C-labeled 1-truss bundle $p \equiv (\underline{p}, \mathrm{lbl}_p)$, its dual is the C^{op}-labeled 1-truss bundle $p^\dagger \equiv (\underline{p^\dagger}, \mathrm{lbl}_{p^\dagger}) = ((\underline{p})^\dagger, (\mathrm{lbl}_p)^{\mathrm{op}})$, whose underlying 1-truss bundle is the dual of the underlying 1-truss bundle of p (i.e., has opposite face order and dimension map, see Construction 2.1.107), and whose labeling is the opposite of the labeling of p.

Given a labeled 1-truss bundle map $F : p \to q$ (consisting of an underlying map $\underline{F} : \underline{p} \to \underline{q}$ and a relabeling functor $\mathrm{lbl}_F : \mathsf{C} \to \mathsf{D}$), its dual is the labeled bundle map $F^\dagger : p^\dagger \to q^\dagger$ with dual underlying 1-truss bundle map $\underline{F^\dagger} := (\underline{F})^\dagger : (\underline{p})^\dagger \to (\underline{q})^\dagger$ (i.e., the same map of sets as the map $\underline{F}$ itself, see Construction 2.1.107), and opposite relabeling functor $\mathrm{lbl}_{F^\dagger} := (\mathrm{lbl}_F)^{\mathrm{op}} : \mathsf{C}^{\mathrm{op}} \to \mathsf{D}^{\mathrm{op}}$.

We therefore have a covariant involutive functor of labeled 1-truss bundles:

$$\dagger : \mathsf{LblTrsBun}_1 \cong \mathsf{LblTrsBun}_1.$$

Note that this functor preserves neither the base nor the labeling category. ⌟

Construction 2.2.79 (Dualization of labeled 1-truss bundles and their bordisms). For a C-labeled 1-truss bundle bordism $u : p \Rightarrow q$, given by a C-labeled 1-truss bundle $u \equiv (\underline{u} : U \to B \times [1], \mathrm{lbl}_u : U \to \mathsf{C})$, its dual is the labeled bundle bordism $u^\dagger : q^\dagger \Rightarrow p^\dagger$ given by the dual labeled bundle $u^\dagger \equiv (\underline{u^\dagger}, \mathrm{lbl}_{u^\dagger}) = ((\underline{u})^\dagger : U^\dagger \to B^{\mathrm{op}} \times [1], (\mathrm{lbl}_u)^{\mathrm{op}} : U^\dagger \to \mathsf{C}^{\mathrm{op}})$.

Dualization of bordisms is thus contravariant, giving an involutive isomorphism:

$$\dagger : \mathsf{TBord}^1(B)_{/\!/\mathsf{C}} \cong (\mathsf{TBord}^1(B^{\mathrm{op}})_{/\!/\mathsf{C}^{\mathrm{op}}})^{\mathrm{op}}.$$

When the labeling category is trivial, this specializes to the dualization of unlabeled bundles and their bordisms from Construction 2.1.108. When instead the base is a point, this specializes to an involutive isomorphism on labeled 1-trusses and their bordisms:

$$\dagger : \mathsf{TBord}^1_{/\!/\mathsf{C}} \cong (\mathsf{TBord}^1_{/\!/\mathsf{C}^{\mathrm{op}}})^{\mathrm{op}}.$$

⌟

Remark 2.2.80 (Dual labeled bundles via classifying functors). The dualization of labeled bundles may be reexpressed using classifying functors as follows. Given a labeled 1-truss bundle p, with classifying functor $\chi_p : B \to \mathsf{TBord}^1_{/\!/\mathsf{C}}$, its dual labeled bundle $p^\dagger$ has classifying functor

$$(\chi_{p^\dagger} : B^{\mathrm{op}} \to \mathsf{TBord}^1_{/\!/\mathsf{C}^{\mathrm{op}}}) = \big(B \xrightarrow{\chi_p} \mathsf{TBord}^1_{/\!/\mathsf{C}} \xrightarrow{\dagger} (\mathsf{TBord}^1_{/\!/\mathsf{C}^{\mathrm{op}}})^{\mathrm{op}}\big)^{\mathrm{op}}.$$

This association of classifying functors $\chi_p \mapsto (\dagger \circ \chi_p)^{\mathrm{op}}$ is functorial and reproduces the involutive isomorphism of the previous Construction 2.2.79. ⌟

Finally, straightforwardly, we have the labeled version of the suspension of 1-truss bundles from Construction 2.1.111.

Remark 2.2.81 (Suspension of labeled 1-truss bundles). Assume that the category C has both initial and terminal objects. The suspension Σp of a C-labeled 1-truss bundle p has, of course, underlying bundle $\underline{\Sigma p}$ being the suspension $\Sigma\underline{p} : \Sigma T \to \Sigma B$ of the underlying 1-truss bundle $\underline{p} : T \to B$; the labeling functor $\mathrm{lbl}_{\Sigma p} : \Sigma T \to \mathsf{C}$ is equal to the labeling functor lbl_p on the equator $T \subset \Sigma T$ and sends the initial and terminal objects of ΣT to the initial and terminal objects of the labeling category C. ⌟

2.3. $\Diamond n$-Trusses, bordisms, bundles, and blocks

1-Trusses have provided a robust combinatorial model of framed stratified 1-dimensional spaces. 1-Truss bundles encode families of such spaces and so appear to model certain multi-dimensional stratified spaces; however the stratified topology of the total spaces of those families is criticially constrained by the nature of the stratifications of the bases. To obtain a faithful, universal combinatorial model of framed stratified n-dimensional spaces we must, as promised, iterate the notion of 1-truss bundles. An *n-truss* is a 1-truss bundle over a 1-truss bundle over a 1-truss bundle, and so forth, over, in the end, a 1-truss. An example of an n-truss is illustrated on the left in Figure 2.41; the base 1-truss poset T_1 has a single (red) singular element, the 2-truss poset T_2 fibers over the 1-truss poset with the singular elements forming an X pattern, and the 3-truss poset T_3 fibers over the 2-truss poset with the singular elements forming a braid pattern that resolves the singular crossing of the 2-truss X. On the right of that figure are corresponding geometric stratifications of the open 1-, 2-, and 3-cubes—corresponding for instance in the sense that the fundamental posets of those stratified cubes are the adjacent truss posets. Needless to say this juxtaposition of n-trusses and stratified spaces is meant to suggestively preview the fact that the theory of n-trusses will indeed, as imagined, provide a resilient combinatorial model of framed stratified spaces of any dimension.

Now, any model of such stratified spaces must account for stratified families thereof, and so there is an attendent basic notion of *n-truss bordism*, which specifies a family of n-trusses over the combinatorial stratified 1-simplex, and furthermore a notion of *n-truss bundle*, which encodes a family of n-trusses over a more general stratified poset. Recall that composition of 1-truss bordism functorial relations between 1-truss posets provided a transparent means of composing 1-truss bordisms. By contrast, even the existence of a composition of n-truss bordisms is neither geometrically nor combinatorially evident. Constructing such a composition will rely critically on the method of truss induction developed in the previous Section 2.2. In practice, that construction will occur in mutually inductive tandem with establishing that n-truss bundles are classified by functors into a *recursive category of n-trusses*, defined as 1-trusses labeled in 1-trusses labeled in 1-trusses labeled in, and so forth, labeled in, finally, 1-trusses.

Outline. In Section 2.3.1, we introduce n-trusses as towers of 1-truss bundles, and n-truss bordisms as such towers over a combinatorial 1-simplex; we also define a recursive category of n-trusses in terms of 1-trusses labeled in 1-trusses labeled in 1-trusses and so on iteratively. In Section 2.3.2, we define general n-truss bundles as towers now over arbitrary posets, and prove that n-truss bundles are classified by functors into the recursive category of n-trusses. Finally in Section 2.3.3, we describe n-truss blocks, the component combinatorial shapes from which all n-trusses are built, and block sets, the presheaves on the category of such blocks.

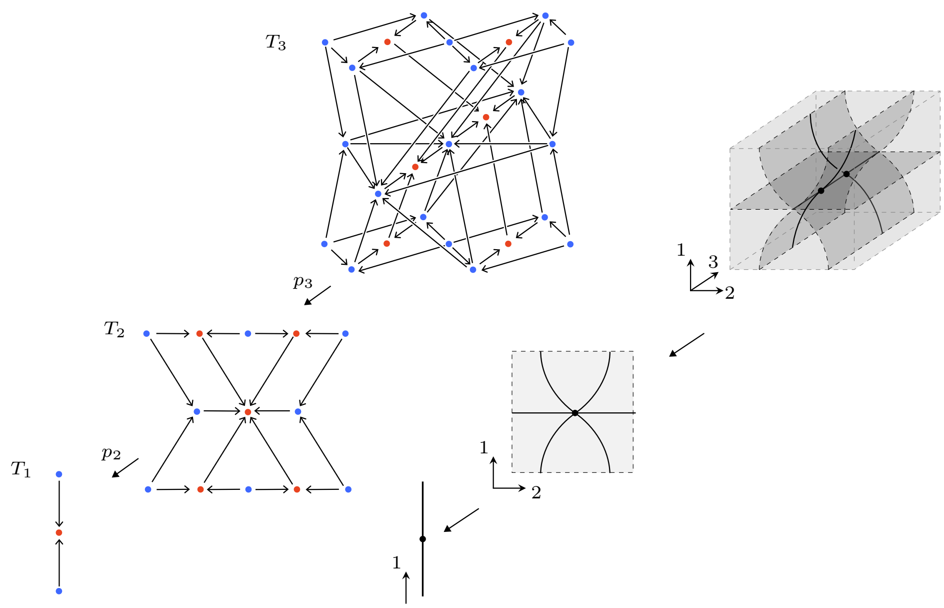


FIGURE 2.41. A 3-truss and its corresponding stratifications.

2.3.1. $\Diamond n$-Trusses and their bordisms.

SYNOPSIS. We define n-trusses as towers of 1-truss bundles, with each bundle having base poset being the total face poset of the previous bundle. We similarly introduce n-truss bordisms as towers of 1-truss bundles over the combinatorial 1-simplex, describe the succession of functorial relations determined by the stages of such towers, and define the composition of n-truss bordisms in terms of the composites of those functorial relations; this will provide a category of n-trusses and their bordisms. We then apply the n-fold iteration of the labeled 1-truss bordism functor to obtain an alternative, recursively-defined version of the category of n-trusses and their bordisms.

2.3.1.1. $\Diamond n$-Trusses as towers of 1-truss bundles. A 1-truss considered just with its face order is, of course, a poset; we have a notion of 1-truss bundle over any poset. A 2-truss is then simply a 1-truss bundle over the 1-truss face poset. The face order of that 1-truss bundle provides the total poset of the 2-truss. A 3-truss is then a 1-truss bundle over the 2-truss total poset. And so on, as follows.

DEFINITION 2.3.1 (n-Trusses). An **n-truss** is a sequence of 1-truss bundles

$$T_n \xrightarrow{p_n} T_{n-1} \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_2} T_1 \xrightarrow{p_1} T_0 = [0]$$

in which the base poset of each bundle p_i is the total poset of the subsequent bundle p_{i-1}. ⌟

NOTATION 2.3.2 (n-Trusses). We typically abbreviate the sequence of bundles $\{T_i \xrightarrow{p_i} T_{i-1}\}$ by an indicative letter, referring to the whole n-truss simply as T. (We will often refer to the sequence informally as a 'tower of bundles' and to its kth element T_k as the 'k-stage' of the tower.) We call the face order poset $(T_n, \trianglelefteq)$ of the first bundle the 'total poset' of the n-truss; we almost always let the face order relation be implicit, denoting the total poset simply T_n. ⌟

TERMINOLOGY 2.3.3 (Open and closed n-trusses). We call an n-truss T 'open', respectively 'closed', when all its constituent 1-truss bundles $p_i : T_i \to T_{i-1}$ are open, respectively closed. ⌟

EXAMPLE 2.3.4 (A 2-truss). In Figure 2.42, on the left we illustrate a 2-truss T. The first bundle $p_1 : T_1 \to T_0$ has base poset $T_0 = [0]$ and so its total poset is simply a 1-truss T_1. The total poset T_2 of the second bundle $p_2 : T_2 \to T_1$, with its 1-truss fibers and bordism transitions between them, evidently has a 2-dimensional character.

On the right of that figure, we illustrate a tower of stratified bundles of stratified intervals, whose fundamental poset tower is the given 2-truss face poset tower. That this juxtaposition comes from a faithful correspondence, between truss towers and towers of appropriately framed suitably stratified bundles, will be of crucial concern, and is established rather later on. ⌟

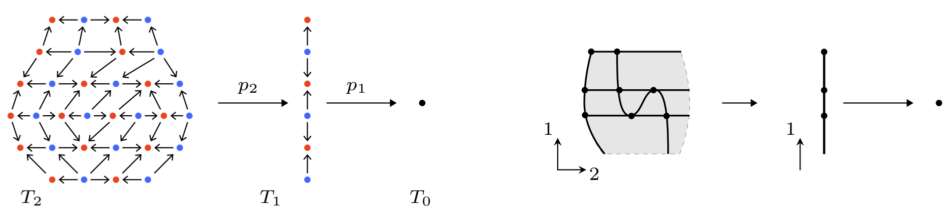


FIGURE 2.42. A 2-truss and corresponding stratifications.

EXAMPLE 2.3.5 (An open 3-truss). Earlier in Figure 2.41, on the left we illustrated an open 3-truss T. As before and as becomes especially prudent in 3-dimensional examples, we only depicted generating arrows of the truss face posets; all other arrows are composites of the given ones. On the right of the figure, we illustrated a corresponding tower of stratifications, of the open 3-cube, 2-cube, and 1-cube; the fundamental poset tower is the given 3-truss face poset tower. Notice that each of these cube stratifications is a refinement of the pullback of the previous cube stratification; the structure of that refinement reflects the geometric relationships among the singular elements in the correlative truss poset. ⌟

Recall that a labeled 1-truss bundle is a 1-truss bundle with a labeling functor from the total poset of the bundle. Similarly, a labeled n-truss is just an n-truss with a labeling functor from its total poset. That labeling will, most immediately, provide a means of encoding yet further truss bundles over that total poset, and, later on and most practically, provide a means of encoding global stratification structures on the total poset.

DEFINITION 2.3.6 (Labeled n-trusses). Given a category C, a **C-labeled n-truss** T is a pair $(\underline{T}, \mathrm{lbl}_T)$ consisting of an n-truss $\underline{T} = (T_n \to T_{n-1} \to \cdots \to T_0 = [0])$ and a functor $\mathrm{lbl}_T : T_n \to \mathsf{C}$ from the total poset of the n-truss to the category. ⌟

We refer as before to the n-truss $\underline{T}$ as the 'underlying n-truss', and to the functor lbl_T as the 'labeling functor'. We can display the data of a labeled n-truss as a 'labeled sequence':

$$\mathsf{C} \xleftarrow{\mathrm{lbl}_T} T_n \xrightarrow{p_n} T_{n-1} \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_2} T_1 \xrightarrow{p_1} T_0 = [0]$$

EXAMPLE 2.3.7 (A labeled 2-truss). Previously in Figure 2.39 we illustrated a 1-truss bundle labeled in a monoid. Since the base poset there happens to be a 1-truss, that in fact is already an example of a labeled 2-truss. ⌟

EXAMPLE 2.3.8 (A labeled 3-truss). In Figure 2.43, we illustrate an open 3-truss labeled in the poset $[1] \times [1]$. As before, the labeling is indicated by color matching the objects of the truss total poset and their images in the labeling poset. ⌟

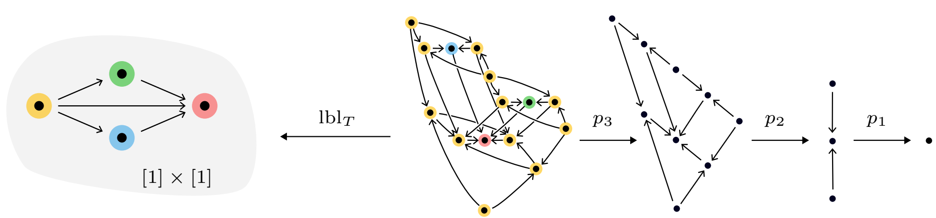


FIGURE 2.43. A labeled open 3-truss.

2.3.1.2. $\lozenge n$-Truss bordisms and their composition. Recall that 1-trusses are a combinatorial model of stratified intervals, and 1-truss bordisms are designed to provide a combinatorial model of constructible bundles of stratified intervals over the stratified 1-simplex; as such, 1-truss bordisms constitute bundles of 1-trusses over the combinatorial 1-simplex. Similarly, n-truss bordisms are, intuitively and functionally speaking, bundles of n-trusses over the combinatorial 1-simplex.

DEFINITION 2.3.9 (n-Truss bordisms). An **n-truss bordism** is a sequence of 1-truss bundles

$$R_n \xrightarrow{p_n} R_{n-1} \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_2} R_1 \xrightarrow{p_1} R_0 = [1]$$

in which the base poset of each bundle is the total poset of the subsequent bundle. ⌟

As in the case of n-trusses, we typically compress the sequence of bundles $\{R_i \xrightarrow{p_i} R_{i-1}\}$ to an indicative letter, referring to the whole n-truss bordism as for instance R. We call the face order poset $(R_n, \trianglelefteq)$ the 'total poset' of the n-truss bordism, and abbreviate it simply if abusively R_n.

Example 2.3.10 (A 2-truss bordism). In Figure 2.44, we illustrate a 2-truss bordism. The portion of the 2-truss bordism tower eventually projecting to $0 \in [1]$ is itself a 2-truss, and the portion eventually projecting to $1 \in [1]$ is similarly a 2-truss; the bordism provides a transition from that domain 2-truss to that codomain 2-truss. ⌟

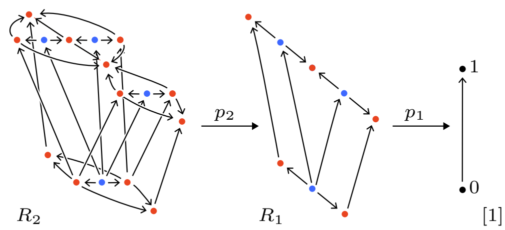


Figure 2.44. A 2-truss bordism.

Definition 2.3.11 (Labeled n-truss bordism). Given a category C, a **C-labeled n-truss bordism** R is a pair $(\underline{R}, \mathrm{lbl}_R)$ consisting of an n-truss bordism $\underline{R} = (R_n \to R_{n-1} \to \cdots \to R_0 = [1])$ and a functor $\mathrm{lbl}_R : R_n \to \mathsf{C}$ from its total poset to the category. ⌟

As expected we refer to the 'underlying n-truss bordism' $\underline{R}$ and the 'labeling functor' lbl_R. We typically display the labeled n-truss bordism as a labeled sequence:

$$\mathsf{C} \xleftarrow{\mathrm{lbl}_R} R_n \xrightarrow{p_n} R_{n-1} \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_2} R_1 \xrightarrow{p_1} R_0 = [1]$$

Of course, n-truss bordisms labeled in the terminal category $\mathsf{C} = *$ are simply n-truss bordisms.

Example 2.3.12 (A labeled 2-truss bordism). In Figure 2.45, we illustrate a labeled 2-truss bordism. Note that the 2-truss bordism tower $R_2 \xrightarrow{p_2} R_1 \xrightarrow{p_1} R_0$ makes up half of the portion $T_3 \xrightarrow{p_3} T_2 \xrightarrow{p_2} T_1$ of the tower of the 3-truss in Figure 2.41. On the left is a labeling functor with poset target. The functor is indicated by color coding the preimages of the objects of the labeling poset; we also color code the preimages of the identity morphisms of the two maximal elements. ⌟

Terminology 2.3.13 (Domain and codomain of a labeled n-truss bordism). Given a C-labeled n-truss bordism $R = (\underline{R}, \mathrm{lbl}_R)$, its 'domain' $\mathrm{dom}(R)$ is the C-labeled n-truss $T^{(0)}$, whose underlying n-truss $\underline{T^{(0)}}$ is obtained by an iterated restriction of the tower of bundles $\underline{R}$ to $0 \in [1]$, and whose labeling $\mathrm{lbl}_{T^{(0)}}$ is the restriction of the labeling lbl_R to the total poset $T^{(0)}_n$ of $\underline{T^{(0)}}$. Similarly the 'codomain' $\mathrm{cod}(R)$ is the C-labeled n-truss $T^{(1)}$ obtained by restricting the tower of bundles to $1 \in [1]$. That is, the domain and codomain

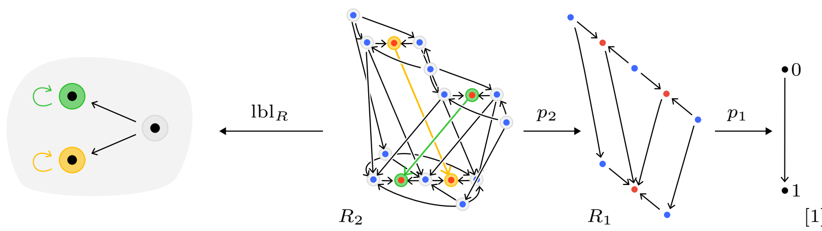


FIGURE 2.45. A labeled 2-truss bordism.

are the top and bottom rows in the following diagram of 1-truss bundle restrictions, as in Construction 2.1.103.

$$\begin{array}{ccccccccccc}
 & & T^{(0)}_n & \longrightarrow & T^{(0)}_{n-1} & \longrightarrow & \cdots & \longrightarrow & T^{(0)}_1 & \longrightarrow & [0] \\
 & \swarrow^{\mathrm{lbl}_{T^{(0)}}} & \downarrow & \lrcorner & \downarrow & \lrcorner & \cdots & & \downarrow & \lrcorner & \downarrow 0 \\
\mathsf{C} & \xleftarrow{\mathrm{lbl}_R} & R_n & \xrightarrow{p_n} & R_{n-1} & \xrightarrow{p_{n-1}} & \cdots & \xrightarrow{p_2} & R_1 & \xrightarrow{p_1} & [1] \\
 & \nwarrow_{\mathrm{lbl}_{T^{(1)}}} & \uparrow & \urcorner & \uparrow & \urcorner & \cdots & & \uparrow & \urcorner & \uparrow 1 \\
 & & T^{(1)}_n & \longrightarrow & T^{(1)}_{n-1} & \longrightarrow & \cdots & \longrightarrow & T^{(1)}_1 & \longrightarrow & [0]
\end{array}$$

We will suggestively denote the labeled n-truss bordism as a morphism $R : \mathrm{dom}(R) \twoheadrightarrow \mathrm{cod}(R)$. ⌟

Needless to say we would like to define a composition of n-truss bordisms (and their labeled counterparts). That is, given n-truss bordisms $R^{(01)} : T^{(0)} \twoheadrightarrow T^{(1)}$ and $R^{(12)} : T^{(1)} \twoheadrightarrow T^{(2)}$, we would like to form a composite n-truss bordism $R^{(02)} : T^{(0)} \twoheadrightarrow T^{(2)}$. As in the case of 1-truss bordisms, the composite n-truss bordism is given in terms of the composition of certain functorial relations, which we describe presently.

TERMINOLOGY 2.3.14 (Functorial relations of an n-truss bordism). For an n-truss bordism $R : T \twoheadrightarrow S$, its '$k$-stage functorial relation'

$$\mathsf{rel}^R_k : (T_k, \trianglelefteq) \twoheadrightarrow (S_k, \trianglelefteq)$$

is defined, for k-stage elements $t \in T_k$ and $s \in S_k$, by declaring

$$\mathsf{rel}^R_k(t, s) \qquad \Longleftrightarrow \qquad (t \trianglelefteq s) \text{ in } (R_k, \trianglelefteq).$$

Note that relation is indeed functorial, simply because $(R_k, \trianglelefteq)$ is a poset. ⌟

Note that in the case of 1-truss bordisms, we did not introduce separate notation for the (interchangable) functorial relation and poset structures; see Terminology 2.1.72 and Notation 2.1.73. By contrast, in the case of n-truss bordisms, it is clarifying to have the notational distinction between the functorial relation rel^R_k and the poset $(R_k, \trianglelefteq)$; however these remain interchangable in the following sense.

Observation 2.3.15 (n-Truss bordisms are determined by their functorial relations). For fixed n-trusses T and S, there is at most one n-truss bordism $R : T \nrightarrow S$ with specified k-stage relations $\mathsf{rel}^R_k : (T_k, \trianglelefteq) \nrightarrow (S_k, \trianglelefteq)$; in other words, an n-truss bordism with given domain and codomain is determined by its associated functorial relations.

Specifically, given n-trusses T and S, along with k-stage relations $\mathsf{rel}^R_k : (T_k, \trianglelefteq) \nrightarrow (S_k, \trianglelefteq)$, the n-truss bordism R is necessarily given as follows: the set R_k is the union of the k-stage sets T_k and S_k, and the projection $R_k \to R_{k-1}$ is the union of the projections $T_k \to T_{k-1}$ and $S_k \to S_{k-1}$; the face order poset $(R_k, \trianglelefteq)$ restricts to the face order posets $(T_k, \trianglelefteq)$ and $(S_k, \trianglelefteq)$, and there is a face poset arrow $(t \in T_k) \trianglelefteq (s \in S_k)$ exactly when there is a relation $\mathsf{rel}^R_k(t, s)$; the frame order poset $(R_k, \preceq)$ is simply the union of the frame order posets $(T_k, \preceq)$ and $(S_k, \preceq)$; finally, the dimension map $\dim : (R_k, \trianglelefteq) \to [1]^{\mathrm{op}}$ is determined element-wise by the dimension maps on the posets $(T_k, \trianglelefteq)$ and $(S_k, \trianglelefteq)$. ⌟

Leveraging this observation, we may now attempt to define composition of n-truss bordisms via the composition of the associated k-stage functorial relations.

Definition 2.3.16 (Composition of n-truss bordisms). Given n-truss bordisms $R^{(01)} : T^{(0)} \nrightarrow T^{(1)}$ and $R^{(12)} : T^{(1)} \nrightarrow T^{(2)}$, the **composite n-truss bordism** $R^{(02)} \equiv R^{(12)} \circ R^{(01)} : T^{(0)} \nrightarrow T^{(2)}$ is the n-truss bordism whose functorial relations are the composites of the functorial relations of the component bordisms; that is, for all $1 \leq k \leq n$, the composite k-stage functorial relation is

$$\mathsf{rel}^{R^{(02)}}_k := \mathsf{rel}^{R^{(12)}}_k \circ \mathsf{rel}^{R^{(01)}}_k. \qquad ⌟$$

Of course, it is not immediately clear that the given collection of composite functorial relations $\{\mathsf{rel}^{R^{(12)}}_k \circ \mathsf{rel}^{R^{(01)}}_k\}$ is in fact the collection of k-stage relations of an n-truss bordism $R^{(02)}$; that is, it remains to show that this definition indeed specifies a composite n-truss bordism, as its phrasing presupposes. Allowing for now that presupposition, we may attempt to define the more general labeled composition, as follows.

Definition 2.3.17 (Composition of labeled n-truss bordisms). Given composable C-labeled n-truss bordisms $R^{(01)} \equiv (\underline{R}^{(01)}, \mathrm{lbl}_{R^{(01)}})$ and $R^{(12)} \equiv (\underline{R}^{(12)}, \mathrm{lbl}_{R^{(12)}})$, the **composite labeled n-truss bordism** $R^{(02)} \equiv R^{(12)} \circ R^{(01)}$ is the labeled n-truss bordism $(\underline{R}^{(02)}, \mathrm{lbl}_{R^{(02)}})$, whose underlying n-truss bordism is $\underline{R}^{(02)} := \underline{R}^{(12)} \circ \underline{R}^{(01)}$ and whose labeling is given by

$$\mathrm{lbl}_{R^{(02)}}(x_0 \trianglelefteq x_2) := \mathrm{lbl}_{R^{(12)}}(x_1 \trianglelefteq x_2) \circ \mathrm{lbl}_{R^{(01)}}(x_0 \trianglelefteq x_1)$$

whenever $x_0 \trianglelefteq x_1$ and $x_1 \trianglelefteq x_2$ are composable arrows in the total posets $\underline{R}^{(01)}_n$ and $\underline{R}^{(12)}_n$ respectively. ⌟

It is by no means evident that the value of the labeling functor $\mathrm{lbl}_{R^{(02)}}(x_0 \trianglelefteq x_2)$ does not depend on the factorization $x_0 \trianglelefteq x_1 \trianglelefteq x_2$; it thus remains to be verified that this definition indeed specifies such a functor, as it implicitly claims to.

LEMMA 2.3.18 (Composition of labeled n-truss bordisms is well defined). *The specification in Definition 2.3.16 provides a well-defined n-truss bordism $R^{(02)}$ with the given k-stage relations $\mathsf{rel}_k^{R^{(02)}}$, and the specification in Definition 2.3.17 provides a well-defined labeling functor $\mathrm{lbl}_{R^{(02)}} : R_n^{(02)} \to \mathsf{C}$ with the given labels $\mathrm{lbl}_{R^{(02)}}(x_0 \trianglelefteq x_2)$.*

A direct proof of this result would involve, among other things, a tower of inductive arguments each stage of which is itself a truss induction. We instead defer the matter until we can give a more nimble proof via an interleaved induction involving the classification of n-truss bundles; Lemma 2.3.18 will be established as part of Lemma 2.3.48.

EXAMPLE 2.3.19 (Composition of 2-truss bordisms). In Figure 2.46 we illustrate two composable 2-truss bordisms

$$R^{(12)} = (R_2^{(12)} \xrightarrow{p_2^{(12)}} R_1^{(12)} \xrightarrow{p_1^{(12)}} [1])$$
$$R^{(01)} = (R_2^{(01)} \xrightarrow{p_2^{(01)}} R_1^{(01)} \xrightarrow{p_1^{(01)}} [1])$$

and their composite $R^{(12)} \circ R^{(01)} =: R^{(02)} = (R_2^{(02)} \xrightarrow{p_2^{(02)}} R_1^{(02)} \xrightarrow{p_1^{(02)}} [1])$. For legibility we have drawn only the generating arrows at all stages of the bordisms. Note that the 1-truss bordisms $R_1^{(01)}$, $R_1^{(12)}$, and their composite $R_1^{(02)}$ are exactly those depicted in Figure 2.16. ⌟

NOTATION 2.3.20 (Categories of labeled n-trusses and their bordisms). Given a category C, the 'category of C-labeled n-trusses and their bordisms', whose objects are C-labeled n-trusses and whose morphisms are C-labeled n-truss bordisms, will be denoted $n\mathsf{TBord}_{/\!/\mathsf{C}}$. ⌟

NOTATION 2.3.21 (The category of n-trusses and their bordisms). Of course, we may and will consider the case where the labeling is in the terminal category and thus carries no information whatsoever. The resulting 'category of n-trusses and their bordisms', with objects n-trusses and morphisms n-truss bordisms, will be denoted $n\mathsf{TBord} \equiv n\mathsf{TBord}_{/\!/*}$. ⌟

Note that forgetting the labeling provides a functor $n\mathsf{TBord}_{/\!/\mathsf{C}} \to n\mathsf{TBord}$.

OBSERVATION 2.3.22 (The terminal and initial n-trusses). The terminal object of $n\mathsf{TBord}$ is the n-truss $\bar{\mathbb{T}}_0^n = (p_n, p_{n-1}, \ldots, p_1)$ in which every bundle p_i is trivial, with fiber the trivial closed 1-truss $\bar{\mathbb{T}}_0$. Similarly, the initial object of $n\mathsf{TBord}$ is the n-truss $\mathring{\mathbb{T}}_0^n = (p_n, p_{n-1}, \ldots, p_1)$ in which every bundle p_i is trivial, with fiber the trivial open 1-truss $\mathring{\mathbb{T}}_0$. ⌟

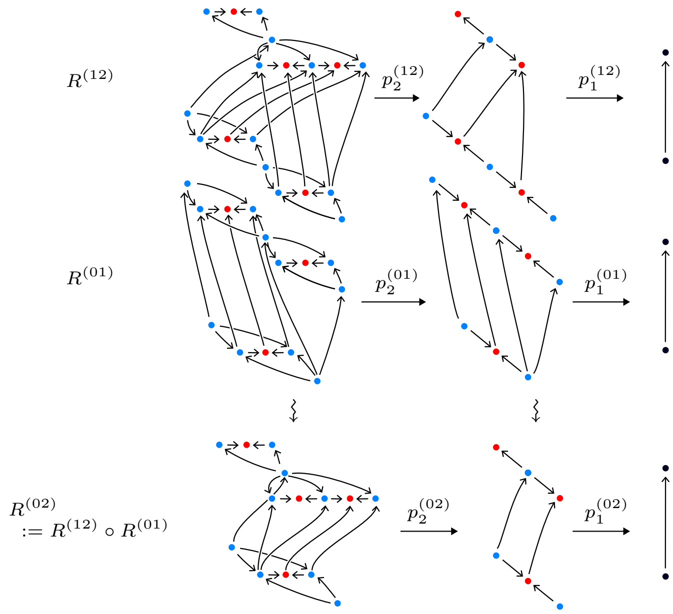


FIGURE 2.46. Composition of 2-truss bordisms.

2.3.1.3. ◊The recursive category of n-trusses and their bordisms. That there is a composition of labeled n-truss bordisms (if not yet obviously the specific one given before in Definitions 2.3.16 and 2.3.17) is an almost unsettlingly slick consequence of reinterpreting n-trusses and their bordisms as 1-truss-labeled $(n-1)$-trusses and their bordisms, and therefore by inductive iteration as 1-trusses labeled in 1-trusses labeled in 1-trusses and so on, as follows.

Recall from Definition 2.2.50 the labeled 1-truss bordism endofunctor $\mathsf{TBord}^1_{/\!/-} : \mathsf{Cat} \to \mathsf{Cat}$, that takes a category C to the category $\mathsf{TBord}^1_{/\!/\mathsf{C}}$ of C-labeled 1-trusses and their bordisms. We promised to iterate that functor; here we go.

DEFINITION 2.3.23 (The iterated labeled 1-truss bordism functor). The **n-fold iterated labeled 1-truss bordism functor**, denoted $\mathsf{TBord}^n_{/\!/-}$, is the composite

$$\mathsf{TBord}^1_{/\!/-} \circ \mathsf{TBord}^1_{/\!/-} \circ \cdots \circ \mathsf{TBord}^1_{/\!/-} : \mathsf{Cat} \to \mathsf{Cat}$$

with n instances of the labeled 1-truss bordism functor. ⌟

Naturally when $n = 0$, we take the functor $\mathsf{TBord}^0_{/\!/-}$ to be the identity functor on Cat. Evaluating the n-fold labeled bordism functor at a specific labeling category $\mathsf{C} \in \mathsf{Cat}$ provides the following category.

NOTATION 2.3.24 (The recursive category of C-labeled n-trusses and their bordisms). Given a category C, the category $\mathsf{TBord}^n_{/\!/\mathsf{C}}$ is called the 'recursive category of C-labeled n-trusses and their bordisms'. ⌟

The name of this category telegraphs an expectation about its objects and morphisms, which we make precise as follows.

Lemma 2.3.25 (Equivalence of recursive and non-recursive categories). *There is an equivalence between the category of labeled n-trusses and their bordisms, and the recursive category of labeled n-trusses and their bordisms:*

$$n\mathsf{TBord}_{/\!/\mathsf{C}} \simeq \mathsf{TBord}^n_{/\!/\mathsf{C}} \quad .$$

We defer a proof until we are in the context of classification of n-truss bundles; Lemma 2.3.25 will be established along with Lemma 2.3.18 as part of Lemma 2.3.48.

Given this equivalence, we may and will refer to objects of $\mathsf{TBord}^n_{/\!/\mathsf{C}}$ as n-trusses, and to morphisms of $\mathsf{TBord}^n_{/\!/\mathsf{C}}$ as n-truss bordisms.

Observation 2.3.26 (n-Trusses as $(n-1)$-truss-labeled 1-trusses). The composition of the component functors in the iterated bordism functor $\mathsf{TBord}^n_{/\!/-} = (\mathsf{TBord}^1_{/\!/-})^{\circ n} : \mathsf{Cat} \to \mathsf{Cat}$ is associative and therefore may be rebracketed variously as convenient. For instance, bracketing together the last $n-1$ instances of the 1-truss bordism endofunctor provides the equality

$$\mathsf{TBord}^n_{/\!/\mathsf{C}} = \mathsf{TBord}^{n-1}_{/\!/\{\mathsf{TBord}^1_{/\!/\mathsf{C}}\}}$$

That is, the (recursive) category of C-labeled n-trusses and their bordisms is the (recursive) category of $(n-1)$-trusses and their bordisms *labeled* in the category of C-labeled 1-trusses and their bordisms. Informally, we express (the unlabeled version of) this equality by saying 'n-trusses are 1-truss-labeled $(n-1)$-trusses'.

By contrast, the opposite bracketing provides the equality

$$\mathsf{TBord}^n_{/\!/\mathsf{C}} = \mathsf{TBord}^1_{/\!/\{\mathsf{TBord}^{n-1}_{/\!/\mathsf{C}}\}}$$

That is, the (recursive) category of C-labeled n-trusses and their bordisms is the category of 1-trusses and their bordisms *labeled* in the (recursive) category of C-labeled $(n-1)$-trusses and their bordisms. Informally, we express this fact by saying 'n-trusses are $(n-1)$-truss-labeled 1-trusses'.

Of course, any intermediate bracketing will do just as well:

$$\mathsf{TBord}^n_{/\!/\mathsf{C}} = \mathsf{TBord}^k_{/\!/\{\mathsf{TBord}^{n-k}_{/\!/\mathsf{C}}\}}$$

That is, n-trusses are $(n-k)$-truss-labeled k-trusses. ⌟

2.3.2. ◊n-Truss bundles and their classification. In the previous section, we developed the notion of n-trusses, providing a combinatorial model of towers of suitably framed stratified bundles, and of n-truss bordisms, providing a corresponding model of such towers over the stratified 1-simplex. Now we describe the natural generalization to n-truss bundles, which will model such framed stratified towers over more general stratified spaces.

SYNOPSIS. We introduce n-truss bundles as towers of 1-truss bundles that begin with an arbitrary base poset. We discuss the classification of n-truss bundles by functors into the recursive category of n-trusses and their bordisms, and use classification constructions to prove that the recursive category of n-trusses and their bordisms is equivalent to the category of n-trusses and their bordisms; in the process we establish that the latter category has a well-defined composition. Finally, we mention pullbacks, a non-commutative product, dualization, and suspension for n-truss bundles.

2.3.2.1. ◊n-Truss bundles and bundle maps. We introduced n-trusses as towers of 1-truss bundles over a point, and n-truss bordisms as towers of 1-truss bundles over a combinatorial 1-simplex; the generalization to towers over arbitrary posets is direct, as follows.

DEFINITION 2.3.27 (n-Truss bundle). An **n-truss bundle** over a base poset B is a sequence of 1-truss bundles

$$T_n \xrightarrow{p_n} T_{n-1} \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_2} T_1 \xrightarrow{p_1} T_0 = B$$

in which the base poset of each bundle is the total poset of the next bundle. ⌟

We typically compress the sequence of bundles $\{T_i \xrightarrow{p_i} T_{i-1}\}$ to a single letter indicative of the maps, referring to the whole n-truss bundle as for instance p. We refer to the face order poset $(T_n, \trianglelefteq)$ as the 'total poset' of the n-truss bundle and abbreviate it simply by T_n.

TERMINOLOGY 2.3.28 (Open and closed n-truss bundles). We call an n-truss bundle p 'open', respectively 'closed', when all its 1-truss bundles $p_i : T_i \to T_{i-1}$ are open, respectively closed. ⌟

EXAMPLE 2.3.29 (The composition of 2-truss bordisms as a 2-truss bundle). Recall the 2-truss bordisms $R^{(01)}$ and $R^{(12)}$ and their composite $R^{(02)}$ from Figure 2.46. Identifying $R_0^{(ij)}$ with the poset $\{i \to j\}$, the union of the posets $R_0^{(ij)}$ yields the poset $T_0 := [2] = (0 \to 1 \to 2)$. The union of the posets $R_1^{(ij)}$ is the total poset T_1 of a 1-truss bundle over T_0; that 1-truss bundle was illustrated previously in Figure 2.23. The union of the posets $R_2^{(ij)}$ is the total poset T_2 of a 1-truss bundle over T_1. Altogether this provides a 2-truss bundle $(T_2 \to T_1 \to T_0 = [2])$ over the 2-simplex, encoding that composition of 2-truss bordisms. ⌟

REMARK 2.3.30 (Generating arrows of n-truss bundles). For an n-truss bundle $p = (T_n \xrightarrow{p_n} T_{n-1} \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_2} T_1 \xrightarrow{p_1} T_0 = B)$ over a base poset B, the covering relations $\mathrm{cov}(T_i) \subset \mathrm{mor}(T_i, \trianglelefteq)$ are determined by inductive application of Construction 2.1.81, which specified the covering relation of the total poset of a 1-truss bundle in terms of the covering relation of its base poset. We refer to the morphisms of the covering relations $\mathrm{cov}(T_i)$ as 'generating arrows' of the n-truss bundle. Note that we have already in some previous illustrations depicted only the generating arrows of n-trusses and n-truss bordisms, and will continue to do so as a matter of course for any n-truss bundles. ⌟

As fundamental as labelings will be for the subsequent combinatorial theory of stratified spaces, we can by now introduce them simply and without fuss as follows.

Definition 2.3.31 (Labeled n-truss bundles). Given a poset B and a category C, a **C-labeled n-truss bundle** p over B is a pair $(\underline{p}, \mathrm{lbl}_p)$ consisting of an n-truss bundle $\underline{p}$, and a functor $\mathrm{lbl}_p : T_n \to \mathsf{C}$ from the total poset of the bundle to the category. ⌟

Of course we refer to the 'underlying n-truss bundle' $\underline{p}$ and the 'labeling functor' lbl_p. We display the labeled n-truss bundle as a labeled sequence:

$$\mathsf{C} \xleftarrow{\mathrm{lbl}_p} T_n \xrightarrow{p_n} T_{n-1} \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_2} T_1 \xrightarrow{p_1} T_0 = B$$

Needless to say, n-truss bundles over base $B = [1]$ are simply n-truss bordisms, and over base $B = [0]$ are simply n-trusses.

Definition 2.3.32 (Maps of labeled n-truss bundles). For categories C and D, let $p = \{T_i \xrightarrow{p_i} T_{i-1}\}$ be a C-labeled n-truss bundle and let $q = \{S_i \xrightarrow{q_i} S_{i-1}\}$ be a D-labeled n-truss bundle. A **map of labeled n-truss bundles** $F : p \to q$ is a pair $(\underline{F}, \mathrm{lbl}_F)$ consisting of (1) a sequence $\underline{F} = (F_n, F_{n-1}, ..., F_1, F_0)$, where $F_0 : T_0 \to S_0$ is a poset map and each pair $(F_i, F_{i-1}) : p_i \to q_i$ is a 1-truss bundle map (as in Definition 2.1.87), and (2) a functor $\mathrm{lbl}_F : \mathsf{C} \to \mathsf{D}$ for which $((F_n, F_{n-1}), \mathrm{lbl}_F)$ is a labeled 1-truss bundle map (as in Definition 2.2.60). ⌟

We display the data of a map $F \equiv (\underline{F}, \mathrm{lbl}_F)$ as a commutative diagram:

$$\begin{array}{ccccccccccc} \mathsf{C} & \xleftarrow{\mathrm{lbl}_p} & T_n & \xrightarrow{p_n} & T_{n-1} & \xrightarrow{p_{n-1}} & \cdots & \xrightarrow{p_2} & T_1 & \xrightarrow{p_1} & T_0 \\ {\scriptstyle \mathrm{lbl}_F}\downarrow & & {\scriptstyle F_n}\downarrow & & {\scriptstyle F_{n-1}}\downarrow & & \cdots & & \downarrow{\scriptstyle F_1} & & \downarrow{\scriptstyle F_0} \\ \mathsf{D} & \xleftarrow[\mathrm{lbl}_q]{} & S_n & \xrightarrow[q_n]{} & S_{n-1} & \xrightarrow[q_{n-1}]{} & \cdots & \xrightarrow[q_2]{} & S_1 & \xrightarrow[q_1]{} & S_0 \end{array}$$

As in the case of labeled 1-truss bundles, we refer to lbl_F as the 'label category functor', and to the sequence $\underline{F}$ as the 'underlying' bundle map. We make explicit the following obvious specializations of the previous definition.

Terminology 2.3.33 (Maps of n-truss bundles). For unlabeled n-truss bundles p and q, a 'map of n-truss bundles' $p \to q$ is a sequence of 1-truss bundle maps $p_i \to q_i$, that is just the first piece of data from the definition of maps of labeled n-truss bundles. Equivalently, a map of unlabeled n-truss bundles is a map of labeled n-truss bundles whose labelings are both in the terminal category. ⌟

Terminology 2.3.34 (Maps of labeled n-trusses). For labeled n-trusses T and S, a 'map of labeled n-trusses' $T \to S$ is simply a map of labeled n-truss bundles whose base posets are both trivial. ⌟

Terminology 2.3.35 (Maps of n-trusses). For n-trusses $T = \{T_i \xrightarrow{p_i} T_{i-1}\}$ and $S = \{S_i \xrightarrow{q_i} S_{i-1}\}$, a 'map of n-trusses' $T \to S$ is simply a sequence of 1-truss bundle maps $p_i \to q_i$. Equivalently, it is a map of n-truss bundles both of whose base posets are trivial, or a map of labeled n-trusses whose labelings are both in the terminal category. ⌟

Various basic conditions on labeled n-truss bundle maps carry over from the corresponding 1-truss versions, as follows.

Terminology 2.3.36 (Label-preserving and base-preserving n-truss bundle maps). A labeled n-truss bundle map is 'label preserving' if the label category functor lbl_F is an identity, and is 'base preserving' if the underlying bundle map $\underline{F}$ has its initial poset map F_0 being an identity. ⌟

Terminology 2.3.37 (Singular, regular, and balanced labeled n-truss bundle maps). A labeled n-truss bundle map $F \equiv (\underline{F} = \{p_i \to q_i\}, \mathrm{lbl}_F)$ is 'singular', 'regular', or 'balanced' if every component 1-truss bundle map $p_i \to q_i$ is such, respectively, in the sense of Terminology 2.1.88. ⌟

Componentwise composition of the sequence $\{p_i \to q_i\}$ of underlying bundle maps, along with composition of the label category functors, provides the following categories.

Notation 2.3.38 (Categories of n-trusses and n-truss bundles). Using the above notions of maps, we have the following four categories:

Trs_n	n-Trusses and their maps.
LblTrs_n	Labeled n-trusses and their maps.
TrsBun_n	n-Truss bundles and their maps.
$\mathsf{LblTrsBun}_n$	Labeled n-truss bundles and their maps.

We will also have particular need of the following subcategory of TrsBun_n:

$\mathsf{Trs}_n(B)$	n-Truss bundles over the poset B and base-preserving maps.

As before, the decoration $\mathring{\mathsf{T}}$ or $\bar{\mathsf{T}}$ will indicate the restriction to the open objects and regular maps, or closed objects and singular maps, respectively. ⌟

Remark 2.3.39 (Enriched categories of n-trusses and n-truss bundles). The hom sets $\mathsf{Trs}_n(T, S)$ in the category Trs_n are a priori, of course, discrete; but we may instead regard $\mathsf{Trs}_n(T, S)$ as a poset, whose objects are n-truss maps and whose arrows are the natural transformations $\nu : E_n \Rightarrow F_n$ of the total poset maps $E_n, F_n : T_n \to S_n$ of the n-truss maps $E, F : T \to S$. (Note that if such a natural transformation exists, it is unique, and such a natural transformation induces natural transformations $E_i \Rightarrow F_i$ at every i-stage of the truss towers.) Altogether this provides a Pos-enrichment of the category Trs_n. Regarding a poset as a topological space via its specialization topology then provides a $k\mathsf{Top}$-enrichment of the category Trs_n. (Here $k\mathsf{Top}$ denotes the category of compactly generated spaces; see Convention B.1.1 and Notation B.1.2 and the intervening discussion.) All the same comments apply to the case of truss bundles, and we therefore have the following two $k\mathsf{Top}$-enriched categories:

$\mathcal{T}rs_n$ n-Trusses and their $k\mathsf{Top}$-space of maps.

$\mathcal{T}rs_n(B)$ n-Truss bundles over the poset B and their $k\mathsf{Top}$-space of base-preserving maps.

These enrichments provide a subtle additional structure beyond the discrete categories of trusses and truss bundles, but critically they should not be considered as being $(\infty, 1)$-categorical models, as the hom spaces are not weak Hausdorff. ⌟

Terminology 2.3.40 (Restriction of labeled n-truss bundles). Given a C-labeled n-truss bundle $p \equiv (\underline{p} = (p_n, p_{n-1}, ..., p_1), \mathrm{lbl}_p)$ over a poset B, and a subposet $A \hookrightarrow B$, the 'restriction' of the labeled bundle to the subposet is the C-labeled n-truss bundle $p|_A$ given by the upper row in the following diagram:

$$\begin{array}{ccccccccccc}
 & \overset{\mathrm{lbl}_{p|_A}}{\swarrow} & T_n|_A & \xrightarrow{p_n|_A} & T_{n-1}|_A & \xrightarrow{p_{n-1}|_A} & \cdots & \xrightarrow{p_2|_A} & T_1|_A & \xrightarrow{p_1|_A} & A \\
\mathsf{C} & & \big\downarrow & \lrcorner & \big\downarrow & \lrcorner & \cdots & & \big\downarrow & \lrcorner & \big\downarrow \\
 & \underset{\mathrm{lbl}_p}{\nwarrow} & T_n & \xrightarrow[p_n]{} & T_{n-1} & \xrightarrow[p_{n-1}]{} & \cdots & \xrightarrow[p_2]{} & T_1 & \xrightarrow[p_1]{} & B
\end{array}$$

Here each square is a pullback, in fact a restriction, of 1-truss bundles. This process provides in particular a functor $-|_A : \mathsf{Trs}_n(B) \to \mathsf{Trs}_n(A)$. ⌟

Remark 2.3.41 (Balanced isomorphism for labeled n-truss bundles). Generalizing Remark 2.2.66, balanced label- and base-preserving n-truss bundle isomorphisms preserve all structural data (face orders, frame orders, dimension maps at all stages, projection towers, labeling functors), and are unique when they exist. As before, there is therefore no need to distinguish between distinct but balanced label- and base-preservingly isomorphic labeled n-truss bundles. ⌟

Given a 1-truss bundle, we could forget everything except the total poset or the base poset. The corresponding constructions for n-truss bundles involve discarding either part of the tail or part of the head of the constituent sequence of 1-truss bundles, as follows.

Construction 2.3.42 (Upper truncation of n-truss bundles). The 'upper $(n-k)$-truncation functor'

$$(-)_{>k} : \mathsf{LblTrsBun}_n \to \mathsf{LblTrsBun}_{n-k}$$

takes a labeled n-truss bundle $p = (\underline{p} = (p_n, p_{n-1}, ..., p_1), \mathrm{lbl}_p)$ to the labeled $(n-k)$-truss bundle $p_{>k} = (\underline{p}_{>k} = (p_n, p_{n-1}, ..., p_{k+1}), \mathrm{lbl}_p)$ given by the first $(n-k)$-many 1-truss bundles in the tower, with the labeling of the total poset. (When $k = n$, we interpret this truncation to yield just the total poset of the n-truss bundle, with its labeling functor.) ⌟

CONSTRUCTION 2.3.43 (Lower truncation of n-truss bundles). The 'lower k-truncation functor'

$$(-)_{\leq k} : \mathsf{LblTrsBun}_n \to \mathsf{TrsBun}_k$$

takes a labeled n-truss bundle $p = (\underline{p} = (p_n, p_{n-1}, ..., p_1), \mathrm{lbl}_p)$ to the unlabeled k-truss bundle $p_{\leq k} = (p_k, p_{k-1}, ..., p_1)$ given by the last k-many 1-truss bundles in the tower. (When $k = 0$, we interpret this truncation to yield just the base poset of the n-truss bundle.) ⌟

2.3.2.2. ◊Classification and totalization for n-truss bundles. Recall from Observation 2.2.73 that C-labeled 1-truss bundles are classified by functors into the category $\mathsf{TBord}^1_{/\!/\mathsf{C}}$ of C-labeled 1-trusses and their bordisms. We now discuss the analogous classification in the n-truss case: C-labeled n-truss bundles are classified by functors into the *recursive* category $\mathsf{TBord}^n_{/\!/\mathsf{C}}$ of C-labeled n-trusses and their bordisms. Along the way we will finally establish that that recursive category is equivalent to the category $n\mathsf{TBord}_{/\!/\mathsf{C}}$ of C-labeled n-trusses and their bordisms.

CONSTRUCTION 2.3.44 (Classifying functors of labeled n-truss bundles). We describe a map

$$p \equiv (\underline{p} = (p_n, p_{n-1}, ..., p_1), \mathrm{lbl}_p : T_n \to \mathsf{C}) \quad \mapsto \quad (\chi_p : B \to \mathsf{TBord}^n_{/\!/\mathsf{C}})$$

that takes a C-labeled n-truss bundle $p \equiv (\underline{p}, \mathrm{lbl}_p)$ over a poset B, with underlying bundle $\underline{p} = (p_n, p_{n-1}, ..., p_1)$ and labeling functor $\mathrm{lbl}_p : T_n \to \mathsf{C}$, to an associated **classifying functor** $\chi_p : B \to \mathsf{TBord}^n_{/\!/\mathsf{C}}$.

We construct the functor χ_p inductively, as follows.

- Specify an initial functor $\chi^n_p := \mathrm{lbl}_p : T_n \to \mathsf{C}$ as the given labeling functor.
- Inductively in descending i, consider the $(\mathsf{TBord}^{n-i}_{/\!/\mathsf{C}})$-labeled 1-truss bundle $(p_i : T_i \to T_{i-1}, \chi^i_p : T_i \to \mathsf{TBord}^{n-i}_{/\!/\mathsf{C}})$; define

$$\chi^{i-1}_p : T_{i-1} \to \mathsf{TBord}^1_{/\!/(\mathsf{TBord}^{n-i}_{/\!/\mathsf{C}})} = \mathsf{TBord}^{n-i+1}_{/\!/\mathsf{C}}$$

  to be the classifying functor of that labeled 1-truss bundle (p_i, χ^i_p).
- Finally set $\chi_p := \chi^0_p : B \to \mathsf{TBord}^n_{/\!/\mathsf{C}}$. ⌟

Notice that in the inductive construction above, we used the fact (see Observation 2.3.26) that $\mathsf{TBord}^1_{/\!/(\mathsf{TBord}^{n-i}_{/\!/\mathsf{C}})}$ is a suitable expression for $\mathsf{TBord}^{n-i+1}_{/\!/\mathsf{C}}$.

Conversely, to a classifying functor we may associate a total labeled bundle, as follows; the construction will simply invert each inductive step of the preceding construction of classifying functors.

CONSTRUCTION 2.3.45 (Total labeled n-truss bundles of classifying functors). We describe a map

$$(\mathsf{F} : B \to \mathsf{TBord}^n_{/\!/\mathsf{C}}) \quad \mapsto \quad \pi_{\mathsf{F}} \equiv (\pi_{\underline{\mathsf{F}}} = (\pi^n_{\underline{\mathsf{F}}}, \pi^{n-1}_{\underline{\mathsf{F}}}, ..., \pi^1_{\underline{\mathsf{F}}}), \mathsf{lbl}_{\mathsf{F}} : \mathrm{Tot}^n \mathsf{F} \to \mathsf{C})$$

that takes a functor $\mathsf{F} : B \to \mathsf{TBord}^n_{/\!/\mathsf{C}}$ to an associated **total C-labeled n-truss bundle** $\pi_{\mathsf{F}} \equiv (\pi_{\underline{\mathsf{F}}}, \mathsf{lbl}_{\mathsf{F}})$.

We construct the labeled n-truss bundle π_{F} inductively, as follows.

› Specify an initial functor $\mathsf{lbl}^0_{\mathsf{F}} : \mathrm{Tot}^0\mathsf{F} \to \mathsf{TBord}^n_{/\!/\mathsf{C}}$ to be the given functor $\mathsf{F} : B \to \mathsf{TBord}^n_{/\!/\mathsf{C}}$.

› Inductively in ascending i, consider the classying functor

$$\mathsf{lbl}^{i-1}_{\mathsf{F}} : \mathrm{Tot}^{i-1}\mathsf{F} \to \mathsf{TBord}^{n-i+1}_{/\!/\mathsf{C}} = \mathsf{TBord}^1_{/\!/(\mathsf{TBord}^{n-i}_{/\!/\mathsf{C}})};$$

define $(\pi^i_{\underline{\mathsf{F}}} : \mathrm{Tot}^i\mathsf{F} \to \mathrm{Tot}^{i-1}\mathsf{F},\ \mathsf{lbl}^i_{\mathsf{F}} : \mathrm{Tot}^i\mathsf{F} \to \mathsf{TBord}^{n-i}_{/\!/\mathsf{C}})$ to be the total $(\mathsf{TBord}^{n-i}_{/\!/\mathsf{C}})$-labeled 1-truss bundle of that classifying functor.

› Finally set $\pi_{\underline{\mathsf{F}}} := (\pi^n_{\underline{\mathsf{F}}}, \pi^{n-1}_{\underline{\mathsf{F}}}, ..., \pi^1_{\underline{\mathsf{F}}})$ and $\mathsf{lbl}_{\mathsf{F}} := \mathsf{lbl}^n_{\mathsf{F}} : \mathrm{Tot}^n\mathsf{F} \to \mathsf{C}$. ⌟

Remark 2.3.46 (Unwinding the classification constructions). Consider the tower of intermediate classifying maps arising in the preceding inductive classification construction; we display, as follows, the form of this tower in the case of a C-labeled 3-truss bundle p with underlying 3-truss bundle $T_3 \xrightarrow{p_3} T_2 \xrightarrow{p_2} T_1 \xrightarrow{p_1} T_0$ and labeling functor $\mathrm{lbl}_p : T_3 \to \mathsf{C}$.

$$\begin{array}{ccl}
T_3 & \xrightarrow{\chi^3_p \,=\, \mathrm{lbl}_p} & \mathsf{C} \\
{\scriptstyle p_3}\big\downarrow & & \\
T_2 & \xrightarrow{\chi^2_p} & \mathsf{TBord}^1_{/\!/\mathsf{C}} \\
{\scriptstyle p_2}\big\downarrow & & \\
T_1 & \xrightarrow{\chi^1_p} & \mathsf{TBord}^2_{/\!/\mathsf{C}} = \mathsf{TBord}^1_{/\!/\mathsf{TBord}^1_{/\!/\mathsf{C}}} \\
{\scriptstyle p_1}\big\downarrow & & \\
T_0 & \xrightarrow{\chi^0_p} & \mathsf{TBord}^3_{/\!/\mathsf{C}} = \mathsf{TBord}^1_{/\!/\mathsf{TBord}^2_{/\!/\mathsf{C}}}
\end{array}$$

Explicitly, the C-labeled 1-truss bundle (p_3, χ^3_p) is classified by χ^2_p, then the labeled 1-truss bundle (p_2, χ^2_p) is classified by χ^1_p, and finally the labeled 1-truss bundle (p_1, χ^1_p) is classified by χ^0_p. Each of the following subsets of the diagram thus determines the entire C-labeled 3-truss bundle: by definition the bundles p_1, p_2, and p_3, together with the functor $\chi^3_p = \mathrm{lbl}_p$; or the bundles p_1 and p_2, together with the functor χ^2_p; or the bundle p_2 together with the functor χ^1_p; or just by itself the functor χ^0_p. ⌟

Example 2.3.47 (Classification and totalization for a labeled 3-truss bundle). As an example of the iterated classification procedure described and displayed in the previous remark, in Figure 2.47 we depict the classifying tower of a 3-truss (with trivial labeling category for simplicity). Consider having, at the outset, the tower of 1-truss bundles $T_3 \xrightarrow{p_3} T_2 \xrightarrow{p_2} T_1 \xrightarrow{p_1} T_0$ (with trivial labeling $\mathrm{lbl}_p : T_3 \to *$).

First, form the classifying functor

$$\chi^2_p := \chi_{(p_3, \mathrm{lbl}_p)} : T_2 \to \mathsf{TBord}^1_{/\!/ *} \equiv \mathsf{TBord}^1$$

of the (trivially labeled) 1-truss bundle $T_3 \xrightarrow{p_3} T_2$. Note that this classifying functor happens to factor as $T_2 \to \mathsf{B}\mathbb{F} \hookrightarrow \mathsf{TBord}^1$, where $\mathsf{B}\mathbb{F}$ is the opposite flip flop monoid described in Example 2.2.59. That factorization allows us to consider the a priori TBord^1-labeled 1-truss bundle (p_2, χ^2_p) as being $\mathsf{B}\mathbb{F}$-labeled, and therefore specified by color-coding the arrows of the poset T_2; in fact, this $\mathsf{B}\mathbb{F}$-labeled bundle is the one that appeared in Figure 2.39.

Second, form the classifying functor

$$\chi^1_p := \chi_{(p_2, \chi^2_p)} : T_1 \to \mathsf{TBord}^1_{/\!/ \mathsf{B}\mathbb{F}} \hookrightarrow \mathsf{TBord}^1_{/\!/ \mathsf{TBord}^1} \equiv \mathsf{TBord}^2$$

of the $\mathsf{B}\mathbb{F}$-labeled 1-truss bundle (p_2, χ^2_p). That functor is similarly specified by color-coding the arrows of the poset T_1; again in fact this $(\mathsf{TBord}^1_{/\!/ \mathsf{B}\mathbb{F}})$-labeled bundle previously appeared in Figure 2.40.

Third, form the classifying functor

$$\chi_p \equiv \chi^0_p := \chi_{(p_1, \chi^1_p)} : T_0 \to \mathsf{TBord}^2_{/\!/ \mathsf{B}\mathbb{F}} \hookrightarrow \mathsf{TBord}^2_{/\!/ \mathsf{TBord}^1} \equiv \mathsf{TBord}^3$$

of the $(\mathsf{TBord}^1_{/\!/ \mathsf{B}\mathbb{F}})$-labeled 1-truss bundle (p_1, χ^1_p). In this case, where the base poset T_0 is a point, that final classifying functor is rather tautological, merely picking out the point of TBord^3 indicating the given 3-truss. But of course for bundles with nontrivial base poset, even this final classifying functor would trace out an informative diagram in the target classifying category TBord^3. ⌟

Equipped with the classification and totalization contructions for labeled n-truss bundles, we may now establish together two facts previously deferred: that the composition of labeled n-truss bordisms (from Definitions 2.3.16 and 2.3.17) is well-defined (as claimed in Lemma 2.3.18), and that that category is equivalent to the recursive category of labeled n-trusses and their bordisms (as claimed in Lemma 2.3.25).

Lemma 2.3.48 (Categories of n-trusses and their bordisms). *The composition of labeled n-truss bordisms is well-defined, and the resulting category of labeled n-trusses and their bordisms is equivalent to the recursive category of labeled n-trusses and their bordisms:*

$$\chi_- : n\mathsf{TBord}_{/\!/ \mathsf{C}} \simeq \mathsf{TBord}^n_{/\!/ \mathsf{C}} : \pi_-$$

This equivalence is by a functor χ_-, that takes n-trusses and bordisms to the images of their classifying functors in the recursive category, and a functor π_-, that takes objects and morphisms of the recursive category to their total n-trusses and bordisms.

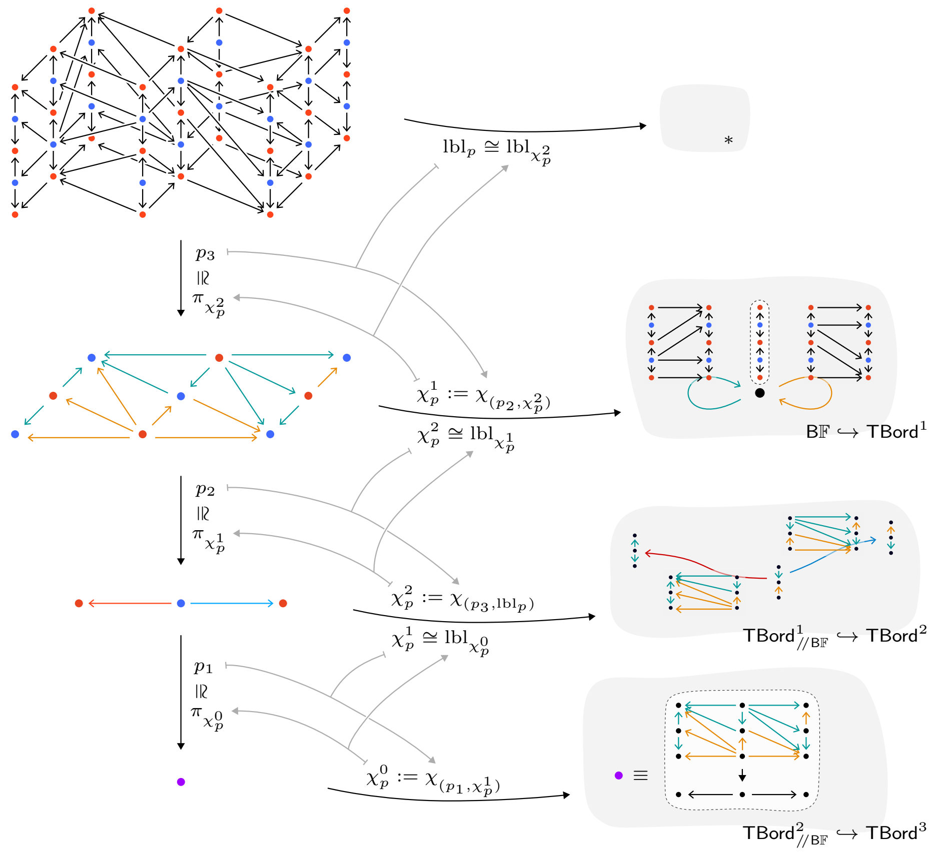


FIGURE 2.47. The classifying tower of a 3-truss.

PROOF. The case $n = 1$ was established previously in Lemma 2.2.43 (since $1\mathsf{TBord}_{/\!/\mathsf{C}}$ and $\mathsf{TBord}^1_{/\!/\mathsf{C}}$ are identical by definition). Assume inductively that $(n-1)\mathsf{TBord}_{/\!/\mathsf{C}}$ has well-defined composition, and that we have the desired equivalence

$$\chi_- : (n-1)\mathsf{TBord}_{/\!/\mathsf{C}} \simeq \mathsf{TBord}^{n-1}_{/\!/\mathsf{C}} : \pi_- \quad .$$

Setting the labeling category itself to be $\mathsf{TBord}^1_{/\!/\mathsf{C}}$, we have in particular that the category $(n-1)\mathsf{TBord}_{/\!/\{\mathsf{TBord}^1_{/\!/\mathsf{C}}\}}$ is well-defined and we have the equivalence

$$\chi_- : (n-1)\mathsf{TBord}_{/\!/\{\mathsf{TBord}^1_{/\!/\mathsf{C}}\}} \simeq \mathsf{TBord}^{n-1}_{/\!/\{\mathsf{TBord}^1_{/\!/\mathsf{C}}\}} : \pi_- \quad .$$

We first show that $n\mathsf{TBord}_{/\!/\mathsf{C}}$ has well-defined composition. We do this in a roundabout fashion by providing a (necessarily well defined) labeled n-truss bordism $R^{(02)}$, and then showing that that labeled bordism $R^{(02)}$ has the underlying k-stage relations specified by Definition 2.3.16 and the labeling functor factorization specified by Definition 2.3.17.

Consider two composable labeled n-truss bordisms $R^{(01)} : T^{(0)} \twoheadrightarrow T^{(1)}$ (with $R^{(01)} = (\underline{R}^{(01)}, \mathrm{lbl}_{R^{(01)}})$) and $R^{(12)} : T^{(1)} \twoheadrightarrow T^{(2)}$ (with $R^{(12)} =$

$(\underline{R}^{(12)}, \mathrm{lbl}_{R^{(12)}}))$. Let $\chi_{R^{(01)}} : [1] \to \mathsf{TBord}^n_{/\!/\mathsf{C}}$ and $\chi_{R^{(12)}} : [1] \to \mathsf{TBord}^n_{/\!/\mathsf{C}}$ be the classifying functors of the labeled n-truss bordisms $R^{(01)}$ and $R^{(12)}$. Considered as morphisms in $\mathsf{TBord}^n_{/\!/\mathsf{C}}$, those functors admit a composite $\chi_{R^{(12)}} \circ \chi_{R^{(01)}} : [1] \to \mathsf{TBord}^n_{/\!/\mathsf{C}}$. We may thus form the total labeled n-truss bundle of that composite:

$$R^{(02)} := \pi_{(\chi_{R^{(01)}} \circ \chi_{R^{(12)}})}$$

For simplicity, we identify bundles up to (label- and base-preserving) balanced isomorphism; observe that $\mathrm{dom}(R^{(02)}) = T^{(0)}$ and $\mathrm{cod}(R^{(02)}) = T^{(2)}$.

We now verify that the underlying bundle $\underline{R}^{(02)}$ has k-stage relations $\mathsf{rel}^{R^{(02)}}_k$ being precisely the composite relations $\mathsf{rel}^{R^{(12)}}_k \circ \mathsf{rel}^{R^{(01)}}_k$, as required by Definition 2.3.16. Let $\underline{R}^{(02)} = (p^{(02)}_n, p^{(02)}_{n-1}, \ldots, p^{(02)}_1)$, $\underline{R}^{(01)} = (p^{(01)}_n, p^{(01)}_{n-1}, \ldots, p^{(01)}_1)$, and $\underline{R}^{(12)} = (p^{(12)}_n, p^{(12)}_{n-1}, \ldots, p^{(12)}_1)$ denote the constituent 1-truss bundles. Inductively we may assume $\mathsf{rel}^{R^{(02)}}_k = \mathsf{rel}^{R^{(12)}}_k \circ \mathsf{rel}^{R^{(01)}}_k$ for $k < n$, and then argue that $\mathsf{rel}^{R^{(02)}}_n = \mathsf{rel}^{R^{(12)}}_n \circ \mathsf{rel}^{R^{(01)}}_n$ as follows.

- If $(x_0, x_2) \in \mathsf{rel}^{R^{(02)}}_n$, then $p^{(02)}_n(x_0, x_2) =: (x'_0, x'_2) \in \mathsf{rel}^{R^{(02)}}_{n-1}$. By induction, there exists some $x'_1 \in T^{(1)}_{n-1}$ with $(x'_0, x'_1) \in \mathsf{rel}^{R^{(01)}}_{n-1}$ and $(x'_1, x'_2) \in \mathsf{rel}^{R^{(12)}}_{n-1}$. Now observe

$$\chi_{p^{(02)}_n}(x'_0 \trianglelefteq x'_2) = \chi_{p^{(12)}_n}(x'_1 \trianglelefteq x'_2) \circ \chi_{p^{(01)}_n}(x'_0 \trianglelefteq x'_1)$$

  It follows from the definition of composition of 1-truss bordisms as composition of relations (see Definition 2.1.51) that there is some $x_1 \in T^{(1)}_n$ with $p^{(02)}_n(x_1) = x'_1$ and with $(x_0, x_1) \in \mathsf{rel}^{R^{(01)}}_n$ and $(x_1, x_2) \in \mathsf{rel}^{R^{(12)}}_n$.
- Conversely, if $\mathsf{rel}^{R^{(01)}}_n(x_0, x_1)$ and $\mathsf{rel}^{R^{(12)}}_n(x_1, x_2)$, then $p^{(01)}_n(x_0, x_1) =: (x'_0, x'_1) \in \mathsf{rel}^{R^{(01)}}_{n-1}$ and $p^{(12)}_n(x_1, x_2) =: (x'_1, x'_2) \in \mathsf{rel}^{R^{(12)}}_{n-1}$. By induction we have $(x'_0, x'_2) \in \mathsf{rel}^{R^{(02)}}_{n-1}$; the previously displayed equality of classifying functors implies that $(x_0, x_2) \in \mathsf{rel}^{R^{(02)}}_n$.

That much verifies that the underlying bundle $\underline{R}^{(02)}$ is well-defined by the composites of the k-stage relations of the underlying bundles $\underline{R}^{(01)}$ and $\underline{R}^{(12)}$.

Next, again by the same displayed equality of classifying functors (and the fact that that equality holds for any factorizing element $x'_1 \in T^{(1)}_{n-1}$), the labeling functor $\mathrm{lbl}_{R^{(02)}}$ satisfies the labeling functor factorization

$$\mathrm{lbl}_{R^{(02)}}(x_0 \trianglelefteq x_2) := \mathrm{lbl}_{R^{(12)}}(x_1 \trianglelefteq x_2) \circ \mathrm{lbl}_{R^{(01)}}(x_0 \trianglelefteq x_1),$$

as specified by Definition 2.3.17; that much ensures that the labeling functor is well-defined and altogether that $n\mathsf{TBord}_{/\!/\mathsf{C}}$ indeed forms a category.

It remains to show that classification and totalization form an equivalence between $n\mathsf{TBord}_{/\!/\mathsf{C}}$ and $\mathsf{TBord}^n_{/\!/\mathsf{C}}$. With the inductive assumption (applied to the labeling category $\mathsf{TBord}^1_{/\!/\mathsf{C}}$), it suffices to check that

$$\chi_{-} : n\mathsf{TBord}_{/\!/\mathsf{C}} \simeq (n-1)\mathsf{TBord}_{/\!/\{\mathsf{TBord}^1_{/\!/\mathsf{C}}\}} : \pi_{-}$$

Observe that this classification construction χ_{-} and totalization construction π_{-} are functorial, and are inverse on objects and morphisms up to label-preserving balanced bundle isomorphism. □

The correspondence, via Constructions 2.3.44 and 2.3.45, between labeled n-truss bundles and functors into the (recursive) category of labeled n-trusses and their bordisms, is functorial, with respect to a notion of bordisms of bundles, generalizing the previous Definitions 2.1.96 and 2.2.70, as follows.

Definition 2.3.49 (Bordisms of labeled n-truss bundles and their composition). Given C-labeled n-truss bundles p and q over a poset B, a **C-labeled n-truss bundle bordism** $u : p \Rightarrow q$ is a C-labeled n-truss bundle u over $B \times [1]$ such that $u|_{B\times\{0\}} = p$ and $u|_{B\times\{1\}} = q$.

The **composition** of two such labeled bordisms $u : p \Rightarrow q$ and $v : q \Rightarrow r$ is the labeled bordism $v \circ u : p \Rightarrow r$ whose iterated restriction $(v \circ u)|_{\{x\}\times[1]}$ is the composite labeled bordism $v|_{\{x\}\times[1]} \circ u|_{\{x\}\times[1]}$, for all elements $x \in B$. ⌟

Note that, given a C-labeled n-truss bundle bordism $u : p \Rightarrow q$, its classifying functor $\chi_u : B \times [1] \to \mathsf{TBord}^n_{/\!/\mathsf{C}}$ may also be considered as a 'classifying natural transformation' $\chi_p \Rightarrow \chi_q : B \to \mathsf{TBord}^n_{/\!/\mathsf{C}}$.

Notation 2.3.50 (Category of labeled n-truss bundles and their bordisms). For a fixed base poset B and a category C, the 'category of C-labeled n-truss bundles and their bordisms', whose objects are C-labeled n-truss bundles over B and whose morphisms are C-labeled n-truss bundle bordisms, will be denoted $n\mathsf{TBord}(B)_{/\!/\mathsf{C}}$. ⌟

Remark 2.3.51 (Isobordisms of n-truss bundles are unique). An invertible n-truss bundle bordism is called an 'n-truss bundle isobordism'. Given two *unlabeled* n-truss bundles, if there is an isobordism between them, then there is a unique such isobordism; this uniqueness follows by iteratively applying Observation 2.1.98. However, for the same reasons as in Remark 2.2.72, this uniqueness does not hold in the labeled case. ⌟

Classification and totalization now provide an equivalence of categories, generalizing the previous Lemma 2.3.48 from n-trusses to n-truss bundles and the earlier Observation 2.2.73 from 1-truss bundles to n-truss bundles.

Observation 2.3.52 (Classification and totalization functors for labeled n-truss bundles). Given a poset B and a category C, there is an equivalence of categories

$$\chi_{-} : n\mathsf{TBord}(B)_{/\!/\mathsf{C}} \rightleftarrows \mathrm{Fun}(B, \mathsf{TBord}^n_{/\!/\mathsf{C}}) : \pi_{-}$$

specified as follows.

The 'classification functor' χ_- takes a C-labeled n-truss bundle p to its classifying functor $\chi_p : B \to \mathsf{TBord}^n_{/\!/\mathsf{C}}$, and a C-labeled n-truss bundle bordism $u : p \Rightarrow q$ (by definition a labeled n-truss bundle over $B \times [1]$) to its classifying functor $\chi_u : B \times [1] \to \mathsf{TBord}^n_{/\!/\mathsf{C}}$ viewed as a natural transformation $\chi_u : \chi_p \Rightarrow \chi_q$.

The 'totalization functor' π_- takes a functor $\mathsf{F} : B \to \mathsf{TBord}^n_{/\!/\mathsf{C}}$ to its total C-labeled n-truss bundle π_{F}, and a natural transformation, represented as a functor $\mathsf{N} : B \times [1] \to \mathsf{TBord}^n_{/\!/\mathsf{C}}$, to its total labeled n-truss bundle π_{N}, considered as a bundle bordism. ⌟

Remark 2.3.53 (Classifying categorical labeled n-truss bundles). Generalizing Remark 2.1.85, a 'categorical n-truss bundle' p over a category B is a tower $\mathsf{T}_n \xrightarrow{p_n} \mathsf{T}_{n-1} \to ... \to \mathsf{T}_1 \xrightarrow{p_1} \mathsf{T}_0 = \mathsf{B}$ of categorical 1-truss bundles. A 'categorical C-labeled n-truss bundle' p over a category B is simply a categorical n-truss bundle $\underline{p}$ over B, together with a labeling functor $\mathrm{lbl}_p : \mathsf{T}_n \to \mathsf{C}$. Generalizing Remark 2.2.74, the above classification and totalization constructions carry over to this categorical case, showing that C-labeled n-truss bundles over a category B (and their bundle bordisms) correspond to functors $\mathsf{B} \to \mathsf{TBord}^n_{/\!/\mathsf{C}}$ (and their natural transformations). ⌟

2.3.2.3. ◊Pullback, product, dualization, and suspension of n-truss bundles. Our usual constructions of pullbacks, duals, and suspensions carry over from 1-truss bundles to n-truss bundles. We also describe a notion of (non-commutative) products for n-trusses and n-truss bundles, based on the construction of pullbacks.

Construction 2.3.54 (Pullbacks of labeled n-truss bundles). Consider a C-labeled n-truss bundle $p = (\underline{p}, \mathrm{lbl}_p)$ over a poset B, with underlying bundle $\underline{p} = (p_n, p_{n-1}, ..., p_1)$. Given a poset map $G : A \to B$, the pullback of the bundle p (along the map G) is the C-labeled n-truss bundle $G^*p \equiv (\underline{G^*p}, \mathrm{lbl}_{G^*p})$, with underlying bundle $\underline{G^*p} = (G^*p_n, G^*p_{n-1}, ..., G^*p_1)$ and labeling functor lbl_{G^*p} constructed as follows.

- Define $\mathrm{Tot}^0 G := G : A \to B$.
- Inductively with ascending i, define $G^*p_i : G^*T_i \to G^*T_{i-1}$ and $\mathrm{Tot}^i G : G^*T_i \to T_i$ by the 1-truss bundle pullback of $p_i : T_i \to T_{i-1}$ along the poset map $\mathrm{Tot}^{i-1} G : G^*T_{i-1} \to T_{i-1}$ (where G^*T_{i-1} is the total poset of G^*p_{i-1}).
- Finally, set the labeling functor lbl_{G^*p} to be the composite $\mathrm{lbl}_p \circ \mathrm{Tot}^n G$.

⌟

We can display the pullback G^*p of the labeled n-truss bundle p, along the base poset map G, as the upper row in the following diagram:

$$\begin{array}{ccccccccccc} & & G^*T_n & \xrightarrow{G^*p_n} & G^*T_{n-1} & \xrightarrow{G^*p_{n-1}} & \cdots & \xrightarrow{G^*p_2} & G^*T_1 & \xrightarrow{G^*p_1} & A \\ & \swarrow^{\mathrm{lbl}_{G^*p}} & \downarrow{\scriptstyle \mathrm{Tot}^n G} & \lrcorner & \downarrow{\scriptstyle \mathrm{Tot}^{n-1} G} & \lrcorner & \cdots & & \downarrow{\scriptstyle \mathrm{Tot}^1 G} & \lrcorner & \downarrow{\scriptstyle \mathrm{Tot}^0 G = G} \\ \mathsf{C} & \nwarrow_{\mathrm{lbl}_p} & T_n & \xrightarrow[p_n]{} & T_{n-1} & \xrightarrow[p_{n-1}]{} & \cdots & \xrightarrow[p_2]{} & T_1 & \xrightarrow[p_1]{} & B \end{array}$$

Note that the poset maps $\mathrm{Tot}^i G$ (together with the labeling category functor $\mathrm{id} : \mathsf{C} \to \mathsf{C}$) assemble into a C-labeled n-truss bundle map $G^*p \to p$, which we call the 'pullback bundle map'. As before, when $G : A \hookrightarrow B$ is a subposet, the pullback recovers the earlier notion of restriction of labeled n-truss bundles, i.e. $G^*p = p|_A$.

As a special case of truss bundle pullbacks, we obtain the following truss products.

Construction 2.3.55 (Products of labeled n-trusses and n-truss bundles). Given a C-labeled n-truss $T = (\underline{T}, \mathrm{lbl}_T)$ and an unlabeled m-truss bundle $q = (q_m, q_{m-1}, ..., q_1)$ over a poset S_0, where $q_i : S_i \to S_{i-1}$; let $G : S_m \to [0]$ be the terminal map, let $(G^*\underline{T}, q)$ denote the tower of 1-truss bundles obtained by concatenating the tower $G^*\underline{T}$ with the tower q, and let $G^*\mathrm{lbl}_T$ be shorthand for the composite $\mathrm{lbl}_T \circ (G \times \mathrm{id}_{T_n}) : S_m \times T_n \to \mathsf{C}$. We define the **truss product** $q \times T$ to be the C-labeled $(m+n)$-truss bundle $((G^*\underline{T}, q), G^*\mathrm{lbl}_T)$ over the poset S_0. ⌟

Remark 2.3.56 (Non-commutativity of products). By omitting labelings, the preceding construction gives a notion of products of trusses. Note well that given an n-truss T and an m-truss S, the product $T \times S$ and the product $S \times T$ differ in general; that is, truss products are non-commutative. Some examples of this non-commutativity can be found in Chapter C. ⌟

Taking duals of the constituent 1-truss bundles provides a dualization of labeled n-truss bundles, as follows.

Construction 2.3.57 (Dualization of labeled n-truss bundles and their maps). Given a C-labeled n-truss bundle $p = (\underline{p}, \mathrm{lbl}_p)$ with underlying n-truss bundle $\underline{p} = (p_n, p_{n-1}, ..., p_1)$, its dual is the C^{op}-labeled n-truss bundle $p^\dagger$, whose underlying n-truss bundle is $\underline{p^\dagger} = (p^\dagger_n, p^\dagger_{n-1}, ..., p^\dagger_1)$ (where $p^\dagger_i$ is the dual of the 1-truss bundle p_i, see Construction 2.1.107), and whose labeling $\mathrm{lbl}_{p^\dagger}$ is the opposite labeling $(\mathrm{lbl}_p)^{\mathrm{op}}$.

Given a labeled n-truss bundle map $F : p \to q$, the dual bundle map $F^\dagger : p^\dagger \to q^\dagger$ has its underlying n-truss bundle map $\underline{F^\dagger} = \underline{F}^\dagger$ given by the same maps of sets as the bundle map $\underline{F}$ itself, and has relabeling functor $\mathrm{lbl}_{F^\dagger} := (\mathrm{lbl}_F)^{\mathrm{op}}$.

We thus have a covariant involutive functor of labeled n-truss bundles:

$$\dagger : \mathsf{LblTrsBun}_n \cong \mathsf{LblTrsBun}_n. \quad ⌟$$

CONSTRUCTION 2.3.58 (Dualization of labeled n-truss bundles and their bordisms). For a C-labeled n-truss bundle bordism $u : p \Rightarrow q$, given as a labeled bundle u over $B \times [1]$, its dual C^{op}-labeled n-truss bundle bordism $u^\dagger : q^\dagger \Rightarrow p^\dagger$ is provided by the dual labeled bundle $u^\dagger$ (over $(B \times [1])^{\mathrm{op}} \cong B^{\mathrm{op}} \times [1]$) of the given labeled bundle u (over $B \times [1]$).

Dualization therefore gives an involutive isomorphism of labeled n-truss bundles and their bordisms:

$$\dagger : n\mathsf{TBord}(B)_{/\!/\mathsf{C}} \cong (n\mathsf{TBord}(B^{\mathrm{op}})_{/\!/\mathsf{C}^{\mathrm{op}}})^{\mathrm{op}}.$$

When the base poset is trivial, this specializes, using the equivalence $n\mathsf{TBord}_{/\!/\mathsf{C}} \simeq \mathsf{TBord}^n_{/\!/\mathsf{C}}$ from Lemma 2.3.48, to an involutive isomorphism:

$$\dagger : \mathsf{TBord}^n_{/\!/\mathsf{C}} \cong (\mathsf{TBord}^n_{/\!/\mathsf{C}^{\mathrm{op}}})^{\mathrm{op}}.$$ ⌟

REMARK 2.3.59 (Duality of closed and open trusses). The dualization of Construction 2.3.57 sends closed n-truss bundles to open n-truss bundles, and singular n-truss bundle maps to regular n-truss bundle maps. Thus, in particular, dualization of n-truss bundles and their maps specializes to a covariant involutive isomorphism

$$\dagger : \bar{\mathsf{Trs}}_n(B) \rightleftarrows \mathring{\mathsf{Trs}}_n(B^{\mathrm{op}}) : \dagger$$

between the category of closed n-truss bundles with singular maps $\bar{\mathsf{Trs}}_n(B)$, over the base poset B, and the category of open n-truss bundles with regular maps $\mathring{\mathsf{Trs}}_n(B^{\mathrm{op}})$, over the opposite base poset (see Notation 2.3.38). Of course, when the base poset is trivial, this specializes to the dualization case of most fundamental concern, between closed trusses and open trusses:

$$\dagger : \bar{\mathsf{Trs}}_n \rightleftarrows \mathring{\mathsf{Trs}}_n : \dagger$$ ⌟

Finally, straightforwardly generalizing the 1-truss bundle case, we mention suspensions of n-truss bundles.

CONSTRUCTION 2.3.60 (Suspension of n-truss bundles). For an (unlabeled) n-truss bundle $p = (p_n, p_{n-1}, ..., p_1)$, its suspension is the n-truss bundle $\Sigma p = (\Sigma p_n, \Sigma p_{n-1}, ..., \Sigma p_1)$, where Σp_i is the suspension bundle of the 1-truss bundle p_i, see Construction 2.1.111.

When the category C has both initial and terminal objects, a C-labeled n-truss bundle $p = (\underline{p}, \mathrm{lbl}_p)$ has a suspension $\Sigma p = (\underline{\Sigma p}, \mathrm{lbl}_{\Sigma p})$ with underlying bundle the suspension $\Sigma\underline{p}$ of the underlying n-truss bundle, and with labeling functor $\mathrm{lbl}_{\Sigma p} : \Sigma T_n \to \overline{\mathsf{C}}$ being simply lbl_p on $T_n \subset \Sigma T_n$ and sending the initial and terminal objects of the suspension to the initial and terminal objects of the labeling category, see Remark 2.2.81. ⌟

2.3.3. ◊n-Truss blocks and block sets. One of the most classical starting points for combinatorial topology is to consider the category Δ of combinatorial simplices; the presheaves on Δ are the *simplicial sets*. Though fundamental, there is nothing exclusive about simplices as a collection of basic shapes, and instead one may consider, for instance, the category $\mathbb{G}$ of combinatorial globes or the category $\square$ of combinatorial cubes; the presheaves on $\mathbb{G}$ are the *globular sets*, and the presheaves on $\square$ are the *cubical sets*.

The theory of trusses provides a new collection of combinatorial basic shapes, namely the *truss blocks*. Truss blocks are by definition the trusses with an initial element, and they assemble into a combinatorially defined category $\mathbb{X}$; presheaves on $\mathbb{X}$ are the *block sets*. Truss blocks simultaneously generalize combinatorial simplices, combinatorial globes, and combinatorial cubes and therefore inherit some of the merits and potentialities of each classical context. More importantly, they are the component building blocks of trusses themselves, and therefore basic for the theory of framed combinatorial topology as such.

SYNOPSIS. We discuss face, degeneracy, embedding, and coarsening maps of trusses, and show that a singular map of closes trusses uniquely factors into a degeneracy and a face, and that a regular map of open trusses uniquely factors into a coarsening and an embedding. We then define truss blocks as closed trusses with an initial element, and introduce the category of blocks and their singular maps. Next we define block sets as presheaves on the category of blocks, and block complexes as presheaves on the category of blocks with just their face maps. Finally, we briefly describe the dual story of truss braces, that is open trusses with a terminal element, the resulting category of braces and their regular maps, and the consequent notion of brace sets.

2.3.3.1. ◊Factorization of truss maps. The classical category $\Delta_{\leq n}$ of combinatorial simplices of dimension at most n (like the category of all combinatorial simplices) has two distinguished classes of maps, namely the injective monotone maps, called face maps, and the surjective monotone maps, called degeneracy maps; the category has the fundamental geometric property that any map factors uniquely as a degeneracy followed by a face map. The category $\bar{\mathsf{Trs}}_n$ of closed n-trusses and singular maps similarly has two distinguished classes of maps, *faces* and *degeneracies*, and a corresponding unique factorization property. Moreover, the dual category $\mathring{\mathsf{Trs}}_n$ of open n-trusses and regular maps also has dual distinguished classes of maps, called *embeddings* and *coarsenings*, and a respective factorization property.

We now introduce the various relevant classes of truss maps.

TERMINOLOGY 2.3.61 (Subtrusses, faces, and embeddings of 1-trusses). A map of 1-trusses $F : T \to S$ is called an 'injection' if it is injective on objects.

› An injection $F : T \to S$ is a **subtruss** map if it is balanced.
› An injection $F : T \to S$ is a **face** map if T and S are closed and the map is singular. (We also refer to these as 'closed face' maps for emphasis.)
› An injection $F : T \to S$ is an **embedding** map if T and S are open and the map is regular. (We also refer to these as 'open embedding' maps for emphasis.) ⌟

Observation 2.3.62 (Characterizing faces and embeddings). Note that a closed face map is necessarily balanced, and an open embedding map is also necessarily balanced. Thus closed faces are exactly the closed subtrusses of closed 1-trusses, and open embeddings are exactly the open subtrusses of open 1-trusses. ⌟

Recall, for use in the following terminology for surjective truss maps, that the endpoint type of a truss refers to the dimensions of the (frame order) minimal and maximal elements; see Terminology 2.1.23. Note that a surjective map of 1-truss preserves endpoints, and so if the trusses have the same endpoint type then the map preserves the dimensions of the endpoints.

Terminology 2.3.63 (Degeneracies and coarsenings of 1-trusses). A map of 1-trusses $F : T \to S$ is called a 'surjection' if it is surjective on objects.

- A surjection $F : T \to S$ is a **degeneracy** map if T and S have the same endpoint type and the map is singular. If furthermore T and S are closed, the map is a **closed degeneracy**.
- A surjection $F : T \to S$ is a **coarsening** map if T and S have the same endpoint type and the map is regular. If furthermore T and S are open, the map is a **open coarsening**. ⌟

Note that if a surjective map of 1-trusses is balanced, then it must be an isomorphism; so there is no new notion of surjective maps corresponding to the notion of subtrusses in the injective map case.

Example 2.3.64 (Faces, embeddings, degeneracies, and coarsenings of 1-trusses). In Figure 2.48 we depict an example of each of the aforementioned types of maps of 1-trusses. ⌟

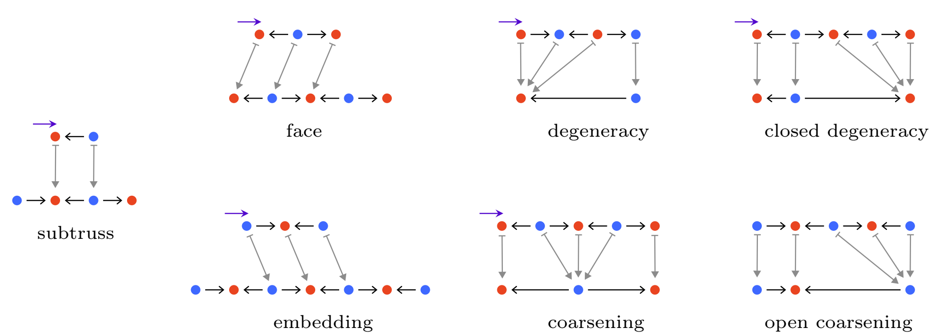


Figure 2.48. Faces, embeddings, degeneracies, and coarsenings.

The preceding terminology for 1-truss maps carries over to the case of n-trusses (and labeled n-trusses and more generally bundles) as follows.

Terminology 2.3.65 (Faces, embeddings, degeneracies, and coarsenings of n-trusses). Given n-trusses $T = (p_n, p_{n-1}, ..., p_1)$ and $S = (q_n, q_{n-1}, ..., q_1)$, an n-truss map $F : T \to S$ is called an 'injection' if each 1-truss bundle map $F_i : p_i \to q_i$ has every one of its fibers being a 1-truss injection. Similarly, the n-truss map $F : T \to S$ is called a 'subtruss', 'face', 'embedding', 'surjection', 'degeneracy', 'closed degeneracy', 'coarsening', or 'open coarsening' exactly when each 1-truss bundle map F_i is fiberwise of the corresponding designation.

The same terms apply, again by simply requiring the condition on every fiber at every stage, to n-truss bundles maps and further to labeled n-truss bundle maps, without further conditions[16] on the given base map. ⌟

Notation 2.3.66 (Categories of truss degeneracies and truss coarsenings). The category of n-trusses and their degeneracies is denoted $\mathsf{Trs}_n^{\mathrm{deg}}$. Similarly the category of n-trusses and their coarsenings is denoted $\mathsf{Trs}_n^{\mathrm{crs}}$. ⌟

Terminology 2.3.67 (Coarsenings versus refinements). Given a coarsening map of n-trusses $F : T \to S$, which grammatically we consider as 'coarsening T to S', we also call the map a **refinement**, and grammatically consider it as 'refining S to (or by) T'. That is, we use the terms 'coarsening' and 'refinement' for the same structure but seen from converse perspectives—a coarsening coarsens the domain to the codomain, while a refinement refines the codomain to the domain. ⌟

Equipped with the notions of face and degeneracy maps, and dually embedding and coarsening maps, we find that both singular maps of closed trusses, and regular maps of open trusses, admit a canonical factorization into an epimorphism and a monomorphism; that is, both the categories $\bar{\mathsf{Trs}}_n$ and $\mathring{\mathsf{Trs}}_n$ have an epi–mono factorizaton property, as follows.

Lemma 2.3.68 (Epi–mono factorization for closed singular and open regular maps). *Any singular map F of closed n-trusses factors uniquely into a degeneracy F^{E} followed by a face F^{M}. Similarly, any regular map F of open n-trusses factors uniquely into a coarsening F^{E} followed by an embedding F^{M}.*

Proof. In both cases the factorization $F = F^{\mathsf{M}} F^{\mathsf{E}}$ is given simply by factoring the ith stage face poset maps $F_i = F_i^{\mathsf{M}} F_i^{\mathsf{E}}$ using the standard epi–mono factorization in the category Pos of posets. □

Example 2.3.69 (Epi–mono factorization of closed singular truss maps). In Figure 2.49 we depict a singular map $F : T \to S$ of closed 2-trusses, together with its epi–mono factorization $F = F^{\mathsf{M}} \circ F^{\mathsf{E}}$. ⌟

Example 2.3.70 (Failure of epi–mono factorization for general truss maps). In Figure 2.50, we depict a map $F : T \to S$ of 2-trusses (neither a closed singular nor an open regular map), which cannot be factored into an epimorphism followed by a monomorphism. ⌟

[16] In fact, when considering posets that are fundamental posets of cellular or cellulable stratifications, it does make sense to enforce corresponding designations on the base map as well (i.e., requiring the map to be the fundamental poset map of a stratified topological

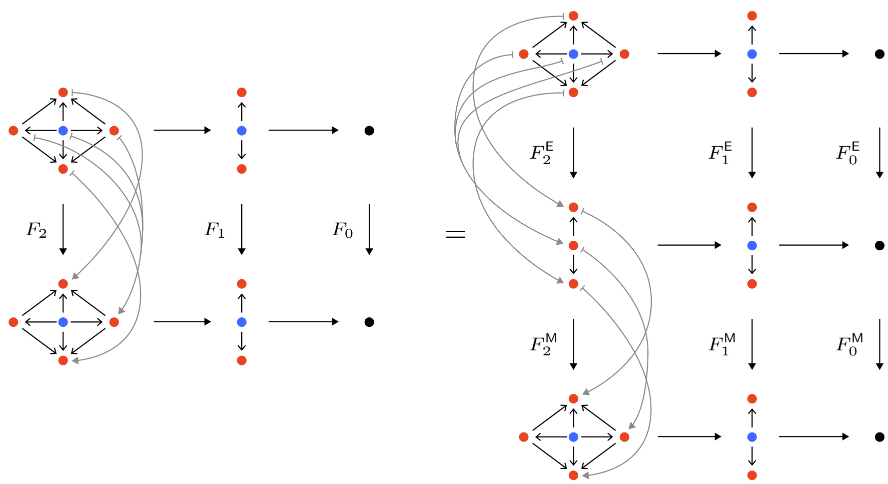


FIGURE 2.49. Epi–mono factorization of a closed singular 2-truss map.

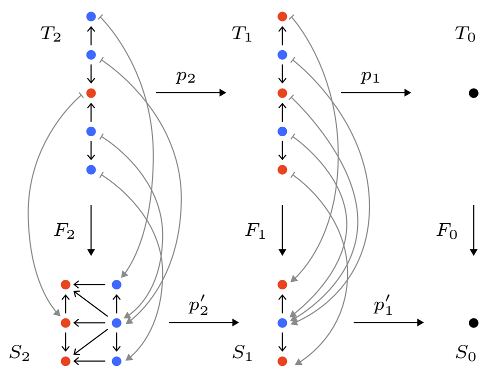


FIGURE 2.50. Failure of epi–mono factorization of a 2-truss map.

The epi–mono factorization property of the category of closed trusses and their singular maps, and similarly of the category of open trusses and their regular maps, provides a rigid formal structure to the morphisms of these categories. The natural transformations of these categories are even more constrained, in that there are no non-identity natural transformations whatsoever; thus the hom posets (consisting of maps and their natural transformations) are in fact discrete.

'face' map, 'degeneracy' map, open stratified 'embedding', or stratified 'coarsening'). For most of our later constructions, this remark will apply.

Lemma 2.3.71 (Rigidity of natural transformations for 1-trusses). *Consider 1-trusses T and S, and 1-truss maps $E, F : T \to S$. Assume one of the following holds:*

- *T and S are open, and E and F are regular (for instance embeddings),*
- *T and S are closed, and E and F are singular (for instance faces),*
- *E and F are coarsenings, or*
- *E and F are degeneracies, or*
- *E and F are balanced.*

Then any natural transformation $\nu : E \Rightarrow F : (T, \trianglelefteq) \to (S, \trianglelefteq)$ must be the identity.

Proof. We discuss the case of open 1-trusses and regular maps. (The closed singular case follows by duality, and the other cases follow by similar arguments.) The maps E and F send a regular value $a \in T$ to regular values $E(a) \in S$ and $F(a) \in S$; because there are no non-identity arrows between regular values in S, we must have $E(a) = F(a)$. Because T is open, a singular value $b \in T$ has two adjacent regular values $b \pm 1$, which are sent to $E(b \pm 1) = F(b \pm 1)$. Since the maps E and F are functorial and monotone, both $E(b)$ and $F(b)$ must be the unique element $y \in S$ such that $E(b-1) = F(b-1) \trianglelefteq y \trianglerighteq E(b+1) = F(b+1)$, and thus $E(b) = F(b)$. The functors E and F are thus identical and the natural transformation ν is necessarily trivial. □

Lemma 2.3.72 (Rigidity of natural transformations for n-trusses). *Consider n-trusses T and S, and n-truss maps $E, F : T \to S$. Assume one of the following holds:*

- *T and S are open and E and F are regular (for instance embeddings),*
- *T and S are closed and E and F are singular (for instance faces)*
- *E and F are coarsenings, or*
- *E and F are degeneracies, or*
- *E and F are balanced.*

Then any natural transformation $\nu : E_n \Rightarrow F_n$ of total poset maps $E_n, F_n : (T_n, \trianglelefteq) \to (S_n, \trianglelefteq)$ must be the identity.

All natural transformations are identities also in the corresponding cases of base-preserving n-truss bundle maps, namely open bundles and regular maps, closed bundles and singular maps, coarsenings of bundles, or degeneracies of bundles.

Proof. We discuss the case of open n-trusses and regular maps. (The case of closed n-trusses and singular maps follows by duality, and the other cases follow by similar arguments.) Let $T = (p_n, p_{n-1}, ..., p_1)$ and $S = (q_n, q_{n-1}, ..., q_1)$ be the constituent 1-truss bundles. Arguing inductively, assume the statement holds for $(n-1)$-trusses; the base case of 1-trusses was shown in the previous lemma. Postcomposing the natural transformation ν with the bundle projection q_n yields a natural transformation $q_n \circ \nu : q_n \circ E_n \Rightarrow q_n \circ F_n$, which may equivalently be considered a natural transformation $E_{n-1} \circ p_n \Rightarrow F_{n-1} \circ p_n$. We must have $q_n \circ \nu = \nu_{n-1} \circ p_n$ for some natural transformation $\nu_{n-1} : E_{n-1} \Rightarrow F_{n-1}$. (In general, given poset maps $E, F : B \to C$ and $G : A \to B$, any natural transformation $\nu : E \circ G \Rightarrow F \circ G$ will be of the form $\mu \circ G$ for some natural transformation $\mu : E \Rightarrow F$.) By the inductive assumption we know that $\nu_{n-1} = \mathrm{id}$. Finally, by applying the rigidity of 1-trusses (from the previous lemma) to the transformation ν, restricted to the fibers of p_n and q_n, we find that ν is itself trivial, as required.

The case of base-preserving n-truss bundle maps follows by applying the same argument to each n-truss fiber of the bundle. □

2.3.3.2. ◊The definition of truss blocks. Any closed truss may be considered as being built by piecing together certain elementary combinatorial building blocks, called straightforwardly *truss blocks* or simply *blocks*. A truss block is a closed truss with an initial element; the existence of an initial element ensures these blocks serve as component combinatorial cells.

Definition 2.3.73 (Truss blocks). An **n-truss block** T is a closed n-truss T whose total poset $(T_n, \trianglelefteq)$ has an initial element $\bot$. It is more specifically an **n-truss m-block** if the initial element has depth m. ⌟

Recall that the depth of an element in a poset is the maximal length of a chain starting at that element.

Example 2.3.74 (Truss blocks). In Figure 2.51 we depict a 2-truss 2-block on the left, together with its (informal) geometric realization on the right. In contrast to our earlier example of a general closed 2-truss in Figure 2.42, note that the realization of this 2-truss 2-block consists of a single 2-cell. In Figure 2.52 we similarly depict a 3-truss 3-block and its realization. A plethora of further truss blocks and their realizations can be found in Chapter C. ⌟

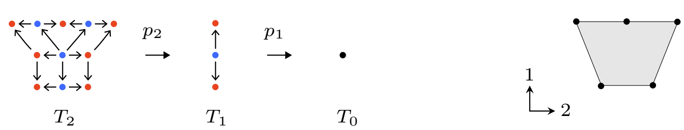


Figure 2.51. A 2-truss 2-block and its corresponding framed 2-cell.

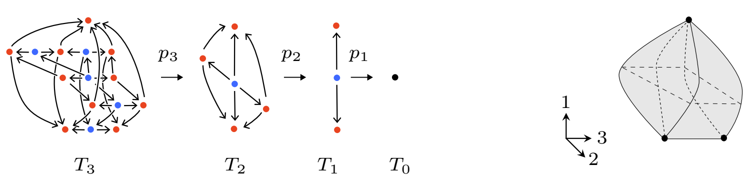


Figure 2.52. A 3-truss 3-block and its corresponding framed 3-cell.

Remark 2.3.75 (Blocks truncate). Given an n-truss block $T = (p_n, p_{n-1}, ..., p_1)$, the truncation $T_{\leq i} = (p_i, p_{i-1}, ..., p_1)$ is an i-truss block; indeed, the initial element in the total poset T_n of the n-truss T projects by the map $p_{>i} = p_{i+1} \circ p_{i+2} \circ ... \circ p_n$ to an initial element in the total poset T_i of the i-truss $T_{\leq i}$. ⌟

Remark 2.3.76 (Blocks stabilize). Given an n-truss block $T = (p_n, p_{n-1}, ..., p_1)$, there is an associated $(n+i)$-truss block $T_{+i} = (\mathrm{id}, ..., \mathrm{id}, p_n, p_{n-1}, ..., p_1)$, whose first i 1-truss bundles are identities with singular fibers. ⌟

Observation 2.3.77 (Dimensions of blocks). Given an n-truss m-block $T = (p_n, p_{n-1}, ..., p_1)$, the block depth m is computed by

$$m = \sum_{i=1}^{n} \dim(p_{>i}(\bot)) \quad .$$

Here, in the ith summand, dim is the dimension functor of the 1-truss bundle p_i. In other words, the depth m is the count of the number of bundles p_i that are non-trivial (i.e. whose underlying poset map is not the identity map). In particular, when $m < n$, at least one bundle p_i must be trivial. Note further that the depth m corresponds to the geometric dimension of the realization of the m-block. ⌟

Notation 2.3.78 (Categories of n-blocks). The category of n-truss blocks, denoted Blk_n, is the full subcategory, of the category $\bar{\mathsf{Trs}}_n$ of closed trusses and singular maps, whose objects are n-truss blocks. ⌟

Notation 2.3.79 (The category of blocks). The category of blocks, denoted $\mathbb{X}$, is the colimit under stabilization of the categories Blk_n of n-truss blocks. ⌟

Remark 2.3.80 (The notation for the category of blocks). The reader who flipped ahead to Appendix Figure C.1 to see more 2-dimensional blocks, will have recognized some cell structures familiar from other shape categories, such as the 2-globe (the unique 2-dimensional shape in the globular category $\mathbb{G}$), the 2-simplex (the unique 2-dimensional shape in the simplicial category Δ), and the 2-cube (the unique 2-dimensional shape in the cubical category $\square$), along with some less standard 2-cell decompositions. The last block in that figure has two 1-cells on both its upper and lower boundaries, and is therefore unequivocally beyond the realm of globular, simplicial, cubical, or opetopic models for higher categorical structures. We take this shape as informally characteristic and let its X configuration of regular values inspire the notation $\mathbb{X}$ for the category of blocks. ⌟

Recall from Terminologies 2.3.61 and 2.3.65 and Observation 2.3.62 that faces of closed trusses are the closed subtrusses. There are distinguished faces, namely those that are blocks, constructed as closures of elements of closed trusses, as follows.

Construction 2.3.81 (Face blocks in closed trusses). Let $T = (p_n, p_{n-1}, ..., p_1)$ be a closed n-truss, and consider an element $x \in T_n$ in the total poset. We construct a subtruss inclusion $T^{\trianglerighteq x} \hookrightarrow T$ such that $T^{\trianglerighteq x} = (p_n^{\trianglerighteq x}, p_{n-1}^{\trianglerighteq x}, ..., p_1^{\trianglerighteq x})$ is a block, called the **face block** of the element x.

For each $i \leq n$, denote by $x_i := p_{>i}x$ the image of the element x under the composite projection $p_{>i} = p_{i+1} \circ ... \circ p_n : T_n \to T_i$. Set $(T_i^{\trianglerighteq x}, \trianglelefteq)$ to be the subposet of $(T_i, \trianglelefteq)$ given by the upper closure of the element x_i in $(T_i, \trianglelefteq)$. Set the 1-truss bundle projection $p_i^{\trianglerighteq x} : T_i^{\trianglerighteq x} \to T_{i-1}^{\trianglerighteq x}$ to be the restriction of $p_i : T_i \to T_{i-1}$ to the upper closure subposets. Altogether these bundles and their inclusions $p_i^{\trianglerighteq x} \hookrightarrow p_i$ form the components of the desired subtruss $T^{\trianglerighteq x} \hookrightarrow T$. Note that $T^{\trianglerighteq x}$ is an m-block, where m is the depth of the element x in the total poset T_n. ⌟

Example 2.3.82 (Face blocks in closed trusses). In Figure 2.53, on the left we depict a 2-truss $T_2 \to T_1 \to T_0$, and highlight three of its face blocks. The total poset T_2 is the fundamental poset of the cell complex on the right; note that face blocks (obtained by taking 'upper closures' in the total poset) correspond to closed cells in the cell complex (obtained by taking 'topological closure' in the complex). ⌟

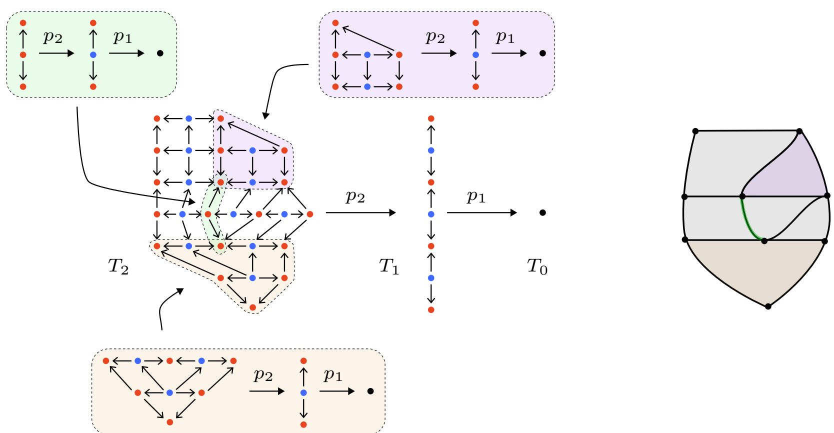


Figure 2.53. Face blocks in a 2-truss.

Remark 2.3.83 (All subtruss blocks are face blocks). Given an n-truss T, the subtrusses of T that are n-truss blocks are in bijective correspondence with the elements of the total poset T_n. Indeed, every such subtruss block of T determines an element of T_n, namely the image of the initial element of the block, and conversely elements of T_n determine face blocks by taking upper closures as in Construction 2.3.81. ⌟

2.3.3.3. ◊Block sets and block complexes. Simplicial sets are presheaves on the category of combinatorial simplicies. We have corresponding notions of presheaves on the category of n-truss blocks, or more generally all blocks, as follows.

DEFINITION 2.3.84 (n-Truss block sets). An **n-truss block set** is a Set-valued presheaf on the category Blk_n of n-truss blocks. ⌟

NOTATION 2.3.85 (Category of n-truss block sets). The 'category of n-truss block sets', i.e. the category of Set-valued presheaves on the category Blk_n of n-truss blocks, will be denoted BlkSet_n. ⌟

DEFINITION 2.3.86 (Block sets). A **block set** is a Set-valued presheaf on the category $\mathbb{X}$ of blocks. ⌟

NOTATION 2.3.87 (Category of block sets). The 'category of block sets', i.e. the category of Set-valued presheaves on the category $\mathbb{X}$ of blocks, will be denoted $\widehat{\mathbb{X}}$. ⌟

We usually abbreviate 'n-truss block set' simply to 'block set', leaving the dimension n implicit and eliding the difference between presheaves on Blk_n for some fixed finite n and presheaves on the colimit category $\mathbb{X}$ (which includes all the categories Blk_n at once). In the subsequent discussion, we restrict attention to n-truss block sets, for fixed n, but everything can be extended to block sets, that is to the context of variable n.

TERMINOLOGY 2.3.88 (Faces and degeneracies in block sets). For a block set $X \in \mathsf{BlkSet}_n$ and a block $B \in \mathsf{Blk}_n$, we refer to elements of the set $X(B)$ as 'blocks of shape B' in the block set X. For a given block $b \in X(B)$ of shape B in the block set X, we may take face map or degeneracy maps of the block, as follows.

- For a specific face map $F : C \to B$ in Blk_n, we call $c := (X(F))(b) \in X(C)$ the 'F-face' of the block $b \in X(B)$.
- For a specific degeneracy map $F : C \to B$ in Blk_n, we call $c := (X(F))(b) \in X(C)$ the 'F-degeneracy' of the block $b \in X(B)$. ⌟

TERMINOLOGY 2.3.89 (Nondegenerate blocks). A block $c \in X(C)$ (of shape C in the block set X) is a 'nondegenerate block' if it is not an F-degeneracy for any non-identity degeneracy map F. ⌟

Recall that semi-simplicial sets, also known as Δ-complexes [**Hat02**], are simplicial sets 'without degeneracy maps'. We have an analogous notion of block complexes as block sets 'without degeneracy maps', as follows.

DEFINITION 2.3.90 (n-Truss block complex). An **n-truss block complex** is a Set-valued presheaf on the injective subcategory $\mathsf{Blk}_n^{\mathrm{inj}} \subset \mathsf{Blk}_n$, i.e. on the wide subcategory containing only face maps. ⌟

Of course one may define a 'block complex' (without fixing n) as a presheaf on the injective subcategory of the category of blocks. We typically abbreviate 'n-truss block complex' to simply 'block complex'.

Recall that a CW-complex is regular when each of its closed cells embeds into the whole complex. Similarly a Δ-complex may be called regular when each of its closed simplices embeds into the complex; concretely, that occurs when all the faces of any given simplex are distinct. Analogously, a regular block complex is one for which all the faces of any given block are distinct, as follows.

DEFINITION 2.3.91 (Regular block complexes). A block complex X is called **regular** when, for every block $b \in X(B)$ of shape B, and every block shape C, all the faces $(X(F))(b) \in X(C)$, for face maps $F : C \to B$, are distinct. ⌟

NOTATION 2.3.92 (Categories of block complexes). The 'category of block complexes', i.e. the category of presheaves on the injective subcategory $\mathsf{Blk}_n^{\mathrm{inj}}$ of n-truss blocks, will be denoted $\mathsf{BlkCplx}_n$. Its full subcategory of regular block complexes will be denoted $\mathsf{RBlkCplx}_n$. ⌟

The next example illustrates the differences between the notions of block set, block complex, and regular block complex: in a regular block complex, all the faces of each block are distinct; in a general block complex, a block may have more than one face coincident; in a block set, a face of a block may be a nontrivial degeneracy of another block.

EXAMPLE 2.3.93 (A block set, a block complex, and a regular block complex). In Figure 2.54, we depict the nondegenerate blocks of a 2-truss block set (on the left) and of a 2-truss block complex (on the right). In each case, we bubble and color-code the individual nondegenerate blocks; we then use the color coding to indicate the face-block relationships. Note that in the block set, the left and right 1-faces of the 2-block are a nontrivial degeneracy of the 0-block (and are bubbled in purple accordingly); the existence of these degenerate blocks prevents this block set from being a block complex. By contrast, in the (nonregular) block complex, the left and right 1-faces of the 2-block are both the red-bubbled nondegenerate 1-block.

In Figure 2.55, we illustrate a regular block complex, by depicting its three 2-truss 2-blocks and the identifications of their face 1-blocks. Specifically, each of the three distinct gray-bubbled 1-blocks is shared between two 2-blocks, as indicated by the given geometric arrangement. The remaining 1-blocks are pairwise identified according to the bubble colors. Altogether, this regular complex has six 0-blocks (not indicated), nine 1-blocks, and three 2-blocks. Note the truss poset towers of all the blocks can be inferred, by projection, from the given total posets and the purple frame arrows. ⌟

We end this discussion of block sets with a few technical observations about the categorical relationship of blocks, trusses, their maps, and their presheaves. (Readers may skip ahead to Section 2.3.3.4 without consequence.)

CONSTRUCTION 2.3.94 (Block nerve of trusses). Given a closed n-truss T, its 'block nerve' is the block set $\bar{\mathsf{Trs}}_n(-, T)$ sending a block B to the hom set $\bar{\mathsf{Trs}}_n(B, T)$. This construction is functorial and gives rise to the 'block nerve functor' $\mathrm{N}_{\mathsf{Blk}} : \bar{\mathsf{Trs}}_n \to \mathsf{BlkSet}_n$. ⌟

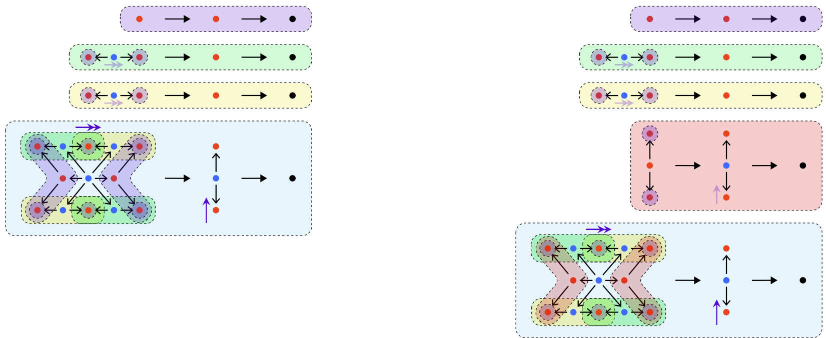


FIGURE 2.54. The blocks of a block set and of a block complex.

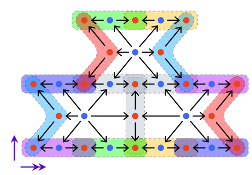


FIGURE 2.55. A regular block complex.

REMARK 2.3.95 (Building trusses from their blocks). We can now make precise the sense in which a closed n-truss can be built from its blocks. Given a closed n-truss $T \in \bar{\mathsf{Trs}}_n$, denote by Blk_n/T the comma category of the inclusion $\mathsf{Blk}_n \hookrightarrow \bar{\mathsf{Trs}}_n$ (that is, objects of Blk_n/T are singular maps $B \to T$ from blocks to T, and morphisms are block maps $B \to B'$ commuting with the given maps to T). The truss T is now given by the colimit

$$T = \mathrm{colim}(\mathsf{Blk}_n/T \to \bar{\mathsf{Trs}}_n)$$

of the forgetful functor $\mathsf{Blk}_n/T \to \bar{\mathsf{Trs}}_n$ mapping $(B \to T)$ to the block B. (The colimit may also be taken to have source the smaller category with objects the face maps from blocks to the truss T, and with morphisms the commuting face maps of blocks.) ⌟

The above remark is equivalent to the statement that the functor $\mathsf{Blk}_n \hookrightarrow \bar{\mathsf{Trs}}_n$ is dense, and also to the statement that the nerve functor $\mathrm{N}_{\mathsf{Blk}}$ is fully faithful.

REMARK 2.3.96 (Block complexes and regularity in block sets). Restricting presheaves along the inclusion $j : \mathsf{Blk}_n^{\mathrm{inj}} \hookrightarrow \mathsf{Blk}_n$ provides the pullback functor $j^* : \mathsf{BlkSet}_n \to \mathsf{BlkCplx}_n$ from block sets to block complexes. That functor has left adjoint $j_!$ and right adjoint $j^!$ given by left and right Kan extension respectively [**Rie14**]. The left adjoint $j_!$ can be thought of as 'freely adjoining degeneracies' to a given block complex.

We say that a block set 'is a block complex' if it lies in the essential image of this free adjunction $j_!$, and we say it 'is regular' if it lies in the essential image of the free adjunction $j_!$ restricted to regular block complexes. ⌟

OBSERVATION 2.3.97 (Block nerves are regular). For any closed n-truss T, its block nerve $\mathrm{N}_{\mathsf{Blk}} T$ is regular. Indeed, any map $B \to T$ from a block B to the truss T factors uniquely into a degeneracy map followed by a face map (see Lemma 2.3.68). Truss face maps are injective on blocks. It follows that $\mathrm{N}_{\mathsf{Blk}} T \cong j_! \bar{\mathsf{Trs}}_n(i-, T)$ for $i : \mathsf{Blk}_n^{\mathrm{inj}} \hookrightarrow \bar{\mathsf{Trs}}_n$, and that $\bar{\mathsf{Trs}}_n(i-, T)$ is regular, as needed. ⌟

Degenerate blocks in block sets can have a rather different character than degenerate simplices in simplices sets, as highlighted by the following final two remarks.

REMARK 2.3.98 (Interior versus boundary degeneracies). Degeneracies of blocks may be separated into two distinct classes:

- A degeneracy map $F : C \to B$ is a 'boundary degeneracy' if the blocks C and B are of the same dimension.
- A degeneracy map $F : C \to B$ is an 'interior degeneracy' if the dimension of the block C is strictly greater than the dimension of the block B.

The second sort of degeneracy is familiar from simplicial sets. Due to the existence of degeneracies of the first sort, a block in a block set may be degenerate in a way that is only visible on its boundary. ⌟

REMARK 2.3.99 (Block sets are not Eilenberg–Zilber). The Eilenberg–Zilber lemma [**EZ50**] states that every simplex in a simplicial set is a degeneracy of a unique nondegenerate simplex. The analogous property fails for block sets that do not satisfy further sheaf conditions. ⌟

2.3.3.4. ◊Truss braces and brace sets. Recall that closed n-trusses and open n-trusses are related by covariant involutive duality isomorphisms $\dagger : \bar{\mathsf{Trs}}_n \cong \mathring{\mathsf{Trs}}_n : \dagger$, which reverse the face orders and interchange the role of singular and regular elements and maps. The whole story of truss blocks and block sets may be transported across this duality, to provide a corresponding story of *truss braces* and *brace sets*. For convenience, we briefly record the most central aspects of that dual story.

DEFINITION 2.3.100 (Truss braces). An **n-truss brace** T is an open n-truss whose total poset $(T_n, \trianglelefteq)$ has a terminal element $\top$. It is more specifically an **n-truss m-brace** if the terminal element has height $(n - m)$. ⌟

Recall the height of an element in a poset is the maximal length of a chain ending at that element.

EXAMPLE 2.3.101 (A truss brace). In Figure 2.56 we depict a 2-truss 0-brace on the left, obtained by dualizing the 2-truss 2-block from Figure 2.51. We depict on the right an (informal) geometric realization of that brace; notice that this realization is geometrically dual to the cell realizing the dual truss block. (Such realizations will be formalized later on using the notion of 'meshes'.) ⌟

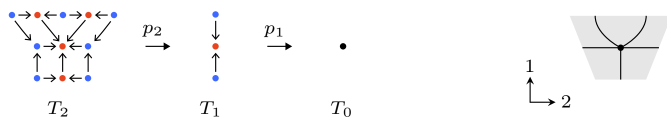


FIGURE 2.56. A 2-truss 0-brace.

REMARK 2.3.102 (Braces stabilize). Given an n-truss brace $T = (p_n, p_{n-1}, ..., p_1)$, there is an associated $(n+i)$-truss brace $T^{+i} = (\mathrm{id}, ..., \mathrm{id}, p_n, p_{n-1}, ..., p_1)$, whose first i 1-truss bundles are identities with regular fibers.

OBSERVATION 2.3.103 (Dimensions of braces). Given an n-truss m-brace $T = (p_n, p_{n-1}, ..., p_1)$, the brace dual height m is computed by

$$m = \sum_{i=1}^{n} \dim(p_{>i}(\top))$$

In particular, when $m > 0$, at least one bundle p_i must be trivial.

Dual to 'face blocks' we have a notion of 'embedding braces' as follows.

CONSTRUCTION 2.3.104 (Embedding braces in open trusses). Let T be an open n-truss, and consider an element $x \in T_n$ in the total poset. There is a subtruss $T^{\trianglelefteq x} \hookrightarrow T$, called the **embedding brace** of the element x, given as the unique open subtruss of T that is a brace whose terminal element maps to x.

NOTATION 2.3.105 (Category of n-truss braces). The category of n-truss braces, denoted Brc_n, is the full subcategory, of the category $\mathring{\mathsf{Trs}}_n$ of open trusses and regular maps, whose objects are n-truss braces.

NOTATION 2.3.106 (The category of braces). The category of braces, denoted $\mathbb{X}$, is the colimit under stabilization of the categories Brc_n of n-truss braces.

DEFINITION 2.3.107 (n-Truss brace sets). An *n-truss brace set* is a Set-valued presheaf on the category Brc_n of n-truss braces.

NOTATION 2.3.108 (Category of n-truss brace sets). The 'category of n-truss brace sets', i.e. the category of Set-valued presheaves on the category Brc_n of n-truss braces, will be denoted BrcSet_n.

DEFINITION 2.3.109 (Brace sets). A *brace set* is a Set-valued presheaf on the category $\mathbb{X}$ of braces.

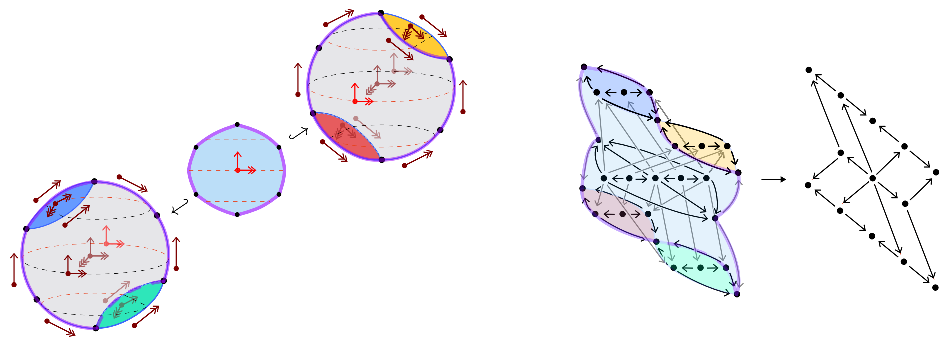
NOTATION 2.3.110 (Category of brace sets). The 'category of brace sets', i.e. the category of Set-valued presheaves on the category $\mathbb{X}$ of braces, will be denoted $\widehat{\mathbb{X}}$.

Note that the dualization isomorphism $\dagger : \bar{\mathsf{Trs}}_n \cong \mathring{\mathsf{Trs}}_n$ restricts to an isomorphism of n-truss blocks and n-truss braces $\mathsf{Blk}_n \cong \mathsf{Brc}_n$ and thus provides isomorphisms of presheaf categories $\mathsf{BlkSet}_n \cong \mathsf{BrcSet}_n$ and altogether an isomorphism

$$\widehat{\mathbb{X}} \cong \widehat{\mathbb{X}}.$$

CHAPTER 3

◊Constructibility of framed combinatorial structures



In Chapter 1, we infused classical combinatorial topology with a concept of framings, eventually defining framed regular cells as combinatorial regular cells equipped with locally collapsible simplicial framings. In Chapter 2, we reimagined combinatorial stratified topology through the prism of inductive constructibility, defining trusses as iterated constructible bundles of entrance path posets of stratified intervals. In this chapter, we will ascertain an equivalence between these independently motivated and a priori rather distinct structures: framed regular cells have corresponding integral truss blocks, and truss blocks have corresponding gradient framed regular cells. This identification provides a computable classification, via constructible combinatorics, of framed cell shapes.

We begin this chapter, in Section 3.1, with an illustrated overview of the three fundamental classifications, of framed cells by truss blocks, of collapsible framed cell complexes by closed trusses, and of framed cell complexes by regular block complexes. We then, in Section 3.2, introduce the requisite intermediate structure of proframed simplicial complexes; suitably cellularized, these proframed complexes will have both framed cell complexes as gradients and trusses as fundamental stratified posets. Finally, Section 3.3, we develop the necessary cellularization techniques and assemble the proofs of the classification results.

3.1. ◊Overview of the classifications

We state and illustrate the three primary classification results in increasing generality: the classification of framed regular cells by truss blocks, the classification of collapsible framed regular cell complexes by trusses, and the classification of framed regular cell complexes by regular truss block complexes.

Theorem 3.1.1 (Truss blocks classify framed regular cells). *n-Framed regular cells are classified by n-truss blocks; that is, there is a canonical equivalence of categories*

$$\mathsf{FrCell}_n \underset{\nabla_{\mathsf{C}}}{\overset{\int_{\mathsf{T}}}{\rightleftarrows}} \mathsf{Blk}_n \quad .$$

In this equivalence, framed regular cells are sent to truss blocks by the 'truss integration' functor $\int_{\mathsf{T}}$, and truss blocks are sent to framed regular cells by the 'cell gradient' functor ∇_{C}, which are both explained and constructed in due course.

We illustrate an instance of this equivalence in Figure 3.1. On the left, we depict a 3-framed regular cell by a framed realization in $\mathbb{R}^3$ (note the same cell appeared earlier in Figure 1.55 and yet earlier in the title picture of Chapter 1); on the right, we depict its corresponding 3-truss block.

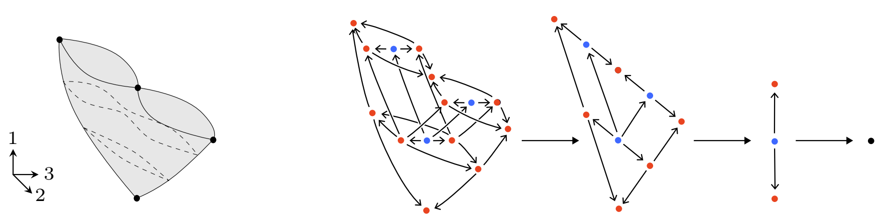


Figure 3.1. A framed regular cell and its corresponding truss block.

Theorem 3.1.2 (Trusses classify collapsible framed regular cell complexes). *Collapsible n-framed regular cell complexes are classified by closed n-trusses and their singular maps; that is, there is a canonical equivalence of categories*

$$\mathsf{CollFrCellCplx}_n \underset{\nabla_{\mathsf{C}}}{\overset{\int_{\mathsf{T}}}{\rightleftarrows}} \bar{\mathsf{T}}\mathsf{rs}_n \quad .$$

In fact, the proof of the previous classification of framed regular cells will rely on this classification of framed regular cell complexes, not vice versa as one might expect.

We illustrate an instance of this equivalence in Figure 3.2. On the left, we depict a collapsible 3-framed regular cell complex (note this complex appeared before in Figure 1.56); on the right, we depict its corresponding closed 3-truss.

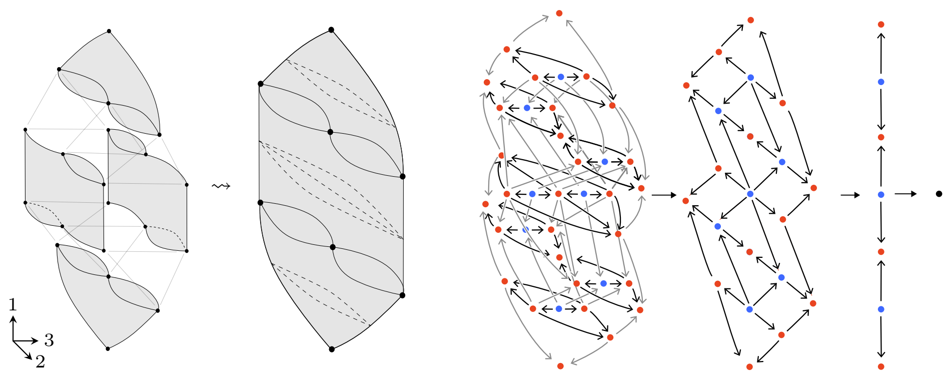


FIGURE 3.2. A collapsible framed regular cell complex and its corresponding closed truss.

THEOREM 3.1.3 (Regular truss block complexes classify framed regular cell complexes). *n-Framed regular cell complexes are classified by regular n-truss block complexes; that is, there is a canonical equivalence of categories*

$$\mathsf{FrCellCplx}_n \underset{\nabla_{\mathsf{C}}}{\overset{\int_{\mathsf{T}}}{\rightleftarrows}} \mathsf{RBlkCplx}_n \quad .$$

We illustrate an instance of this equivalence in Figure 3.3. On the left, we depict a 2-framed regular cell complex, consisting of two 0-cells, connected by two 1-cells, which together are bounded by two distinct 2-cells (note that this complex appeared earlier in Figure 1.43). On the right, the corresponding truss block complex is depicted by its six nondegenerate truss blocks, each in its own bubble, color-coded to the cells on the left; the face relations are indicated by colored subbubbles.

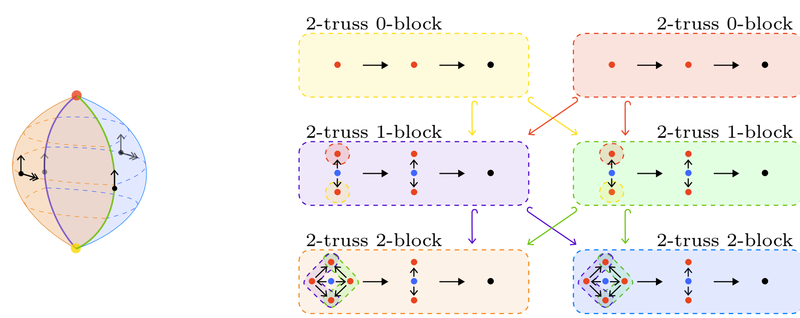


FIGURE 3.3. A framed regular cell complex and its corresponding truss block complex.

3.2. ◊Proframed combinatorial structures

We would like to construct a correspondence, in particular, between framed regular cells and truss blocks. This correspondence (and its generalization to cell complexes and trusses) will proceed via a crucial intermediate structure, namely *proframed simplicial complexes*. Such a complex is a suitable tower of simplicial complexes; see Figure 3.4 for an illustrative example.

Recall that a framed regular cell is in particular a cell-wise collapsible framed simplicial complex. Given such a cell, for instance the one on the left of Figure 3.1, consider the framing as providing a collection of infinitesimal vector fields on the underlying simplicial complex. Imagine *integrating* the maximal frame vector field to a foliation, and then quotienting the complex (and its frame) by that foliation. One may hope that the quotient is itself a framed simplicial complex representing a framed regular cell, and therefore one may iterate the integration process to obtain a tower of regular cells—the underlying simplicial complexes of that tower will form a proframed simplicial complex. Finally, the fundamental posets of the regular cells of that tower will assemble into a truss block, for instance the one on the right of Figure 3.1. Altogether we call this the process of forming the *truss integral*.

To reverse the process, given a truss block, one may realize it to a tower of simplicial complexes, that is to a proframed simplicial complex. The kernels of the projections in that tower provide 1-dimensional foliations of the complexes and one may imagine vector fields tangent to those foliations, along with *gradient-like* vector fields for the projections to the leaves of the foliations, altogether forming a framing structure on the total simplicial complex of the proframe tower. One may hope that the total complex with its framing forms a framed regular cell, providing altogether an inverse process of forming the *cell gradient*.

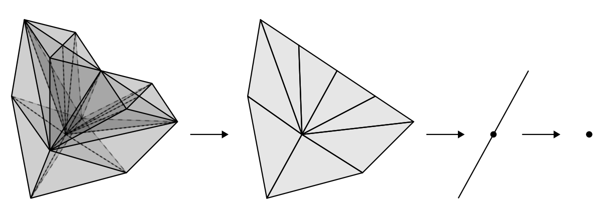


FIGURE 3.4. A proframed simplicial complex.

OUTLINE. In Section 3.2.1, we introduce proframed simplices, their realizations, and maps, and we construct the gradient framed simplex of a proframed simplex and the integral proframed simplex of a framed simplex. In Section 3.2.2, we define proframed simplicial complexes, define the notions of gradient and integral for complexes, introduce a collapsibility condition on proframed complexes, and show that collapsible framings always integrate to collapsible proframings.

3.2.1. ◊Proframed simplices. Recall a frame on the standard simplex is a numeral labeling of its spine. Quotienting by the spine vectors in reverse order of their label numbers provides a 'proframe' tower of simplicial projections, each with 1-dimensional affine kernel. We imagine the frame as consisting of *differential* or infinitesimal data of (simplicial) tangent vectors; by contrast we imagine the proframe as consisting of *integral* or global data of (simplicial) projections. This heuristic dichotomy will become more vivid and defensible later in the context of complexes: there the frames remain locally defined simplex by simplex, whereas the proframes will be globally defined via unified projections on the entire complex.

Synopsis. We define proframed simplices, and their partial and embedded generalizations, as towers of simplicial projections with controlled affine kernels. We then describe proframed realizations as affine embeddings of these towers into the standard euclidean proframe. We specify proframed and subproframed maps as suitable transformations of simplicial towers. Finally, we construct the gradient functor taking proframed simplices to framed simplices and the inverse integral functor taking framed simplices to proframed simplices.

3.2.1.1. ◊The definition of proframed simplices. As when we defined framed simplices, we begin with the basic case of proframes, and then generalize to partial, embedded, and embedded partial proframes.

Definition 3.2.1 (Proframe on a simplex). A **proframe** of an m-simplex S is an isomorphism $S \cong [m]$ together with a sequence $\mathcal{P} = (p_m, p_{m-1}, \ldots, p_1)$ of surjective simplicial maps of the form

$$[m] \xrightarrow{p_m} [m-1] \xrightarrow{p_{m-1}} [m-2] \xrightarrow{p_{m-2}} \cdots \xrightarrow{p_2} [1] \xrightarrow{p_1} [0]. \qquad \lrcorner$$

We usually denote proframes on S by pairs $(S \cong [m], \mathcal{P})$; we may also keep the isomorphism $S \cong [m]$ implicit, especially when the simplex S was already ordered, writing the proframe as simply $(S, \mathcal{P})$ or $([m], \mathcal{P})$ or just $\mathcal{P}$ depending on context and convenience.

Example 3.2.2 (Proframes on simplices). In Figure 3.5 and Figure 3.6 we illustrate four proframed simplices. The arrows indicate a spine of the simplex (and thus its isomorphism with a standard simplex), and each simplicial degeneracy is indicated by highlighting its affine kernel. $\lrcorner$

Recall that a partial frame of a simplex S is a degeneracy $S \twoheadrightarrow [k]$ and a frame on the target simplex $[k]$. A partial proframe is defined analogously, as follows.

Definition 3.2.3 (Partial proframe on a simplex). A **k-partial proframe** on an m-simplex S is a degeneracy $p_\perp : S \twoheadrightarrow [k]$ together with a proframe $\mathcal{P} = (p_k, p_{k-1}, \ldots, p_1)$ of the simplex $[k]$. $\lrcorner$

We denote k-partial proframes on a simplex S by pairs $(S \twoheadrightarrow [k], \mathcal{P})$. As in the case of partial frames, we refer to the affine kernel $U = \ker^{\mathrm{aff}}(S \twoheadrightarrow$

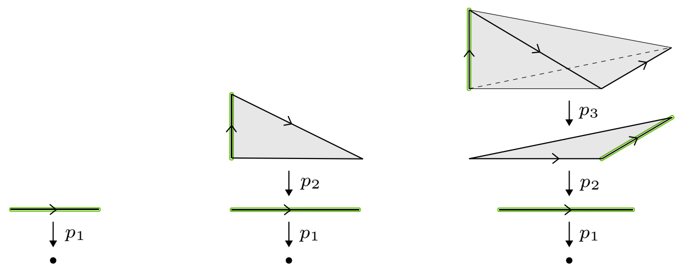


FIGURE 3.5. Proframed simplices.

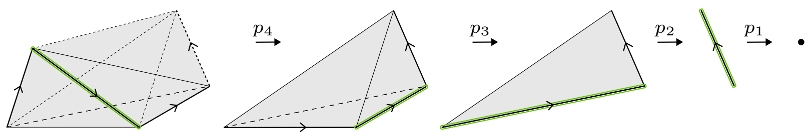


FIGURE 3.6. A proframed 4-simplex.

$[k]$) as the 'unframed subspace' of the partially proframed simplex $(S \twoheadrightarrow [k], \mathcal{P})$. Note that in an m-partial proframe of an m-simplex $(S \twoheadrightarrow [m], \mathcal{P})$, the degeneracy $S \twoheadrightarrow [m]$ must be an isomorphism; thus m-partial proframes of m-simplices are simply proframes of m-simplices.

EXAMPLE 3.2.4 (Partial proframes on simplices). In Figure 3.7 we illustrate several partially proframed simplices. As before, each degeneracy is indicated by highlighting its affine kernel; we distinguish in red the 'unframed subspace' kernel of the initial degeneracy $p_\perp$ and in green the kernels of the other projections p_i. ⌟

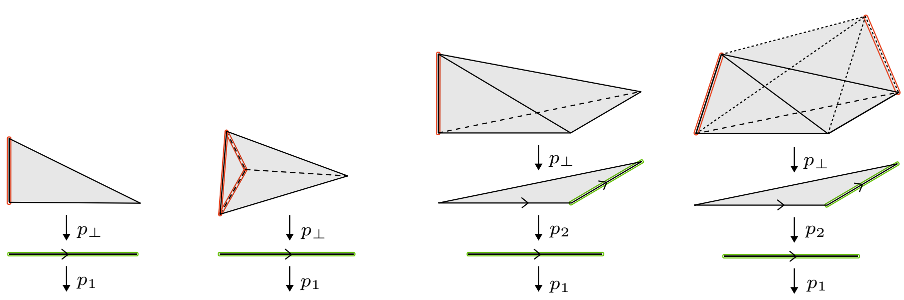


FIGURE 3.7. Partially proframed simplices.

Recall that an embedded frame of a simplex S is an isomorphism $S \cong [m]$ together with a labeling of the vectors of $\mathsf{spine}[m]$ by numerals in $\{1, 2, \ldots, n\}$. Quotienting the spine vectors with label n, then those with label $n-1$, and so on, provides a series of simplicial degeneracies, each of which has affine kernel either containing a single vector or being empty. Such a series provides the notion of embedded proframe, as follows.

Definition 3.2.5 (Embedded proframe on a simplex). An n-**embedded proframe** of an m-simplex S is an isomorphism $S \cong [m]$ together with a sequence $\mathcal{P} = (p_n, p_{n-1}, \ldots, p_1)$ of surjective simplicial maps of the form

$$[m] = [m_n] \xrightarrow{p_n} [m_{n-1}] \xrightarrow{p_{n-1}} [m_{n-2}] \xrightarrow{p_{n-2}} \cdots \xrightarrow{p_2} [m_1] \xrightarrow{p_1} [m_0] = [0]$$

where for each i, either $m_{i-1} = m_i - 1$ or $m_{i-1} = m_i$. ⌟

We usually denote n-embedded proframed simplices by pairs $(S \cong [m], \mathcal{P})$, abbreviated to $(S, \mathcal{P})$ or $([m], \mathcal{P})$ or just $\mathcal{P}$ depending. Note that m-embedded proframes of an m-simplex are simply ordinary (non-embedded) proframes.

Example 3.2.6 (Embedded proframes). In Figure 3.8 we illustrate a few n-embedded proframed m-simplices. As before, each degeneracy is indicated by highlighting its affine kernel. ⌟

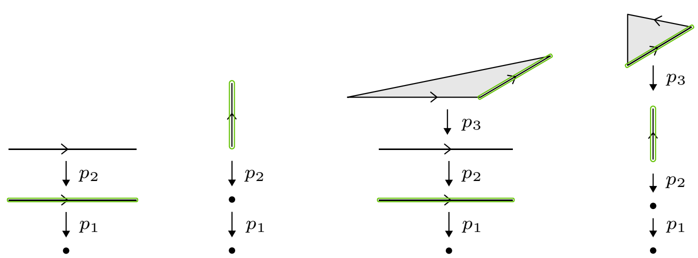


Figure 3.8. Embedded proframes on simplices.

Of course we have the conceptual pushout of partial and embedded proframes.

Definition 3.2.7 (Embedded partial proframe on a simplex). An n-**embedded k-partial proframe** on an m-simplex S is a degeneracy $p_\perp : S \twoheadrightarrow [k]$ together with an n-embedded proframe $\mathcal{P} = (p_n, p_{n-1}, \ldots, p_1)$ of the simplex $[k]$. ⌟

However, neither partiality nor embedded partiality will be needed for our principal concerns, and so we leave it without illustration or discussion.

3.2.1.2. ◊Proframed realizations. Recall the classical linear algebraic notion of the standard euclidean proframe. From the standard frame $(e_1, e_2, ..., e_n)$ on $\mathbb{R}^n$, we may form the spans $\langle e_{n-k+1}, e_{n-k+2}, ..., e_n \rangle \cong \mathbb{R}^k$; these assemble into the standard euclidean indframe, i.e., linear flag,

$$\mathbb{R}^0 \hookrightarrow \mathbb{R}^1 \hookrightarrow \mathbb{R}^2 \hookrightarrow \cdots \hookrightarrow \mathbb{R}^n$$

where $\mathbb{R}^{i-1} \hookrightarrow \mathbb{R}^i$ is the inclusion by adding a leading zero coordinate. The quotients of the total space $\mathbb{R}^n$ by these subspaces $\mathbb{R}^k$ provides a corresponding proframe, i.e., tower of linear projections, as follows.

TERMINOLOGY 3.2.8 (The standard euclidean proframe). The 'standard euclidean proframe' of $\mathbb{R}^n$, denoted $\mathcal{P}^n_{\mathbb{R}}$, is the sequence of projections

$$\mathbb{R}^n \xrightarrow{\pi_n} \mathbb{R}^{n-1} \xrightarrow{\pi_{n-1}} \mathbb{R}^{n-2} \xrightarrow{\pi_{n-2}} \cdots \xrightarrow{\pi_2} \mathbb{R}^1 \xrightarrow{\pi_1} \mathbb{R}^0$$

where $\pi_i : \mathbb{R}^i \twoheadrightarrow \mathbb{R}^{i-1}$ forgets the last coordinate of $\mathbb{R}^i$. ⌟

See Section A.1 for an explication of classical linear frames, indframes, and proframes, and their embedded generalizations.

Recall that a framed realization of a (possibly embedded) framed simplex was an embedding of the simplex in euclidean space, that suitably respected the frame structure on spine vectors. Analogously a proframed realization of a (possibly embedded) proframed simplex is an embedding of the proframe tower into the standard euclidean proframe tower, that suitably respects the proframe structure on spine vectors.

DEFINITION 3.2.9 (Proframed realization of an embedded proframed simplex). A **proframed realization** of an n-embedded proframed simplex $(S \cong [m], \mathcal{P} = (p_n, p_{n-1}, \ldots, p_1))$ is a sequence of linear embeddings $r^{\mathcal{P}}_i : \Delta^{m_i} \hookrightarrow \mathbb{R}^i$, giving a commutative diagram,

$$\begin{array}{ccccccccccc} |S| & \xrightarrow{\cong} & \Delta^{m_n} & \xrightarrow{p_n} & \Delta^{m_{n-1}} & \xrightarrow{p_{n-1}} & \cdots & \xrightarrow{p_2} & \Delta^{m_1} & \xrightarrow{p_1} & \Delta^{m_0} \\ & \searrow & \downarrow r^{\mathcal{P}}_n & & \downarrow r^{\mathcal{P}}_{n-1} & & & & \downarrow r^{\mathcal{P}}_1 & & \downarrow r^{\mathcal{P}}_0 \\ & & \mathbb{R}^n & \xrightarrow{\pi_n} & \mathbb{R}^{n-1} & \xrightarrow{\pi_{n-1}} & \cdots & \xrightarrow{\pi_2} & \mathbb{R}^1 & \xrightarrow{\pi_1} & \mathbb{R}^0 \end{array}$$

such that, for any spine vector $v \subset \Delta^{m_i}$ that is degenerated by the projection p_i, the image $r^{\mathcal{P}}_i(v) \subset \mathbb{R}^i$ is a positive vector in the fiber $\pi_i^{-1}(r^{\mathcal{P}}_{i-1}(p_i(v)))$. ⌟

The definition specializes, of course, to proframed realization of (non-embedded) proframed m-simplices, in which case $m_i = i$ throughout. It also straightforwardly generalizes to the partial proframed and embedded partial proframed cases: for the n-embedded k-partial case, simply replace the isomorphism $|S| \xrightarrow{\cong} \Delta^{m_n}$ in the diagram by the degeneracy $|S| \xrightarrow{p_\perp} \Delta^k$, and replace Δ^{m_i} by Δ^{k_i} throughout (with $k_n := k$); for the (non-embedded) k-partial case, furthermore note $k_i = i$.

EXAMPLE 3.2.10 (Proframed realizations). In Figure 3.9 we illustrate a proframed realization of each of a proframed, partial proframed, embedded proframed, and embedded partial proframed simplex, respectively. ⌟

3.2.1.3. ◊Proframed maps. Recall that a framed map is a map of simplices that, for each vector of the source, either preserves the frame label of the vector or else degenerates the vector. Analogously, a proframed map will be a map of simplicial towers, that for each vector in the total simplex of the source, either preserves the whole proframe restricted to that vector or else degenerates that vector, as follows.

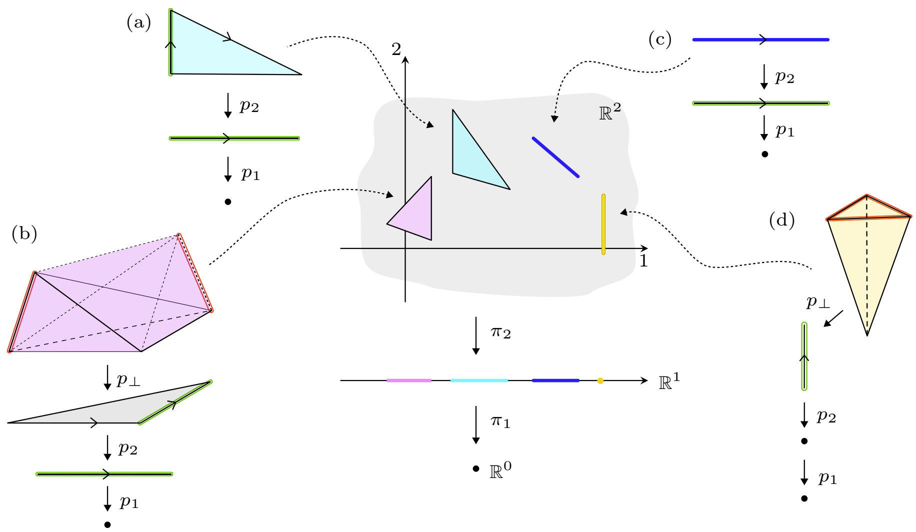


FIGURE 3.9. Proframed realizations of proframed simplices.

DEFINITION 3.2.11 (Proframed maps). Given n-embedded proframed simplices $(S \cong [l], \mathcal{P} = (p_n, \ldots, p_1))$ and $(T \cong [m], \mathcal{Q} = (q_n, \ldots, q_1))$, a **proframed map** $F : (S \cong [l], \mathcal{P}) \to (T \cong [m], \mathcal{Q})$ is a map of sequences

$$\begin{array}{ccccccccc} [l] = [l_n] & \xrightarrow{p_n} & [l_{n-1}] & \xrightarrow{p_{n-1}} & \cdots & \xrightarrow{p_2} & [l_1] & \xrightarrow{p_1} & [l_0] = [0] \\ \downarrow{\scriptstyle F_n} & & \downarrow{\scriptstyle F_{n-1}} & & \cdots & & \downarrow{\scriptstyle F_1} & & \downarrow{\scriptstyle F_0} \\ [m] = [m_n] & \xrightarrow[q_n]{} & [m_{n-1}] & \xrightarrow[q_{n-1}]{} & \cdots & \xrightarrow[q_2]{} & [m_1] & \xrightarrow[q_1]{} & [m_0] = [0] \end{array}$$

such that for every vector $v : [1] \to [l]$, either its proframe is preserved, i.e., $F : \mathcal{P}|_v \cong \mathcal{Q}|_{F_n \circ v}$, or the vector is degenerated, i.e., $F_n \circ v : [1] \to [m]$ is constant. ⌟

NOTATION 3.2.12 (Category of proframed simplices). The category of n-embedded proframed simplices and their proframed maps is denoted $\mathsf{ProFrSimp}_n$. ⌟

REMARK 3.2.13 (Subproframed maps). Recall from Remark 1.1.69 and Definition 1.1.70 that, unlike a framed map, a subframed map of framed simplices may send a vector to a vector with a more specialized frame label. The corresponding notion of subproframed map of proframed simplices is rather natural, as follows. Given n-embedded proframed simplices $(S \cong [l], \mathcal{P} = (p_n, ..., p_1))$ and $(T \cong [m], \mathcal{Q} = (q_n, ..., q_1))$, a subproframed map $F : (S \cong [l], \mathcal{P}) \to (T \cong [m], \mathcal{Q})$ is a map of sequences of unordered simplices,

$$\begin{array}{ccccccccc} [l_n]^{\mathsf{un}} & \xrightarrow{p_n} & [l_{n-1}]^{\mathsf{un}} & \xrightarrow{p_{n-1}} & \cdots & \xrightarrow{p_2} & [l_1]^{\mathsf{un}} & \xrightarrow{p_1} & [l_0]^{\mathsf{un}} \\ \downarrow{\scriptstyle F_n} & & \downarrow{\scriptstyle F_{n-1}} & & \cdots & & \downarrow{\scriptstyle F_1} & & \downarrow{\scriptstyle F_0} \\ [m_n]^{\mathsf{un}} & \xrightarrow[q_n]{} & [m_{n-1}]^{\mathsf{un}} & \xrightarrow[q_{n-1}]{} & \cdots & \xrightarrow[q_2]{} & [m_1]^{\mathsf{un}} & \xrightarrow[q_1]{} & [m_0]^{\mathsf{un}} \end{array}$$

such that any ordered vector $v : [1] \to [l_i]$ with $p_i \circ v : [1] \to [l_{i-1}]$ constant, is sent to an ordered vector $F_i \circ v : [1] \to [m_i]$. The structure of the sequence itself controls the specialization of the proframed vectors, without mention of frame labels or the standard stratification of euclidean frame vectors. ⌟

The notions of proframed and subproframed maps generalize straightforwardly to the case of embedded partial proframes, but we omit such a discussion.

◊ Image of proframed map and subproframed map – see old numbered pix

3.2.1.4. ◊Gradients and integrals for simplices. Recall that we informally think of frames as infinitesimal data, concerning tangential vectors, and of proframes as global data, concerning foliations. We now describe the translation between these structures: we will refer to the process of taking a proframe and constructing a frame as forming a 'gradient', and we will refer to the converse passage from a frame to a proframe as 'integration'.[17]

We begin with the gradient frame of a proframe. (For convenience, we will mainly work with ordered simplices $[m]$ rather than unordered simplices with a chosen order $S \cong [m]$ as before.)

Notation 3.2.14 (Composite projections in proframes). For an n-embedded proframe $\mathcal{P} = (p_n, \ldots, p_1)$ of the simplex $[m]$, we abbreviate the composite $p_{i+1} \cdots p_n : [m] \to [m_i]$ by $p_{\to i}$. ⌟

Definition 3.2.15 (Gradient frame). Given an n-embedded proframed m-simplex $([m], \mathcal{P} = (p_n, \ldots, p_1))$, its **gradient frame** $\nabla\mathcal{P}$ is the n-embedded framed m-simplex $([m], \nabla\mathcal{P} : \mathsf{spine}[m] \hookrightarrow \underline{n})$ with $\nabla\mathcal{P}$ mapping, for all i, any spine vector in $\ker^{\mathrm{aff}}(p_{\to i-1}) \setminus \ker^{\mathrm{aff}}(p_{\to i})$ to the frame label $i \in \underline{n}$. ⌟

In other words, if the spine vector $v \in \mathsf{spine}[m]$ projects to a spine vector $p_{\to i} v \in \mathsf{spine}[m_i]$ and the projection $p_i : [m_i] \to [m_{i-1}]$ degenerates that vector $p_{\to i} v$ to a constant $p_{\to i-1} v$, then the spine vector $v \in \mathsf{spine}[m]$ is given the label i.

As in classical geometric situations, in general the converse process of integration is less procedural and any potential construction is less assured to work. As such, we define the integral as a formal right inverse to the gradient.

Definition 3.2.16 (Integral proframe). Given an n-embedded framed m-simplex $([m], \mathcal{F} : \mathsf{spine}[m] \hookrightarrow \underline{n})$, an **integral proframe** $\int\mathcal{F}$ is an n-embedded proframe on the simplex $[m]$, whose gradient $\nabla \int\mathcal{F}$ is the given frame $\mathcal{F}$. ⌟

However, in the special case of simplices, we do have an effective construction of integral proframes, as follows.

Construction 3.2.17 (Integral proframe of an embedded frame). For an n-embedded framed m-simplex $([m], \mathcal{F} : \mathsf{spine}[m] \hookrightarrow \underline{n})$, an integral proframe is given by the n-embedded proframed m-simplex $([m], \int\mathcal{F} = (p_n, \ldots, p_1))$

[17] The reference relationship between frames and proframes in the classical linear and affine algebraic case is discussed in Chapter A.

obtained by inductively setting $p_i : [m_i] \to [m_{i-1}]$ to be the simplicial map collapsing the spine vector $p_{\to i}(\mathcal{F}^{-1}(i))$, i.e., the spine vector with frame label i; if there is no spine vector with frame label i, then p_i is set to be the identity. ⌟

Observation 3.2.18 (The gradient and integral for simplices are inverse). For any n-embedded proframe $\mathcal{P}$ of a simplex and any n-embedded frame $\mathcal{F}$ of a simplex, we have

$$\nabla \textstyle\int \mathcal{F} = \mathcal{F} \quad \text{and} \quad \textstyle\int \nabla \mathcal{P} = \mathcal{P}.$$ ⌟

Thus in the case of simplices, the integral always exists and is a two-sided inverse to the gradient; but later in the more general case of simplicial complexes, we will find that, though all proframes are differentiable, some frames fail to be uniquely integrable or even integrable at all.

Example 3.2.19 (Gradient frame and integral proframe). In Figure 3.10 we illustrate a 4-embedded framed 3-simplex and its integral 4-embedded proframed 3-simplex; equivalently, that proframed simplex has gradient that framed simplex. ⌟

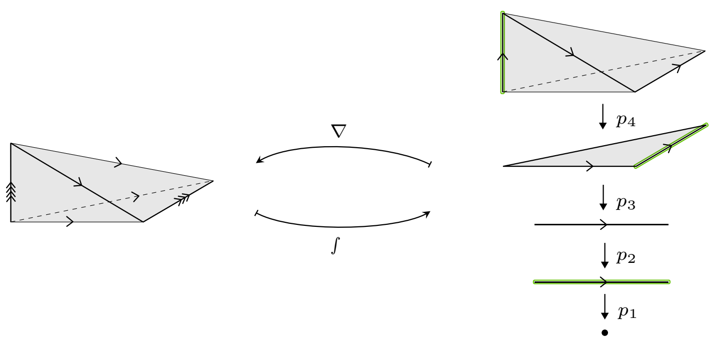


Figure 3.10. The gradient frame and the integral proframe.

As one can expect, gradients and integrals generalize from the embedded case to the embedded partial case.

To promote the above gradients and integrals to functors, we next construct them on framed and proframed maps.

Observation 3.2.20 (Gradients and integrals respect restriction). Given an n-embedded proframed simplex $(S \cong [m], \mathcal{P})$ and an n-embedded framed simplex $(S \cong [m], \mathcal{F})$, and a face $f : [j] \to [m]$, the gradient of the restriction to the face is the restriction of the gradient, and similarly for the integrals:

$$\nabla(\mathcal{P}|_f) = (\nabla \mathcal{P})|_f \quad \text{and} \quad \textstyle\int(\mathcal{F}|_f) = (\int \mathcal{F})|_f.$$ ⌟

Terminology 3.2.21 (Gradient framed map of a proframed map). Given an n-embedded proframed simplex $(S \cong [j], \mathcal{P})$, an n-embedded proframed simplex $(T \cong [k], \mathcal{Q})$, and a proframed map $F : (S \cong [j], \mathcal{P}) \to (T \cong [k], \mathcal{Q})$, the 'gradient' ∇F is simply the framed map $(S \cong [j], \nabla \mathcal{P}) \to (T \cong [k], \nabla \mathcal{Q})$ determined by the simplicial map $F : S \to T$. ⌟

Terminology 3.2.22 (Integral proframed map of a framed map). Given an n-embedded framed simplex $(S \cong [j], \mathcal{F})$, an n-embedded framed simplex $(T \cong [k], \mathcal{G})$, and a framed map $F : (S \cong [j], \mathcal{F}) \to (T \cong [k], \mathcal{G})$, an 'integral' $\int F$ is a proframed map $(S \cong [j], \int \mathcal{F}) \to (T \cong [k], \int \mathcal{G})$ whose gradient is the framed map F. ⌟

Just as there exists a unique integral proframe of any framed simplex, there exists a unique integral proframed map of any framed map; that integral proframed map is constructed by setting its top component $F_n : [j] \to [k]$ to be the given framed map $F : [j] \to [k]$, and observing that the condition that F is framed ensures the map F_n descends to maps $F_i : [j_i] \to [k_i]$ as required. This yields gradient and integral functors, which assemble into an equivalence of categories as follows.

Observation 3.2.23 (Correspondence of frames and proframes). The gradient and integral functors are inverse equivalences between the category of n-embedded framed simplices with framed maps and the category of n-embedded proframed simplices with proframed maps:

$$\nabla : \mathsf{ProFrSimp}_n \cong \mathsf{FrSimp}_n : \int$$ ⌟

Remark 3.2.24 (Correspondence of framed and proframed realizations). Given an n-embedded proframed simplex $(S \cong [m], \mathcal{P})$ with corresponding gradient framed simplex $(S \cong [m], \mathcal{F} = \nabla \mathcal{P})$, any proframed realization $\{r^{\mathcal{P}}_i : \Delta^{m_i} \hookrightarrow \mathbb{R}^i\}$ determines and is determined by a framed realization $r_{\mathcal{F}} : |S| \cong \Delta^{m_n} \hookrightarrow \mathbb{R}^n$ by equating $r^{\mathcal{P}}_n = r_{\mathcal{F}}$. ⌟

3.2.2. ◊Proframed simplicial complexes. Recall a framing of a simplicial complex is a local notion: it is simply a framing of each of its simplices, compatible with restriction. A proframing of a simplicial complex will be by contrast a global notion, namely a suitable tower of projections of complexes. As anticipated, there will always be a gradient framed complex associated to a proframed complex, but only certain framed complexes, namely the collapsible one, will be integrable.

Synopsis. We defined proframings of simplicial complexes as towers of projections that restrict to proframings on every simplex. We then define gradients for proframed complexes and discuss integrability of framed complexes. Finally we introduce the notion of collapsible proframing and show that collapsible framed complexes always have an integral collapsible proframing.

3.2.2.1. ◊The definition of proframed simplicial complexes. Recall from Alternative Definition 1.2.13 a framing of a simplicial complex may be considered as an ordering on the complex and a compatible collection of framings on its ordered simplices. We take a similar approach to defining proframed complexes.

Definition 3.2.25 (Proframings of simplicial complexes). An n-**proframing** of a simplicial complex K is an ordering of K together with a sequence $\mathcal{P} = (p_n, p_{n-1}, ..., p_1)$ of ordered simplicial surjections

$$K = K_n \xrightarrow{p_n} K_{n-1} \xrightarrow{p_{n-1}} ... \xrightarrow{p_2} K_1 \xrightarrow{p_1} K_0 = [0]$$

such that on each simplex $x : [m] \hookrightarrow K$, the restricted sequence $\mathcal{P}|_x$ is an n-embedded proframe of that simplex $[m]$. ⌟

Naturally we will refer to the pair $(K, \mathcal{P})$ of a simplicial complex with an n-proframing $\mathcal{P} = (p_n, p_{n-1}, ..., p_1)$ as an 'n-proframed simplicial complex'. For convenience we will keep the ordering implicit; henceforth, we simply say 'simplicial complex' in place of 'simplicial complex with a choice of ordering', and we assume all simplicial maps are order preserving.

Example 3.2.26 (Proframings of complexes). In Figure 3.11 we illustrate three 2-proframed simplicial complexes. Each projection p_i is suggested as a geometric projection, but we also highlight the affine kernels of every p_i on each simplex, as before. ⌟

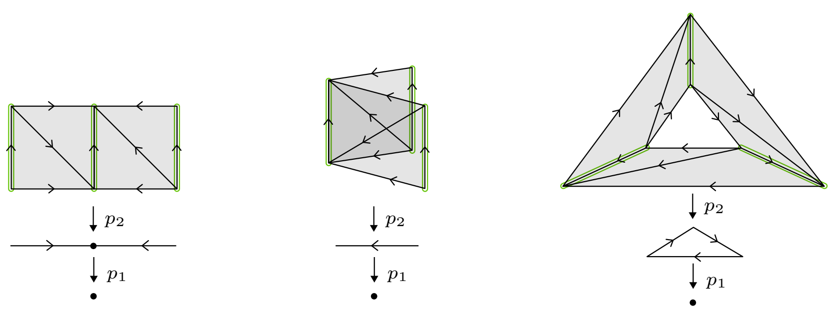


Figure 3.11. 2-Proframings of simplicial complexes.

Definition 3.2.27 (Maps of proframings). Given n-proframed simplicial complexes $(K, \mathcal{P} = (p_n, ..., p_1))$ and $(L, \mathcal{Q} = (q_n, ..., q_1))$, a **proframed map** $F : (K, \mathcal{P}) \to (L, \mathcal{Q})$ is a map of sequences

$$\begin{array}{ccccccccc} K = K_n & \xrightarrow{p_n} & K_{n-1} & \xrightarrow{p_{n-1}} & \cdots & \xrightarrow{p_2} & K_1 & \xrightarrow{p_1} & K_0 = [0] \\ \downarrow{\scriptstyle F_n} & & \downarrow{\scriptstyle F_{n-1}} & & \cdots & & \downarrow{\scriptstyle F_1} & & \downarrow{\scriptstyle F_0} \\ L = L_n & \xrightarrow[q_n]{} & L_{n-1} & \xrightarrow[q_{n-1}]{} & \cdots & \xrightarrow[q_2]{} & L_1 & \xrightarrow[q_1]{} & L_0 = [0] \end{array}$$

such that, on every simplex $x : [k] \hookrightarrow K$, with image $y = \mathrm{im}(F_n \circ x) : [l] \hookrightarrow L$, the sequence restricts to a proframed map $F : \mathcal{P}|_x \to \mathcal{Q}|_y$ of n-embedded proframed simplices. ⌟

NOTATION 3.2.28 (Category of proframings). The category of n-proframed simplicial complexes and their proframed maps will be denoted by $\mathsf{ProFrSimpCplx}_n$. ⌟

NOTATION 3.2.29 (Truncations of proframings). Given an n-proframing $\mathcal{P} = (K_n \xrightarrow{p_n} K_{n-1} \xrightarrow{p_{n-1}} ... \xrightarrow{p_1} K_0)$ of the simplicial complex K, its (lower) **i-truncation** $\mathcal{P}_{\leq i}$ is the i-proframing $(K_i \xrightarrow{p_i} K_{i-1} \xrightarrow{p_{i-1}} ... \xrightarrow{p_1} K_0)$ of the simplicial complex K_i. By similarly truncating maps, we obtain i-truncation functors $(-)_{\leq i} : \mathsf{ProFrSimpCplx}_n \to \mathsf{ProFrSimpCplx}_i$. ⌟

3.2.2.2. ◊Gradients and integrals for simplicial complexes. From a proframed simplicial complex, we can constructively form the associated gradient framed simplicial complex, as follows.

DEFINITION 3.2.30 (Gradients of proframed simplicial complexes). Given an n-proframing $\mathcal{P}$ of a simplicial complex K, the **gradient framing** $\nabla \mathcal{P}$ is the n-framing of K with the same ordering as $\mathcal{P}$ and with the n-embedded frame $(\nabla \mathcal{P})_x$ on each simplex $x : [m] \hookrightarrow K$ given by the gradient frame $\nabla(\mathcal{P}|_x)$ of the restricted proframe $\mathcal{P}|_x$. ⌟

The fact that the frames $(\nabla \mathcal{P})_x$ are compatible with face restrictions, as required, follows from the compatibility of gradients with face restriction, as in Observation 3.2.20.

DEFINITION 3.2.31 (Gradients of proframed maps). Given a proframed map $F = (F_n, F_{n-1}, ..., F_1, F_0) : (K, \mathcal{P}) \to (L, \mathcal{Q})$ of n-proframed simplicial complexes, the **gradient framed map** $\nabla F : (K, \nabla \mathcal{P}) \to (L, \nabla \mathcal{Q})$ is the framed map given by the simplicial map $F_n : K \to L$. ⌟

TERMINOLOGY 3.2.32 (The gradient framing functor). The construction of gradients on proframings and their maps yields the 'gradient framing' functor

$$\nabla : \mathsf{ProFrSimpCplx}_n \to \mathsf{FrSimpCplx}_n.$$ ⌟

Going the other way, we would like to take a framing and produce an integral proframing, which is to say something whose gradient is the original framing, as follows.

DEFINITION 3.2.33 (Integral proframings). Given an n-framed simplicial complex $(K, \mathcal{F})$, an **integral proframing** of $(K, \mathcal{F})$ is an n-proframed simplicial complex $(K, \mathcal{P})$ whose gradient framing $\nabla \mathcal{P}$ is the given framing $\mathcal{F}$. ⌟

However, not all framings are *integrable*, and even for an integrable framing, the integral proframing may not be unique.

EXAMPLE 3.2.34 (Non-uniqueness of integral proframings). In Figure 3.12 we illustrate a 2-proframed simplicial complex and its gradient, a 2-framed simplicial complex. Note that the proframing is not the unique integral of the framing: another integral proframing could be obtained by modifying p_2 to have only a single 1-simplex in its image. ⌟

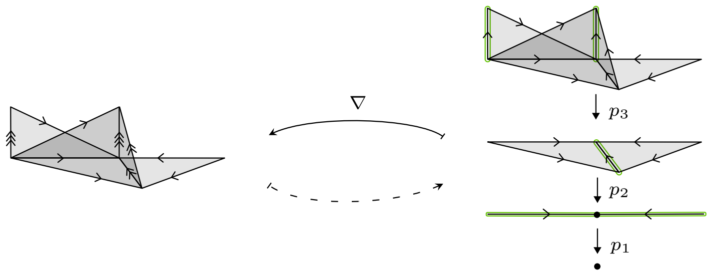


FIGURE 3.12. Proframed simplicial complex with its gradient framing.

EXAMPLE 3.2.35 (Non-integrable framings). In Figure 3.13 we depict two 2-framings of the boundary $\partial[2]^{\mathsf{un}}$ of the unordered 2-simplex $[2]^{\mathsf{un}}$. Neither framing admits an integral proframing. ⌟

FIGURE 3.13. Two framings without integral proframings.

REMARK 3.2.36 (Integrating simplex boundary framings). The failures of integrability in the previous example may be seen in the context of the following more general observation. An n-framing $\mathcal{F}$ of the unordered simplex boundary $\partial[m]^{\mathsf{un}}$ is integrable if and only if it is the restriction of some n-framing $\overline{\mathcal{F}}$ of the unordered m-simplex $[m]^{\mathsf{un}}$; that is, $\mathcal{F} = \overline{\mathcal{F}}\big|_{\partial[m]^{\mathsf{un}}}$. ⌟

3.2.2.3. ◊Collapsible proframings. Recall that a collapsible framing is a framing that admits a sequence of elementary simplicial collapses degenerating all the frame vectors in descending order, such that suitable collapse subsequences satisfy unique lifting properties. We will define an analogous notion of collapsibility for proframings, and then show that collapsible framings have unique integral collapsible proframings.

The notion of collapsible proframing will be formulated in terms of fiber categories, which we develop presently, of the component projections of the proframing.

TERMINOLOGY 3.2.37 (Fiber set). Given a simplicial map $p : K \to K'$, and a simplex $z : [m] \hookrightarrow K'$, the 'fiber set' over z is the set of all simplices $x : [k] \hookrightarrow K$ such that the composite $p \circ x$ has image identical to the image of z; denote, leaving the map p implicit, the fiber set over z by K_z. ⌟

Simplices in fiber sets K_z of proframing projections $p_i : K_i \to K_{i-1}$ fall into only two classes, as follows.

Definition 3.2.38 (Section and spacer simplices in proframings). Given an n-proframed simplicial complex $(K, \mathcal{P})$ and a simplex $x : [k] \hookrightarrow K_i$ in the fiber set over $z : [m] \hookrightarrow K_{i-1}$, then the simplex x is a **section simplex** if $k = m$ and a **spacer simplex** if $k = m + 1$. ⌟

Terminology 3.2.39 (Upper and lower sections of spacers). Consider an n-proframed simplicial complex $(K, \mathcal{P})$ and a spacer simplex $x : [k] \hookrightarrow K_i$. Let $v : [1] \to [k]$ be the unique simplicial vector in the affine kernel of $p_i \circ x$.

(1) The 'upper section' $\partial^+ x : [k-1] \hookrightarrow K_i$ of x is the face of x not containing $x \circ v(0)$.
(2) The 'lower section' $\partial^- x : [k-1] \hookrightarrow K_i$ of x is the face of x not containing $x \circ v(1)$. ⌟

Given a spacer simplex x, its upper and lower sections $\partial^\pm x$ are, in particular, section simplices in the previous sense.

Definition 3.2.40 (Fiber categories). Consider an n-proframed simplicial complex $(K, \mathcal{P} = (p_n, ..., p_1))$ and a simplex $z : [m] \hookrightarrow K_{i-1}$. The **fiber category** over z, denoted $\Phi_\mathcal{P}(z)$, is the free category whose objects are section simplices $y \in K_z$, and whose generating morphisms $y_- \to y_+$ are spacer simplices $x \in K_z$ with $y_\pm = \partial^\pm x$. ⌟

Construction 3.2.41 (Transition functors of fiber categories). For an n-proframed simplicial complex $(K, \mathcal{P})$, consider simplices $z : [m] \hookrightarrow K_{i-1}$ and $w : [l] \hookrightarrow K_{i-1}$ such that w is a face of z. Note that each simplex $x \in K_z$ in the fiber set over z, has a face simplex $x|_{w \subset z} \in K_w$ in the fiber set over w. Moreover, this restriction $x \mapsto x|_{w \subset z}$ takes sections to sections, but takes spacers either to spacers or to sections. The restriction thus induces a 'transition functor' $-|_{w \subset z} : \Phi_\mathcal{P}(z) \to \Phi_\mathcal{P}(w)$. ⌟

Example 3.2.42 (Fiber categories and transition functors). In Figure 3.14, for the indicated 3-proframing $K_3 \xrightarrow{p_3} K_2 \to \cdots$, we depict the fiber categories and transition functors for selected simplices in K_2. Note that each fiber category object (indicated by a colored circle) corresponds to a section simplex, and each generating morphism (indicated by a colored arrow) corresponds to a spacer simplex. The transition functors between fiber categories are indicated by dotted arrows. ⌟

We now have the components in place to define collapsibility for proframings.

Definition 3.2.43 (Collapsible proframings). An n-proframed simplicial complex $(K, \mathcal{P} = \{K_i \xrightarrow{p_i} K_{i-1}\})$ is **collapsible** if, either $n = 0$ and K is the point $*$, or $n > 0$ and the following two conditions hold.

(1) *Fibers are linear*: For any simplex $z : [m] \hookrightarrow K_{i-1}$, the fiber category $\Phi_\mathcal{P}(z)$ is a total order.
(2) *Fiber transitions are endpoint-preserving*: For simplices $w \subset z$ in K_{i-1} the transition functor $-|_{w \subset z}$ is endpoint-preserving, that is it preserves least and greatest elements as a map of total orders. ⌟

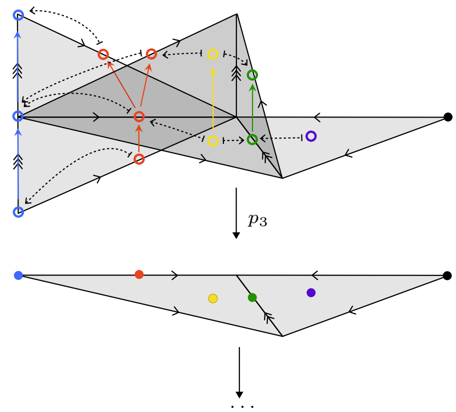


FIGURE 3.14. Fiber categories and transition functors.

REMARK 3.2.44 (Linear complexes). We will refer to a simplicial complex consisting of 0- and 1-simplices, whose geometric realization is either a point or a closed interval, as a 'linear complex'. For a collapsible proframing $(K, \mathcal{P} = (p_n, p_{n-1}, ..., p_1))$, the preimages $p_i^{-1}(x)$ of 0-simplices x are linear subcomplexes of K_i. ⌟

NOTATION 3.2.45 (Category of collapsible proframings). The full subcategory of the category of n-proframed simplicial complexes, with objects the collapsible proframings, is denoted by $\mathsf{CollProFrSimpCplx}_n$. ⌟

PROPOSITION 3.2.46 (Integrals of collapsible framings). *Collapsible n-framings have unique (up to isomorphism) integral collapsible proframings.*

PROOF. Let $(K, \mathcal{F})$ be a collapsible n-framed simplicial complex. Recall from the Definition 1.2.31 of collapsible framed simplicial complexes, that there is a map $q_n : K = K_n \to K_{n-1}$ collapsing all the n-frame vectors and that K_{n-1} itself carries inductively a collapsible framing, so there is another map $q_{n-1} : K_{n-1} \to K_{n-2}$ collapsing all the $(n-1)$-frame vectors, and so on. This provides a tower of simplicial quotient maps:

$$K = K_n \xrightarrow{q_n} K_{n-1} \xrightarrow{q_{n-1}} \cdots \xrightarrow{q_2} K_1 \xrightarrow{q_1} K_0 = [0]$$

By construction, the gradient of this tower is the original framing, and so we have produced an n-proframing $\mathcal{Q}$ integrating $\mathcal{F}$.

However, we must still verify that $\mathcal{Q}$ is collapsible in the sense of Definition 3.2.43. Arguing inductively, it suffices to check this for the fibers and fiber transitions of q_n. Observe that for any simplex $z : [m] \hookrightarrow K_{n-1}$, the fiber category $\Phi_{\mathcal{Q}}(z)$ must be connected, since q_n is a quotient of n-vectors, and q_n maps all simplices in the fiber over z to the single simplex z. Now, using the frame flow continuation uniqueness of collapsible framings, one checks that $\Phi_{\mathcal{Q}}(z)$ must, in fact, be a total order: indeed, no object in that category can have two or more generating arrows pointing to it or from it.

From the flow section existence property it follows that fiber transitions are endpoint-preserving.

It remains to show that $\mathcal{Q}$ is the unique collapsible integral proframing of $\mathcal{F}$. Assume there exists another integral n-proframing $(K, \mathcal{P} = (p_n, p_{n-1}, ..., p_2, p_1))$ of $\mathcal{F}$. Since p_k must degenerate all k-frame vectors, the universal property of quotients yields a surjective simplicial map of towers $F : \mathcal{Q} \to \mathcal{P}$. Arguing by contradiction, take the lowest index i such that F_i is not a simplicial isomorphism. In particular, F_i fails to be an isomorphism of the fibers $q_i^{-1}(x)$ and $p_i^{-1}(y)$, where $y = F_{i-1}(x)$, for some $x \in K_{i-1}$. Since F_i is a simplicial surjection, $p_i^{-1}(y)$ must be a strictly smaller linear complex, and thus $F_i : q_i^{-1}(x) \to p_i^{-1}(y)$ degenerates at least one 1-simplex in $q_i^{-1}(x)$. By inductively lifting that simplex to a 1-simplex of $K = K_n$ using the flow section existence property, one derives a contradiction: that lifted simplex cannot have the same frame label in $\mathcal{Q}$ and $\mathcal{P}$, which contradicts the assumption that they have the same gradient. □

Remark 3.2.47 (Gradient collapsibility is insufficient). Note that given an n-proframing $\mathcal{P}$, requiring that the gradient framing $\nabla \mathcal{P}$ be collapsible, does not ensure that the proframing itself is collapsible. ⌟

Observation 3.2.48 (Integration as a functor). Given an n-framed map $F : (K, \mathcal{F}) \to (L, \mathcal{G})$ of collapsible framed simplicial complexes, the integral n-proframed map $\int F : (K, \int \mathcal{F}) \to (L, \int \mathcal{G})$ is inductively constructed by setting $F = F_n$ and then defining F_{i-1} such that $p_i \circ F_i = F_{i-1} \circ q_i$ where q_i and p_i are the ith maps in the proframings $\int \mathcal{F}$ and $\int \mathcal{G}$, respectively. The association from collapsible framings to their unique integral collapsible proframings thus provides a functor

$$\int : \mathsf{CollFrSimpCplx}_n \to \mathsf{CollProFrSimpCplx}_n \quad .$$ ⌟

This observation, together with previous definitions and constructions, assembles into the following result.

Proposition 3.2.49 (Gradient and integral equivalence). *The gradient and integral functors yield an equivalence of categories*

$$\nabla : \mathsf{CollProFrSimpCplx}_n \simeq \mathsf{CollFrSimpCplx}_n : \int \quad .$$ □

3.3. ◊Proofs of the classifications

Recall the lullaby from the introduction to Section 3.2: take your framed regular cell and consider the underlying framed simplicial complex, inductively quotient by the integral foliations of the highest frame vectors to obtain a proframed simplicial complex, then take the fundamental poset to deliver a truss block; conversely realize your truss block to a proframed simplicial complex, and assemble gradient-like vector fields for the various subquotients in the proframing, to produce a framed simplicial structure supporting a framed cell. It is now time to actually establish the correspondence so adumbrated.

In detailing and making precise the necessary processes of forming the integral truss and conversely the gradient cell, a crucial matter arises, which is to demonstrate that the face-order posets of trusses are actually cellular. By definition this requires the component truss blocks to have spherical boundaries; in a bit of excess we show this by proving that truss block posets actually have *shellable* piecewise-linear spherical boundaries. Roughly speaking, a spherical complex is shellable when its facets can be removed in some order one by one such that, after each removal, the complex is always a ball. Recall the regular cell and corresponding truss block from Figure 3.1, and the proframed simplicial complex realization of that truss block shown in Figure 3.4. In Figure 3.15 we illustrate a shelling of the boundary of the top complex of that proframing, thus of the boundary of that regular cell; in fact, as we will see in the construction, that shelling is obtained inductively via shelling each layer of the proframed complex in turn.

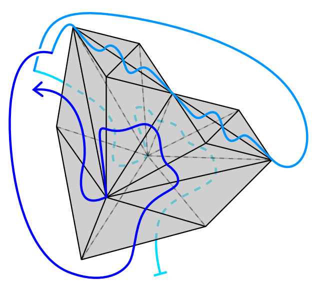


FIGURE 3.15. A shelling of a regular cell boundary.

OUTLINE. In Section 3.3.1, we introduce the section–spacer dichotomy for framed cells, and describe for any framed cell an associated integral proframed cell tower. In Section 3.3.2, we construct the gradient functor from closed trusses to collapsible framed cell complexes, via in particular a proof that the component truss blocks are shellable, and we present the converse integral functor from collapsible framed cell complexes to closed trusses. Finally in Section 3.3.3, we assemble the proofs of the framed-cell–truss correspondences and record corollaries regarding enumerabililty and piecewise-linearity of framed cells.

3.3.1. ◊Integrating framed regular cells. Recall that a framed regular cell has, forgetting the cellular poset structure, an underlying collapsible framed simplicial complex. By the results of the previous section, that collapsible framed simplicial complex has an associated integral collapsible proframed simplicial complex. What is not, a priori, clear is that the layers of that proframed simplicial complex admit cellular poset structures for which they are framed regular cells, and that the projection maps in the proframed complex are framed cellular maps with respect to those poset structures. In this section we show that that is indeed the situation: any framed regular cell has an associated *integral proframed regular cell*, that is a tower of framed regular cells and framed cellular projections.

Synopsis. We differentiate the cells of an n-framed regular cell into section and spacer cells, and show that every spacer cell has distinguished lower and upper section cells in its boundary. Using the section–spacer dichotomy, we then show that the top simplicial projection of the integral simplicial proframing of a framed regular cell admits the structure of a cellular poset map, and thereby inductively construct an integral proframed regular cell tower for any framed regular cell.

3.3.1.1. ✲ Central cell structure. We discuss the distinction of framed cells into *section cells* and *spacer cells*; a spacer cell consistutes a bulk region with lower and upper sections cells in its boundary. (Section and spacer cells are analogous to the section and spacer simplices previously discussed in the context of simplices in proframed simplicial complexes, see Definition 3.2.38.)

Consider an n-framed regular cell $(X, \mathcal{F})$; in a substantive abuse of notation we will not introduce separate notation for the underlying framed simplicial complex of a framed regular cell, and will rely on the reader to distinguish when we are referring to a cellular structure, i.e., to a simplicial complex together with its cellular poset order, or merely to a simplicial complex structure. Recall the framed regular cell $(X, \mathcal{F})$ gives in particular the following structures: (1) its cellular poset X, (2) the associated ordered simplicial complex NX, (3) the distinct framing-induced order on the unordered simplicial complex NX^{un}; see Notation 1.3.28 and Remark 1.3.34. Recall further that in illustrations of a framed regular cell, we typically draw the simplicial complex realizing the cellular poset and indicate the order recording the cellular structure by small blue arrows emanating from vertices, and then indicate the framing and its order by frame arrows on edges; see Figures 1.40 and 1.41.

The underlying collapsible framed simplicial complex of the framed cell $(X, \mathcal{F})$ has an associated integral collapsible proframed simplicial complex denoted $\mathcal{P} = \int \mathcal{F} = (p_n, ..., p_1)$, with $p_i : X_i \to X_{i-1}$. As before let $\perp$ denote the initial element of the cellular poset X, and write $\perp_{n-1} = p_n(\perp) \in X_{n-1}$ for the image of this initial element in the next simplicial layer. Framed cells are distinguished by the nature of the fibers of the projected element $\perp_{n-1}$, as follows.

Definition 3.3.1 (Section and spacer cells). An n-framed regular k-cell $(X, \mathcal{F})$ is a **section cell** if the fiber category $\Phi_{\mathcal{P}}(\bot_{n-1})$ is trivial. An n-framed regular k-cell is a **spacer cell** if the fiber category $\Phi_{\mathcal{P}}(\bot_{n-1})$ is isomorphic to the category $(\bot^- \to \bot \to \bot^+)$ (by an isomorphism taking the initial element $\bot$ to the middle element of the linear fiber). ⌟

Terminology 3.3.2 (Central fiber bounds). For a framed regular cell, we refer to the fiber category $\Phi_{\mathcal{P}}(\bot_{n-1})$ as the 'central fiber'. When it is a spacer cell, we call the 0-simplices $\bot^{\pm}$ the (upper resp. lower) 'central fiber bounds' of the cell. ⌟

Example 3.3.3 (Section and spacer cells). In Figure 3.16 we illustrate a 3-framed regular section 2-cell and a 3-framed regular spacer 3-cell. We indicate the cellular poset structure with blue arrows, and highlight the central fiber elements in red. ⌟

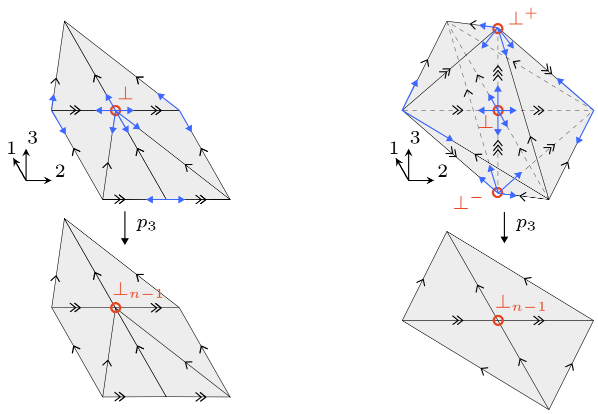


Figure 3.16. Framed regular section and spacer cells.

Lemma 3.3.4 (Framed cells are sections or spacers). *For any n-framed regular k-cell $(X, \mathcal{F})$, either the cell is a section, in which case the map $p_n : X_n \to X_{n-1}$ is an isomorphism of frame-ordered simplicial complexes, or the cell is a spacer.*

Proof. Since $\bot$ is initial in the poset X, and since fibers over 0-simplices are linear complexes (see Remark 3.2.44), $\Phi_{\mathcal{P}}(\bot_{n-1})$ must have ≤ 3 objects.

If $\Phi_{\mathcal{P}}(\bot_{n-1})$ has 1 object, we show $p_n : X_n \to X_{n-1}$ is an isomorphism. This holds iff X_n has no spacer simplices. Arguing by contradiction, let $x : [m] \hookrightarrow X_n$ be a spacer simplex with maximal m. Since $\bot$ is initial in the poset X, $\bot$ must already be a vertex in x. Maximality of m guarantees no other simplex contains x as a face, and so m must equal the dimension k of the cell. As a spacer simplex, x represents a morphism in some fiber category $\Phi_{\mathcal{P}}(z)$ over $z = p_n(x)$. Pick the initial or terminal object in that category, which represents a $(k-1)$-simplex $y_{\pm}$. This simplex must also contains $\bot$, since $z = p_n(y_{\pm})$. However, y lies in the boundary of the cell $|X|$ and thus so must $\bot$, which contradicts the assumption that X is a cellular poset with initial object $\bot$.

If $\Phi_{\mathcal{P}}(\bot_{n-1})$ has 2 objects, a similar argument applies: note, that one of $y_\pm$ contains $\bot$, and now a spacer simplex must exist. Thus, this case is impossible.

This leaves only the case of $\Phi_{\mathcal{P}}(\bot_{n-1})$ having 3 objects, in which case the fiber category must be of the required form $(\bot_- \to \bot \to \bot_+)$; and so the cell is a spacer. □

Construction 3.3.5 (Upper and lower sections). For an n-framed regular cell $(X, \mathcal{F})$, define the frame-ordered simplicial maps $\gamma^- : X_{n-1} \to X_n$ resp. $\gamma^+ : X_{n-1} \to X_n$, called the 'lower section' resp. 'upper section' of the cell, by mapping each j-simplex z of the complex X_{n-1} to the j-simplex of the complex X_n that is initial resp. terminal in the fiber category $\Phi_{\mathcal{P}}(z)$ over z. ⌟

Note that $\gamma^- = \gamma^+$ exactly when $(X, \mathcal{F})$ is a section cell. When $(X, \mathcal{F})$ is a spacer cell, then $\bot^\pm \in \mathrm{im}(\gamma^\pm)$.

We now show that the images of upper and lower sections are exactly the cells with initial elements $\bot^\pm$.

Lemma 3.3.6 (Section images are cells). *If the framed regular cell* $(X, \mathcal{F})$ *is a spacer k-cell, with lower and upper sections $\gamma^\pm$, then the posets $X^{\geq \bot^\pm}$ are regular $(k-1)$-cells, whose simplices are exactly those in the image of the sections $\gamma^\pm$.*

Proof. We argue in the case of the lower section γ^- (the case of the upper section is similar). First observe that any simplex in X containing $\bot^-$ but not $\bot$ is a section simplex for p_n. (Otherwise, we could pick some spacer simplex y containing $\bot^-$ but not $\bot$. Then there must be some k-simplex y' containing y, which itself is a spacer not containing $\bot$, and that is impossible.)

We show that $X^{\geq \bot^-}$ is a $(k-1)$-cell in ∂X. We start by picking some $x \in \partial X$ such that $X^{\geq x}$ is a $(k-1)$-cell and such that $\bot^- \in X^{\geq x}$. This implies that the framed cell $(X^{\geq x}, \mathcal{F}|_{X^{\geq x}})$ must be a section cell (indeed, $X^{\geq x}$ will contain a $(k-1)$-simplex containing $\bot^-$ which, as we've just observed, must be a section simplex). In fact, each $(k-1)$-simplex in $X^{\geq x}$ must either contain $\bot^-$ or $\bot^+$: this follows, since taking the cone of the section $(k-1)$-simplices in $X^{\geq x}$ with cone point $\bot$ must yield spacer k-simplices. Observe that $X^{\geq x}$ cannot, however, contain both $\bot^-$ and $\bot^+$, without contradicting the collapsibility of the framing $\mathcal{F}$ restricted to $X^{\geq x}$. It follows that all $(k-1)$-simplices of the $(k-1)$-cell $X^{\geq x}$ must contain the vertex $\bot^-$. But this is only possible if $x = \bot^-$.

Finally, we check $\mathrm{im}(\gamma^-)$ contains the same simplices as $X^{\geq \bot^-}$. This follows since any simplex in X_{n-1} lies in a simplex containing $\bot_{n-1}$. □

Terminology 3.3.7 (Central section cells). Given a spacer cell $(X, \mathcal{F})$, we refer to the subposets $X^{\geq \bot^\pm}$, determined by the images of the sections

$\gamma^{\pm}$, as the 'lower central section cell' resp. 'upper central section cell' of $(X, \mathcal{F})$. ⌟

3.3.1.2. ✻ Integral proframed cells. Equipped with the dichotomy between section and spacer cells and knowing that a spacer cell has in its boundary lower and upper central section cells, we may now inductively construct cellular structures on the layers of the proframed simplicial complex associated to a framed regular cell.

As in the previous section we fix an n-framed regular k-cell $(X, \mathcal{F})$, and consider its associated proframed simplicial complex $\mathcal{P} = \int \mathcal{F} = (p_n, p_{n-1}, ..., p_1)$, with $p_i : X_i \to X_{i-1}$. Recall that by definition, for any $x \in X$, the restriction of the framing $\mathcal{F}$ to the subcell $X^{\geq x} \hookrightarrow X$ provides a framed cell $(X^{\geq x}, \mathcal{F}|_{X^{\geq x}})$; in particular, the restriction $\mathcal{F}|_{X^{\geq x}}$ is a collapsible framing.

Notation 3.3.8 (The proframing of subcells). For brevity we denote the integral proframed simplicial complex of the subcell $(X^{\geq x}, \mathcal{F}|_{X^{\geq x}})$ by $\mathcal{P}^x = \int(\mathcal{F}|_{X^{\geq x}}) = (p^x_n, p^x_{n-1}, ..., p^x_1)$, with $p^x_i : X^x_i \to X^x_{i-1}$. ⌟

Observation 3.3.9 (Integral restrictions are restricted integrals). From the construction of integral proframings for collapsible framings, it follows that the integral of the restricted framing $\int(\mathcal{F}|_{X^{\geq x}})$ is simply the restriction of the integral $(\int \mathcal{F})|_{X^{\geq x}}$. In particular, the first projection p^x_n of the subcell proframe is the restriction of the global projection p_n to the subcell $X^{\geq x}$. ⌟

Lemma 3.3.10 (Cellular structure on projected complexes). *For an n-framed k-cell $(X, \mathcal{F})$ with integral proframe simplicial projection $p_n : X = X_n \to X_{n-1}$, there exists a unique cellular poset structure on the simplicial complex X_{n-1} such that the projection p_n is a cellular map of regular cells.*

Proof. If $(X, \mathcal{F})$ is a section cell, then p_n is a simplicial isomorphism and thus we must have $p_n : X_n \cong X_{n-1}$ as cellular posets.

If $(X, \mathcal{F})$ is a spacer cell, we define the poset structure on X_{n-1} by identifying $\gamma^- : X_{n-1} \cong X^{\geq \perp^-}$ as posets via γ^- (of course, we could equivalently use γ^+). One checks that $p_n : X = X_n \to X_{n-1} \cong X^{\geq \perp^-}$ is a cellular poset map, which follows by induction (in the cell dimension k), and using Observation 3.3.9, projecting boundary cells onto their respective lower section cells. □

Remark 3.3.11 (Isomorphism of lower and upper central section cells). The preceding result provides a cellular poset isomorphism $\gamma^+ \circ p_n : X^{\geq \perp^-} \cong X^{\geq \perp^+}$ between the lower and upper central section cells. ⌟

Construction 3.3.12 (Integral of a framed cell). Applying Lemma 3.3.10 inductively, we obtain a tower of regular cells

$$X = X_n \xrightarrow{p_n} X_{n-1} \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_2} X_1 \xrightarrow{p_1} X_0 = [0]$$

(As before we quite abuse notation and level the cell structure implicit.) The proframed simplicial complex $\mathcal{P} = \int \mathcal{F} = (p_n, p_{n-1}, ..., p_1)$ truncates to a proframe $\mathcal{P}_{\leq i}$, and the gradient $\mathcal{F}_i := \nabla\,\mathcal{P}_{\leq i}$ provides a framing of the complex X_i. In fact the cell structure and that framing gives a framed regular cell $(X_i, \mathcal{F}_i)$, and so we obtain a tower of framed regular cells:

$$(X, \mathcal{F}) = (X_n, \mathcal{F}_n) \xrightarrow{p_n} (X_{n-1}, \mathcal{F}_{n-1}) \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_2} (X_1, \mathcal{F}_1) \xrightarrow{p_1} (X_0, \mathcal{F}_0) = [0]$$

⌟

Note that each projection p_i in this cell tower is either a cell isomorphism (if its domain is a section cell) or a cell projection (if its domain is a spacer cell).

Terminology 3.3.13 (Integral proframed cell). We refer to the tower of framed cells in the previous construction as the 'integral proframed cell' of the framed regular cell $(X, \mathcal{F})$. ⌟

Remark 3.3.14 (Integral proframed cell complex). The preceding construction generalizes to the case of collapsible framed regular cell complexes $(X, \mathcal{F})$, yielding their associated 'integral proframed cell complexes'. ⌟

Example 3.3.15 (Projected framed cell structure). In Figure 3.17 we illustrate the 3-framed 3-globe cell and the induced projection to the 2-framed 2-globe cell that forms the next stage of its integral proframed cell. We emphasize with the green arrows the cellular poset structure on the image 2-complex, provided by Lemma 3.3.10. ⌟

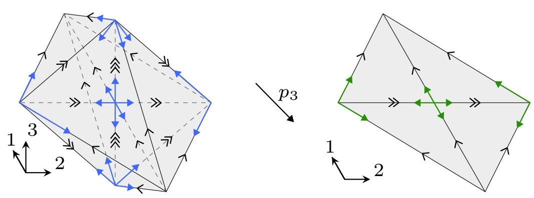


Figure 3.17. The projected 2-framed cell of a 3-framed cell.

Observation 3.3.16 (Highest frame vectors form a framed 1-cell). Given an n-framed k-cell $(X, \mathcal{F})$, recall from Terminology 1.3.48 that $\mathsf{axl}\,\bot$ denotes the subcomplex of the frame-ordered simplicial complex X spanned by the highest frame vectors of the cell. As stated earlier in Remark 1.3.49, provided the cell is not 0-dimensional, this complex $\mathsf{axl}\,\bot$ is the concatenation of two 1-simplices; that complex corresponds to a cellular subposet of the cellular poset X that is canonically isomorphic to the fundamental poset of a 1-cell.

Given a framed cell $(X, \mathcal{F})$, then by Lemma 3.3.4, it is either a spacer cell or a section cell. If it is a spacer cell, then $\mathsf{axl}\,\bot \cong (\bot^- \to \bot \to \bot^+)$ as needed. If it is a section cell, then $\mathsf{axl}\,\bot \cong \mathsf{axl}\,\bot_{n-1}$, where $\bot_{n-1}$ is the initial object of the next cell $(X_{n-1}, \mathcal{F}_{n-1})$ in the integral cell tower given in Construction 3.3.12. The observation follows by induction. ⌟

3.3.2. ◊Functors between collapsible cell complexes and trusses. Now equipped with a better understanding of the cellular structure of framed regular cells and their associated integral proframed simplicial complexes, we can construct the relevant equivalence functors between collapsible framed cell complexes and closed trusses. At a distance, the translations are clear: framed cell complexes integrate to proframed cell complexes (by the work of the previous section), which then have associated fundamental poset trusses; and trusses realize to proframed simplicial complexes which have associated gradient framed cell complexes. The remaining substantive issue is establishing that the gradient complex of the proframed complex of a truss is in fact cellular. We take the occasion to prove the rather stronger fact that these complexes are locally shellable PL cellular posets.

Synopsis. We construct the gradient functor from closed trusses to collapsible framed cell complexes, by building a proframed simplicial complex from the face orders of the truss and then taking its associated gradient framed simplicial complex; to see that the resulting complex is cellular, we prove that the component truss blocks are pure, shellable, and thin. We then present the converse integral functor from collapsible framed cell complexes to closed trusses, by taking the associated integral proframed cell complex previously constructed and then passing to its fundamental poset truss.

3.3.2.1. ✻ From trusses to collapsible cell complexes. We will now construct the 'gradient cell' functor from closed n-trusses to collapsible framed regular cell complexes:

$$\nabla_{\mathsf{C}} : \bar{\mathsf{Trs}}_n \to \mathsf{CollFrCellCplx}_n.$$

We first give the construction on objects, and then separately on morphisms.

Construction 3.3.17 (Gradient cell complexes of closed trusses). Given a closed n-truss $T = (T_n \xrightarrow{p_n} T_{n-1} \xrightarrow{p_{n-1}} ... \xrightarrow{p_1} T_0)$, we need to produce a collapsible framed cell complex $\nabla_{\mathsf{C}} T = (X, \mathcal{F})$. Recall by definition such a complex is a cellular poset X together with an n-framing $\mathcal{F}$ of its underlying simplicial complex, such that the framed simplicial complex $(X, \mathcal{F})$ is collapsible and each closed cell $(X^{\geq x}, \mathcal{F}|_{X^{\geq x}})$ is itself a collapsible framed simplicial complex.

By the correspondence, established in Proposition 3.2.49, between collapsible framed simplicial complexes and collapsible proframed simplicial complexes, it suffices to produce a cellular poset X together with a collapsible proframed simplicial complex $(X, \mathcal{P})$ (beginning with the underlying simplicial complex of the poset X), whose restriction to each closed cell $(X^{\geq x}, \mathcal{P}|_{X^{\geq x}})$ is also collapsible. Explicitly, the desired framing is the gradient $\mathcal{F} := \nabla \mathcal{P}$ of the given proframing.

It furthermore suffices to provide, a priori more, the following: (1) a tower of cellular poset maps

$$X_n \xrightarrow{q_n} X_{n-1} \xrightarrow{q_{n-1}} \cdots \xrightarrow{q_2} X_1 \xrightarrow{q_1} X_0$$

together with (2) orderings on the underlying simplicial complexes X_i, with respect to which $\mathcal{P} = (q_n, q_{n-1}, ..., q_1)$ becomes a proframed simplicial complex, that is (3) collapsible and whose restriction to each closed cell is again collapsible. We now construct in turn (1) that tower and (2) those orderings, and then verify (3) the collapsibility conditions.

(1) We fix the poset X_i to be the face-order truss poset $(T_i, \trianglelefteq)$, and set the poset map $q_i : X_i \to X_{i-1}$ to be the projection $p_i : (T_i, \trianglelefteq) \to (T_{i-1}, \trianglelefteq)$. The heart of the matter is showing that the posets X_i are *cellular*; we excise that to Lemma 3.3.23 below, which will in turn depend on the subsequent Lemmas 3.3.24, 3.3.25, and 3.3.26. That the maps q_i are cellular follows from the fact that 1-truss bundles have lifts in the sense of Observation 2.1.83.

(2) Inductively assume we have defined the order on the simplicial complex X_{i-1}. To provide an order on the simplicial complex X_i, we need to consistently order the vertices of each of the k-simplices $x : [k]^{\mathsf{un}} \hookrightarrow X_i$ in the unordered simplicial complex X_i. To give such a consistent order it suffices to do so for the 1-simplices. Such a 1-simplex x either projects to an object y of X_{i-1}, or else it projects to an (ordered) 1-simplex $z : z(0) \to z(1)$ in X_{i-1}. In the first case, order $x = x(0) \to x(1)$ such that $x(0) \prec x(1)$ in the frame order $(T_i, \preceq)$. In the second case, order x such that its projection to z is order preserving. By construction, the poset map q_i is a simplicial map $q_i : X_i \to X_{i-1}$ of ordered simplicial complexes, and the collection $\{q_i\}$ forms a proframed simplicial complex.

(3) Recall that the given proframing $\mathcal{P} = (q_n, q_{n-1}, ..., q_1)$ is collapsible if its fiber categories are linear and the fiber transitions are endpoint-preserving. Since the maps in the proframing are the truss poset projections, those two conditions are exactly the ones verified via truss induction in Observation 2.2.38. Applying that same observation to the 1-truss bundles in each truss block $T^{\trianglerighteq x}$ implies that the cell-restricted proframings $(X^{\geq x}, \mathcal{P}|_{X^{\geq x}})$ are also collapsible, as required.

$\lrcorner$

Construction 3.3.18 (Gradient cellular maps of singular truss maps). Given a singular n-truss map $F : T \to S$, we provide the poset map $\nabla_{\mathsf{C}} F : \nabla_{\mathsf{C}} T \to \nabla_{\mathsf{C}} S$ by setting $\nabla_{\mathsf{C}} F = F_n : T_n \to S_n$. By the construction of the gradients $\nabla_{\mathsf{C}} T$ and $\nabla_{\mathsf{C}} S$ via proframe towers, we also have a poset map $\nabla_{\mathsf{C}} F_{\leq n-1} : \nabla_{\mathsf{C}} T_{\leq n-1} \to \nabla_{\mathsf{C}} S_{\leq n-1}$, which by induction we may assume is framed cellular (that is, is a cellular poset map and preserves highest frame vectors). That $\nabla_{\mathsf{C}} F$ is then framed cellular follows by investigating the map $\nabla_{\mathsf{C}} F \to \nabla_{\mathsf{C}} F_{\leq n-1}$, using that F is singular. $\lrcorner$

The crucial matter remains, to prove that the truss posets are cellular. We take a roundabout approach by showing that these posets are moreover *PL cellular*, i.e., that the realizations of the strict upper closure of any element is PL homeomorphic to the standard PL sphere (see Definition 1.3.30). And in fact, along the way we will demonstrate the yet stronger claim that those strict upper closure PL spheres are *shellable*.

We will utilize the following convenient condition for PL cellular sphericality.

PROPOSITION 3.3.19 (See [**Bjö84**, Prop. 4.5 ff.]). *If a poset X is pure of dimension m, shellable, and thin, then its realization $|X|$ is a regular cell complex that is PL homeomorphic to the PL m-sphere.* □

TERMINOLOGY 3.3.20 (Pure poset). A simplicial complex is called 'pure of dimension m' if its facets (that is, nondegenerate simplices that are not the face of any other nondegenerate simplex) are all of the same dimension m. Similarly, a poset X is called pure of dimension m if its nerve simplicial complex NX is pure of dimension m. ⌟

TERMINOLOGY 3.3.21 (Shellable poset). A poset X is called 'shellable' if the simplicial complex NX is pure of dimension m and its facets admit an ordering $K_0, K_1, K_2, ..., K_j$, such that, for all $0 < l \leq j$, the subcomplex $(\cup_{i<l} K_i) \cap K_l$ (obtained by intersecting the simplex K_l with the union of the preceding simplices K_i, $i < l$) is a pure simplicial complex of dimension $(m - 1)$. ⌟

TERMINOLOGY 3.3.22 (Thin poset). Finally, a poset X is called 'thin' if for every non-refinable length-2 chain $x < y < z$ in X there is exactly one $y' \neq y$ such that $x < y' < z$. (This is also sometimes called the 'diamond property'.) ⌟

We proceed to the cellularity result.

LEMMA 3.3.23 (Cellularity of closed trusses). *For a closed n-truss T, each face order poset $(T_i, \trianglelefteq)$ is a PL cellular poset.*

PROOF. As the condition of PL cellularity applies to the strict upper closures $T^{>x}$ of elements, it suffices to assume that the truss T is in fact an n-truss block, with initial element $\bot = x$, and to show that the boundary $\partial T_n = T_n^{\triangleright \bot}$ of the truss block realizes to a PL m-sphere. By Proposition 3.3.19, it is enough to establish that the boundary ∂T_n is pure, shellable, and thin.

Needless to say we proceed by inductively assuming that the boundary ∂T_{n-1} is itself pure (of dimension $k-1$), shellable, and thin. Let $\bot_{n-1} = p_n(\bot)$ denote the projection of $\bot \in T_n$ to T_{n-1}. If the element $\bot$ is singular in the fiber over $\bot_{n-1}$, then the 1-truss bundle $p_n : T_n \to T_{n-1}$ is an isomorphism of (face order) posets; thus the boundary ∂T_n is itself pure (of dimension $k - 1$), shellable, and thin.

For the case when instead the element $\bot$ is regular, we parcel out the proofs to the following Lemmas 3.3.24, 3.3.25, and 3.3.26. □

LEMMA 3.3.24 (Truss blocks are pure). *The boundary of every truss block T is of pure dimension.*

PROOF. From discussion in the proof of Lemma 3.3.23, we assume that the boundary ∂T_{n-1} is pure of dimension $k - 1$, and that $\bot$ is a regular element over $\bot_{n-1}$.

Observe that facets of the block T_n project to facets of the block T_{n-1}; this follows since the projection p_n is surjective on simplices, and the fiber transition maps are also surjective, see Observation 2.2.38. Each facet of T_{n-1} must contain the vertex $\bot_{n-1}$. Since by assumption the fiber over $\bot_{n-1}$ has spacers, there must also be spacers in the fiber over each facet of T_{n-1}. Thus facets in T_n must themselves be spacers. From the inductive assumption, we know that all facets in T_{n-1} have dimension k, so the facets of T_n have dimension $k+1$, and thus finally the facets of ∂T_n are of dimension k, as required. □

◊ Above proof not carefully treated

LEMMA 3.3.25 (Truss blocks are shellable). *The boundary of every truss block T is shellable.*

PROOF. From discussion in the proof of Lemma 3.3.23, we assume that the boundary ∂T_{n-1} is shellable, and that $\bot$ is a regular element over $\bot_{n-1}$. Initiality of the element $\bot$ implies the fiber $p_n^{-1}(\bot_{n-1})$ must be of the form $\bot^- \rhd \bot \lhd \bot^+$.

Let t_{n-1} be the number of facets in ∂T_{n-1}, and similarly t_n the number of facets in ∂T_n. Consider, by the inductive assumption, a shelling $K_1, K_2, ..., K_{t_{n-1}}$ order of the facets of ∂T_{n-1}. This order induces a shelling $K_\bullet = (K_1^\bot, K_2^\bot, ...K_{t_{n-1}}^\bot)$ of T_{n-1}, where $K_i^\bot$ is obtained from K_i by adjoining a new first vertex $\bot_{n-1}$. Now build a shelling $L_\bullet = (L_1, L_2, ...L_{t_n})$ of ∂T_n in the following three steps.

(1) *Lower section shelling*: We define the first t_{n-1} facets
$$L_1, L_2, ..., L_{t_{n-1}}$$
in the sequence $L_\bullet$, by setting L_i to be the lowest section lying over $K_i^\bot$. Note that these facets have $L_i(0) = \bot^-$.

(2) *Side shelling*: We next define the subsequence
$$L_{t_{n-1}+1}, L_{t_{n-1}+2}, ..., L_{t_n - t_{n-1}}$$
of $L_\bullet$ to be the sequence
$$\begin{aligned} &L_{(1,1)}, L_{(1,2)}, ...L_{(1,j_1)}, \\ &L_{(2,1)}, L_{(2,2)}, ..., L_{(2,j_2)}, ..., \\ &L_{(t_{n-1},1)}, L_{(t_{n-1},2)} ..., L_{(t_{n-1},j_{n-1})} \end{aligned}$$
where $L_{(i,j)}$ is the jth spacer (in the scaffold order!) lying over K_i.

(3) *Upper section shelling*: Finally, we define the last t_{n-1} facets
$$L_{t_n - t_{n-1}+1}, L_{t_n - t_{n-1}+2}, ..., L_{t_n}$$
in the sequence $L_\bullet$ by setting $L_{t_n - t_{n-1}+i}$ to be the top section lying over $K_i^\bot$. Note that these facets have $L_{t_n - t_{n-1}+i}(0) = \bot^+$.

Altogether, this constructs a shelling of ∂T_n. (In Figure 3.18 we illustrate an example of the resulting shelling in the case $n = 2$.) □

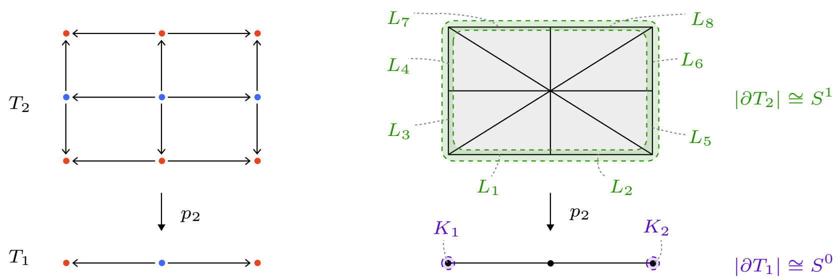


FIGURE 3.18. Inductive shelling of the boundary of a truss block

LEMMA 3.3.26 (Truss blocks are thin). *The boundary of every truss block T is thin.*

PROOF. From discussion in the proof of Lemma 3.3.23, we assume that $\bot$ is a regular element over $\bot_{n-1}$. We also assume by induction that the block T_{n-1} is thin. We will show that the block T_n is thin, which implies that the boundary ∂T_n is thin, since the block is the boundary with an initial element adjoined. Consider a 2-simplex $K : [2] \to T_n$ such that the chain $\mathrm{im}(K) = (x \to y \to z)$ is non-refinable. There are two cases, distinguished based on the dimension of the base projection simplex $J : [j] \hookrightarrow T_{n-1}$ with $\mathrm{im}(J) = \mathrm{im}(p_n K)$; either $j = 2$ or $j = 1$.

First assume $j = 2$. Then the base projection J is a chain $(x_{n-1} \to y_{n-1} \to z_{n-1})$ in T_{n-1}. Note this chain J must be non-refinable (otherwise, K would be refinable). Thinness of T_{n-1} implies there is exactly one other non-refinable chain $J' : x_{n-1} \to y'_{n-1} \to z_{n-1}$. Since the 1-truss bordisms lying over the chain J compose to the same 1-truss bordism as the 1-truss bordisms lying over the chain J', there must be at least one chain K' from x to z lying over J'. Now, there cannot be a third chain K'' from x to z, since that would have to lie over either J or J'; assume, without loss of generality, that it lies over J and that $K \prec K''$ in the scaffold order of sections over J; all spacers over J between K and K'' must now have fiber morphisms in the fiber over y_{n-1}. Thus the 3-spacer containing the 2-section K as its lower section, has a spine that refines K, contradicting that K was non-refinable. It follows that a third chain K'' cannot exist.

Next assume $j = 1$. In this case, the base projection J is a 1-simplex $(x_{n-1} \to z_{n-1})$ in T_{n-1}. Thus K must be a spacer over J. Arguing by truss induction on the 1-truss bundle p_n over J, we find that exactly two non-refinable chains from x to z must exist. Namely, either the lower or the upper section of K must have a jump morphism that lies over J (this follows from the arguments in the proof of Lemma 2.2.29, or can be seen by thinking of the section order as a directed path through jump morphisms, see Figure 2.31). In the former case the two non-refinable chains are given by the spine of K and the spine of its predecessor, and in the latter case, by the spine of K and the spine of its successor. □

◊ The above proof has not been seriously scrubbed

3.3.2.2. ✻ From collapsible cell complexes to trusses. Conversely to the gradient cell construction in the previous section, we now construction the 'integral truss' functor from collapsible framed regular cell complexes to trusses:

$$\textstyle\int_{\mathsf{T}} : \mathsf{CollFrCellCplx}_n \to \bar{\mathsf{T}}\mathsf{rs}_n$$

We first give the construction on objects, and then briefly mention the case of morphisms. The construction can be relatively succinct because the real work here already happened in the earlier construction of integral proframed cell structures.

Construction 3.3.27 (Integral trusses of collapsible framed cell complexes). Given a collapsible n-framed regular cell complex $(X, \mathcal{F})$, we need to produce a closed n-truss $\int_{\mathsf{T}}(X, \mathcal{F}) = T = (T_n \xrightarrow{q_n} T_{n-1} \xrightarrow{q_{n-1}} \ldots \xrightarrow{q_1} T_0)$.

Using crucially Construction 3.3.12 and its generalization Remark 3.3.14, construct the integral proframed cell complex of the framed cell complex $(X, \mathcal{F})$; this entails having cellular posets X_i, cellular poset maps $p_i : X_i \to X_{i-1}$, and orderings on the simplicial complexes X_i such that the maps p_i are also ordered simplicial and form the proframed simplicial complex $\mathcal{P} = \int \mathcal{F} = (p_n, ..., p_1)$.

We will now define (1) the face-order posets $(T_i, \trianglelefteq)$ and poset maps $q_i : (T_i, \trianglelefteq) \to (T_{i-1}, \trianglelefteq)$; (2) a dimension functor $\dim : (T_i, \trianglelefteq) \to [1]^{\mathrm{op}}$; and (3) a frame order $(T_i, \preceq)$; and then we verify that (4) the fibers of q_i over objects are closed 1-trusses, and (5) the fibers over morphisms are 1-truss bordisms.

(1) Define the face order poset $(T_i, \trianglelefteq)$ to be the cellular poset X_i, and set the projection $q_i := p_i : X_i \to X_{i-1}$.

(2) Define $\dim : (T_i, \trianglelefteq) \to [1]^{\mathrm{op}}$ to map $x \in T_i$ to 0 if $X^{\geq x}$ is a section cell, and to 1 if $X_i^{\geq x}$ is a spacer cell in X_i (see Definition 3.3.1). Since section cells can only contain other section cells in their closure, this defines a poset map as required.

(3) Define two elements x, y in T_i to be related in the frame order $(T_i, \preceq)$ by $x \prec y$ if and only if they are in the same fiber of q_i and there is a linear subcomplex $x \to \cdots \to y$ in the frame-ordered simplicial complex X_i.

(4) Using our characterizations of collapsibility, section, and spacer cells, one checks that the structures $\trianglelefteq$, $\dim$, and $\preceq$ restrict on fibers $q_i^{-1}(z)$ over objects z in T_{i-1}, to give closed 1-trusses $T_z = (q_i^{-1}(z), \trianglelefteq, \dim, \preceq)$.

(5) Let $f : z \to w$ be an arrow in T_{i-1}. Denote by $R : T_z \nrightarrow T_w$ the functorial relation $q_i^{-1}(f)$ coming from the face order poset T_i. Since the proframing $\mathcal{P}$ is collapsible, it follows that $R \subset (T_z, \preceq) \times (T_w, \preceq)$ is bimonotone. Since the fiber transition functors are surjective, it follows that R fully relates elements (and thus preserves singular endpoints). Moreover, if $x \in \mathsf{sing}(T_z)$ there is a unique $y \in \mathsf{sing}(T_w)$ such that $R(x, y)$: indeed, the projection p_i restricts on section cells $X^{\geq x}$ to poset isomorphisms $p_i : X^{\geq x} \cong X^{\geq z}$, and

thus $R(x,y)$ holds if and only if $X^{\geq y} = p_i^{-1}(X^{\geq w})$. The statement that the relation R is a 1-truss bordism now follows from Corollary 2.1.66. ⌟

◊ Did not check, or scrubb, the above proof, particularly the end.

CONSTRUCTION 3.3.28 (Integral truss maps of framed cellular maps). Given a framed cellular map $F : (X, \mathcal{F}) \to (Y, \mathcal{G})$ of collapsible framed regular cell complexes, we construct the singular truss map $\textstyle\int_{\mathsf{T}} F : \int_{\mathsf{T}}(X,\mathcal{F}) \to \int_{\mathsf{T}}(Y,\mathcal{G})$. The ith truss component of $\int_{\mathsf{T}} F$ is defined to be the ith component of the integral proframed simplicial map $\int F$ of the framed simplicial map associated to the cellular map F. The resulting truss map $\int_{\mathsf{T}} F$ is singular, i.e., maps singular objects to singular objects, because in fact the simplicial map F_i sends section cells in $(X_i, \nabla((\int_{\mathsf{T}} \mathcal{F})_{\leq i}))$ to section cells in $(Y_i, \nabla((\int_{\mathsf{T}} \mathcal{G})_{\leq i}))$. ⌟

◊ The above construction remains sketchy / pushing things under the rug

3.3.3. ◊Equivalences of framed cell and truss structures. Finally, we can record that truss integration and cell gradient assemble into the following equivalences of categories.

PROOF OF THEOREM 3.1.1 AND THEOREM 3.1.2. Given the functors $\nabla_{\mathsf{C}} : \bar{\mathsf{T}}\mathsf{rs}_n \to \mathsf{CollFrCellCplx}_n$ and $\int_{\mathsf{T}} : \mathsf{CollFrCellCplx}_n \to \bar{\mathsf{T}}\mathsf{rs}_n$ defined in the preceding sections, observe that there are unique natural isomorphisms $\mathrm{id} \cong \int_{\mathsf{T}} \circ \nabla_{\mathsf{C}}$ and $\mathrm{id} \cong \nabla_{\mathsf{C}} \circ \int_{\mathsf{T}}$. (Cf. the rigidity of natural transformations of trusses from Lemma 2.3.71.) Furthermore this equivalence restricts to an equivalence of the subcategories $\mathsf{FrCell}_n \hookrightarrow \mathsf{CollFrCellCplx}_n$ and $\mathsf{Blk}_n \hookrightarrow \bar{\mathsf{T}}\mathsf{rs}_n$. □

PROOF OF THEOREM 3.1.3. We have an equivalence between the categories of framed regular cells FrCell_n and of truss blocks Blk_n, and we want an equivalence between the categories of framed regular cell complexes and of regular block complexes. Regular block complexes are by definition 'regular presheaves' on the category of blocks (and their injections); here regularity demands that each block maps injectively into the complex. It remains only to observe that framed regular cell complexes can be recast as regular presheaves on the category of framed regular cells (and their inclusions); again regularity demands that each cell maps injectively into the complex. □

Of course, we could have considered more general classes of not-necessarily-regular presheaves on framed cells and truss blocks.

Recall that regular cells are algorithmically unrecognizable among posets, and so in particular it is impossible to decidably enumerate regular cells (cf. Section 1.3.2.5). These computability issues evaporate in the framed context, in the following sense.

COROLLARY 3.3.29 (Framed regular cells are decidably enumerable). *There is an algorithm for decidably enumerating framed regular cells among all framed posets.*

Proof. By virtue of their inductive combinatorial definition, we can certainly exhaustively enumerate truss blocks up to a given bound on the size of their underlying poset. By Theorem 3.1.1, we can therefore decidably enumerate framed regular cells. □

Remark 3.3.30 (Efficient enumeration of blocks and framed cells). Unlike for instance convex polytopes, which are only enumerable by an exceptionally expensive search, the enumeration of truss blocks is quite efficient in the size of the truss. ⌟

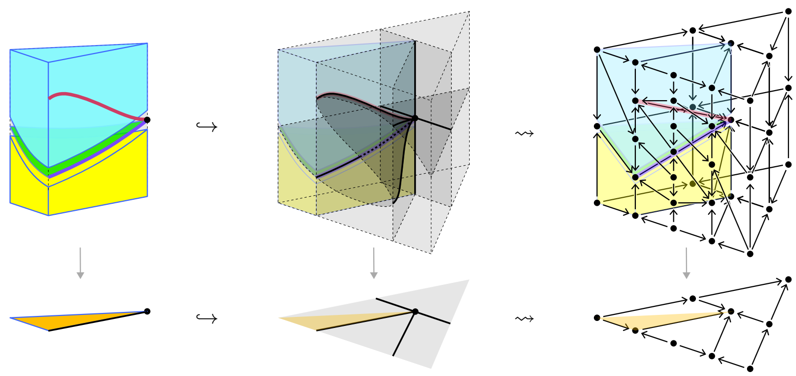
Finally, let us record a consequence, anticipated back in Remark 1.3.62, of the fact that the boundaries of truss blocks are shellable posets—namely that the notions of cellular and PL cellular coincide in the context of framed regular cells.

Corollary 3.3.31 (Framed regular cell complexes are piecewise linear). *For any n-framed regular cell complex $(X, \mathcal{F})$, the cellular poset X is PL cellular, and every cell $X^{\geq x}$ and its boundary $X^{>x}$ are shellable.*

Proof. By Construction 3.3.27, we know the cellular poset of our framed regular cell complex is the face poset of the corresponding closed truss. And by Lemma 3.3.23, we know that the face poset of a closed truss is PL cellular. Any cell $X^{\geq x}$ inherits a framed regular cell structure, thus corresponds to a truss block, whose boundary, and therefore also the whole block, is shellable by Lemma 3.3.25. □

CHAPTER 4

⧫Constructible framed topology: meshes



In this chapter, we develop the theory of meshes. Meshes are iterated constructible stratified bundles of framed stratified intervals. In Chapter 2, we introduced the combinatorial counterpart, namely trusses, as iterated constructible bundles of framed fence posets. The stratified geometric realization of a truss is a mesh, and conversely the stratified fundamental poset of a mesh is a truss. Those geometric realization and fundamental poset operations constitute an equivalence of the combinatorial and geometric theories, and so in particular provide a combinatorial model of the local structures of constructibly framed stratified spaces. As an application, leveraging the equivalence of truss blocks and framed regular cells from Chapter 3, we obtain a constructive classification of framed subdivisions of framed regular cells. In the subsequent Chapter 5, we introduce tame stratifications, as the comprehensive class of stratifications that are refinable by a mesh, and prove that all tame stratifications are combinatorially classified by stratified trusses.

The first half of this chapter, namely Section 4.1, introduces 1-meshes, 1-mesh bundles, and n-meshes. The second half of this chapter, namely Section 4.2, builds the fundamental truss of a mesh and the mesh realization of a truss, proves that those constructions are inverse equivalences, and discusses applications thereof.

4.1. ⧫1-Meshes, 1-mesh bundles, and n-meshes

Arbitrary stratifications, even after imposing strong local regularity properties, and even after restricting attention to stratifications of euclidean space, are drastically and uncontrollably complicated. Needless to say at this point, one would like to identify a tractable class of stratifications of euclidean space, that on the one hand can be combinatorially classified, and on the other hand are sufficiently general. Here by 'sufficiently general' we mean, for instance, that they coarsen to a class of stratifications that encodes all topological phenomena we might reasonably care about in a finitary context. Our fundamental contention is that such a tractable class is obtained by insisting that the stratifications behave well with respect to the standard framing of euclidean space, where by 'standard framing' we really mean a complete flag of foliations by standard euclidean subspaces, and where by 'behave well' we mean that the stratification projects along the foliations, constructibly and inductively, to a stratification of the same type in lower dimension.

We already have, of course, a reasonable class of stratifications of 1-dimensional euclidean spaces, namely finite stratifications by points and open intervals, i.e. by contractible submanifolds without boundary. Such a stratification of the euclidean space $\mathbb{R}^1$, or more generally a connected submanifold thereof or yet more generally of another manifold framed by an embedding in $\mathbb{R}^1$, is the essence of the notion of a *1-mesh*. An example 1-mesh is illustrated on the lower left in Figure 4.1; this open interval in $\mathbb{R}^1$ is stratified by two point strata and three open interval strata.

The decisive subtleties arise in considering stratified families of such euclidean stratifications. We certainly want to restrict attention to stratified bundles of 1-mesh stratifications, but that by itself is insufficient to provide a controllably iterable theory. We insist then that the boundaries of the fiber 1-meshes vary continuously in the base, and, critically, that the bundle is *constructible* in the sense that, roughly speaking, entrance paths in the base stratification lift uniquely to singular entrance paths in the total stratification. Such a continuous, constructible stratified bundle of 1-meshes will be called a *1-mesh bundle*. An example 1-mesh bundle is illustrated, for instance, by the left map in Figure 4.1; the fiber 1-meshes are closed intervals, stratified by two or three points and one or two open intervals, with the point strata constructibly wandering as indicated.

The crucial, if swift and obvious, maneuver of the whole theory is iterating this operation: consider a 1-mesh bundle over a 1-mesh bundle over a 1-mesh bundle over ... a 1-mesh. Such a sequence defines our concept of *n-mesh*. Seen, not as built up by bundles on bundles from the base 1-mesh, but conversely and as advertised from the perspective of the total stratification, an n-mesh is a stratification in (or suitably embedded in) n-dimensional euclidean space that projects constructibly, with fiber 1-meshes, to an $(n-1)$-mesh stratification in $(n-1)$-dimensional euclidean space. An example

3-mesh is illustrated by the whole of Figure 4.1; the 1-mesh fibers of the top projection are generically closed intervals but degenerate to point fibers along the left and right seams.

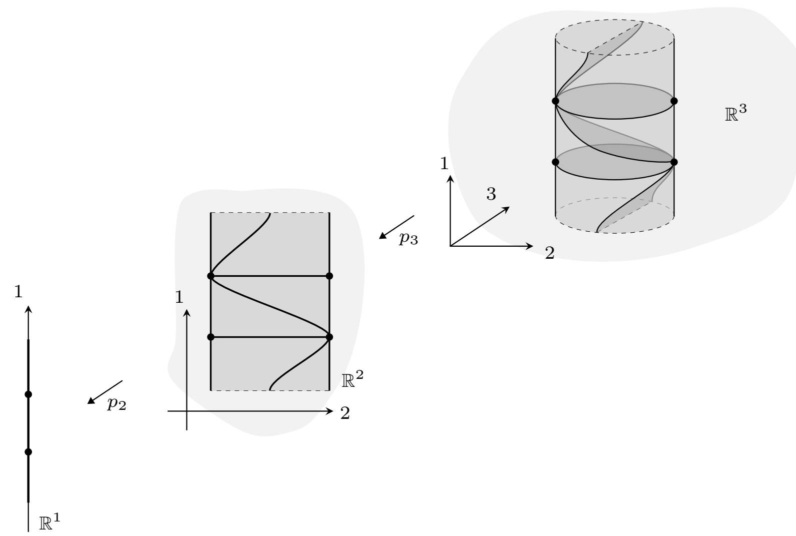


FIGURE 4.1. A 3-mesh.

OUTLINE. In Section 4.1.1, we describe 1-framed realizations of manifolds into standard 1-dimensional euclidean space, and define 1-meshes as finitely stratified manifolds with 1-framed realizations. In Section 4.1.2, we then introduce 1-mesh bundles as boundary-continuous, constructible stratified bundles of 1-meshes, and illustrate various local phenomena in 1-mesh bundles. Finally in Section 4.1.3, we define n-meshes as towers of 1-mesh bundles, discuss maps of such towers, and present the resulting categories and ∞-categories of meshes and their maps.

4.1.1. ⧫1-Meshes.

SYNOPSIS. We introduce 1-framed realizations of manifolds as embeddings into standard 1-framed 1-manifolds. We then define 1-meshes as stratified manifolds with a 1-framed realization, and distinguish linear, circular, and trivial 1-meshes. Finally we describe maps of 1-meshes, as those respecting both the stratification and the framing, delineate the notions of singular, regular, and balanced maps, and define submeshes and mesh degeneracies and coarsenings.

4.1.1.1. ⧫1-Framed realizations. Classically, a tangential framing of a smooth manifold is a trivialization of the tangent bundle. When the manifold does not have a smooth structure, we can ask instead for a trivialization of the tangent microbundle. Whether we work in the smooth or topological category, any sufficiently nice codimension-k embedding (or immersion) of a manifold

M, into a framed target n-manifold, induces a framing of the k-stabilized tangent (micro)bundle of M; we call such a map a *framed realization*.

In the case of 1-dimensional manifolds, we will focus on the following standard targets, endowed with their respective standard framings.

TERMINOLOGY 4.1.1 (Standard 1-framed target manifolds). The 'standard 1-framed euclidean space' is the standard real line $\mathbb{R}$, equipped with its positive orientation. The 'standard 1-framed circle' is the standard circle $S^1 \subset \mathbb{C}$, equipped with its counterclockwise orientation. ⌟

CONVENTION 4.1.2 (Manifolds are topological). Unless mentioned otherwise, the term 'manifold' will mean connected topological manifold, with or without boundary. ⌟

DEFINITION 4.1.3 (1-Framed realizations of manifolds). We distinguish two types of 1-framed realizations, as follows:

- A **1-framed linear realization** of a manifold M is an embedding $\gamma : M \to \mathbb{R}$.
- A **1-framed circular realization** of a manifold M is a homeomorphism $\gamma : M \to S^1$. ⌟

The pair (M, γ), consisting of a manifold M and a 1-framed (linear or circular) realization γ, will be called a '1-framed realized manifold', or simply a '1-realized manifold', for short.

TERMINOLOGY 4.1.4 (Support and boundedness of realizations). Given a 1-framed linearly realized manifold (M, γ), we refer to $\gamma(M) \subset \mathbb{R}$ as the 'support' of M. We call the realization 'bounded' if the support is a bounded subset of $\mathbb{R}$. ⌟

TERMINOLOGY 4.1.5 (Normal versus tangential framings). For a 1-realized manifold (M, γ), we refer to the structure provided by the 1-framed realization differently depending on the dimension:

- When $\dim(M) = 0$, we say that M obtains a 'normal 1-framing' from the target standard framed $\mathbb{R}$.
- When $\dim(M) = 1$, we say that M obtains a 'tangential 1-framing' from the target $\mathbb{R}$ or S^1. ⌟

REMARK 4.1.6 (Framed realizations up to homotopy). For a 1-manifold M, the space of linear or circular 1-realizations of M (as a subspace of $\mathrm{Map}(M, X)$, for X either $\mathbb{R}$ or S^1) is homotopy equivalent to $\mathbb{Z}_2$. That is, up to homotopy there are exactly two 1-framed realizations of any 1-manifold. ⌟

Next we may consider *framed maps* between framed realized manifolds, as maps that preserve the frame structure of the realization target, in the following sense.

TERMINOLOGY 4.1.7 (Framed maps of standard framed targets). For X and Y both being either $\mathbb{R}$ or S^1, a 'framed map' $F : X \to Y$ is an orientation preserving map. (More generally, we may allow either source or target to be a connected 1-dimensional submanifold of $\mathbb{R}$.) Concretely, we have the following cases.

- A framed map $F : \mathbb{R} \to \mathbb{R}$ is a monotone map.
- A framed map $F : \mathbb{R} \to S^1$ is a map of the form $x \mapsto e^{i\phi(x)}$, where $\phi : \mathbb{R} \to \mathbb{R}$ is monotone.
- A framed map $F : S^1 \to S^1$ is a map of the form $e^{ix} \mapsto e^{i\phi(x)}$, where $\phi : \mathbb{R} \to \mathbb{R}$ is monotone.
- A framed map $F : S^1 \to \mathbb{R}$ is a constant map. ⌟

Definition 4.1.8 (Framed maps of 1-realized manifolds). Given 1-framed realized manifolds (M, γ) and (N, ρ), a **framed map of 1-realized manifolds** is a map $F : M \to N$ that induces a framed map $F : \gamma(M) \to \rho(N)$ of the realization images. ⌟

4.1.1.2. ♦The definition of 1-meshes. A 1-mesh is a suitably stratified 1-framed manifold, as follows.

⋄ Could use some more words here, cf truss section

Definition 4.1.9 (General 1-meshes). A **1-mesh** (M, f, γ) is a manifold M, with a finite stratification f whose strata are manifolds without boundary, and a 1-framed realization γ. ⌟

Terminology 4.1.10 (Linear, circular, and trivial 1-meshes). A 1-mesh (M, f, γ) is called 'linear' when γ is linear, and 'circular' when γ is circular; it is called 'trivial' when the stratification has a single stratum. ⌟

Terminology 4.1.11 (Closed and open 1-meshes). A linear 1-mesh (M, f, γ) is called 'closed' or 'open' when the image $\gamma(M) \subset \mathbb{R}$ is closed or open, respectively, as a subspace of $\mathbb{R}$. ⌟

Example 4.1.12 (1-Meshes). In Figure 4.2, we illustrate 1-meshes of various types. In each case, we color the 0-dimensional strata in red, and the 1-dimensional strata in blue. For linear 1-meshes, we depict the euclidean space target $\mathbb{R}$ of the realization; later on, we will omit illustration of that target, and instead include a small purple 'coordinate arrow', indicating the orientation direction of the realization target. Similarly, we indicate the realization of circular meshes by an arrow giving the orientation of the realization target S^1. Note that the figure distinguishes three types of trivial 1-meshes: the trivial 0-dimensional mesh, the trivial linear 1-dimensional mesh, and the trivial circular 1-dimensional mesh.[18] ⌟

Though much of the theory of meshes can be developed in parallel for the linear and circular cases, our concern will be (as in the case of 1-trusses)

[18]By contrast, we had distinguished only two trivial 1-trusses in Figure 2.4. This discrepancy indicates that the trivial 1-truss, whose element is of dimension 1, should have two distinct combinatorial incarnations: the 'trivial linear' and the 'trivial circular' 1-truss. We will not bother rectifying the combinatorial situation to accommodate this distinction, since we are ultimately interested predominately in the linear case.

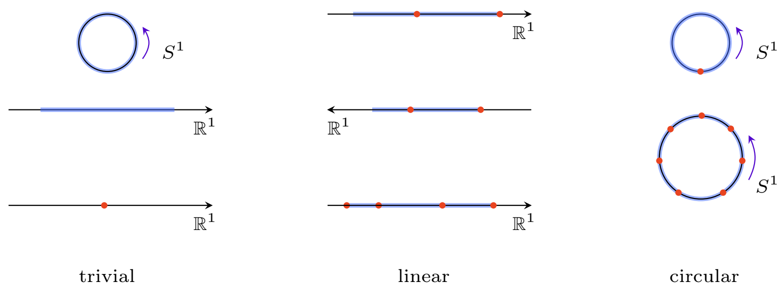


FIGURE 4.2. Trivial, linear, and circular 1-meshes.

almost exclusively with the linear case, and we therefore adopt the following convention.

CONVENTION 4.1.13 (Linear 1-meshes by default). Henceforth, we will use the term '1-meshes' to mean 'linear 1-meshes' unless otherwise noted. ⌟

Moreover, it will be technically convenient to, and we will, assume that all our linear realizations are bounded. Note that we can always obtain a bounded linear realization from a general linear realization by post-composing with a bounded framed embedding $\mathbb{R} \hookrightarrow \mathbb{R}$.

CONVENTION 4.1.14 (Bounded linear realizations by default). We will assume that the realization of any linearly realized 1-mesh (M, f, γ) is bounded. ⌟

NOTATION 4.1.15 (Realization bounds). For a 1-mesh (M, f, γ) with realization $\gamma : M \hookrightarrow \mathbb{R}$, we refer to the lower and upper bounds of the subspace $\gamma(M) \subset \mathbb{R}$ as the 'lower realization bound' γ^- and 'upper realization bound' γ^+. ⌟

REMARK 4.1.16 (Contractible choice of equivalent 1-realizations). We say two 1-realizations of a manifold are 'framed homeomorphic' if they differ by post-composition with a framed homeomorphism of $\mathbb{R}$. The theory of 1-meshes could be developed by taking only a framed homeomorphism class of 1-realizations (rather than a specific 1-realization) to be part of the data of the 1-mesh. Indeed, for a given 1-mesh (M, f, γ), the space of 1-realizations framed homeomorphic to the given 1-realization is contractible. ⌟

4.1.1.3. ♦Maps of 1-meshes. A map of 1-meshes preserves both the stratification and the framing, as follows. Recall the notion of framed map from Definition 4.1.8.

DEFINITION 4.1.17 (Maps of 1-meshes). A **map of 1-meshes** $F : (M, f, \gamma) \to (N, g, \rho)$ is a continuous map $F : M \to N$ that is both a stratified map $F : (M, f) \to (N, g)$ and a framed map $F : (M, \gamma) \to (N, \rho)$. ⌟

Corresponding to our earlier definitions of 'singular', 'regular', and 'balanced' maps of 1-trusses (see Definition 2.1.17), we have the following terminology for maps of 1-meshes.

Definition 4.1.18 (Singular, regular, and balanced maps of 1-meshes). Let $F : (M, f, \gamma) \to (N, g, \rho)$ be a map of 1-meshes.

- The map F is **singular** if it maps point strata to point strata.
- The map F is **regular** if it maps interval strata to interval strata.
- The map F is **balanced** if it is both singular and regular. ⌟

Furthermore, in parallel with our earlier definitions of 'subtruss', 'truss degeneracy', and 'truss coarsening' (see Terminology 2.3.61 and Terminology 2.3.63), we have the following properties of mesh maps.

Terminology 4.1.19 (Submeshes of 1-meshes). A map of 1-meshes $F : (M, f, \gamma) \to (N, g, \rho)$ is called a 'submesh' when $F : (M, f) \to (N, g)$ is a substratification (see Definition B.2.6). ⌟

Terminology 4.1.20 (Degeneracies and coarsenings of 1-meshes). A map of 1-meshes $F : (M, f, \gamma) \to (N, g, \rho)$ may be of one of the following types.

- The 1-mesh map F is called a 'mesh degeneracy' when it is surjective, singular, and maps each interval stratum either homeomorphically onto its image stratum or to a point stratum.
- The 1-mesh map F is called a 'mesh coarsening' if it is a coarsening of stratifications $F : (M, f) \to (N, g)$ (see Definition B.2.4). (Note that mesh coarsenings are necessarily surjective regular 1-mesh maps, and homeomorphisms of underlying spaces by definition.) ⌟

Example 4.1.21 (Maps of 1-meshes). In Figure 4.3, in the first row, we depict a singular, a regular, and a balanced map of 1-meshes. In the second row, we depict a mesh degeneracy, a mesh coarsening, and a submesh (which are, respectively, themselves singular, regular, and balanced maps by definition). In the third row, the map is a 'mixed' 1-mesh map, which is neither singular nor regular. ⌟

4.1.2. ♦1-Mesh bundles.

Synopsis. We introduce families of 1-meshes as stratified bundles whose fibers are 1-meshes, together with parametrized 1-framed realizations. We then define 1-mesh bundles as families of 1-meshes satisfying a boundary continuity condition and a constructibility condition. We illustrate various local phenomena that occur in 1-mesh bundles, along with an assortment of families of 1-meshes that fail to be 1-mesh bundles, due to disparate failures of continuity or constructibility. Next we briefly describe maps of 1-mesh bundles, along with pullback and compactification constructions. Finally, we show that cellularity and cellulability properties lift from the base to the total stratifications of 1-mesh bundles.

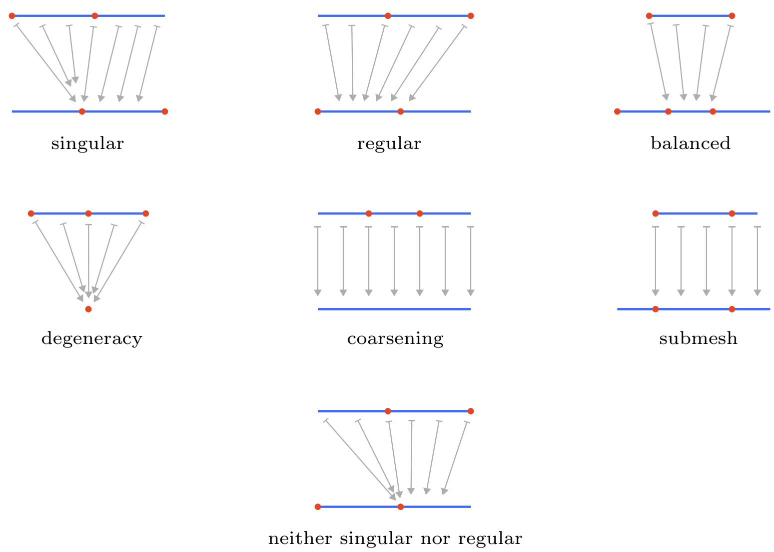


FIGURE 4.3. Types of maps of 1-meshes.

4.1.2.1. ⧫1-Framed realizations of families. As defined, a 1-mesh is a stratified manifold with a 1-framed realization. We now describe a notion of 1-framed realization parametrized by a stratified base space, and use that notion to define parametrized families of 1-meshes. For brevity we adopt the following convention.

CONVENTION 4.1.22 (Finiteness). We henceforth assume all stratifications are finite, i.e. have finitely many strata, unless otherwise noted. ⌟

DEFINITION 4.1.23 (1-Framed realizations of manifold families). Given a 'family of manifolds' $p : M \to B$ indexed by a space B (meaning that p is a continuous map and each fiber $p^{-1}(b)$, $b \in B$, is a manifold), a **1-framed realization** of the family is a bundle embedding $\gamma : M \hookrightarrow B \times \mathbb{R}$ into the trivial bundle $\pi : B \times \mathbb{R} \to B$ (i.e. an embedding such that $\pi \circ \gamma = p$). ⌟

Recall from Definition B.2.25 that a stratified bundle is a stratified map that is a locally trivial bundle within each stratum of the base.

DEFINITION 4.1.24 (Family of 1-meshes). A **family of 1-meshes** $(p, \gamma) : (M, f) \to (B, g)$, indexed by the stratified space (B, g), is a stratified bundle $p : (M, f) \to (B, g)$, together with a 1-framed realization γ of the underlying family of manifolds $p : M \to B$, such that the stratification f and the realization γ restrict to give every fiber $(M_b := p^{-1}(b), f_b, \gamma_b)$ the structure of a 1-mesh. ⌟

NOTATION 4.1.25 (Realization bounds for families). For a family of 1-meshes $(p, \gamma) : (M, f) \to (B, g)$, we denote by $\gamma^{\pm} : B \to B \times \mathbb{R}$ the (not necessarily continuous) functions given by the fiberwise lower and upper realization bounds $b \mapsto (b, \gamma_b^\pm)$ (see Notation 4.1.15). (Abusing notation we let $\gamma^\pm$ also refer to the composite $B \to B \times \mathbb{R} \to \mathbb{R}$.) ⌟

Consider a stratified bundle $p : (E, f) \to (B, g)$ such that any stratum of f intersects each fiber $p^{-1}(b)$ in a manifold. Note that, for a given stratum s of f, the dimension of the fiber $s \cap p^{-1}(b)$ is, when non-empty, independent of which fiber is considered. We refer to that dimension as the 'fiber dimension' of the stratum. In a family of 1-meshes, each stratum of the family has fiber dimension 0 or 1; we distinguish the strata as singular or regular accordingly.

Terminology 4.1.26 (Regular and singular strata). For a family of 1-meshes, we call a stratum of the total space **singular** when its fiber dimension is 0, and we call it **regular** when its fiber dimension is 1. ⌟

4.1.2.2. ⧫The definition of 1-mesh bundles. 1-Mesh bundles will be particularly well-behaved families of 1-meshes, namely those whose realization bounds are continuous and whose stratified bundle is *constructible* in an appropriate sense. The constructibility condition will be formulated in stratified-topological terms, but, leveraging later results of Section 4.2, it will be the case that the stratified-topological constructibility condition implies constructibility in the usual categorical sense, namely that bundles can be constructed by pullback along functors into a suitable classifying category.

To formulate the constructibility condition for 1-mesh bundles, we recall the notions of formal entrance paths and fundamental posets of stratifications.

Terminology 4.1.27 (Fundamental posets of stratifications). Given a stratification (X, f) and two strata $r, s \in f$, we say there exists a 'formal entrance path' from r to s, when $\overline{r} \cap s$, the intersection of the closure of r with s, is nonempty. The 'fundamental poset' $\sqcap(X, f)$ (also written simply $\sqcap f$) is the poset whose objects are the strata of (X, f), and whose morphisms are the transitive closure of the formal entrance path relation. Note that the fundamental poset provides a functor from the category of stratifications to the category of posets. See Definitions B.1.7 and B.1.11, Construction B.2.15, and surroundings for further discussion of these notions. ⌟

Definition 4.1.28 (1-Mesh bundles). A **1-mesh bundle** $(p, \gamma) : (M, f) \to (B, g)$ over a base stratification (B, g) is a family of 1-meshes satisfying the following conditions.

1. *Continuity*: The realization bounds $\gamma^\pm : B \to \mathbb{R}$ are continuous.
2. *Constructibility*: For every arrow $r \to s$ in the fundamental poset $\sqcap(B, g)$, and every lift of the stratum r to a singular stratum t of the total stratification (M, f), there exists a unique lift of $r \to s$ to an arrow $t \to u$ in the fundamental poset $\sqcap(M, f)$ and u is itself singular. ⌟

Roughly speaking, the constructibility condition ensures that point strata in the fibers behave functionally, during fiber transitions that cover entrance paths in the base. Notice that the condition does not refer to regular strata at all, but the behavior of those strata is nevertheless constrained by the functionality of their boundary singular strata.

TERMINOLOGY 4.1.29 (Open and closed 1-mesh bundles). A 1-mesh bundle is called 'closed' or 'open' when all its fibers are, respectively, closed or open 1-meshes. ⌟

EXAMPLE 4.1.30 (A 1-mesh bundle over the stratified 1-simplex). In Figure 4.4, we depict a 1-mesh bundle over the stratified 1-simplex $\|[1]\|$. (The stratified 1-simplex is the stratified realization of the combinatorial 1-simplex, cf. Section B.1.5.) The fibers over points of the base stratum $[0,1)$ are open, but the fiber over $\{1\}$ is half-open, and thus the bundle altogether is neither open nor closed.

The fundamental poset of the total stratification of this bundle is the 1-truss bordism previously illustrated in Figure 2.9, and the fundamental poset functor applied to the bundle projection is the projection of that 1-truss bordism, considered as a 1-truss bundle over the combinatorial 1-simplex. ⌟

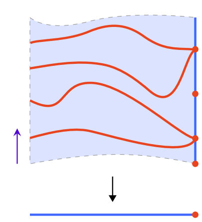


FIGURE 4.4. A 1-mesh bundle over the stratified 1-simplex.

EXAMPLE 4.1.31 (A 1-mesh bundle over a poset realization). In Figure 4.5, we depict a 1-mesh bundle (p,γ) over the stratified realization $(B,g) = \|P\|$ of the poset $P = (a \leftarrow b \leftarrow c \to d)$. The bundle is drawn together with its 1-framed realization as a subbundle of the projection bundle $B \times \mathbb{R} \to B$; the orientation of the fiber $\mathbb{R}$ is indicated by an adjacent purple arrow.

Note that the fibers over the points in the stratum $\mathsf{str}(c) \subset \|P\|$ are open 1-meshes, while the fibers over the points of the strata $\mathsf{str}(a)$ and $\mathsf{str}(d)$ are closed 1-meshes, and the fibers over the points of the stratum $\mathsf{str}(b)$ are neither open nor closed. The fundamental poset of this 1-mesh bundle is the 1-truss bundle previously illustrated in Figure 2.19. ⌟

REMARK 4.1.32 (Omitting orientations). When depicting 1-mesh bundles as subbundles of the standard projection bundle $B\times\mathbb{R} \to B$, we will sometimes forgo indicating an orientation of the fiber $\mathbb{R}$, as we did for 1-truss bundles, cf. Remark 2.1.82. This omission leaves, of course, a $\mathbb{Z}_2$ ambiguity, which though is typically without consequence. ⌟

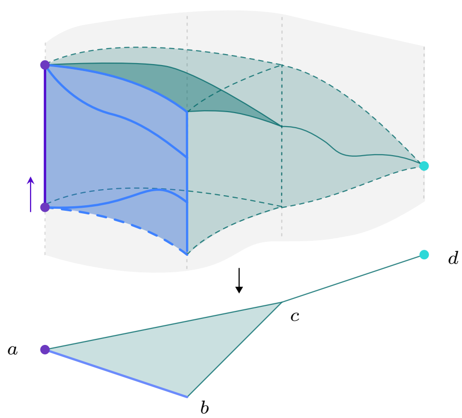


FIGURE 4.5. A 1-mesh bundle over a poset realization.

A general stratified bundle is only locally trivial over each base stratum, and so admits some stratified monodromy. By contrast, due to the rigid nature of their fibers, 1-mesh bundles are in fact globally trivial over each base stratum, as follows.

NOTATION 4.1.33 (Restriction to base strata). Let $p : (M, f) \to (B, g)$ be a 1-mesh bundle. The restriction of this bundle to a stratum $s \in g$ is a 1-mesh bundle denoted $p|_s : (p^{-1}(s), f) \to s$. ⌟

OBSERVATION 4.1.34 (Trivialization over base strata). Because the automorphism space of every 1-mesh is contractible, the restricted 1-mesh bundle $p|_s$ (of any 1-mesh bundle p to any base stratum s) is isomorphic to a trivial 1-mesh bundle:

$$\begin{array}{ccc} (p^{-1}(s), f) & \xrightarrow{\cong} & s \times \mathsf{fib}(s) \\ & {\scriptstyle p|_s} \searrow \quad \swarrow & \\ & s & \end{array}$$

Here $\mathsf{fib}(s)$ denotes a 1-mesh, called the 'fiber 1-mesh', over the stratum s. ⌟

REMARK 4.1.35 (Mapping cylinders as bundles). Recall from Section 2.1.2.5 that suitable singular 1-truss maps had associated mapping cylinder 1-truss bordisms, and suitable regular 1-truss maps had associated mapping cocylinder 1-truss bordisms. We have an analogous relationship for 1-meshes: the mapping cylinder of a suitable singular 1-mesh map is a 1-mesh bundle over the stratified 1-simplex, and similarly the mapping cocylinder of a suitable regular 1-mesh map is again a 1-mesh bundle over the stratified 1-simplex. These cylinder and cocylinder constructions are discussed later on, in Remark 4.2.6, as a consequence of the constructions in the truss case and the equivalence of meshes and trusses. ⌟

As given above, the constructibility condition for a 1-mesh bundle refers to a lifting condition involving arrows in the fundamental posets; such arrows are generated by the transitive closure of the formal entrance path relations, and the condition is in principle a bit unwieldy as a result. However, with mild assumptions on the behavior of the base stratification, we can refine the constructibility condition as follows.

Terminology 4.1.36 (Frontier-constructibility and local path-connectedness). We may impose the following conditions on a stratification (X, f).

- The stratification (X, f) is 'frontier-constructible' if $(\overline{r} \cap s \neq \emptyset) \Rightarrow (s \subset \overline{r})$ for any two strata $r, s \in f$.
- The stratification (X, f) is 'pairwise locally path-connected' if the union $s \cup r$ is locally path-connected, for any two strata $r, s \in f$.
- We refer to a stratification as 'reasonably regular' if it satisfies both of the preceding conditions. ⌟

In a reasonably regular stratification, the fundamental poset has an arrow $r \to s$ precisely when there is an entrance path from r to s; see Lemma B.1.30 and Observation B.1.31. We can therefore, in that case, rephrase the constructibility condition for 1-mesh bundles in terms of entrance paths, as follows.

Observation 4.1.37 (1-Mesh bundle over a reasonably regular base). Let $(p, \gamma) : (M, f) \to (B, g)$ be a family of 1-meshes over a reasonably regular base (B, g). This family is a 1-mesh bundle if it satisfies the 'continuity' condition from Definition 4.1.28, as well as the following condition.

(2′) *Path-independent constructibility*: For every entrance path $\alpha : r \to s$ in the base (B, g), and every lift of the stratum r to a singular stratum t of the total stratification (M, f), there exists a unique lift of $\alpha : r \to s$ to an entrance path $\beta : t \to u$ in (M, f) and u is itself a singular stratum. Furthermore, the resulting singular stratum u, that is the target of the lifted entrance path β, is independent of which entrance path α from r to s was chosen initially. ⌟

Since eventually we will be considering 1-mesh bundles over 1-mesh bundles iteratively, it is worth noting that if the base stratification of a 1-mesh bundle is reasonably regular, then it follows that the total stratification is also reasonably regular; see later Observations 4.1.67 and 4.1.68. However, various elementary constructions may break reasonable regularity; for instance, restricting the standard stratification of the realized simplex $\|[k]\|$ to its boundary yields a stratification that is not frontier-constructible.

The definition of 1-mesh bundles has a natural generalization, that allows the base to be a category, not just a poset. (This generalization will not play a substantive role later and can be safely skipped.) However, this categorical version requires a stronger regularity condition on the base stratification, namely 'conicality' (see Section B.3.1); conical stratifications are, in particular, reasonably regular. In the following categorical discussion, we will implicitly assume stratifications are conical as needed.

To begin, in place of the fundamental poset of a stratification, we need a notion of the fundamental ∞-category and fundamental 1-category of a stratification, as follows.

TERMINOLOGY 4.1.38 (Fundamental ∞-category). The 'fundamental ∞-category' $\sqcap_\infty f$ of a stratification (X, f) is the quasicategory whose n-simplices are the stratified maps $\|[n]\| \to (X, f)$. ⌟

TERMINOLOGY 4.1.39 (Fundamental 1-category). The 'fundamental category' $\sqcap_1 f$ of a stratification (X, f) is the truncation of the fundamental ∞-category to a 1-category. (See Definition B.3.10 and Construction B.3.14 and the intervening discussion of truncation.) ⌟

Moreover, the fundamental poset $\sqcap f$ of a stratified space (X, f), given by Terminology 4.1.27, is equivalent to the 0-truncation of the fundamental ∞-category $\sqcap_\infty f$; see Terminology B.3.12 and Observation B.3.13.[19]

EXAMPLE 4.1.40 (Base posets versus base categories). In Figure 4.6, in the middle we depict a bundle that is locally a well-behaved 1-mesh bundle. However, it is not a 1-mesh bundle in the sense of Definition 4.1.28, because it fails the constructibility condition; the relevant failure of unique lifts is illustrated on the left. The bundle will be, though, a categorical 1-mesh bundle, in the sense that the constructibility condition will be restored when we consider not the fundamental poset but the fundamental categories of the stratifications; the relevant uniqueness of lifts is illustrated on the right. The necessary categorical notion is formalized in the next remark. (Note that the fundamental category of the total stratification here was illustrated as a categorical 1-truss bundle in Figure 2.20.) ⌟

We are now equipped to describe the anticipated categorical version of 1-mesh bundles.

REMARK 4.1.41 (Categorical 1-mesh bundles). A 'categorical 1-mesh bundle' $p : (M, f) \to (B, g)$ is a family of 1-meshes, over a reasonably regular base (B, g), satisfying the 'continuity' condition from Definition 4.1.28, as well as the following condition.

(2′) *Path-dependent constructibility*: For every entrance path $\alpha : r \to s$ in the fundamental category $\sqcap_1 g$, and every lift of the stratum r to a singular stratum t of the total stratification (M, f), there exists a unique lift of $\alpha : r \to s$ to an entrance path $\beta : t \to u$ in the

[19]There is a useful analogy that sets are to spaces what posets are to stratified spaces. For instance, the fundamental ∞-category of a stratified space admits a conservative functor to its (0-truncated) fundamental poset, as the fundamental ∞-groupoid of a space admits a conservative functor to its (0-truncated) connected component set; see Remark B.3.15.

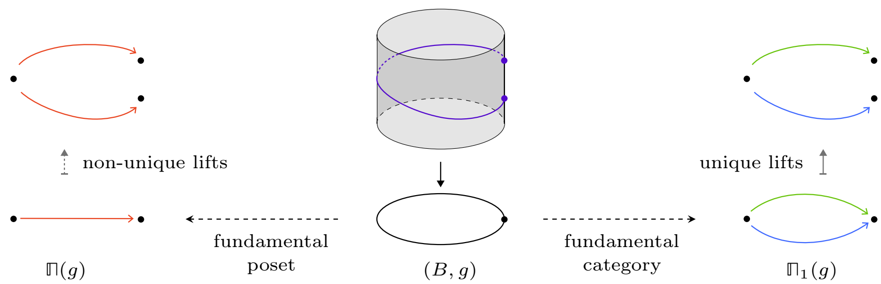


FIGURE 4.6. A categorical 1-mesh bundle.

fundamental category $\sqcap_1 f$, and furthermore u is itself a singular stratum. ⌟

We could have formulated the path-dependent constructibility condition by merely asking every literal entrance path $r \to s$ in the base stratification (not in the quotient fundamental category) to lift uniquely to a suitable literal entrance path $t \to u$ in the total stratification. In fact, the resulting notion is unchanged: a homotopy α_t of entrance paths provides a homotopy β_t of lifts, which gives a path of singular strata u_t; since the fibers in a family of 1-meshes are 1-meshes and therefore have discrete sets of singular points, any such path of singular strata is constant. Henceforth, we will freely work with either version of categorical 1-mesh bundles, as convenient.

To emphasize the distinction from the case of categorical 1-mesh bundles, we sometimes refer to 1-mesh bundles in the sense of Definition 4.1.28 as 'posetal 1-mesh bundles'. As discussed, the bundle shown in Figure 4.6 is categorical but not posetal.

REMARK 4.1.42 (Categorical versus posetal 1-mesh bundles). Every posetal 1-mesh bundle over a reasonably regular base is, of course, a categorical 1-mesh bundle. Conversely, if the base stratification is posetal, meaning its fundamental ∞-category is 0-truncated (or, in other words, the stratification is 'stratified homotopy equivalent' to the stratified realization of a poset P), then any categorical 1-mesh bundle is in fact posetal. ⌟

We give an explicit statement and proof of the converse implication in that last remark, in the concrete case where the base is a realization of a poset.

PROPOSITION 4.1.43 (Categorical bundles over posets are posetal). *A categorical 1-mesh bundle over the stratified realization of a poset (or a constructible substratification thereof) is a posetal 1-mesh bundle.*

PROOF. Given a poset X with stratified realization $\|X\|$, consider a categorical 1-mesh bundle $p : (M, f) \to \|X\|$. Let $r \equiv \mathsf{str}(x)$ be a stratum in $\|X\|$ with a lift to a singular stratum t, and let $\alpha : r \to s$ be an entrance path. Since the stratum r, its closure, the stratum s, and its closure are all contractible, every entrance path $r \to s$ is homotopic to α and so equivalent to it in the fundamental category. Thus no matter the entrance path, there is altogether a unique lift to an entrance path $\beta : t \to u$, and u is singular, as required for a posetal 1-mesh bundle. □

Remark 4.1.44 (Classification of categorical 1-mesh bundles). We will later that the posetal constructibility condition in Definition 4.1.28 precisely ensures that posetal 1-mesh bundles over (sufficiently regular) base stratifications (B, g) are classified up to bundle isomorphism by functors $\sqcap g \to \mathsf{TBord}^1$ from the fundamental poset of the base to the classifying category of 1-trusses and their bordisms.

The categorical case is similar: the categorical constructibility condition in Remark 4.1.41 ensures that categorical 1-mesh bundles are classified up to bundle isomorphism by ∞-functors $\sqcap_\infty(g) \to \mathsf{TBord}^1$, and so (since the codomain is a 1-category) by 1-categorical functors $\sqcap_1(g) \to \mathsf{TBord}^1$. ⌟

Remark 4.1.45 (Categorical 1-mesh bundles are higher constructible). Entrance paths are stratified maps from the stratified 1-simplex. The path constructibility condition for a categorical 1-mesh bundle, given in Remark 4.1.41, is thus a lifting condition for the stratified 1-simplex. One might imagine that there is a $(k \in \mathbb{N})$-indexed family of 'higher constructibility' conditions, given by analogous lifting conditions for maps out of the stratified k-simplex. However, in the context of conical stratifications, all those higher constructibility conditions are automatically satisfied by categorical 1-mesh bundles, as defined just with the '1-constructibility' requirement. ⌟

4.1.2.3. ⧫Local phenomena in families of 1-meshes. We describe and illustrate various local phenomena that occur in families of 1-meshes: first examples of families that are indeed 1-mesh bundles, then examples that violate either or both of the continuity and constructibility conditions, to different extents and in sundry ways, and so fail to be 1-mesh bundles.

Example 4.1.46 (Local forms of 1-mesh bundles). In Figure 4.7 we illustrate some local behaviors in 1-mesh bundles. The top three are 'collisions' in the sense that two singular strata in the generic fiber converge into a single singular stratum of the special fiber. The bottom three are 'creations' in the sense that a new singular stratum appears in the special fiber, with no singular stratum of the generic fiber converging to it. The right two are also 'collapses' in the sense that the interval of the generic fiber degenerates into a point of the special fiber.

The 1-truss bordisms obtained as the fundamental posets of these 1-mesh bundles were illustrated in Figure 2.12. ⌟

Example 4.1.47 (Families of 1-meshes that are almost continuous and constructible). In Figure 4.8 we illustrate three families of 1-meshes that are not 1-mesh bundles, because they fail one or both of the continuity and constructibility conditions. In the first case, the upper realization bound has an upward discontinuity at the special fiber. In the second case, the upper realization bound has a downward discontinuity at the special fiber; this

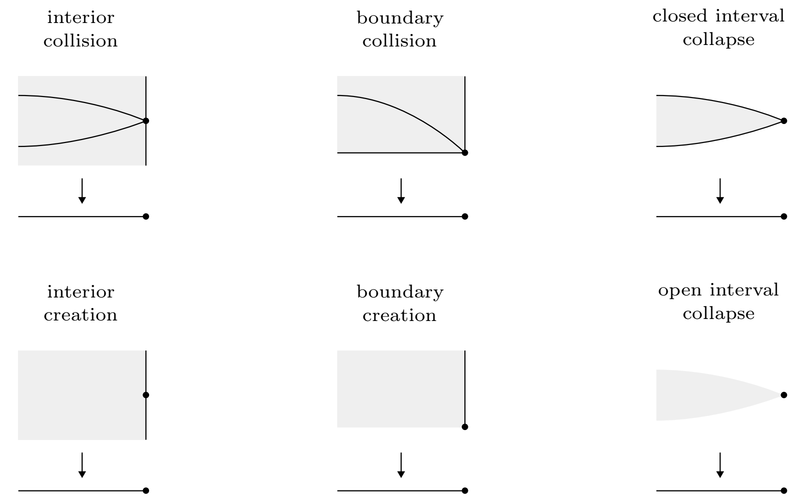


Figure 4.7. Local forms of 1-mesh bundles.

case also fails constructibility since the singular stratum of the generic fiber does not converge to any stratum of the special fiber. In the third case, the realization bounds are continuous, but again the generic singular stratum does not converge to any special stratum, and so this case fails constructibility. However, all three of these failures are rather mild in the sense that, in each case, either the generic or special fiber can be extended to resolve the issue; in other words, these families of 1-meshes embed as subfamilies of actual 1-mesh bundles.[20]

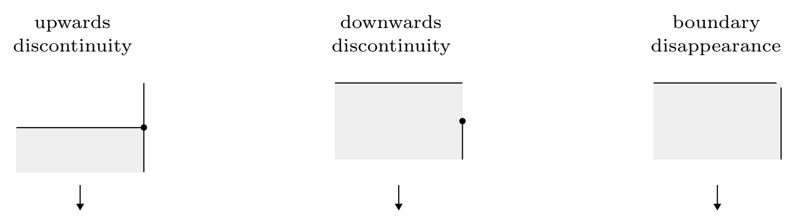


Figure 4.8. Mild mesh mishaps of constructible continuity.

The combinatorial counterparts of each of these families, namely the relations obtained by taking fundamental posets, were illustrated in Figure 2.13. ⌟

Example 4.1.48 (Families of 1-meshes that fail constructible lift existence). In Figure 4.9 we illustrate three families of 1-meshes that are not

[20]One could enlarge the class of valid 1-mesh bundles to include families that allow certain boundary discontinuities or certain boundary disappearances, but we forego that generalization here.

1-mesh bundles, again because they fail one or both of the continuity and constructibility conditions. In the first case, the generic fiber singular stratum converges, but to a regular stratum of the special fiber, violating constructibility. That same violation occurs in the second case, though with a boundary singular stratum; this case also evidently fails realization continuity. The third case appears to violate constructibility in the same way as the other two cases. However, this case is in fact, in a sense, even worse, because there are two lifts (of the entrance path in the base) starting at the generic fiber singular stratum—namely the entrance path converging to the regular stratum, and the transitive composite of that entrance path and the entrance path into the singular stratum of the special fiber.

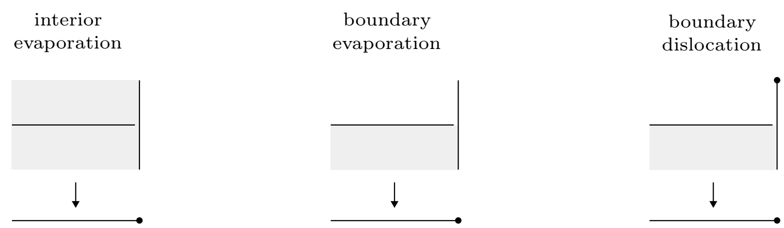


FIGURE 4.9. Lamentable lapses of liftability.

None of these three families are subfamilies of mesh bundles, as the constructibility failure is intrinsic. Nevertheless, one could refine the special fiber by adding a singular stratum inside the regular stratum, in order to obtain either a mesh bundle or a subfamily of a mesh bundle.

The combinatorial counterparts of these families were illustrated earlier, the first two cases as the first two relations in Figure 2.14, and the third case as the first relation in Figure 2.15. ⌟

EXAMPLE 4.1.49 (Families of 1-meshes with divergent realization bounds). In Figure 4.10 we illustrate two families of 1-meshes that fail both continuity and constructibility in a most serious and irresolvable manner. In both cases, the upper realization bound is not only discontinuous but also fails to have a limit as it approaches the special fiber. Furthermore, in the first case, there are distinct formal entrance paths from the generic fiber singular stratum to all three of the special fiber strata, and in the second case to both of the special fiber strata, contravening constructibility. The second case has a yet more serious pathology, namely that there is a formal entrance path from the generic fiber regular stratum to the special fiber regular stratum, which crosses (in the framing direction) the formal entrance path from the generic fiber singular stratum to the special fiber singular stratum.

The combinatorial counterparts of these families were illustrated earlier, the first case as the last relation in Figure 2.14, and the second case as the last relation in Figure 2.15. ⌟

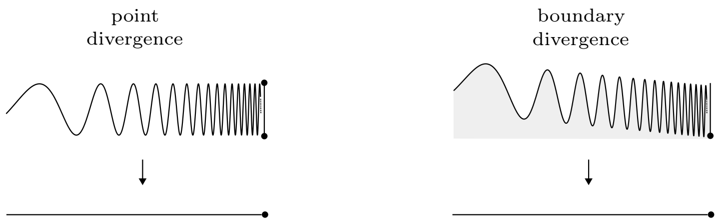


FIGURE 4.10. Brazen breaches of boundaries.

EXAMPLE 4.1.50 (Families of 1-meshes that fail constructible lift uniqueness). In Figure 4.11 we illustrate two families of 1-meshes that fail constructibility in a more subtle, if no less serious, manner. Both families have a conical total stratification and a conical base stratification, and so demonstrate that even in the better-behaved context of conical stratifications, the constructibility condition remains crucial.

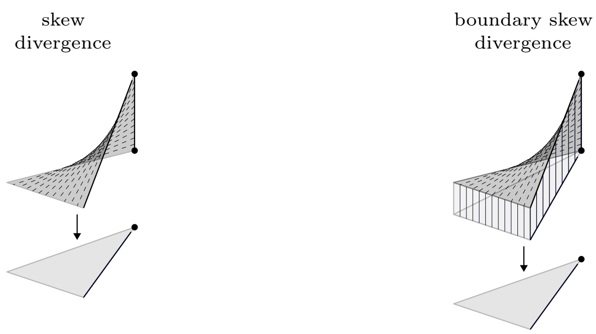


FIGURE 4.11. Subtle skews of mesh lift uniqueness.

These are both families over the stratified 2-simplex. The first case has a stratified closed interval fiber over the 0-simplex, a single stratum over the 1-simplex and a single stratum over the 2-simplex. Though the stratum over the 1-simplex converges uniquely to a singular stratum over the 0-simplex, the (singular) stratum over the 2-simplex admits entrance paths to all three of the strata over the 0-simplex, violating the uniqueness of lifts in the constructibility condition. In the second case, every fiber of the family is the same standard stratified closed interval; when restricted to the 0- and 1-simplex the family is the trivial (in particular, constructible) bundle, and when restricted to the 1- and 2-simplex the family is again the trivial (in particular, constructible) bundle; nevertheless, the entrance paths over the entrance path from the 2- to 0-simplex fail constructibility as in the first case.

The fundamental poset of the first case here contains the fundamental poset of the first case of Figure 4.10 (namely the last relation of Figure 2.14). The fundamental poset of the second case here contains the fundamental posets of both cases of Figure 4.10 (the second of which is the last relation of

Figure 2.15) and also the fundamental poset of the last case of Figure 4.9 (namely the first relation of Figure 2.15); the inherited constructibility failures are pervasive. ⌟

4.1.2.4. ♦Maps of 1-mesh bundles. Maps of 1-mesh bundles are simply stratified bundle maps that restrict to maps of 1-meshes on each fiber, and require little fanfare.

DEFINITION 4.1.51 (Maps of 1-mesh bundles). For 1-mesh bundles $(p, \gamma) : (M, f) \to (B, g)$ and $(p', \gamma') : (M', f') \to (B', g')$, a **map of 1-mesh bundles** $F : p \to p'$ is a stratified map $F : (M, f) \to (M', f')$ that descends along p and p' to a stratified map $G : (B, g) \to (B', g')$, such that the restriction of F to each fiber $M_b := p^{-1}(b)$, $b \in B$, is a 1-mesh map $(M_b, f_b, \gamma_b) \to (M'_{G(b)}, f'_{G(b)}, \gamma'_{G(b)})$. ⌟

TERMINOLOGY 4.1.52 (Singular, regular, and balanced 1-mesh bundle maps). We call a 1-mesh bundle map 'singular' or 'regular' or 'balanced' if it is respectively singular or regular or balanced on every fiber, in the sense of Definition 4.1.18. ⌟

TERMINOLOGY 4.1.53 (Degeneracies and coarsenings of 1-mesh bundles). We call a 1-mesh bundle map a 'degeneracy' or 'coarsening' when it is such on every fiber, in the sense of Terminology 4.1.20. ⌟

NOTATION 4.1.54 (Implicit realizations). Henceforth, we often keep the 1-framed realization γ of 1-mesh bundles (p, γ) implicit, denoting the 1-mesh bundle by simply $p : (M, f) \to (B, g)$. ⌟

EXAMPLE 4.1.55 (1-Mesh bundle map). In Figure 4.12, we depict a 1-mesh bundle map, neither singular nor regular as it happens. ⌟

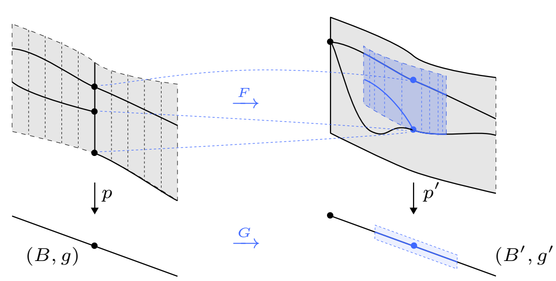


FIGURE 4.12. 1-Mesh bundle map.

REMARK 4.1.56 (Mapping 1-realized bundles). We often think about the total space M of 1-mesh bundle (p, γ) being more or less identified with its embedded image $\gamma(M)$ under its 1-framed realization $\gamma : M \hookrightarrow B \times \mathbb{R}$. That convenient identification is compatible with bundle maps in the sense that every 1-mesh bundle map $F : (p, \gamma) \to (p', \gamma')$ induces a commutative diagram of continuous maps as follows:

$$\begin{array}{ccccc} \gamma(M) & \xleftarrow{\gamma} & M & \xrightarrow{p} & B \\ \downarrow{\scriptstyle \tilde{F}} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle G} \\ \gamma'(M') & \xleftarrow{\gamma'} & M' & \xrightarrow{p'} & B' \end{array}$$

A useful construction in the context of 1-mesh bundle maps is pullback bundles, as follows.

Construction 4.1.57 (Pullbacks of 1-mesh bundles). Given a 1-mesh bundle $p : (M, f) \to (B, g)$ with 1-framed realization $\gamma : M \hookrightarrow B \times \mathbb{R}$, and a stratified map $G : (B', g') \to (B, g)$, the 'pullback 1-mesh bundle' $(G^*p, G^*\gamma)$ is given as follows. The stratified bundle $G^*p : (G^*M, G^*f) \to (B', g')$ is the stratified pullback of the bundle p along the map G (see Definition B.2.27):

$$\begin{array}{ccc} (G^*M, G^*f) & \xrightarrow{\mathrm{Tot}G} & (M, f) \\ \downarrow{\scriptstyle G^*p} & \lrcorner & \downarrow{\scriptstyle p} \\ (B', g') & \xrightarrow{G} & (B, g) \end{array}$$

The 1-framed realization $G^*\gamma : G^*M \hookrightarrow B' \times \mathbb{R}$ for G^*p is defined by $(G^*\gamma)(x) := ((G^*p)(x), (\pi_{\mathbb{R}} \circ \gamma \circ \mathrm{Tot}G)(x)) \in B' \times \mathbb{R}$, where $\pi_{\mathbb{R}} : B \times \mathbb{R} \to \mathbb{R}$ is the projection.

Another construction of 1-mesh bundle maps is the inclusion of a 1-mesh bundle into its fiberwise compactification, as follows. (This canonical compactification will be particularly pertinent in our later comparison of meshes and trusses, allowing us to reduce certain statements to the case of closed bundles.)

Construction 4.1.58 (Fiberwise compactifications of 1-mesh bundles). Given a 1-mesh bundle $p : (M, f) \to (B, g)$ with 1-framed realization $\gamma : M \hookrightarrow B \times \mathbb{R}$, the 'fiberwise compactification' 1-mesh bundle $\overline{p} : (\overline{M}, \overline{f}) \to (B, g)$ is given as follows. Set the space $\overline{M}$ to be the closure of $\gamma(M)$ in $B \times \mathbb{R}$, with projection $\overline{p} : \overline{M} \to B$ being the restriction of $\pi : B \times \mathbb{R} \to B$ to $\overline{M}$. The stratification $(\overline{M}, \overline{f})$ has each stratum being either an image $\gamma(r)$ of a stratum $r \in f$ or an image $\gamma^{\pm}(s)$ of a stratum $s \in g$. Of course the 1-framed realization $\overline{\gamma} : \overline{M} \hookrightarrow B \times \mathbb{R}$ is simply the inclusion.

Example 4.1.59 (1-Mesh bundle compactification). In Figure 4.13, we depict the compactification $\overline{p}$ of a 1-mesh bundle p over the standard stratified 1-simplex.

Observation 4.1.60 (Pullbacks preserve fiberwise compactifications). Consider a 1-mesh bundle $p : (M, f) \to (B, g)$, together with a stratified map $G : (B', g') \to (B, g)$. The pullback $G^*\overline{p}$ of the fiberwise compactification of p is the fiberwise compactification $\overline{G^*p}$ of the pullback G^*p.

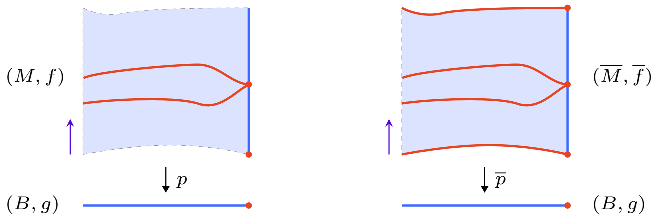


FIGURE 4.13. The compactification of a 1-mesh bundle.

4.1.2.5. ♦✠ Lifting regularity to total stratifications. When considering 1-mesh bundles, we usually assume that the base stratification satisfies some regularity properties. These regularity properties often 'lift' from the base stratification to the total stratification of the bundle, as we will discuss presently. These lifting properties are particularly useful when we begin to iterate 1-mesh bundles.

Eventually, we will be comparing iterated mesh bundles to the purely combinatorial iterated truss bundles; we cannot expect to combinatorialize mesh bundles unless the base stratification is itself combinatorializable. Recall from Section 1.3.1 that regular cell complexes (stratifications by open disks whose closures are closed disks) are faithfully modeled by their fundamental posets, and so are a suitable class of base stratifications. We broaden our attention to stratifications that refine to constructible substratifications (see Definition B.2.9) of regular cell complexes, as follows.

TERMINOLOGY 4.1.61 (Cellular stratifications, cf. Definition B.3.22). A 'cellular stratification' is a constructible substratification of a locally finite regular cell complex (stratified by its cells). ⌟

TERMINOLOGY 4.1.62 (Cellulable stratifications, cf. Definition B.3.20). A 'cellulable stratification' is a stratification that admits a refinement to a cellular stratification. ⌟

See Section B.3.3 for a more extensive discussion of cellular and cellulable stratifications.[21]

Cellulability provides a sufficiently broad class of sufficiently combinatorializable and sufficiently regular stratifications; this class admits the following lifting property, as desired.

PROPOSITION 4.1.63 (Cellulability lifts). *Let $p : (M, f) \to (B, g)$ be a 1-mesh bundle. If the base stratification (B, g) is cellulable then the total stratification (M, f) is as well.*

This property will follow from the liftability of cellularity, as follows.

[21]In fact, it is often convenient to restrict attention to PL cellular stratifications, i.e. constructible substratifications of locally finite regular cell complexes whose fundamental posets are PL cellular, see Definition 1.3.30. However, we will not insist on piecewise-linearity as a matter of course.

LEMMA 4.1.64 (Cellularity lifts). *Let $p : (M, f) \to (B, g)$ be a 1-mesh bundle. If the base stratification (B, g) is cellular then the total stratification (M, f) is too.*

The proof of this lemma will use the following technical observation.

OBSERVATION 4.1.65 (Cells bundled over cells). Let D^m be the closed m-disk, and S^{m-1} its boundary sphere. Let $p : X \to D^m$ be a subbundle of the projection $\pi : D^m \times \mathbb{R} \to D^m$, whose fibers over $x \in D^m$ are subsets of $\mathbb{R}$ of the form $[\gamma^-_x, \gamma^+_x]$, where $\gamma^\pm : D^m \to D^m \times \mathbb{R}$ are continuous sections of the form $\gamma^\pm(x) = (x, \gamma^\pm_x)$. If $\gamma^-_x < \gamma^+_x$ for all $x \in D^m$, except possibly when $x \in S^{m-1}$, then X is a closed $(m+1)$-disk. (Construct a bundle isomorphism to a compact convex set with non-empty interior, which is then necessarily a disk.) A similar observation holds in the PL case. ⌟

To prove Lemma 4.1.64, it will be convenient to use the correspondence, established later, of 1-mesh bundles and 1-truss bundles; specifically we perform constructions in terms of 1-truss bundles and translate them to 1-mesh bundles by realization.

PROOF OF LEMMA 4.1.64. Let the base stratification (B, g) be cellular. By definition, there is a constructible substratification inclusion $(B, g) \hookrightarrow X$ into some regular cell complex X. By removing cells in $X \setminus \overline{B}$, we may assume B is dense in X. Write $Y = \sqcap(B, g)$ for the fundamental poset of the stratification, and (abusing notation) $X = \sqcap X$ for the fundamental poset of the cell complex. Note that the stratified realization of the poset X recovers the original regular cell complex X; in particular that cell complex is the realization of a simplicial complex.

We now claim that we can choose the cell complex X such that for any $x \in X \setminus Y$ there exists a unique $y \in Y$ such that $y <^{\mathrm{cov}} x$ is a covering in $Y \cup \{x\}$ (see Notation B.1.33). Indeed, pick a cell $x \in X \setminus Y$ of highest dimension that fails the desired uniqueness, i.e. is covered by distinct $y, y' \in Y$; observe that $\dim(y) = \dim(y') = \dim(x) + 1$. Then 'blow-up' X at x, replacing x by the simplices in the boundary of its simplicial star in NX, and thereby constructing a new regular cell complex X' (that still contains B as a constructible substratification). This process does not create any new cells that fail the unique covering condition and are of dimension $\dim(x)$, and it does remove the cell x as such a failing cell; the claim follows by induction. (The blow-up process could instead be described using cellular stars, see Remark B.3.28.)

Consider a 1-mesh bundle $p : (M, f) \to (B, g)$, and take fundamental posets to form its fundamental 1-truss bundle $q : T \to Y$ (as described later in Construction 4.2.11). Pick a complex X satisfying the above unique covering condition. We can extend q to a 1-truss bundle $\tilde{q} : \tilde{T} \to X$ as follows. Define the fiber of $\tilde{q}$ over the unique arrow $y <^{\mathrm{cov}} x$ to be the identity $\mathrm{id}_{q^{-1}(y)}$; one checks this uniquely extends to a well-defined 1-truss bundle $\tilde{q}$. Take the mesh realization (as described later in Section 4.2.5.4), to form a 1-mesh bundle $\tilde{p} : (\tilde{M}, \tilde{f}) \to X$ such that $\tilde{p}|_B \cong p$. Furthermore forming the fiberwise compactification of the 1-mesh bundle $\tilde{p}$ constructs a closed 1-mesh bundle over X whose total stratification is a regular cell complex (by Observation 4.1.65). Since that compactified bundle contains p as a constructible substratification, the total stratification of p is cellular as required. □

Proof of Proposition 4.1.63. Suppose the base stratification (B, g) is cellulable; refine it by a cellular stratification $G : (B, c) \to (B, g)$. The pullback bundle $G^*p : (M, d) \to (B, c)$ has cellular total stratification (M, d) by Lemma 4.1.64. The coarsening $\mathrm{Tot}G : (M, d) \to (M, f)$ exhibits (M, f) as cellulable, as required. □

Though our standard regularity condition will be cellulability, we mention that various other regularity properties lift from the base to the total stratification of 1-mesh bundles, as follows.

Observation 4.1.66 (Finiteness and local finiteness lifts). Though we have assumed our stratifications are finite by convention, note that the definition of 1-mesh bundles generalizes to the setting of infinite stratifications. In that broader context, consider a 1-mesh bundle $p : (M, f) \to (B, g)$ in which the base stratification is finite or locally finite. It follows (because the fibers in 1-mesh bundles are finite stratifications) that the total stratification is, respectively, finite or locally finite as well. ⌟

Observation 4.1.67 (Frontier-constructibility lifts). Consider a 1-mesh bundle $p : (M, f) \to (B, g)$, and suppose the base (B, g) is frontier-constructible. It follows that the total stratification (M, f) is again frontier-constructible. ⌟

Observation 4.1.68 (Pairwise locally path-connectedness lifts). Consider a 1-mesh bundle $p : (M, f) \to (B, g)$, and suppose the base (B, g) is pairwise locally path-connected. Then the total stratification (M, f) is also pairwise locally path-connected. ⌟

Together, the preceding two observations imply that reasonable regularity lifts from the base to the total stratifications of 1-mesh bundles.

4.1.3. ♦n-Meshes and their bundles.

Synopsis. We define n-meshes as towers of 1-mesh bundles, and describe their realizations in the standard euclidean n-proframe. We then introduce, more generally, n-mesh bundles as towers over a stratified base space. Next we discuss maps of n-meshes and n-mesh bundles, and delineate notions of singular, regular, balanced, submesh, degeneracy, and coarsening such maps. Finally, we mention various categories and ∞-categories of n-meshes and their bundles.

4.1.3.1. ⧫The definition of n-meshes. As n-trusses were towers of 1-truss bundles, so n-meshes will be towers of 1-mesh bundles, as follows.

Definition 4.1.69 (n-Meshes). An **n-mesh** M is a sequence of 1-mesh bundles

$$(M_n, f_n) \xrightarrow{p_n} (M_{n-1}, f_{n-1}) \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_2} (M_1, f_1) \xrightarrow{p_1} (M_0, f_0) = *$$

in which the base stratification of each bundle p_i is the total stratification of the subsequent bundle p_{i-1}.

A 1-mesh comes equipped with a 1-framed realization, embedding the mesh in $\mathbb{R}$; as a byproduct of the 1-framed realizations of its constituent 1-mesh bundles, an n-mesh will have an 'n-framed realization', embedding the mesh in $\mathbb{R}^n$, as follows. Recall from Terminology 3.2.8 that the standard euclidean n-proframe $\mathcal{P}^n_{\mathbb{R}} = (\pi_n, \pi_{n-1}, \ldots, \pi_1)$ is the tower of projections $\pi_i : \mathbb{R}^i = \mathbb{R}^{i-1} \times \mathbb{R} \to \mathbb{R}^{i-1}$, forgetting the last coordinate.

Construction 4.1.70 (n-Framed realizations of n-meshes). Consider an n-mesh M, consisting of the 1-mesh bundles $p_i : (M_i, f_i) \to (M_{i-1}, f_{i-1})$ with 1-framed realizations $M_i \hookrightarrow M_{i-1} \times \mathbb{R}$. Define a map $\gamma = (\gamma_n, \gamma_{n-1}, \ldots, \gamma_0)$ of towers of spaces

$$\begin{array}{ccccccccc} M_n & \xrightarrow{p_n} & M_{n-1} & \xrightarrow{p_{n-1}} & \cdots & \xrightarrow{p_2} & M_1 & \xrightarrow{p_1} & M_0 = * \\ \downarrow{\scriptstyle\gamma_n} & & \downarrow{\scriptstyle\gamma_{n-1}} & & \cdots & & \downarrow{\scriptstyle\gamma_1} & & \downarrow{\scriptstyle\gamma_0} \\ \mathbb{R}^n & \xrightarrow[\pi_n]{} & \mathbb{R}^{n-1} & \xrightarrow[\pi_{n-1}]{} & \cdots & \xrightarrow[\pi_2]{} & \mathbb{R}^1 & \xrightarrow[\pi_1]{} & \mathbb{R}^0 \end{array}$$

by inductively setting γ_i to be the composite of the realization $M_i \hookrightarrow M_{i-1} \times \mathbb{R}$ with the product $\gamma_{i-1} \times \mathbb{R} : M_{i-1} \times \mathbb{R} \hookrightarrow \mathbb{R}^{i-1} \times \mathbb{R}$.

We refer to the embedding (of towers of spaces) $\gamma : M \hookrightarrow \mathcal{P}^n_{\mathbb{R}}$ as the 'n-framed realization', or simply 'n-realization', of the n-mesh M. Note that the n-realization γ is determined by its top component $\gamma_n : M_n \hookrightarrow \mathbb{R}^n$ and, abusing terminology, we may refer to that top embedding itself as the n-realization. ⌟

Terminology 4.1.71 (Support of n-realized meshes). Given an n-mesh M with n-realization γ, we refer to $\gamma_n(M_n) \subset \mathbb{R}^n$ as the 'support' of the (n-realized) mesh. ⌟

Given an n-mesh M with n-realization γ, note that the components γ_i of γ may either be considered as subspace embeddings $\gamma_i : M_i \hookrightarrow \mathbb{R}^i$ or as stratified maps $\gamma_i : (M_i, f_i) \to \mathbb{R}^i$; as stratified maps they are coarsenings onto their images.

Terminology 4.1.72 (Closed and open n-meshes). An n-mesh is called 'closed' or 'open' if each of the constituent 1-mesh bundles in its tower is closed or open, respectively, and is called 'mixed' if it is neither closed nor open. ⌟

Example 4.1.73 (2-Meshes). In Figure 4.14, we depict two 2-meshes via their 2-framed realizations in the standard proframe $\mathbb{R}^2 \to \mathbb{R}^1 \to \mathbb{R}^0$. The first 2-mesh (on the left) has half-open half-closed fibers in both the mesh bundles p_1 and p_2, and so is mixed. Note that the 2-truss, obtained as the tower of fundamental posets of this 2-mesh, was illustrated in Figure 2.42.

The second 2-mesh (on the right) has open fibers in both its bundles, and thus is an open 2-mesh. The fundamental 2-truss of this 2-mesh is illustrated later on the lower right of Figure 5.28. ⌟

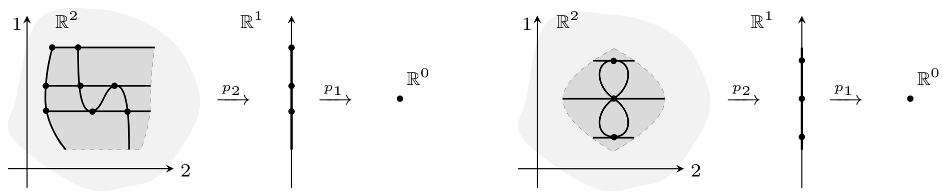


Figure 4.14. 2-Meshes and their framed realizations.

Example 4.1.74 (3-Meshes). Earlier in Figure 4.1, we depicted a 3-mesh via its realization in the standard proframe $\mathbb{R}^3 \to \mathbb{R}^2 \to \mathbb{R}^1$ (we usually omit the last projection $\mathbb{R}^1 \to \mathbb{R}^0$). In this mesh, the bundle p_1 is open, while the bundles p_2 and p_3 are both closed. The 3-truss arising as the fundamental poset tower of this 3-mesh is illustrated later in Figure 5.29. (That figure also indicates a subposet of the total poset of the 3-truss tracing out a combinatorial Dehn-twist, and depicts the corresponding geometric Dehn-twist stratification of the cylinder; the 3-mesh depicted here is in fact the coarsest mesh refining that geometric stratification, as illustrated on the left in Figure 5.18.)

In Figure 4.15, we depict another 3-mesh, again via its realization. Here, all three 1-mesh bundles p_1, p_2, and p_3 are open. The 3-truss given by the fundamental poset tower of this 3-mesh was illustrated at the beginning of this Chapter 4 and is illustrated again later in Figure 5.31. (That latter figure also indicates a subposet of the total poset delineating a combinatorial cusp configuration, and depicts, roughly speaking, a corresponding geometric cusp stratification; the 3-mesh depicted here is in fact the coarsest mesh refining a version of that geometric stratification. That refinement is illustrated, though in a partially compactified form, on the right in Figure 5.18. Notice that the top slice of that partially compactified 3-mesh is, up to frame reflection, the first 2-mesh from Figure 4.14.) ⌟

Observation 4.1.75 (n-Meshes are cellular). By Lemma 4.1.64, cellularity lifts along 1-mesh bundles; that property implies that for n-meshes M, each stratification (M_i, f_i) is cellular, and thus also conical (see Observation B.3.26). ⌟

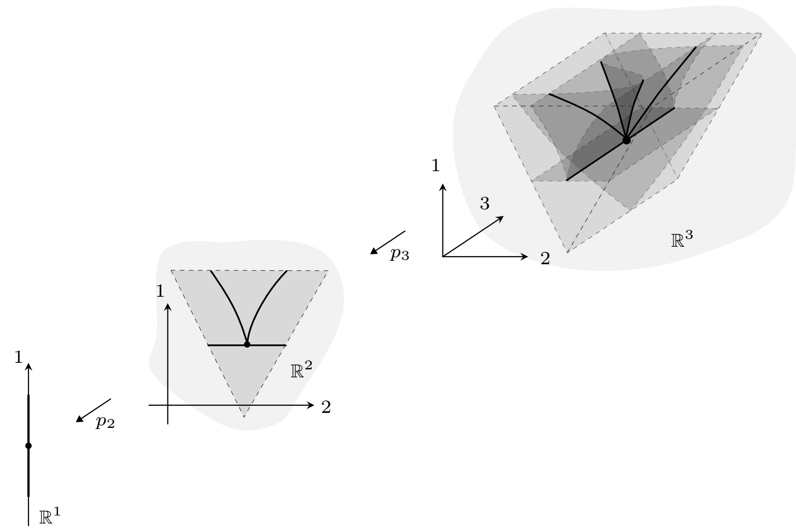


FIGURE 4.15. An open 3-mesh.

4.1.3.2. ⧫n-Mesh bundles. Of course we may consider suitably parametrized families of n-meshes, and those are most simply and conveniently encoded as towers of 1-mesh bundles over a nontrivial base stratification, as follows.

DEFINITION 4.1.76 (n-Mesh bundles). An **n-mesh bundle** over a stratification (B, g) is a sequence of 1-mesh bundles

$$(M_n, f_n) \xrightarrow{p_n} (M_{n-1}, f_{n-1}) \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_2} (M_1, f_1) \xrightarrow{p_1} (M_0, f_0) = (B, g)$$

in which the base stratification of each bundle is the total stratification of the next bundle.

We call a bundle 'closed' or 'open' if all its constituent 1-mesh bundles are, respectively.

CONSTRUCTION 4.1.77 (n-Framed realizations of bundles). Consider an n-mesh bundle $p = (p_n, p_{n-1}, \ldots, p_1)$ over a base stratification (B, g). Replacing, in Construction 4.1.70, the standard projections π_i by the products $B \times \pi_i$, define a map of towers

$$\begin{array}{ccccccccc} M_n & \xrightarrow{p_n} & M_{n-1} & \xrightarrow{p_{n-1}} & \cdots & \xrightarrow{p_2} & M_1 & \xrightarrow{p_1} & M_0 = B \\ \downarrow{\scriptstyle \gamma_n} & & \downarrow{\scriptstyle \gamma_{n-1}} & & \cdots & & \downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \mathrm{id}_B} \\ B \times \mathbb{R}^n & \xrightarrow[B \times \pi_n]{} & B \times \mathbb{R}^{n-1} & \xrightarrow[B \times \pi_{n-1}]{} & \cdots & \xrightarrow[B \times \pi_2]{} & B \times \mathbb{R}^1 & \xrightarrow[B \times \pi_1]{} & B \times \mathbb{R}^0 \end{array}$$

We refer to the map $\gamma = (\gamma_n, \gamma_{n-1}, \ldots, \gamma_0)$ as the 'n-framed realization' or 'n-realization' of the bundle; as before we also refer similarly just to the top map $\gamma_n : M_n \hookrightarrow B \times \mathbb{R}^n$. ⌟

Terminology 4.1.78 (Support of n-realized mesh bundles). The 'support' of a mesh bundle M is the image $\gamma_n(M_n) \subset B \times \mathbb{R}^n$ of its n-realization. ⌟

Recall from Remark 4.1.16 that there is a contractible space of suitable 1-realizations of a 1-mesh; the same applies to 1-mesh bundles. The situation for n-meshes and n-mesh bundles is similar, as follows.

Remark 4.1.79 (Contractible choice of equivalent n-realizations). As discussed in Remark 4.1.16, we could have defined 1-meshes to come equipped with a framed-homeomorphism class of 1-realizations, rather than a specific 1-realization, and such a class is contractible. By applying that shift in perspective to every fiber of all the 1-mesh bundles in the tower of an n-mesh bundle, Construction 4.1.77 produces, for any n-mesh, a contractible space of n-framed realizations, all of which are framed homeomorphic on every fiber. In fact, that contractible space is exactly the space of maps (of towers into the standard proframe) that are obtained from any given n-framed realization by post-composing with an n-framed homeomorphism of euclidean space; that notion of n-framed homeomorphism is made precise shortly in Definition 4.1.86. We thus may and will implicitly conceive of n-mesh realizations up to n-framed homeomorphism when convenient. ⌟

Terminology 4.1.80 (Truncations). Given an n-mesh bundle $p = (p_n, p_{n-1}, \ldots, p_1)$ over the base stratification (B, g), its (lower) 'k-truncation' $p_{\leq k}$ is the k-mesh bundle $(p_k, p_{k-1}, ..., p_1)$ over the same base, obtained by preserving only the k lowest 1-mesh bundles of the tower. ⌟

Recall from Remark 4.1.41 the notion of categorical 1-mesh bundle, in which the entrance path structure of the total stratification is allowed to depend on the base entrance path (and thus on the fundamental category, not just fundamental poset, of the base). The corresponding notion in the context of n-meshes is as follows.

Terminology 4.1.81 (Categorical n-mesh bundles). A 'categorical n-mesh bundle' p over a stratification (B, g) is a sequence of categorical 1-mesh bundles $(M_n, f_n) \xrightarrow{p_n} (M_{n-1}, f_{n-1}) \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_2} (M_1, f_1) \xrightarrow{p_1} (M_0, f_0) = (B, g)$. . ⌟

Remark 4.1.82 (Posetal refinements of categorical bundles). Every categorical n-mesh bundle over a posetal base stratification (B, g) is necessarily a posetal n-mesh bundle (see also Proposition 4.1.43). From this it follows that any categorical n-mesh bundle over a cellulable base stratification can be refined by a posetal mesh bundle: indeed, cellular stratifications are posetal (see Observation B.3.30), and the required refinement can be obtained by a pullback (see Construction 4.1.93). ⌟

Remark 4.1.83 (Unified classification of posetal and categorical bundles). We will see later that (posetal) n-mesh bundles over (sufficiently nice) base stratifications (B, g) are classified by functors $⊓g \to \mathsf{TBord}^n$ from the fundamental poset of the base to the classifying category of n-trusses and their bordisms. In the case of categorical mesh bundles the same classifying category applies (see also Remark 4.1.44), in that categorical n-mesh bundles are classified by ∞-functors $⊓_\infty(g) \to \mathsf{TBord}^n$, or equivalently by 1-categorical functors $⊓_1(g) \to \mathsf{TBord}^n$. ⌟

4.1.3.3. ♦Maps of n-meshes and their bundles. The notion of map of 1-mesh bundles from Definition 4.1.51 straightforwardly provides a notion of map of n-meshes and n-mesh bundles, as follows.

Definition 4.1.84 (Maps of n-mesh bundles). Consider an n-mesh bundle $p = (p_n, p_{n-1}, \ldots, p_1)$ over (B, g) and an n-mesh bundle $q = (q_n, q_{n-1}, \ldots, q_1)$ over (C, h). A **map of n-mesh bundles** $F : p \to q$ is a map of towers

$$\begin{array}{ccccccccc}
(M_n, f_n) & \xrightarrow{p_n} & (M_{n-1}, f_{n-1}) & \xrightarrow{p_{n-1}} & \cdots & \xrightarrow{p_2} & (M_1, f_1) & \xrightarrow{p_1} & (M_0, f_0) = (B, g) \\
\downarrow{\scriptstyle F_n} & & \downarrow{\scriptstyle F_{n-1}} & & \cdots & & \downarrow{\scriptstyle F_1} & & \downarrow{\scriptstyle F_0} \\
(N_n, g_n) & \xrightarrow{q_n} & (N_{n-1}, g_{n-1}) & \xrightarrow{q_{n-1}} & \cdots & \xrightarrow{q_2} & (N_1, g_1) & \xrightarrow{q_1} & (N_0, g_0) = (C, h)
\end{array}$$

where F_0 is a stratified map, and each F_i (for $i > 0$) is a 1-mesh bundle map $p_i \to q_i$. When the base stratifications are trivial, i.e. $(B, g) = (C, h) = *$, this definition provides a notion of **map of n-meshes**. ⌟

Example 4.1.85 (A 3-mesh map). In Figure 4.16, we depict a map of open 3-meshes, which is a substratification on each stage. The source 3-mesh was depicted earlier in Figure 2.41 along with its fundamental 3-truss. (That 3-truss is shown again later in Figure 5.30, with an indication of a subposet of the total poset tracing out a combinatorial braid. That same figure depicts, roughly speaking, a corresponding geometric braid stratification; the 3-mesh depicted here is in fact the coarsest mesh refining a version of that geometric stratification.)

Note that the target 3-mesh has as its 2-mesh truncation the one previously shown on the right in Figure 4.14. Though we do not anywhere illustrate the fundamental 3-truss of the target 3-mesh, the dual of that fundamental 3-truss appeared in Figure 3.2. ⌟

In the previous example, we implicitly illustrated a mesh map, via its realization, in terms of the effect on the mesh supports in the standard proframed euclidean space. The maps of subspaces of euclidean spaces that arise in this way are extremely constrained by respecting the proframe structure, and we describe them precisely as follows.

⋄ It would be good to have an example of a map that's not an embedding

Definition 4.1.86 (Framed maps of euclidean subspaces). For euclidean subspaces $Z \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^n$, an **n-framed map** is a map $F : Z \to W$ that, for every $0 \leq i < n$, descends along the standard projection $\pi_{>i} = \pi_{i+1} \circ \ldots \circ \pi_n : \mathbb{R}^n \to \mathbb{R}^i$, to a map $F_i : \pi_{>i}(Z) \to \pi_{>i}(W)$; i.e. the map F_i satisfies $F_i \circ \pi_{>i} = \pi_{>i} \circ F$.

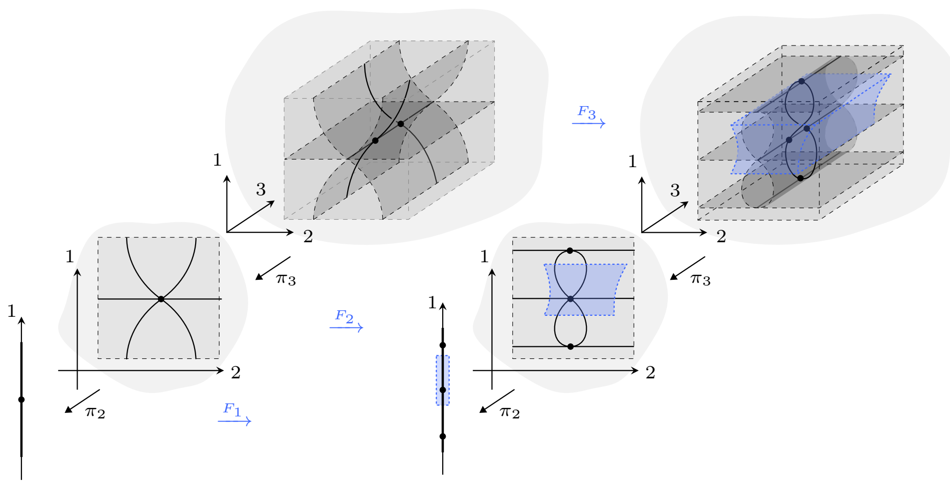


FIGURE 4.16. A 3-mesh map.

Similarly, given subspaces $Z \subset B \times \mathbb{R}^n$ and $W \subset C \times \mathbb{R}^n$ and a map $G : B \to C$, an **n-framed map over the base map** G is a map $F : Z \to W$ that, for every $0 \leq i < n$, descends to a map $F_i : (\mathrm{id}_B \times \pi_{>i})(Z) \to (\mathrm{id}_C \times \pi_{>i})(W)$, with $F_0 = G$; i.e. the map F_i satisfies $F_i \circ (\mathrm{id}_B \times \pi_{>i}) = (\mathrm{id}_C \times \pi_{>i}) \circ F$. ⌟

Note that the notion of an n-framed map $F : Z \to W$ may equivalently be specified by asking for there to exist a tower of maps $F_i : \pi_{>i}(Z) \to \pi_{>i}(W)$, with $F = F_n$, such that $F_{i-1} \circ \pi_i = \pi_i \circ F_i$.

TERMINOLOGY 4.1.87 (Framed maps for n-realized spaces and bundles). Given surjective maps $p : M \to B$ and $q : N \to C$ equipped with base-preserving 'n-realizations' $\gamma_p : M \hookrightarrow B \times \mathbb{R}^n$ and $\gamma_q : N \hookrightarrow C \times \mathbb{R}^n$, a 'framed bundle map' $F : M \to N$ is a bundle map (i.e., $q \circ F = G \circ p$ for some $G : B \to C$) that induces an n-framed map $F : \gamma_p(M) \to \gamma_q(N)$ over the base map G. When B and C are trivial, we simply speak of a 'framed map' $F : M \to N$. ⌟

OBSERVATION 4.1.88 (Framed maps and mesh bundle maps). Consider n-mesh bundles $p = \big((M_n, f_n) \to \cdots \to (M_0, f_0) = (B, d)\big)$ and $q = \big((N_n, g_n) \to \cdots \to (N_0, g_0) = (C, e)\big)$ with respective realizations $\gamma_M : M_n \hookrightarrow \mathbb{R}^n$ and $\gamma_N : N_n \hookrightarrow \mathbb{R}^n$.

(1) Any stratified map $F : \gamma_p(M_n, f_n) \to \gamma_q(N_n, g_n)$ that is framed over a base map F_0 in the sense of Definition 4.1.86 determines a mesh bundle map $\tilde{F} : p \to q$ (such that $\gamma_q \circ \tilde{F}_n = F \circ \gamma_q$ and $\tilde{F}_0 = F_0$).

(2) Conversely, given a mesh bundle map $F : p \to q$ the top component F_n is a framed bundle map in the sense of Terminology 4.1.87 on underlying spaces. Thus, F determines a framed stratified map $\tilde{F} : \gamma_p(M_n, f_n) \to \gamma_q(N_n, g_n)$ (such that $\gamma_q \circ F_n = \tilde{F} \circ \gamma_q$). ⌟

The above remark could probably be clearer. Option: both ways. Recharacterize mesh maps in terms of framed maps. Ie given mesh map, get a framed map that respects the stratification; given framed map that's stratified, We record the following obvious extensions of 1-mesh bundle terminology to the case of n-mesh bundle maps.

Terminology 4.1.89 (Singular, regular, and balanced maps of n-mesh bundles). We call an n-mesh bundle map F 'singular' or 'regular' or 'balanced' if all of its constituent 1-mesh bundle maps F_i (for $1 \leq i \leq n$) are respectively singular or regular or balanced, in the sense of Terminology 4.1.52, i.e. if every bundle map F_i satisfies the respective condition on every fiber. ⌟

Terminology 4.1.90 (Subbundles and submeshes). We call an n-mesh bundle map F a 'subbundle' if the map F_0 is a substratification and all the maps F_i (for $i > 0$) are fiberwise submesh inclusions of 1-meshes in the sense of Terminology 4.1.19. A subbundle map between n-meshes is called simply a 'submesh'. ⌟

Terminology 4.1.91 (Mesh degeneracies and coarsenings). An n-mesh bundle map F is called a 'mesh degeneracy' or a 'mesh coarsening' when all the constituent 1-mesh bundle maps F_i (for $i > 0$) are, on every fiber, 1-mesh degeneracies or 1-mesh coarsenings as defined in Terminology 4.1.20, and the stratified base map F_0 is of a corresponding designation (i.e. a quotient map or a homeomorphism on underlying spaces).[22] ⌟

Note that any n-mesh coarsening induces a tower of homeomorphisms of underlying spaces, and thus provides a tower of coarsenings of stratifications in the usual sense (see Definition B.2.4).

Terminology 4.1.92 (Base preserving maps). We call an n-mesh bundle map F 'over the base' (B, g) or 'base preserving' when the 0-stage map F_0 is an identity id_B. ⌟

We next generalize pullbacks of 1-mesh bundles (see Construction 4.1.57) to n-mesh bundles. Note that we may pullback not only along a map of base stratifications, but along a truncated mesh bundle map, as follows.

Construction 4.1.93 (Pullbacks of mesh bundles). Consider an n-mesh bundle $p = ((M_n, f_n) \xrightarrow{p_n} \cdots \xrightarrow{p_1} (M_0, f_0))$, and a stratification (N_0, g_0) or an i-mesh bundle $q = ((N_i, g_i) \xrightarrow{q_i} \cdots \xrightarrow{q_1} (N_0, g_0))$ for some fixed $0 < i < n$. Given an i-mesh bundle map $G : q \to p_{\leq i}$ from the bundle q to the truncation $p_{\leq i}$, apply inductively the pullback of 1-mesh bundles, from Construction 4.1.57, to obtain the tower of maps

$$\begin{array}{ccccccccccc}
(G^*M_n, G^*f_n) & \xrightarrow{G^*p_n} & \cdots & \xrightarrow{G^*p_{i+2}} & (G^*M_{i+1}, G^*f_{i+1}) & \xrightarrow{G^*p_{i+1}} & (N_i, g_i) & \xrightarrow{q_i} & \cdots & \xrightarrow{q_1} & (N_0, g_0) \\
{\scriptstyle \mathrm{Tot}^{n-i}G}\downarrow & \lrcorner & & \lrcorner & {\scriptstyle \mathrm{Tot}^1 G}\downarrow & \lrcorner & \downarrow{\scriptstyle G_i} & & & & \downarrow{\scriptstyle G_0} \\
(M_n, f_n) & \xrightarrow{p_n} & \cdots & \xrightarrow{p_{i+2}} & (M_{i+1}, f_{i+1}) & \xrightarrow{p_{i+1}} & (M_i, f_i) & \xrightarrow{p_i} & \cdots & \xrightarrow{p_1} & (M_0, f_0)
\end{array}$$

Here, for $1 \leq j \leq n - i$, the maps G^*p_{i+j} and $\mathrm{Tot}^j G$ are defined by the pullback of p_{i+j} along $\mathrm{Tot}^{j-1}G$ (with $\mathrm{Tot}^0 G = G_i$). We call the top tower

[22] Analogously to the truss case discussed in Terminology 2.3.67, we refer to 'mesh coarsenings' as 'mesh refinements', albeit with a grammatical contravariance.

the 'pullback n-mesh bundle' G^*p, and we call the vertical map of towers the 'pullback n-mesh bundle map' $\mathrm{Tot}G : G^*p \to p$. ⌟

TERMINOLOGY 4.1.94 (Bundle restrictions). In the special case of the previous construction where $i = 0$, i.e. the pullback is along simply a stratified map, and when $G_0 : (C,h) \equiv (N_0, g_0) \hookrightarrow (M_0, f_0) \equiv (B, g)$ is a substratification, we call the pullback $G^*p \to p$ the 'restriction of the n-mesh' along the base map $C \hookrightarrow B$, and we denote the restriction by $p|_C \hookrightarrow p$. Note that restricting an n-mesh bundle to a point in its base provides a notion of the 'fiber n-mesh' at that point. ⌟

4.1.3.4. ♦Categories of n-meshes and their bundles. Equipped with the notions of n-meshes, n-mesh bundles, and their maps, we can now introduce various categories of meshes and mesh bundles.

NOTATION 4.1.95 (Categories of n-meshes and n-mesh bundles). Using the previously defined notions of maps, we have the following categories:

Mesh_n	n-Meshes and their maps.
$\mathsf{MeshBun}_n$	n-Mesh bundles and their maps.
$\mathsf{Mesh}_n(B,g)$	n-Mesh bundles over (B,g) and their base-preserving maps.

In each case, the decoration $\mathring{\mathsf{M}}$ or $\bar{\mathsf{M}}$ will indicate the restriction to the open objects and regular maps, or closed objects and singular maps, respectively. ⌟

The set of mesh maps, between any two meshes, has a natural topology, and hence the category of meshes (or mesh bundles) is topologically enriched [**Kel82**], as follows.

CONVENTION 4.1.96 (∞-Categories modeled by Top-enriched categories). Unless otherwise indicated, we will use the term '∞-category' to refer to a Top-enriched category (see [**Lur09a**, §1]). In certain circumstances, for instance involving posets with the specialization topology, we also use $k\mathsf{Top}$-enriched categories (see Notation B.1.2) as a model of ∞-categories, but in those cases will refer to the enrichment specifically. We use the term 'quasicategory' to refer to a simplicial set with inner horn fillers (see [**Joy02**]).

The primary contravention of that convention is that we use the term 'fundamental ∞-category' to refer to a quasicategory. Note also that for suitable emphasis we let '∞-functor' refer either to a Top-enriched functor in the context of ∞-categories, to a $k\mathsf{Top}$-enriched functor in the context of $k\mathsf{Top}$-enriched categories, or to a simplicial map in the context of quasicategories. Of course, we may and sometimes will consider ordinary 1-categories as ∞-categories, by giving their hom sets the discrete topology, or as quasicategories, by taking their simplicial nerve. ⌟

NOTATION 4.1.97 (∞-Categories of n-mesh bundles). By topologizing the hom sets $\mathsf{Mesh}_n(M,N)$ as subspaces of the hom spaces $\mathrm{Map}(M_n, N_n) \in \mathsf{Top}$

(and similarly for mesh bundles and mesh bundles with a fixed base), we obtain the following ∞-categories:

$\mathcal{M}esh_n$	n-Meshes and their spaces of maps.
$\mathcal{M}eshBun_n$	n-Mesh bundles and their spaces of maps.
$\mathcal{M}esh_n(B,g)$	n-Mesh bundles over (B,g) and their spaces of base-preserving maps.

As before, in each case, the decoration $\mathring{\mathcal{M}}$ or $\bar{\mathcal{M}}$ will indicate the restriction to the (Top-enriched) subcategory of open objects and regular maps, or closed objects and singular maps, respectively. We further write $\mathcal{M}esh_n^{\mathrm{bal}}(B,g)$ for the wide subcategory of $\mathcal{M}esh_n(B,g)$ comprising only balanced mesh maps. ⌟

Remark 4.1.98 (Truncating is continuous). Truncating from an n-mesh to a k-mesh provides a Top-enriched functor $(-)_{\leq k} : \mathcal{M}esh_n(B,g) \to \mathcal{M}esh_k(B,g)$ of the ∞-categories of mesh bundles over the base (B,g). ⌟

Notation 4.1.99 (∞-Categories with degeneracies and coarsenings). By restricting the morphisms to be degeneracies or coarsenings, we obtain two wide sub-Top-categories of $\mathcal{M}esh_n$, namely the ∞-category $\mathcal{M}esh_n^{\mathrm{deg}}$ of n-meshes and degeneracies and the ∞-category $\mathcal{M}esh_n^{\mathrm{crs}}$ of n-meshes and coarsenings. ⌟

Recall from Remark 2.2.75 that there is a quasicategory $\mathcal{T}Bord^1$ of '1-trusses and their bordisms', which has 0-simplices being the 1-trusses, 1-simplices being the 1-truss bordisms, and more generally k-simplices being 1-truss bundles over the combinatorial k-simplex. There is an analogous quasicategory $\mathcal{M}Bord_1$ of '1-meshes and their bordisms', which has 0-simplices being the 1-meshes, 1-simplices being the 1-mesh bundles over the standard stratified 1-simplex, and k-simplices being 1-mesh bundles over the standard stratified k-simplex. The same scheme provides the following quasicategory in the n-mesh case.

Definition 4.1.100 (Quasicategories of n-meshes and their bordisms). The **quasicategory of n-meshes and their bordisms** $\mathcal{M}Bord_n$ has k-simplices being the n-mesh bundles over the stratified k-simplex $\|[k]\|$; simplicial maps $f : [k] \to [l]$ operate by bundle pullback along the stratified maps $\|f\| : \|[k]\| \to \|[l]\|$. ⌟

Echoing the truss terminology, we sometimes refer to the 1-simplices of $\mathcal{M}Bord_n$, i.e. the n-mesh bundles over the stratified 1-simplex, as 'n-mesh bordisms'. However, once we establish an equivalence between the quasicategory $\mathcal{M}Bord_n$ and the 1-category TBord^n, we will for the most part work with the latter truss category, and so mesh bordisms will not play a substantial direct role henceforth.

Remark 4.1.101 (Categorical mesh bordisms). Recall from Proposition 4.1.43 that categorical bundles over posets are posetal. As a consequence, if we replaced posetal mesh bundles by categorical mesh bundles in Definition 4.1.100, the resulting 'quasicategory of categorical n-meshes and their bordisms' would be identical to the quasicategory of n-meshes and their bordisms. This coincidence is also encoded in the fact that posetal and categorical mesh bundles will have the same classifying category. ⌟

4.2. ♦Weak equivalence of meshes and trusses

From the outset, the notion of 1-trusses was designed to model the fundamental posets of certain stratifications of intervals, namely 1-meshes, and the notion of 1-truss bundles was formulated to model the fundamental posets of suitable constructible bundles of stratified intervals, namely 1-mesh bundles; thus the whole theory of n-trusses is motivated, in retrospect, from the structures arising in the fundamental posets of n-meshes. It will come as no surprise, then, that the fundamental poset induces a *fundamental truss functor* $\sqcap_{\mathsf{T}}$ from meshes to trusses. An example of a mesh and its associated fundamental truss is illustrated in Figure 4.17. The total stratification of the 3-mesh is depicted on the left, and the total poset of the associated 3-truss is depicted on the right; the lower stages of the mesh and the truss are obtained by successively (from highest to lowest) projecting out the realization or frame vectors, respectively.

What is less immediate than the existence of a fundamental truss associated to a mesh, is that the fundamental truss is a faithful combinatorial encoding of the topological mesh. The encoding is faithful in the sense that meshes with isomorphic fundamental trusses are themselves isomorphic, and moreover mesh maps inducing the same fundamental truss map are themselves homotopic. Furthermore, every truss arises as the fundamental truss of some mesh. Indeed, at least for closed trusses, the stratified realization of the truss posets induces a *mesh realization functor* $\|-\|_{\mathsf{M}}$ from trusses to meshes. (And the fundamental truss of the mesh realization of a truss is isomorphic to the original truss.) Read now from right to left, Figure 4.17 also provides an example of a truss and (up to homotopic artistic license) its associated mesh realization. Notice that the two horizontal 2-disc strata of the 3-mesh are visible in the 3-truss as the 2-truss fibers over the two singular points of the projected 1-truss; similarly the central skew 2-disc stratum of the 3-mesh is visible in the 3-truss as the downward closure of the 1-truss fiber over the central singular point of the projected 2-truss.

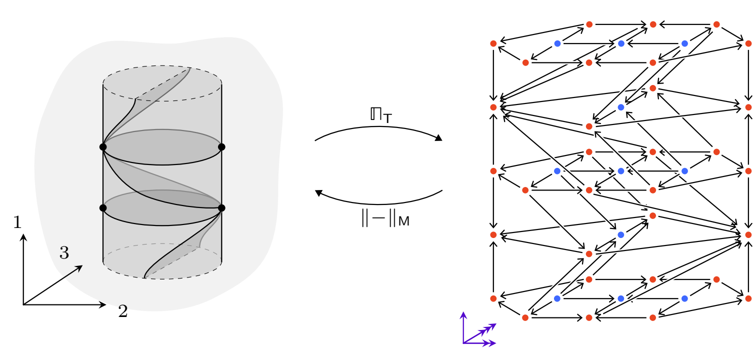


FIGURE 4.17. A corresponding 3-mesh and 3-truss.

Outline. In Section 4.2.1, we overview the equivalences between meshes and trusses, emphasizing the cases of closed meshes and trusses, of open meshes and trusses, and of mesh and truss bordisms. In Section 4.2.2, we construct the fundamental truss functor from meshes to trusses, first as an ordinary functor and then as an ∞-functor. Then in Section 4.2.3, we prove the fundamental truss functor is essentially injective, in the sense that meshes with isomorphic fundamental trusses are isomorphic. Furthermore in Section 4.2.4, we prove the fundamental truss ∞-functor is weakly faithful, in that the hom fibers of that functor are either empty or contractible. Next in Section 4.2.5, we construct the mesh realization functor from trusses to meshes, as a right inverse to the fundamental truss functor. Finally in Section 4.2.6, we assemble the proof of the equivalences of meshes and trusses, and present two applications, namely to the classification of framed subdivisions of framed cells and to the dualization equivalence of open and closed meshes.

4.2.1. ⧫Overview of the equivalences. We state and sketch the context and relationships among several incarnations of equivalences between meshes and trusses, specifically for closed meshes and trusses, open meshes and trusses, closed or open mesh bundles and truss bundles, suitably enriched categories of general mesh bundles and truss bundles, and quasicategories of mesh bordisms and truss bordisms. We then preview two applications of these equivalences, namely to a classification of framed subdivisions of framed regular cells by truss subdivisions of truss blocks, and to a dualization equivalence between closed and open meshes.

The equivalences between meshes and trusses will be witnessed by a 'fundamental truss functor' $\sqcap_{\mathsf{T}}$ and conversely by a 'mesh realization functor' $\|-\|_{\mathsf{M}}$. As the names suggest, the former is a variation of the fundamental poset functor, and the latter is a variation of the stratified realization functor.[23]

The *fundamental truss functor* will take an n-mesh M, given by a tower of 1-mesh bundles, to an n-truss $\sqcap_{\mathsf{T}} M$, given by a tower of 1-truss bundles, defined by applying the fundamental poset functor to the mesh tower. The *mesh realization functor* will take an n-truss T, given by a tower of 1-truss bundles, to an n-mesh $\|T\|_{\mathsf{M}}$, given by a tower of 1-mesh bundles, defined (roughly speaking) by applying the stratified realization functor to the truss tower. For both the fundamental truss and mesh realization functors, the 1-mesh structure of every fiber in the mesh tower induces or is induced by the 1-truss structure of every fiber in the truss tower.

[23] In general the mesh realization will not strictly preserve identities, and so will be just a semifunctor, but we suppress that subtlety in our overview.

Theorem 4.2.1 (Weak equivalence of meshes and trusses). *The fundamental truss and mesh realization functors provide weak equivalences*

$$\bar{\mathcal{M}}esh_n \underset{\|-\|_{\mathsf{M}}}{\overset{\sqcap_{\mathsf{T}}}{\rightleftarrows}} \bar{\mathsf{T}}\mathsf{rs}_n \qquad \mathring{\mathcal{M}}esh_n \underset{\|-\|_{\mathsf{M}}}{\overset{\sqcap_{\mathsf{T}}}{\rightleftarrows}} \mathring{\mathsf{T}}\mathsf{rs}_n$$

between the ∞-category of closed n-meshes and the 1-category of closed n-trusses, and correspondingly for open meshes and open trusses.

That core theorem, worth demarcating, is simply the specialization to the trivial base of the corresponding statement for mesh and truss bundles, as follows.

Theorem 4.2.2 (Weak equivalence of mesh and truss bundles). *Given a cellulable stratification (B, g), the fundamental truss and mesh realization functors provide weak equivalences*

$$\bar{\mathcal{M}}esh_n(B,g) \underset{\|-\|_{\mathsf{M}}}{\overset{\sqcap_{\mathsf{T}}}{\rightleftarrows}} \bar{\mathsf{T}}\mathsf{rs}_n(\sqcap g) \qquad \mathring{\mathcal{M}}esh_n(B,g) \underset{\|-\|_{\mathsf{M}}}{\overset{\sqcap_{\mathsf{T}}}{\rightleftarrows}} \mathring{\mathsf{T}}\mathsf{rs}_n(\sqcap g)$$

between the ∞-category of closed n-mesh bundles over the base stratification (B, g) and the 1-category of closed n-truss bundles over the base poset $\sqcap g$, and correspondingly for open mesh bundles and open truss bundles.

These results are established, following the development of the necessary tooling, in Section 4.2.6.

Remark 4.2.3 (Equivalence of categorical bundles). The preceding result generalizes to an equivalence between the ∞-category of closed (or open) categorical n-mesh bundles (see Terminology 4.1.81) and the 1-category of closed (or open) categorical n-truss bundles (see Remark 2.3.53). Thereby, a categorical n-mesh bundle over a stratification (B, g) corresponds to a categorical n-truss bundle over the fundamental category $\sqcap_1(B, g)$. (The proofs given in the posetal case will carry over to the categorical case, keeping in mind that certain key steps presume the base stratification is cellular, therefore 0-truncated, in which case the notions of categorical and posetal mesh bundles coincide, and the fundamental category and fundamental poset are identical.) ⌟

The above theorems restrict attention to closed or open meshes and trusses, and thereby avoid complications arising from non-invertible higher morphisms in the case of mixed meshes and trusses; those complications may be encoded obliquely in an enriched version of the fundamental truss functor, as follows.

Remark 4.2.4 (Enriched fundamental truss functor). Recall from Remark 2.3.39 that natural transformations between truss bundle maps provide a poset structure on the hom sets in the category of truss bundles, and passing to the specialization topology yields the $k\mathsf{Top}$-enriched category $\mathcal{T}rs_n(X)$

of n-truss bundles over the base poset X. The (mixed) fundamental truss functor $\mathbb{\Pi}_{\mathsf{T}}$ will be suitably continuous, giving an ∞-functor on all meshes:

$$\mathcal{M}esh_n(B,g) \underset{\|-\|_{\mathsf{M}}}{\overset{\mathbb{\Pi}_{\mathsf{T}}}{\rightleftarrows}} \mathcal{T}rs_n(\mathbb{\Pi}g)$$

This functor will be recorded in Proposition 4.2.16. However, the (mixed) mesh realization functor $\|-\|_{\mathsf{M}} : \mathsf{Trs}_n(\mathbb{\Pi}g) \to \mathcal{M}esh_n(B,g)$ (which will be given in Notation 4.2.60 and Construction 4.2.73) is not suitably continuous, and so does not provide an ∞-functor, as indicated (see Observation 4.2.61).[24] ⌟

Complementary to the above primary results relating meshes and their maps to trusses and their maps, there is a relationship of meshes and their bordisms (see Definition 4.1.100) to trusses and their bordisms (see Notations 2.3.20 and 2.3.24 and Lemma 2.3.25), as follows.

THEOREM 4.2.5 (Weak equivalence of mesh and truss bordisms). *The fundamental truss functor induces a trivial fibration of quasicategories*

$$\mathcal{M}Bord_n \xrightarrow{\mathbb{\Pi}_{\mathsf{T}}} \mathsf{TBord}^n$$

and thus provides an equivalence between the quasicategory of n-meshes and their bordisms and the 1-category of n-trusses and their bordisms.

Here we regard the 1-category TBord^n of trusses and their bordisms as a quasicategory by implicitly taking its simplicial nerve. The indicated fibration of quasicategories is given on k-simplices by the fundamental truss functor on mesh bundles over the stratified k-simplex. Crucially, this result concerns only mesh and truss *bordisms* and so is unaffected by the non-invertible higher morphisms that arise in considering mesh and truss *maps* (see Footnote 24). This result is established after the mesh and truss map equivalences in Section 4.2.6.

REMARK 4.2.6 (Mapping cylinders as mesh bordisms). Recall from Construction 2.1.68 that to certain truss maps, there were associated 'mapping cylinder' truss bordisms. The above results and discussion combine to provide a relationship between mesh maps and mesh bordisms, parallel to that earlier relationship between truss maps and truss bordisms. To an appropriate mesh

[24]The mixed fundamental truss ∞-functor is nevertheless a weak equivalence in an appropriate sense. That sense, accounting for *non-invertible* higher morphisms, is $(\infty,2)$-categorical: the category of stratified spaces, and therefore meshes, is secretly $(\infty,2)$-categorical, and the category of posets, and therefore trusses, is not-so-secretly 2-categorical. Indeed, the $k\mathsf{Top}$-enrichment (crucially not a Top-enrichment) of the category of posets is a topological simulacrum of the presence of non-invertible 2-morphisms—*entrance* paths in non-Hausdorff spaces need not be invertible. To avoid the technicalities and diversions of $(\infty,2)$-categories, we must restrict attention to the rigid cases of open or closed meshes and trusses.

map, we expect to be able to form the geometric mapping cylinder to obtain a mesh bordism; formally, we combine the homotopy coherent nerve N^{hc} (see [**Qui06**, §II.3] and [**Joy07**]) of the fundamental truss ∞-functor (from Remark 4.2.4), with the nerve of the truss mapping cylinder, and the inverse of the bordism fundamental truss equivalence (from Theorem 4.2.5), to obtain the 'mesh mapping cylinder' $\mathcal{C}yl$, as follows:

$$\begin{array}{ccc} N\mathsf{Trs}_1^{\mathsf{s},\partial} & \xrightarrow{N\mathrm{Cyl}} & N\mathsf{TBord}^1 \\ {\scriptstyle N^{\mathrm{hc}}\sqcap_{\mathsf{T}}}\uparrow & & \uparrow{\scriptstyle \sqcap_{\mathsf{T}}}\,\sim \\ N^{\mathrm{hc}}\mathcal{M}esh_1^{\mathsf{s},\partial} & \overset{\mathcal{C}yl}{\dashrightarrow} & \mathcal{M}Bord_1 \end{array}$$

Here, $\mathcal{M}esh_1^{\mathsf{s},\partial}$ denotes the wide subcategory of $\mathcal{M}esh_1$ whose morphisms are the singular maps that preserve singular endpoints. (That the functor $N^{\mathrm{hc}}\sqcap_{\mathsf{T}}$ restricted to $\mathcal{M}esh_1^{\mathsf{s},\partial}$ indeed lands in $N\mathsf{Trs}_1^{\mathsf{s},\partial}$ requires consideration, cf. Lemma 2.3.71 and Proposition 4.2.18 and its corollaries.) A similar diagram constructs the 'mesh mapping cocylinder' functor $co\mathcal{C}yl : N^{\mathrm{hc}}(\mathcal{M}esh_1^{\mathsf{r},\partial})^{\mathrm{op}} \to \mathcal{M}Bord_1$. ⌟

We conclude by previewing two applications of the equivalences of meshes and trusses, namely to the classification of subdivisions of framed cells, and to the dualization of meshes.

Composing the weak equivalence of closed meshes and closed trusses from Theorem 4.2.1, with the equivalence of closed trusses and collapsible framed regular cell complexes from Theorem 3.1.2, we obtain the composite weak equivalence

$$\bar{\mathcal{M}}esh_n \underset{\|-\|_{\mathsf{M}}}{\overset{\sqcap_{\mathsf{T}}}{\rightleftarrows}} \bar{\mathsf{T}}\mathsf{rs}_n \underset{\int_{\mathsf{T}}}{\overset{\nabla_{\mathsf{C}}}{\rightleftarrows}} \mathsf{CollFrCellCplx}_n \qquad \left(\nabla_{\mathsf{C}} \circ \sqcap_{\mathsf{T}},\ \|-\|_{\mathsf{M}} \circ \textstyle\int_{\mathsf{T}}\right)$$

Terminology 4.2.7 (Mesh-to-cell gradient and cell-to-mesh realization). We denote the composite functors in the above equivalence by

$$\nabla_{\mathsf{MC}} := \nabla_{\mathsf{C}} \circ \sqcap_{\mathsf{T}}$$
$$\|-\|_{\mathsf{CM}} := \|-\|_{\mathsf{M}} \circ \textstyle\int_{\mathsf{T}}$$

and call them the 'mesh-to-cell (gradient) functor' and the 'cell-to-mesh (realization) functor', respectively. ⌟

◇ Transport this arrow fix to other similar asymmetric curved arrows.

◇ The label positioning in the above display isn't great, but small tweaking does not fix it.

We can leverage the cell-to-mesh realization functor to provide a notion of framed subdivision of a framed regular cell: a framed cell complex framed subdivides a framed cell when equipped with a stratified coarsening between their cell-to-mesh realizations, which on every cell of the complex is a map of meshes (see Definition 4.2.87). Quite unlike non-framed subdivisions, these framed subdivisions are combinatorially classifiable, as follows.

THEOREM 4.2.8 (Classifying subdivisions of framed cells). *A framed regular cell complex Y framed subdivides a framed regular cell X exactly when the framed complex Y is the cell gradient $\nabla_{\mathsf{C}} T$ of a truss T that combinatorially subdivides a truss block B whose cell gradient $\nabla_{\mathsf{C}} B$ is the framed cell X.*

This result is explained, illustrated, and established in Section 4.2.6.1.

Finally, the duality of closed and open trusses may be transported across the equivalence of meshes and trusses to provide a duality of closed and open meshes.

COROLLARY 4.2.9 (Dualization of meshes). *There is a dualization weak equivalence between the ∞-categories of closed n-meshes and open n-meshes:*

$$\dagger : \bar{\mathcal{M}}esh_n \simeq \mathring{\mathcal{M}}esh_n : \dagger \quad .$$

This result is established and discussed in Section 4.2.6.2. Notice that framed regular cell complexes provided a target context for topological realization of *closed* trusses, but there was no evident corresponding cell-like topological realization of *open* trusses. The constructible stratified framework of meshes, by contrast, conveniently accommodates realizations of both closed and open trusses and of course the duality between them.

4.2.2. ♦Fundamental trusses. We will now construct the fundamental truss functors from various categories of n-meshes to corresponding categories of n-trusses. We first address the foundational case of the functor of 1-categories

$$\sqcap_{\mathsf{T}} : \mathsf{Mesh}_n(B, g) \to \mathsf{Trs}_n(\sqcap g).$$

We then observe that that functor is suitably continuous on hom spaces, and so provides an ∞-functor

$$\sqcap_{\mathsf{T}} : \mathcal{M}esh_n(B, g) \to \mathcal{T}rs_n(\sqcap g).$$

Of course, the functor $\mathcal{T}rs_n(\sqcap g) \to \mathsf{Trs}_n(\sqcap g)$ (from the $k\mathsf{Top}$-enriched category to the discrete category) is not continuous on hom spaces, but it is continuous when restricted either to the subcategory of closed trusses and singular maps, or to the subcategory of open trusses and regular maps. We will therefore obtain, as composites, fundamental truss ∞-functors, from the ∞-category of (closed or open) meshes to the discrete category of (closed or open) trusses:

$$\sqcap_{\mathsf{T}} : \bar{\mathcal{M}}esh_n(B, g) \to \bar{\mathsf{Trs}}_n(\sqcap g)$$
$$\sqcap_{\mathsf{T}} : \mathring{\mathcal{M}}esh_n(B, g) \to \mathring{\mathsf{Trs}}_n(\sqcap g).$$

The construction of all these fundamental truss functors is mainly straightforward—take the fundamental poset of each stage—but we will need to check that the resulting towers of posets indeed satisfy the conditions for being trusses, truss bordisms, and truss bundles accordingly.

The following examples, collected from past figures, serve as an informal visual guide to the correspondence of meshes and their fundamental trusses, and can be kept in mind during the detailed constructions and arguments to follow.

Example 4.2.10 (Fundamental trusses and truss bundles). Recall from Figure 4.2 the three linear 1-meshes in the middle and the two trivial 1-meshes on the left that are embedded in $\mathbb{R}^1$; the corresponding fundamental 1-trusses are the three linear 1-trusses in the middle and the two trivial 1-trusses on the left of Figure 2.4. Further recall from Figure 4.3 the various types of maps of 1-meshes; for the singular, regular, and balanced cases, the induced maps of fundamental 1-trusses are those shown in Figure 2.6.

Next recall from Figure 4.4 a 1-mesh bordism (i.e. a 1-mesh bundle over the stratified 1-simplex); the corresponding fundamental 1-truss bordism was depicted in Figure 2.9. The six local forms of 1-mesh bordisms depicted in Figure 4.7 have as corresponding fundamental 1-truss bordisms the six local forms in Figure 2.12. More generally, recall from Figure 4.5 the 1-mesh bundle over the stratified realization of a poset; the corresponding fundamental 1-truss bundle was shown in Figure 2.19.

Furthermore, recall from Figure 2.42 both a 2-mesh and its corresponding fundamental 2-truss, and finally, recall from Figure 2.41 both a 3-mesh and its corresponding fundamental 3-truss. ⌟

Synopsis. We construct the fundamental n-truss bundle associated to an n-mesh bundle, and observe that association provides a functor from the 1-category of mesh bundles to the 1-category of truss bundles. We then show that functor is continuous with respect to a topological enrichment, and, after restriction to closed or open meshes, yields an ∞-functor from the ∞-category of meshes to the discrete category of trusses.

4.2.2.1. ♦Fundamental trusses as an ordinary functor. We detail the construction of the fundamental truss functor as an ordinary functor, from the 1-category of mesh bundles to the 1-category of truss bundles.

Construction 4.2.11 (Fundamental 1-truss bundles). Given a 1-mesh bundle $p : (M, f) \to (B, g)$, we will equip the fundamental poset map $\sqcap p : \sqcap f \to \sqcap g$ with the structure of a 1-truss bundle, yielding the 'fundamental 1-truss bundle' $\sqcap_{\mathsf{T}}(p)$ of the 1-mesh bundle p.

We first describe the 1-truss structure on the fibers of the poset map $\sqcap p$. Trivialize the 1-mesh bundle p over a base stratum s, by an isomorphism $p^{-1}(s) \cong s \times \mathsf{fib}(s)$; here $\mathsf{fib}(s)$ is the fiber 1-mesh over the stratum s, see Observation 4.1.34. Note that the fiber of the fundamental poset map over the stratum s is simply, and canonically, the fundamental poset of the fiber 1-mesh: $(\sqcap p)^{-1}(s) \cong \sqcap(p^{-1}(s)) \cong \sqcap(\mathsf{fib}(s))$. Equip this fiber $(\sqcap p)^{-1}(s)$ with a frame order $\preceq$ that orders strata according to the 1-framing of the 1-mesh $\mathsf{fib}(s)$, and equip the fiber with a dimension map $\dim : (\sqcap p)^{-1}(s) \to [1]^{\mathrm{op}}$ that sends each element $t \in (\sqcap p)^{-1}(s)$ to the fiber dimension $\mathrm{fibdim}(\tilde{t})$ of the corresponding fiber stratum $\tilde{t} \in \mathsf{fib}(s)$. We subsequently will tend to elide, even notationally, the distinction between a poset element $t \in (\sqcap p)^{-1}(s)$ and its corresponding stratum $t \equiv \tilde{t} \in \mathsf{fib}(s)$.

We next and finally check that the poset map $\sqcap p$, now with its 1-truss point fibers, in fact has 1-truss bordism fibers over arrows of the base poset. Specifically, given an entrance path $r \to s$ in the base poset $\sqcap g$, we need that the functorial relation $R = (\sqcap p)^{-1}(r \to s) : (\sqcap p)^{-1}(r) \nrightarrow (\sqcap p)^{-1}(s)$ is a 1-truss bordism. Constructibility of the mesh bundle constrains this relation as follows. Let $t \in (\sqcap p)^{-1}(r)$ be an element in the generic fiber. When t is singular, by constructibility there is a unique element $u \in (\sqcap p)^{-1}(s)$ of the special fiber, such that $R(t, u)$ holds, and u is moreover singular. Otherwise for t a regular element of the general fiber and u an element of the special fiber, the relation $R(t, u)$ holds exactly when the following two implications both hold: first, when the stratum t is bounded above by a singular stratum t^+, then there is a frame order relation $u \preceq u^+$ in the special fiber, for a stratum u^+ with relation $R(t^+, u^+)$; second, when the stratum t is bounded below by a singular stratum t^-, then there is a frame order relation $u^- \preceq u$ in the special fiber, for a stratum u^- with relation $R(t^-, u^-)$. (If the stratum t is bounded neither above nor below, then it is in fact related to every element of the special fiber.) Altogether, this relation is specified as in the construction of singular-determined 1-truss bordisms in Lemma 2.1.63, and in particular is indeed a 1-truss bordism as required. ⌟

Definition 4.2.12 (Fundamental truss bundles). For an n-mesh bundle p over the stratification (B, g), given by the sequence of 1-mesh bundles

$$(M_n, f_n) \xrightarrow{p_n} (M_{n-1}, f_{n-1}) \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_2} (M_1, f_1) \xrightarrow{p_1} (M_0, f_0) = (B, g),$$

its **fundamental truss bundle** $\sqcap_{\mathsf{T}}(p)$ is the n-truss bundle over $\sqcap g$, given by the sequence of 1-truss bundles

$$\sqcap(f_n) \xrightarrow{\sqcap_{\mathsf{T}}(p_n)} \sqcap(f_{n-1}) \xrightarrow{\sqcap_{\mathsf{T}}(p_{n-1})} \cdots \xrightarrow{\sqcap_{\mathsf{T}}(p_2)} \sqcap(f_1) \xrightarrow{\sqcap_{\mathsf{T}}(p_1)} \sqcap(f_0) = \sqcap(g),$$

where each $\sqcap_{\mathsf{T}}(p_i)$ is the fundamental 1-truss bundle of the 1-mesh bundle p_i. ⌟

Definition 4.2.13 (Fundamental truss bundle maps). For a mesh bundle map $F : p \to q$ between an n-mesh bundle p over (B, g) and an n-mesh bundle q over (C, h), with components $F_i : (M_i, f_i) \to (N_i, g_i)$, the **fundamental truss bundle map** $\sqcap_{\mathsf{T}} F : \sqcap_{\mathsf{T}}(p) \to \sqcap_{\mathsf{T}}(q)$ is the truss bundle map with components $(\sqcap_{\mathsf{T}} F)_i = \sqcap(F_i) : \sqcap(f_i) \to \sqcap(g_i)$. ⌟

Notation 4.2.14 (The fundamental truss functor). The previous two definitions together give the fundamental truss functor $\sqcap_{\mathsf{T}} : \mathsf{MeshBun}_n \to \mathsf{TrsBun}_n$ from mesh bundles to truss bundles. That functor restricts to the fundamental truss functor $\sqcap_{\mathsf{T}} : \mathsf{Mesh}_n(B, g) \to \mathsf{Trs}_n(\sqcap g)$ from mesh bundles over the stratification (B, g) to truss bundles over the poset $\sqcap g$. ⌟

Observation 4.2.15 (The fundamental truss functor preserves pullbacks). Recall from Construction 4.1.93 and Construction 2.3.54 the notions of pullback of n-mesh bundles and pullback of n-truss bundles respectively. The fundamental truss functor preserves pullbacks in the sense that the fundamental truss of the pullback is the pullback of the fundamental truss: $\sqcap_{\mathsf{T}}(G^* p) = (\sqcap G)^* \sqcap_{\mathsf{T}}(p)$; here p is an n-mesh bundle over (B, g), and $G : (C, f) \to (B, g)$ is a map of stratifications. ⌟

4.2.2.2. ♦Fundamental trusses as an ∞-functor. We now upgrade the fundamental truss functor to an ∞-functor, with target either the $k\mathsf{Top}$-enriched category of trusses or, when restricted to closed or open mesh bundles, the corresponding discrete categories of trusses.

Recall from Notation 4.1.97 that $\mathcal{M}esh_n(B, g)$ denotes the Top-enriched category of n-mesh bundles over the stratification (B, g), with the hom sets topologized as subspaces of the literal mapping spaces of the total spaces of the mesh bundle towers. Recall further from Remark 2.3.39 that $\mathcal{T}rs_n(X)$ denotes the $k\mathsf{Top}$-enriched category of n-truss bundles over the poset X, with the hom sets given the specialization topology of the poset of truss bundle maps and natural transformations of their total posets.

The fundamental truss provides an ∞-functor (between $k\mathsf{Top}$-enriched categories) as follows.

Proposition 4.2.16 (The fundamental truss as an enriched functor). *The fundamental truss induces an ∞-functor $\sqcap_{\mathsf{T}} : \mathcal{M}esh_n(B, g) \to \mathcal{T}rs_n(\sqcap g)$.*

Proof. One needs to check that the hom space map $\sqcap_{\mathsf{T}} : \mathcal{M}esh_n(B, g)(M, N) \to \mathcal{T}rs_n(\sqcap g)(\sqcap_{\mathsf{T}} M, \sqcap_{\mathsf{T}} N)$ is continuous. That follows from the proof of the continuity on hom spaces of the functor $\sqcap : \mathcal{S}\mathsf{trat}_{\mathrm{lf}} \to \mathcal{P}os_{\mathrm{lf}}$, detailed in Construction B.2.21. (Note as in Remark B.2.20 that the $k\mathsf{Top}$-enrichment of $\mathcal{P}os_{\mathrm{lf}}$ is indeed the specialization topology on poset maps and natural transformations.) □

Remark 4.2.17 (Non-invertible 2-morphisms of trusses and meshes). As mentioned, the preceding functor is only $k\mathsf{Top}$-enriched and not Top-enriched, since the hom spaces in the category $\mathcal{T}rs_n$ (and similarly $\mathcal{T}rs_n(X)$) are not weak Hausdorff. These hom $k\mathsf{Top}$ spaces, qua spaces, do not accurately represent the higher categorical structure of trusses: natural transformations of maps of trusses provide in general *non-invertible* 2-morphisms, which are not especially well represented in the associated specialization topology.

Though we introduced $\mathcal{M}esh_n$ (and similarly $\mathcal{M}esh_n(B, g)$) as an ∞-category (i.e. $(\infty, 1)$-category), in fact entrance path deformations between stratified maps provide non-invertible 2-morphisms of meshes, as well.[25] Thus, while one could show that the functor $\sqcap_{\mathsf{T}}$ in Proposition 4.2.16 is a weak equivalence in some appropriate sense, a more principled approach would be

[25]Similarly, entrance path deformations provide higher morphisms for stratifications more generally; see Remark B.3.19.

to establish an equivalence $\mathcal{M}esh_n \simeq \mathcal{T}rs_n$ of $(\infty, 2)$-categories; we quite forgo that here.

We are at luxury to defer an $(\infty, 2)$-categorical treatment because, in the cases of our primary attention, namely closed trusses and singular maps or open trusses and regular maps, non-invertible 2-morphisms are conspicuously absent; see the next proposition. ⌟

Proposition 4.2.18 (Rigidity of closed or open truss bundles). *The subspace of singular maps in any hom space between closed truss bundles, in the $k\mathsf{Top}$-enriched category $\mathcal{T}rs_n(X)$, is discrete. Similarly, the subspace of regular maps in any hom space between open truss bundles, again in the $k\mathsf{Top}$-enriched category $\mathcal{T}rs_n(X)$, is discrete.*

Proof. These statements follow from the fact, established in Lemma 2.3.72, that singular maps of closed truss bundles, similarly regular maps of open truss bundles, over a fixed base poset, do not admit any non-trivial natural transformations. □

Corollary 4.2.19 (Rigidity of the fundamental truss for closed or open mesh bundles). *The fundamental truss functor restricts to ∞-functors $\mathbb{\Pi}_{\mathsf{T}} : \bar{\mathcal{M}}esh_n(B, g) \to \bar{\mathsf{Trs}}_n(\mathbb{\Pi} g)$ and $\mathbb{\Pi}_{\mathsf{T}} : \mathring{\mathcal{M}}esh_n(B, g) \to \mathring{\mathsf{Trs}}_n(\mathbb{\Pi} g)$.*

Corollary 4.2.20 (Rigidity of the fundamental truss for closed or open meshes). *The fundamental truss functor restricts to ∞-functors $\mathbb{\Pi}_{\mathsf{T}} : \bar{\mathcal{M}}esh_n \to \bar{\mathsf{Trs}}_n$ and $\mathbb{\Pi}_{\mathsf{T}} : \mathring{\mathcal{M}}esh_n \to \mathring{\mathsf{Trs}}_n$.*

Lemma 2.3.72 suggests two other settings for ∞-categorical fundamental truss functors. Note that the fundamental truss construction sends mesh degeneracies to truss degeneracies, and mesh coarsenings to truss coarsenings; thus there are ordinary functors $\mathbb{\Pi}_{\mathsf{T}} : \mathsf{Mesh}_n^{\mathrm{deg}} \to \mathsf{Trs}_n^{\mathrm{deg}}$ and $\mathbb{\Pi}_{\mathsf{T}} : \mathsf{Mesh}_n^{\mathrm{crs}} \to \mathsf{Trs}_n^{\mathrm{crs}}$ (see Notation 2.3.66). Again by the rigidity of hom posets for these truss categories, we have the following consequence.

Observation 4.2.21 (Rigidity of the fundamental truss for degeneracies and coarsenings). The fundamental truss functor restricts to ∞-functors $\mathbb{\Pi}_{\mathsf{T}} : \mathcal{M}esh_n^{\mathrm{deg}} \to \mathsf{Trs}_n^{\mathrm{deg}}$ and $\mathbb{\Pi}_{\mathsf{T}} : \mathcal{M}esh_n^{\mathrm{crs}} \to \mathsf{Trs}_n^{\mathrm{crs}}$ (see Notation 4.1.99). ⌟

The restricted functors in Corollary 4.2.19 (and similarly in Corollary 4.2.20 and Observation 4.2.21) are in fact weak equivalences of ∞-categories. In the next two sections we establish the two core ingredients for those equivalences, namely that the fundamental truss functor is *essentially injective* and *weakly faithful*.

4.2.3. ⧫Essential injectivity of the fundamental truss functor. Of course, given n-meshes M and M', by the functoriality of the fundamental truss functor $\mathbb{\Pi}_{\mathsf{T}}$, if the meshes are isomorphic $M \cong M'$ then the fundamental trusses are isomorphic $\mathbb{\Pi}_{\mathsf{T}}(M) = \mathbb{\Pi}_{\mathsf{T}}(M')$. (We write equality for (balanced) isomorphism since truss isomorphisms are necessarily unique.) Working toward establishing that the fundamental truss functor, suitably restricted, is an equivalence, we will next show that when the fundamental trusses are isomorphic, the meshes were too, i.e. the functor $\mathbb{\Pi}_{\mathsf{T}}$ is 'essentially injective'. We record this result, also for mesh bundles, as follows.

Proposition 4.2.22 (Essential injectivity). *For a cellulable stratification (B, g), the functor $\mathbb{\Pi}_{\mathsf{T}} : \mathsf{Mesh}_n(B, g) \to \mathsf{Trs}_n(\mathbb{\Pi} g)$ is essentially injective; that is, given n-mesh bundles p and p' over the stratification (B, g), if the fundamental trusses are isomorphic, $\mathbb{\Pi}_{\mathsf{T}}(p) = \mathbb{\Pi}_{\mathsf{T}}(p')$, then the mesh bundles are isomorphic, $p \cong p'$.*

The proof of this proposition will occupy the whole of this subsection. The properties established during the construction of the mesh bundle isomorphism (specifically the continuity of the isomorphism for families of mesh bundles) will be reused in our subsequent proof of the weak faithfulness of the fundamental truss functor.

Observation 4.2.23 (Essential injectivity for degeneracies and coarsenings). Note that any isomorphism of n-meshes is both a mesh degeneracy and a mesh coarsening. Therefore the following proof of essentially injective of the fundamental truss functor specializes to give the essential injectivity of the restrictions $\mathbb{\Pi}_{\mathsf{T}} : \mathsf{Mesh}_n^{\mathrm{deg}} \to \mathsf{Trs}_n^{\mathrm{deg}}$ and $\mathbb{\Pi}_{\mathsf{T}} : \mathsf{Mesh}_n^{\mathrm{crs}} \to \mathsf{Trs}_n^{\mathrm{crs}}$. ⌟

Synopsis. We show that it suffices to establish essential injectivity of the fundamental truss functor for bundles over cellular bases, then for 1-mesh bundles, and finally for closed bundles. We then introduce regular contours of 1-mesh bundles, and catchment areas and radial catchment paths of cellular stratifications. Equipped with those technical notions, we construct the desired closed 1-mesh bundle isomorphism via affine interpolations. Finally, we observe the construction of these mesh bundle isomorphisms is continuous in families.

4.2.3.1. ✠ Reduction to closed 1-mesh bundles over a cellular base. By several reduction steps, we show that it suffices to prove essential injectivity for closed 1-mesh bundles over a cellular base. We begin with the reduction from the general n-mesh bundle case to that case over a cellular base.

Observation 4.2.24 (Reduction to cellular base). For cellulable (B, g), pick a refinement $G : (B, c) \to (B, g)$ by a cellular stratification (B, c). Pullback (see Construction 4.1.93) the bundles p and p' to n-mesh bundles G^*p and G^*p' over (B, c). The assumption $\mathbb{\Pi}_{\mathsf{T}}(p) = \mathbb{\Pi}_{\mathsf{T}}(p')$ (in the statement of Proposition 4.2.22) implies that $\mathbb{\Pi}_{\mathsf{T}}(G^*p) = \mathbb{\Pi}_{\mathsf{T}}(G^*p')$ (see Observation 4.2.15). Any n-mesh bundle isomorphism $G^*p \cong G^*p'$ that fixes the base stratification (B, c) will induce an n-mesh bundle isomorphism $p \cong p'$. Thus, it suffices to prove Proposition 4.2.22 for cellular base stratifications. ⌟

We will therefore now assume our base stratification (B, g) is cellular.

Next, arguing inductively, we find that it further suffices to prove essential injectivity for the case of 1-mesh bundles.

Observation 4.2.25 (Reduction to 1-mesh bundles). Consider n-mesh bundles p and p', with component 1-mesh bundles $p_i : (M_i, f_i) \to (M_{i-1}, f_{i-1})$ and $p'_i : (M'_i, f'_i) \to (M'_{i-1}, f'_{i-1})$, respectively, where $(M_0, f_0) = (B, g) = (M'_0, f'_0)$. Suppose we have $\mathbb{\Pi}_{\mathsf{T}}(p) = \mathbb{\Pi}_{\mathsf{T}}(p')$, as assumed in Proposition 4.2.22. That implies $(\mathbb{\Pi}_{\mathsf{T}}(p))_{<n} = (\mathbb{\Pi}_{\mathsf{T}}(p'))_{<n}$, and so, assuming the proposition inductively, we have an isomorphism $G : p_{<n} \cong p'_{<n}$ of truncated mesh bundles. Set the n-mesh bundle G^*p' to be the pullback of p' along G (see Construction 4.1.93); denote the top 1-mesh bundle of G^*p' by $\tilde{p}_n : (\tilde{M}_n, \tilde{f}_n) \to (M_{n-1}, f_{n-1})$ and denote the canonical map $(\tilde{M}_n, \tilde{f}_n) \to (M'_n, f'_n)$ by F, as shown in the next diagram.

$$\begin{array}{ccc} \tilde{f}_n & \xrightarrow{F} & f'_n \\ {\scriptstyle \tilde{p}_n}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle p'_n} \\ f_{n-1} & \xrightarrow{G_{n-1}} & f'_{n-1} \end{array}$$

Since by assumption G is an isomorphism, note that F is also an isomorphism. To prove Proposition 4.2.22, it remains only to construct a bundle isomorphism $\kappa^{p_n}_{\tilde{p}_n}$, as shown below.

$$\begin{array}{ccc} f_n & \xrightarrow[\sim]{\kappa^{p_n}_{\tilde{p}_n}} & \tilde{f}_n \\ & {\scriptstyle p_n}\searrow \quad \swarrow{\scriptstyle \tilde{p}_n} & \\ & f_{n-1} & \end{array}$$

The required bundle isomorphism $\kappa^{p_n}_{\tilde{p}_n}$ is provided by the next proposition. (Note that since cellularity lifts (see Lemma 4.1.64) and since (B, g) is cellular, the stratification f_{n-1} itself is cellular.) $\lrcorner$

Proposition 4.2.26 (Essential injectivity for 1-mesh bundles). *For a cellular stratification (B, g) and 1-mesh bundles $p : (M, f) \to (B, g)$ and $\tilde{p} : (\tilde{M}, \tilde{f}) \to (B, g)$ such that $\mathbb{\Pi}_{\mathsf{T}}(p) = \mathbb{\Pi}_{\mathsf{T}}(\tilde{p})$, there is a 1-mesh bundle isomorphism $\kappa^p_{\tilde{p}} : p \cong \tilde{p}$ that fixes the base (B, g).*

The proof of this statement will take the remainder of this subsection. As a preliminary matter we further reduce to the closed case as follows.

Observation 4.2.27 (Reduction to closed bundles). Fiberwise compactifying both bundles in the preceding proposition, we obtain closed 1-mesh bundles $\bar{p}$ and $\bar{\tilde{p}}$ (see Construction 4.1.58). These bundles certainly still admit an isomorphism $\mathbb{\Pi}_{\mathsf{T}}(\bar{p}) = \mathbb{\Pi}_{\mathsf{T}}(\bar{\tilde{p}})$. Moreover, if we find a bundle isomorphism $\kappa : \bar{p} \cong \bar{\tilde{p}}$, then we obtain a bundle isomorphism $\kappa : p \cong \tilde{p}$ by restriction, as required. Therefore it suffices to prove Proposition 4.2.26 for closed 1-mesh bundles over a cellular base stratification (B, g). $\lrcorner$

Fix closed 1-mesh bundles p and $\tilde{p}$ as in Proposition 4.2.26. The construction of the prospective bundle isomorphism $\kappa^p_{\tilde{p}} : f \cong \tilde{f}$ requires care; as motivation for our approach, we first discuss how *not* to construct $\kappa^p_{\tilde{p}}$. One

Minor: exact logic of above arg [Fixed the logic flow a little, otherwise confirmed as sound] would like to define a stratified homeomorphism $\kappa^p_{\tilde{p}} : f \cong \tilde{f}$ fiberwise; naively one might imagine, fiber by fiber, mapping point strata to point strata and extending linearly to obtain the map on the intervening interval strata (recall the fibers in 1-mesh bundles inherit, via their 1-framed realization embedding, a linear structure from the standard linear structure of $\mathbb{R}$). However, when traversing an entrance path between two strata $r \to s$ in the base (B, g), new singular strata can appear in the special fiber (of either the source or target bundle) over s, that were not present in the generic fiber over r. Because of these creation paths, the rudimentary linear interpolation construction fails; we illustrate the issue in the following example.

Example 4.2.28 (Failure of continuity in fiberwise linear interpolation). Consider the bundles $p : (M, f) \to (B, g)$ and $\tilde{p} : (\tilde{M}, \tilde{f}) \to (B, g)$ shown in Figure 4.18, whose fundamental 1-truss bundles coincide. We depict these bundles, via their 1-framed realizations, as embedded in $B \times \mathbb{R}$; we indicate by a green squiggle an entrance path in the base (B, g), together with a generic fiber (also in green) and a special fiber (in purple) in both p and $\tilde{p}$. If we were to build a bundle isomorphism fiberwise by first identifying the point strata of fibers (as indicated by the mappings on the right) and then linearly interpolating these mappings on interval strata, we would end up with a *discontinuous* bundle isomorphism between p and $\tilde{p}$. ⌟

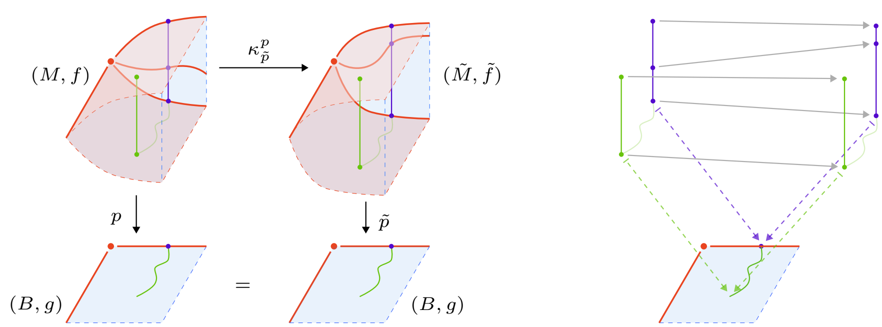


Figure 4.18. Failure of continuity of fiberwise interpolation of mesh bundle isomorphisms.

By contrast, our strategy to ensure continuity in the construction of the bundle isomorphism $\kappa^p_{\tilde{p}}$ will be to use *affine combinations* of maps on generic and special fibers when traversing certain entrance paths.

4.2.3.2. ✲ Regular contours and catchment areas. We first introduce a notion of 'regular contours', which will delineate the boundary strata in the base stratification over which we need to use an affine combination (as opposed to a simple linear interpolation) for the desired mesh bundle isomorphism. We then describe 'catchment areas', which function as a sort of tubular neighborhoods in the base stratification; we will later on define the bundle isomorphism via a combination of maps over a generic fiber on the boundary of the catchment area and a special fiber at the core of the area. Recall that we assume the base stratification (B, g) is cellular, and fix a closed 1-mesh bundle $p : (M, f) \to (B, g)$.

CONSTRUCTION 4.2.29 (Regular contours). Consider a regular stratum s of the stratification f, lying over a stratum $r = p(s)$ of the base stratification g (note, by the assumption on (B, g), that r is a cell). The 'regular contour of the stratum s', denoted $\mathsf{c}_s \subset \partial r$ (where $\partial r = \overline{r} \setminus r$), is the union of strata t in the boundary of r, for which there exists a regular stratum u lying over t, such that u lies in the boundary of s. ⌟

EXAMPLE 4.2.30 (Regular contours). In Figure 4.19, we highlight the regular contour of a chosen stratum s (for the bundle p from Figure 4.18). ⌟

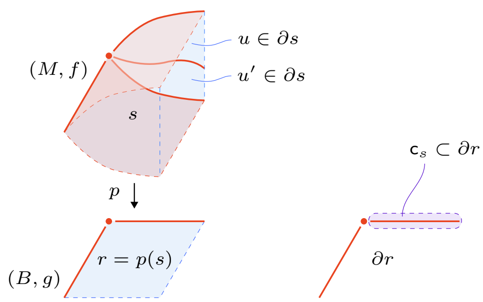


FIGURE 4.19. The regular contour of a regular stratum.

OBSERVATION 4.2.31 (Regular contours only depend on truss structure). If $\tilde{p} : (\tilde{M}, \tilde{f}) \to (B, g)$ is another closed 1-mesh bundle over (B, g) and $\mathbb{\Pi}_{\mathsf{T}}(p) = \mathbb{\Pi}_{\mathsf{T}}(\tilde{p})$, we may identify strata s of f with strata $\tilde{s}$ of $\tilde{f}$, and then the regular countours c_s and $\mathsf{c}_{\tilde{s}}$ of corresponding strata coincide as subspaces of the base B. ⌟

With the notion of regular contours established, we turn to the separate matter of catchment areas.

CONSTRUCTION 4.2.32 (Catchment areas and radial catchment paths). Let r be a stratum in (B, g). Since (B, g) is cellular, it includes as a constructible substratification into a regular cell complex X. Consider the closed cell R obtained as the closure of the cell r in X (and stratify R by its cells); note that $\partial r \subset \partial R$ where $\partial R = R \setminus r$ and again $\partial r = \overline{r} \setminus r$. We endow R with simplicial structure via an identification $R \cong \|\mathbb{\Pi} R\|$. We say $x \in r$ lies in the 'catchment area C_b of an open cell $b \subset \partial r$' if it lies in the open simplicial star of the vertex corresponding to b, and does not lie in the open simplicial star of a vertex corresponding to a higher-dimensional open cell $b' \subset \partial r$. Set the 'closed catchment area $\overline{C}_b$ of b' to be the closure $\overline{C}_b \subset r$ of C_b inside the open cell r; note that the stratum b is not contained in its closed catchment area $\overline{C}_b$. The radial projections of cellular stars[26] (of cells b in ∂r) restrict to 'catchment projections' $\pi_b : \overline{C}_b \to b$. The radial lines of such a projection decompose $\overline{C}_b$ into 'radial catchment path' families $\overline{C}_b \cong F_b \times [0,1) \hookrightarrow R$, where F_b is the boundary of the closed cellular star around b, but with the boundary ∂b removed. (Note that as each radial catchment path approaches 1 in the decomposition $F_b \times [0,1)$, the path in $\overline{C}_b$ is approaching the stratum b.) See the next example and figure for an illustration. ⌟

Example 4.2.33 (Catchment areas and catchment paths). The previous construction of catchment areas and radial catchment paths is illustrated in Figure 4.20, in three cases. The left case is the stratification (B, g) with the stratum r from Figure 4.19. In the middle case, the closed cell R is again the square with its indicated stratification, and the stratum r is again the open 2-cell, but the stratification (B, g) is the union of the stratum r and the three colored boundary strata. Similarly in the right case, the stratification (B, g) is the union of the stratum r and the single colored boundary stratum. In each case, we highlight the (open) catchment areas and their decomposition into catchment path families, with catchment paths oriented from 0 to 1. ⌟

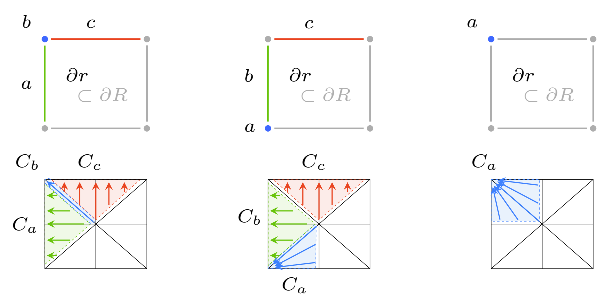


Figure 4.20. Catchment areas for cells and their decomposition into radial catchment paths.

Remark 4.2.34 (Choice of catchment structure). Note that the preceding construction depends on certain choices, namely a choice of regular cell complex X and the identification of closed cells with the simplicial realization of the cells' fundamental posets. Henceforth when working with cellular stratifications (B, g), we will implicitly fix such a regular complex and identifications, and conceive of these as providing the stratification with a 'regular simplicial structure'. ⌟

4.2.3.3. ✠ Constructing the bundle isomorphism. Equipped with the notions of catchment areas and radial catchment paths, we can proceed to construct the bundle isomorphism $\kappa^p_{\tilde{p}}$, and complete the proofs of both Proposition 4.2.26 and Proposition 4.2.22.

[26]See Remark B.3.28 for a definition and discussion of cellular stars. Note that $\mathsf{star}(b) \setminus \partial b$ is the product $\overline{\mathsf{cone}}(\mathsf{link}(b)) \times b$. The radial projection is simply induced by the radial projection $\overline{\mathsf{cone}}(\mathsf{link}(b)) \to \{1\}$ of the closed cone to its cone point; see Terminology B.3.1.

Construction 4.2.35 (The bundle isomorphism for essential injectivity). Consider closed 1-mesh bundles $p : (M, f) \to (B, g)$ and $\tilde{p} : (\tilde{M}, \tilde{f}) \to (B, g)$, with a cellular base (B, g), and such that $\sqcap_{\mathsf{T}}(p) = \sqcap_{\mathsf{T}}(\tilde{p})$. We will define a bundle isomorphism $\kappa^p_{\tilde{p}}$ fiberwise by maps

$$\kappa^p_{\tilde{p}}(x, -) : p^{-1}(x) \to \tilde{p}^{-1}(x)$$

over points $x \in r$, where r is a stratum in the base g. We provide the definition inductively in $\dim(r)$.

If $\dim(r) = 0$ then the fiber isomorphism $\kappa^p_{\tilde{p}}(x, -)$ is simply defined by mapping point strata of $p^{-1}(x)$ monotonically to corresponding point strata of $\tilde{p}^{-1}(x)$ (where 'corresponding' refers to the identification provided by the truss isomorphism $\sqcap_{\mathsf{T}}(p) = \sqcap_{\mathsf{T}}(\tilde{p})$), and then extending the mapping linearly to the interval strata in between those point strata.

Next, if $\dim(r) > 0$, again define the fiber isomorphism $\kappa^p_{\tilde{p}}(x, -)$ to map point strata of $p^{-1}(x)$ monotonically to corresponding point strata of $\tilde{p}^{-1}(x)$. Interval strata s_x in $p^{-1}(x)$ are restrictions of regular strata s in (M, f) to the fiber $p^{-1}(x)$, and canonically correspond to interval strata $\tilde{s}_x$ in $\tilde{p}^{-1}(x)$. Now we define $\kappa^p_{\tilde{p}}(x, -)$ via a collection of maps $s_x \to \tilde{s}_x$, each depending on the local structure around the regular stratum s.

› For all x which do not lie in a catchment area C_b of some cell $b \subset \partial r$, define the required map $s_x \to \tilde{s}_x$ simply by linear interpolation.
› Now proceed inductively in the increasing cell dimension $\dim(b)$ of the cell b for which x is in the catchment area C_b. When $x \in C_b$, the point x is of the form $(u, t) \in \overline{C}_b \cong F_b \times [0, 1)$ for $t \in (0, 1)$. When b is not in the regular contour of the regular stratum s, i.e. $b \notin \mathsf{c}_s$, we define $s_x \to \tilde{s}_x$ again by linear interpolation. When b is in the regular contour of the regular stratum s, i.e. $b \in \mathsf{c}_s$, more care is required and we define

$$\kappa^p_{\tilde{p}}(x, -) = (1 - t)\kappa^p_{\tilde{p}}((u, 0), -) + t\kappa^p_{\tilde{p}}(\pi_b(x), -)$$

where $\pi_b(x)$ is the catchment projection from Construction 4.2.32. Both the isomorphisms $\kappa^p_{\tilde{p}}$ used in that interpolation are already defined by the inductive assumption.

Observe that this induction exhaustively and continuously extends the definition of the bundle isomorphism $\kappa^p_{\tilde{p}}$ to all fibers over the stratum r, as needed. ⌟

Proof of Proposition 4.2.26. The preceding construction provides the required isomorphism $\kappa^p_{\tilde{p}}$, when the 1-mesh bundles are closed. By Observation 4.2.27, that implies the existence of such an isomorphism in the case of general 1-mesh bundles. □

Proof of Proposition 4.2.22. By Observations 4.2.24 and 4.2.25, the case of 1-mesh bundles over a cellular stratification implies the case of n-mesh bundles over a cellulable stratification. □

4.2.3.4. ✲ Continuity of the construction. We make a few useful-later observations about the Construction 4.2.35 of the mesh bundle isomorphisms $\kappa^p_{\tilde{p}}$.

Firstly, though evident, we record the fact that the bundle isomorphism only depends on the realized structure of the mesh bundles, as follows.

Observation 4.2.36 (The bundle isomorphism preserves strict identities). Consider 1-mesh bundles $p : (M, f) \to (B, g)$ and $\tilde{p} : (\tilde{M}, \tilde{f}) \to (B, g)$ with a cellular base (B, g) and such that $\mathbb{\Pi}_{\mathsf{T}}(p) = \mathbb{\Pi}_{\mathsf{T}}(\tilde{p})$. Identify M and $\tilde{M}$ as subspaces of $B \times \mathbb{R}$ using their 1-framed realizations $\gamma : M \hookrightarrow B \times \mathbb{R}$ and $\tilde{\gamma} : \tilde{M} \hookrightarrow B \times \mathbb{R}$. If (M, f) and $(\tilde{M}, \tilde{f})$ have identical realizations in $B \times \mathbb{R}$, then the inductive construction of $\kappa^p_{\tilde{p}}$ (in Construction 4.2.35) yields the bundle identity map $\mathrm{id} : p = \tilde{p}$ on the bundle realizations. ⌟

Secondly, the construction of the mesh bundle isomorphism is continuous in families, as follows.

Definition 4.2.37 (Families of 1-mesh bundles). Given a space Z, a **Z-family** of 1-mesh bundles over (B, g) is a 1-mesh bundle $p : (M, f) \to Z \times (B, g)$. For $z \in Z$, the **z-slice** of p, denoted $p_z : (M_z, f_z) \to (B, g)$, is the restriction of p to the subspace $B \cong \{z\} \times B \hookrightarrow Z \times B$. ⌟

Observation 4.2.38 (The mesh bundle isomorphism for families). For a cellular stratification (B, g), consider Z-families of 1-mesh bundles $p : (M, f) \to Z \times (B, g)$ and $\tilde{p} : (\tilde{M}, \tilde{f}) \to Z \times (B, g)$, such that $\mathbb{\Pi}_{\mathsf{T}}(p) = \mathbb{\Pi}_{\mathsf{T}}(\tilde{p})$. Choose a catchment structure for (B, g) (see Remark 4.2.34). This choice provides a catchment structure for the bundles p_z and $\tilde{p}_z$ for all $z \in Z$, and we may thus construct the 1-mesh bundle isomorphisms $\kappa^{p_z}_{\tilde{p}_z} : p_z \cong \tilde{p}_z$, using Construction 4.2.35. The fiberwise bundle isomorphisms $\kappa^{p_z}_{\tilde{p}_z}$ immediately assemble into a single continuous bundle isomorphism $\kappa^p_{\tilde{p}} : p \cong \tilde{p}$. ⌟

Finally, we mention a means of constructing Z-families of 1-mesh bundles, namely by pullback along Z-families of stratified maps.

Remark 4.2.39 (Families of bundles from pullback along families of maps). Consider cellular stratifications (B, g) and $(B, \tilde{g})$, and let $F : Z \to \mathsf{Strat}_{\mathrm{lf}}(g, \tilde{g})$ be a continuous map from a space Z to the space of stratified maps between g and $\tilde{g}$, such that F is constant on entrance path posets (that is, $\mathbb{\Pi} \circ F : Z \to \mathcal{P}os_{\mathrm{lf}}(\mathbb{\Pi}g, \mathbb{\Pi}\tilde{g})$ is constant). By the tensoredness of stratified spaces (see Construction B.2.24), we can consider F as a stratified map $F : Z \times g \to \tilde{g}$. Given a 1-mesh bundle $\tilde{p} : (\tilde{M}, \tilde{f}) \to (B, \tilde{g})$, we can therefore construct a Z-family of 1-mesh bundles as the pullback $F^*\tilde{p}$ of $\tilde{p}$ along $F : Z \times g \to \tilde{g}$ (see Construction 4.1.57). ⌟

4.2.4. ⧫Weak faithfulness of the fundamental truss functor. We now show that the fundamental truss functor $\mathbb{\Pi}_{\mathsf{T}}$ is a 'weakly faithful' functor of ∞-categories.

As usual, for $m \in \mathbb{N}$, denote by D^{m+1} the closed $(m+1)$-ball and by S^m its boundary. Recall that a topological space U is weakly contractible if every map $\zeta : S^m \to U$ has an extension to a map $\theta : D^{m+1} \to U$ (we will refer to such an extension θ as a 'filler' for ζ).

Proposition 4.2.40 (Weak faithfulness, closed case). *Given closed n-mesh bundles p and p', with cellulable base (B, g), the fundamental truss functor hom space map*

$$\sqcap_{\mathsf{T}} : \bar{\mathcal{M}}esh_n(B,g)(p,p') \to \bar{\mathsf{Trs}}_n(\sqcap g)(\sqcap_{\mathsf{T}}(p), \sqcap_{\mathsf{T}}(p'))$$

has empty or weakly contractible preimages.

Remark 4.2.41 (Weak faithfulness in other rigid cases). Though we will give the explicit statement and proof of weak faithfulness only in the case of closed n-mesh bundles, note that the following cases are immediate analogs of our proof:

› The case of open n-mesh bundles and regular maps ($\mathring{\mathcal{M}}esh_n(B,g)$)
› The case of general n-mesh bundles and balanced maps ($\mathcal{M}esh_n^{\mathrm{bal}}(B,g)$)
› The case of n-meshes and their degeneracies ($\mathsf{Mesh}_n^{\mathrm{deg}}$)
› The case of n-meshes and their coarsenings ($\mathsf{Mesh}_n^{\mathrm{crs}}$)

The thread that ties these different cases together are the rigidity results of Lemma 2.3.72. ⌟

The proof of Proposition 4.2.40 will occupy the whole of this subsection.

Remark 4.2.42 (The fundamental truss functor is weakly fully faithful). Once we have constructed the weak inverse of the fundamental truss functor $\sqcap_{\mathsf{T}}$, it will follow that the fibers of the hom space maps of $\sqcap_{\mathsf{T}}$ are, in fact, never empty. ⌟

Synopsis. We observe that it suffices to prove weak faithfulness of the fundamental truss functor for bundles over cellular bases. We then reduce weak faithfulness for n-mesh bundles to a filler lifting condition for 1-mesh bundles. Finally, we prove that lifting condition by explicitly constructing a filler via suitable fiberwise convex combinations of mesh maps.

✠ **Proof of weak faithfulness.** We first observe that it suffices to prove weak faithfulness for a cellular base.

Remark 4.2.43 (Reduction to cellular base, for weak faithfulness). Given closed n-mesh bundles p and p' as in Proposition 4.2.40, fix a refinement $G : (B,c) \to (B,g)$ of the cellulable stratification (B,g) by a cellular stratification (B,c). Using Construction 4.1.93, we may pullback both p and p' to n-mesh bundles G^*p and G^*p' over (B,c). Let $F : \sqcap_{\mathsf{T}}(p) \to \sqcap_{\mathsf{T}}(p')$ be a map of the fundamental n-truss bundles. This map pulls back, along $\sqcap G : \sqcap c \to \sqcap g$, to an n-truss bundle map $(\sqcap G)^*F : (\sqcap G)^*\sqcap_{\mathsf{T}}(p) \to (\sqcap G)^*\sqcap_{\mathsf{T}}(p')$ (see Construction 2.3.54). The fiber of $\sqcap_{\mathsf{T}}$ in $\bar{\mathcal{M}}esh_n(B,c)(G^*p, G^*p')$ over $(\sqcap G)^*F$ is homeomorphic to the fiber of $\sqcap_{\mathsf{T}}$ in $\bar{\mathcal{M}}esh_n(B,g)(p,p')$ over F. Thus it is sufficient to prove Proposition 4.2.40 in the case of a cellular base. ⌟

We next tackle the proof of Proposition 4.2.40 for a cellular base (B, g). Let p and p' be closed n-mesh bundles consisting of 1-mesh bundles $p_i : (M_i, f_i) \to (M_{i-1}, f_{i-1})$ and $p'_i : (M'_i, f'_i) \to (M'_{i-1}, f'_{i-1})$, respectively, with $(M_0, f_0) = (B, g) = (M'_0, f'_0)$. Consider a map $\zeta : S^m \to \bar{\mathcal{M}}esh_n(B, g)(p, p')$ such that $\sqcap_{\mathsf{T}}(\zeta)$ is constant (in other words, ζ maps into a single fiber of $\sqcap_{\mathsf{T}}$). Note that, by rigidity of singular truss maps of closed trusses (see Lemma 2.3.72), this constancy condition is satisfied automatically except when $m = 0$.

Recall that truncation of meshes is an ∞-functor (see Remark 4.1.98). Truncating the map ζ to degrees below n, we obtain the map $\beta := \zeta_{<n} : S^m \to \bar{\mathcal{M}}esh_{n-1}(B, g)(p_{<n}, p'_{<n})$. Arguing by induction, we may assume that β has a filler $\eta : D^{m+1} \to \bar{\mathcal{M}}esh_{n-1}(B, g)(p_{<n}, p'_{<n})$. Using the tensoredness of stratified spaces (see Construction B.2.24), we may consider the map ζ as a stratified map $S^m \times f_n \to f'_n$, the map β as a stratified map $S^m \times f_{n-1} \to f'_{n-1}$, and the map η as a stratified map $D^{m+1} \times f_{n-1} \to f'_{n-1}$. To show that ζ has a filler it will therefore be sufficient to prove the following.

Proposition 4.2.44 (Lifting fillers in closed 1-mesh bundles). *Consider closed 1-mesh bundles $p : (M, f) \to (B, g)$ and $\tilde{p} : (\tilde{M}, \tilde{f}) \to (\tilde{B}, \tilde{g})$, with cellular bases, and maps $\zeta : S^m \times f \to \tilde{f}$ and $\beta : S^m \times g \to \tilde{g}$ such that, for each $e \in S^m$, the restriction $(\zeta(e, -), \beta(e, -)) : p \to \tilde{p}$ is a 1-mesh bundle map. (If $m = 0$, further assume $\sqcap_{\mathsf{T}}(\zeta(e, -), \beta(e, -))$ is independent of $e \in S^0$).*

Then any filler $\eta : D^{m+1} \times g \to \tilde{g}$ of β lifts to a filler $\theta : D^{m+1} \times f \to \tilde{f}$ of ζ such that, for each $e \in D^{m+1}$, the restriction $(\theta(e, -), \eta(e, -)) : p \to \tilde{p}$ is a 1-mesh bundle map.

Proof. It will be convenient to consider D^{m+1} as the quotient of $[0, 1] \times S^m$ by the subset $\{1\} \times S^m$. As such, we will construct the required filler θ as a mapping $[0, 1] \times S^m \times f \to \tilde{f}$, such that $\theta(1, -)$ is constant in the S^m component. To construct the required filler θ of ζ, lifting the filler η of β, we proceed in two steps.

First, by pulling back along the base filler η, we will construct a 'homotopy #1' map $\theta_1 : [0, 1] \times S^m \times f \to \tilde{f}$, which homotopes $\theta_1(0, -) = \zeta$ into a map $\theta_1(1, -) : S^m \times f \to \tilde{f}$ that descends to a map of base stratifications $S^m \times g \to \tilde{g}$ that is constant in the first component S^m.

Second, using 'fiberwise contractions', we will construct a 'homotopy #2' map $\theta_2 : [0, 1] \times S^m \times f \to \tilde{f}$, which homotopes $\theta_2(0, -) = \theta_1(1, -)$ into a map $\theta_2(1, -) : S^m \times f \to \tilde{f}$ that is itself constant in the first component S^m. Concatenating the homotopies θ_1 and θ_2 will provide the required filler θ of ζ.

(1) Define a closed 1-mesh bundle $\beta^*\tilde{p}$ by pulling back $\tilde{p}$ along β and define a map $\widehat{\zeta}$ as the factorization of ζ through this pullback as shown below.

$$\begin{array}{ccccc} & & \overset{\zeta}{\frown} & & \\ S^m \times f & \xrightarrow[\widehat{\zeta}]{} & \beta^*\tilde{f} & \longrightarrow & \tilde{f} \\ {\scriptstyle S^m\times p}\downarrow & & {\scriptstyle \beta^*\tilde{p}}\downarrow & \lrcorner & \downarrow{\scriptstyle \tilde{p}} \\ S^m \times g & \underset{\mathrm{id}}{=\!=} & S^m \times g & \xrightarrow[\beta]{} & \tilde{g} \end{array}$$

Note $\beta^*\tilde{p}$ is an S^m-family of closed 1-mesh bundles over g (see Remark 4.2.39). We may modify this into a $([0,1] \times S^m)$-family by simply taking the product $[0,1] \times -$. The resulting closed 1-mesh bundle $[0,1] \times \beta^*\tilde{p}$ is bundle isomorphic to the closed 1-mesh bundle $\eta^*\tilde{p}$ defined by the pullback on the right below.

$$\begin{array}{ccccc} [0,1] \times \beta^*\tilde{f} & \xrightarrow[\sim]{\kappa} & \eta^*\tilde{f} & \longrightarrow & \tilde{f} \\ {\scriptstyle [0,1]\times\beta^*\tilde{p}}\downarrow & & {\scriptstyle \eta^*\tilde{p}}\downarrow & \lrcorner & \downarrow{\scriptstyle \tilde{p}} \\ [0,1] \times S^m \times g & \xrightarrow[\mathrm{id}]{} & [0,1] \times S^m \times g & \xrightarrow[\eta]{} & \tilde{g} \end{array}$$

Here, the isomorphism κ can be constructed using Observation 4.2.38, since g is assumed to be cellular. The homotopy #1 map $\theta_1 : [0,1] \times S^m \times f \to \tilde{f}$ is now defined as the composite

$$[0,1] \times S^m \times f \xrightarrow{[0,1]\times\widehat{\zeta}} [0,1] \times \beta^*\tilde{f} \xrightarrow{\kappa} \eta^*\tilde{f} \to \tilde{f} \quad .$$

Since the κ construction preserves identities (see Observation 4.2.36) and since $\beta = \eta(0,-)$, we find that $\kappa(0,-)$ is the identity on $\beta^*\tilde{f}$. Thus, homotopy #1 satisfies $\theta_1(0,-) = \zeta$, and lifts η in the sense that

$$\begin{array}{ccc} [0,1] \times S^m \times f & \xrightarrow{\theta_1} & \tilde{f} \\ {\scriptstyle [0,1]\times S^m\times p}\downarrow & & \downarrow{\scriptstyle \tilde{p}} \\ [0,1] \times S^m \times g & \xrightarrow[\eta]{} & \tilde{g} \end{array}$$

This completes the first half of the construction of θ.

(2) It remains to construct the homotopy #2 map $\theta_2 : [0,1] \times S^m \times f \to \tilde{f}$ such that $\theta_2(0,-) = \theta_1(1,-)$. Recall that for a stratification (Y,h) and a subspace $X \subset Y$, we will use (X,h) to denote the restricted stratification (see Definition B.2.8). We may define the homotopy θ_2 by convexly combining fiberwise maps $\theta_1(1,e,-) : (p^{-1}(y), f) \to (\tilde{p}^{-1}\eta(1,e,y), \tilde{f})$, for $e \in S^m$ and $y \in B$ (note that $\eta(1,e,y)$ is in fact independent of $e \in S^m$). Specifically, pick any $e_0 \in S^m$, and for $t \in [0,1]$, $e \in S^m$, and $y \in B$, define the restriction of $\theta_2(t,e,-)$ to the fiber over y to be the map

$$\begin{aligned} \theta_2(t,e,-) : (p^{-1}(y), f) &\to (\tilde{p}^{-1}\eta(1,e,y), \tilde{f}) \\ x &\mapsto (1-t) \cdot \theta_1(1,e,x) + t \cdot \theta_1(1,e_0,x) \end{aligned}$$

Note that $\theta_2(t, e, -)$ is indeed a 1-mesh map for all t and e, because $\mathbb{\Pi}_{\mathsf{T}}(\theta_1(1, e, -)) = \mathbb{\Pi}_{\mathsf{T}}(\theta_1(1, e_0, -))$, i.e. the convexly combined factors induce the same maps on 1-trusses (that in turn is the case since, by assumption, $\mathbb{\Pi}_{\mathsf{T}}(\zeta(e, -), \beta(e, -))$ is independent of $e \in S^m$). Note that at $t = 1$, the map $\theta_2(t, e, -)$ becomes independent of $e \in S^m$. We can finally chain the homotopies θ_1 and θ_2 into a single homotopy

$$\theta := \theta_1 * \theta_2 : [0, 1] \times S^m \times f \to \tilde{f}$$

which defines the filler θ of ζ, lifting the filler η of β, as required. □

Proof of Proposition 4.2.40. By Remark 4.2.43, it suffices to address the case of cellular base. The discussion preceeding the statement of Proposition 4.2.44 shows that the desired weak faithfulness for closed n-mesh bundles follows inductively from the lifting property of closed 1-mesh bundles established in that proposition. □

CLD did not check the above long proof.

2 : Will recheck one last time

4.2.5. ♦Mesh realizations. We will now construct the mesh realization functors from various categories of n-trusses to corresponding categories of n-meshes.[27] In the first and most foundational instance, we will have the mesh realization functor

$$\|-\|_{\mathsf{M}} : \mathsf{Trs}_n \to \mathcal{M}esh_n$$

Note that this functor is necessarily an ∞-functor because the hom spaces of its domain have discrete topology. More generally, for a fixed cellulable base stratification (B, g), we will have the mesh bundle realization functor

$$\|-\|_{\mathsf{M}} : \mathsf{Trs}_n(\mathbb{\Pi} g) \to \mathcal{M}esh_n(B, g)$$

Restricted to the subcategories of closed or open trusses and meshes, this functor provides weak inverses to the corresponding previously constructed fundamental truss functors. However, the construction of mesh realization will not provide an enriched functor $\mathcal{T}rs_n(\mathbb{\Pi} g) \mathrel{\rightarrow\!\!\!\!\!\times\!\!\!\rightarrow} \mathcal{M}esh_n(B, g)$, and so no candidate inverse, in any case, for the enriched fundamental truss functor $\mathcal{M}esh_n(B, g) \to \mathcal{T}rs_n(\mathbb{\Pi} g)$.

The simplest and most direct construction of mesh realizations occurs for closed trusses; in this case, we will obtain the mesh realization $\|-\|_{\mathsf{M}}$ by a direct application of the stratified realization $\|-\|$. For non-closed trusses, the stratified realization does not always produce the correct mesh topological type; that difficulty arises essentially because the ordinary geometric realization cannot tell the difference between the point poset of a trivial closed 1-truss (which ought to realize to a trivial closed 1-mesh, i.e. a point) and the point poset of a trivial open 1-truss (which ought to realize to a trivial open 1-mesh, i.e. an open interval). We will construct mesh realizations for general trusses by taking the not-necessarily-closed truss, compactifying

[27] In fact, aside from the case of closed trusses, the mesh realization will be only semifunctorial; we elide the 'semi' as immaterial.

it to a closed truss, forming the mesh realization, and then extracting the appropriate submesh as a constructible substratification.

Synopsis. We define mesh realizations for closed trusses via the stratified realization. We then construct the mesh realization for a general truss as a constructible substratification of the realization of the cubical compactification of the truss. We build the required cubical compactification by adjoining singular endpoints fiberwise and inductively throughout the truss tower. We then construct the realization of truss bundles using a compactified cellulation of the base stratification. Finally, we formulate a distinct realization for truss coarsenings, which is, unlike the usual mesh realization, a mesh coarsening.

4.2.5.1. Realizations of closed trusses. We construct mesh realizations of closed trusses using the ordinary stratified geometric realization. We first address the case of 1-truss bundles; the case of n-trusses will follow immediately by iteration. Recall the stratified realizations of posets and poset maps from Constructions 1.3.4 and 1.3.5.

Construction 4.2.45 (1-Mesh bundle realizations of closed 1-truss bundles). Given a closed 1-truss bundle $p : T \to X$, we will endow the realized stratified map $\|p\| : \|T\| \to \|X\|$ with the structure of a closed 1-mesh bundle; we refer to the result as the **closed 1-mesh bundle realization** and denote it $\|p\|_{\mathsf{M}} : \|T\| \to \|X\|$.

We construct a 1-framed realization $\gamma : |T| \hookrightarrow |X| \times \mathbb{R}$ for $\|p\|$. Define γ to map 0-simplices of $|T|$ such that fibers of p land in the corresponding fibers of $|X| \times \mathbb{R} \to |X|$ in a frame-order-preserving manner (e.g., mapping the 0-simplex $i \in p^{-1}(x)$ to $(x, i) \in |X| \times \mathbb{N} \hookrightarrow |X| \times \mathbb{R}$). Then extend the mapping linearly to the remaining simplices of the realization $|T|$.

To see that this defines the data of a 1-mesh bundle, one checks using an induction on the scaffold order of simplices (see Section 2.2.2) that γ is an embedding with continuous upper and lower realization bounds. Constructibility of the family of meshes follows directly from the singular functionality of the constituent truss bordisms of the given truss bundle. ⌟

Example 4.2.46 (1-Mesh bundle realizations of 1-truss bundles). In Figure 4.21 we depict on the left a closed 1-truss bundle $p : T \to X$ (note that we only depict generating arrows, see Construction 2.1.81), and on the right its closed 1-mesh bundle realization $\|p\|_{\mathsf{M}} : \|T\| \to \|X\|$, shown via the 1-framed realization $\gamma : |T| \hookrightarrow |X| \times \mathbb{R}$. ⌟

Construction 4.2.47 (Mesh realizations of closed n-trusses). For a closed n-truss T, consisting of 1-truss bundles $p_i : T_i \to T_{i-1}$, the **closed n-mesh realization** $\|T\|_{\mathsf{M}}$ is the closed n-mesh defined by the tower of 1-mesh bundles

$$\|T_n\| \xrightarrow{\|p_n\|_{\mathsf{M}}} \|T_{n-1}\| \xrightarrow{\|p_{n-1}\|_{\mathsf{M}}} ... \xrightarrow{\|p_2\|_{\mathsf{M}}} \|T_1\| \xrightarrow{\|p_1\|_{\mathsf{M}}} \|T_0\|$$

where each $\|p_i\|_{\mathsf{M}}$ is the 1-mesh bundle realization of the 1-truss bundle p_i, given by Construction 4.2.45. ⌟

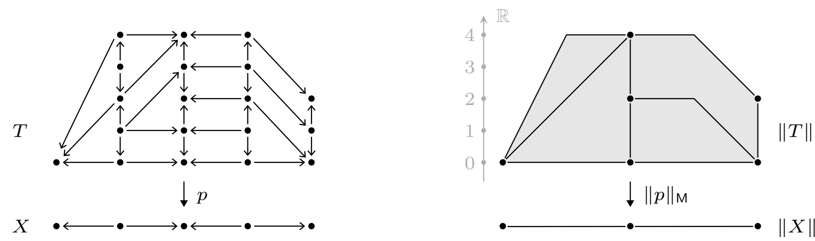


FIGURE 4.21. 1-Mesh bundle realization of a 1-truss bundle.

CONSTRUCTION 4.2.48 (Mesh map realizations of maps of closed n-trusses). Given closed n-trusses T and S, and any n-truss map $F : T \to S$, consisting of the tower of poset maps $F_i : T_i \to S_i$, the **closed n-mesh map realization** $\|F\|_{\mathsf{M}}$ is the map of closed n-meshes consisting of the tower of stratified realizations $\|F_i\| : \|T_i\| \to \|S_i\|$. ⌟

We typically denote the tower $\{\|F_i\|\}$ of stratified realizations by simply $\|F\|$; this is (for maps of closed n-trusses) the same tower as the mesh map realization $\|F\|_{\mathsf{M}}$, but considered merely as a sequence of stratified bundles rather than a sequence of mesh bundles.

As in general the stratifications of the base space will warrant more careful attention, we will return to the case of n-truss bundles in due course.

4.2.5.2. Realizations of general trusses and maps. We turn to the construction of mesh realizations for general n-trusses, and also realizations of maps thereof. This case requires more care because, as mentioned earlier, the naive geometric realization of posets (taking their nerve and applying the usual geometric realization of simplicial sets) inappropriately degenerates trivial open 1-truss fibers. The construction of mesh realizations will proceed by taking a general truss, compactifying it to a closed truss, realizing that to a closed mesh, and finally taking a suitable submesh.

Fiberwise compactifications of 1-truss bundles are a combinatorial analog of the fiberwise compactifications of 1-mesh bundles given in Construction 4.1.58. However, when applying fiberwise compactifications inductively to a tower of bundles, there remains at each stage a choice of how to extend a bundle to the compactification of its base. There are several reasonable possibilities for such extensions; we make a particular choice, the 'cubical compactification', which is suitably *initial* among retractible compactifications, and which will admit a useful explicit construction. We describe these compactifications in the general context of truss bundles.

DEFINITION 4.2.49 (Retractable compactifications). For an n-truss bundle p, a **retractable compactification** is a closed n-truss bundle q, together with a pair of base-preserving bundle maps

$$\iota : p \rightleftarrows q : \rho$$

where the map ι is balanced, the composite $\rho \circ \iota$ is the identity id_p, and the composite $\iota \circ \rho$ admits a natural transformation to the identity id_q. ⌟

DEFINITION 4.2.50 (Cubical compactification, universal property). For an n-truss bundle p over a poset X, the **cubical compactification** is the unique retractable compactification

$$\mathsf{ci} : p \rightleftarrows \overline{p} : \mathsf{cr} \quad ,$$

consisting of the 'cubical inclusion' ci and the 'cubical retraction' cr, such that, for any other retractable compactification $\iota : p \rightleftarrows q : \rho$ there exists a unique n-truss bundle r over $X \times [1]$ subject to the following conditions:

(1) $r|_{X\times\{0\}} = \overline{p}$ and $r|_{X\times\{1\}} = q$;
(2) the restriction of the bundle r, to the union of the images of the inclusions ci in $\overline{p}$ and ι in q, is the product bundle $p \times [1]$ over $X \times [1]$. ⌟

EXAMPLE 4.2.51 (Cubical compactification). In Figure 4.22, we depict the inclusion $\mathsf{ci} : T \hookrightarrow \overline{T}$ of an open 2-truss T (in black) into its cubical compactification $\overline{T}$ (extending T by the gray structure). ⌟

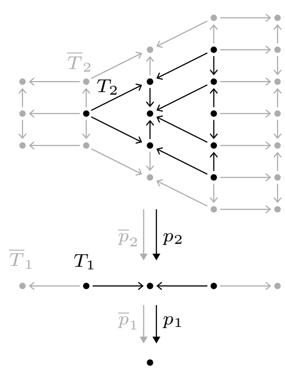


FIGURE 4.22. Cubical compactification of an open 2-truss.

We defer the explicit construction of cubical compactifications, and first complete the construction of mesh realizations for general n-trusses. Recall that a constructible substratification (see Definition B.2.9) is determined by its fundamental poset mapping.

CONSTRUCTION 4.2.52 (Mesh realizations of general n-trusses). For an n-truss T, the n**-mesh realization** $\|T\|_{\mathsf{M}}$ is the constructible submesh

$$\|\mathsf{ci}\|_{\mathsf{M}} : \|T\|_{\mathsf{M}} \hookrightarrow \|\overline{T}\|_{\mathsf{M}}$$

whose fundamental poset subtruss is

$$\sqcap_{\mathsf{T}}(\|\mathsf{ci}\|_{\mathsf{M}}) = \mathsf{ci} : T \hookrightarrow \overline{T}$$

That is, the mesh realization $\|T\|_{\mathsf{M}}$ is the submesh of the closed n-mesh $\|\overline{T}\|_{\mathsf{M}}$, whose stages are the constructible substratifications $(\|\mathsf{ci}\|_{\mathsf{M}})_i : (\|T\|_{\mathsf{M}})_i \hookrightarrow$

$(\|\overline{T}\|_{\mathsf{M}})_i$ that have fundamental poset maps $\sqcap((\|\mathsf{ci}\|_{\mathsf{M}})_i)$ being the i-th stages $\mathsf{ci}_i : T_i \hookrightarrow \overline{T}_i$ of the cubical compactification inclusion map ci. ⌟

The fact that the mesh realization $\|T\|_{\mathsf{M}}$ indeed forms an n-mesh follows from the later explicit inductive construction of cubical compactifications. Note that when the truss T is closed, the preceding construction of $\|T\|_{\mathsf{M}}$ specializes to the earlier Construction 4.2.47.

Notation 4.2.53 (Abbreviation for mesh realization stages). For an n-truss T, just to have a slightly more concise notation, we will denote the i-th stage $(\|T\|_{\mathsf{M}})_i$ of the mesh realization $\|T\|_{\mathsf{M}}$ by simply $\|T_i\|_{\mathsf{M}}$. Though the poset T_i does not by itself have a mesh realization $\|-\|_{\mathsf{M}}$, one should think of taking the mesh realization of that stage in the ambient context of the whole truss. Similarly, we will denote the i-th stage $(\|\mathsf{ci}\|_{\mathsf{M}})_i$ of the compactification inclusion by simplify $\|\mathsf{ci}_i\|_{\mathsf{M}}$. ⌟

Example 4.2.54 (Mesh realization). Recall the open 2-truss T from Figure 4.22. In Figure 4.23, we depict the closed mesh realization $\|\overline{T}\|_{\mathsf{M}}$ of the cubical compactification $\overline{T}$, together with the resulting open mesh realization $\|T\|_{\mathsf{M}}$ of the open truss T, as a tower of constructible substratifications. ⌟

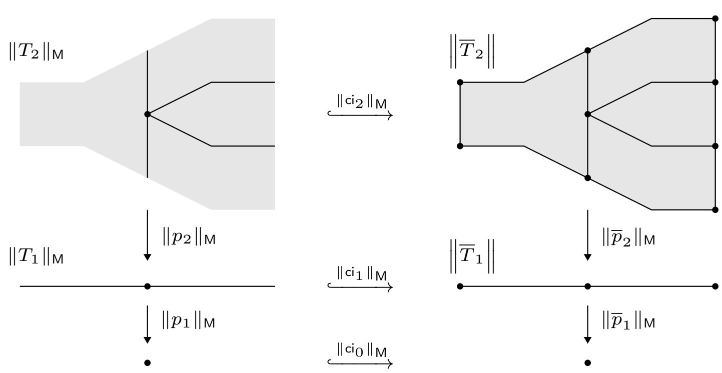
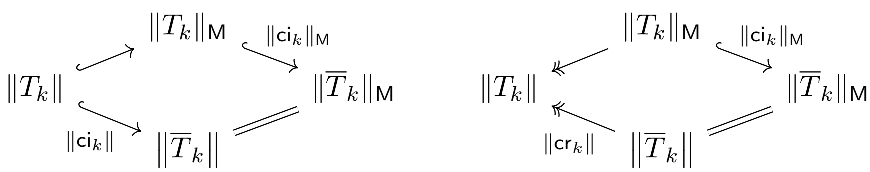


Figure 4.23. The mesh realization of an open 2-truss.

Remark 4.2.55 (Stratified realizations versus mesh realizations). For a non-closed n-truss T, the stratified realization $\|T_k\|$ (of the k-stage poset) and the mesh realization $\|T_k\|_{\mathsf{M}} \equiv (\|T\|_{\mathsf{M}})_k$ are, in general, distinct stratifications. Nevertheless, the stratified realization includes into the mesh realization, and the mesh realization retracts to the stratified realization, as follows.

To describe the inclusion and retraction between the stratified realization $\|T_k\|$ and the mesh realization $\|T_k\|_{\mathsf{M}}$, we must consider the realization of the compactification $\left\|\overline{T}_k\right\| = \|\overline{T}_k\|_{\mathsf{M}}$; of course for the compactification, the stratified and mesh realizations agree. The stratified-to-mesh inclusion $\|T_k\| \hookrightarrow \|T_k\|_{\mathsf{M}}$ and mesh-to-stratified retraction $\|T_k\|_{\mathsf{M}} \twoheadrightarrow \|T_k\|$ are defined by the following squares:



That is, the stratified-to-mesh inclusion $\|T_k\| \hookrightarrow \|T_k\|_{\mathsf{M}}$ is the factorization of the stratified inclusion $\|\mathsf{ci}_k\|$ through the constructible substratification $\|\mathsf{ci}_k\|_{\mathsf{M}}$; and the mesh-to-stratified retraction $\|T_k\|_{\mathsf{M}} \twoheadrightarrow \|T_k\|$ is the composite of the constructible substractification $\|\mathsf{ci}_k\|_{\mathsf{M}}$ with the stratified retraction $\|\mathsf{cr}_k\|$. ⌟

It remains to define mesh realizations for n-truss maps. The realization of general n-trusses in Construction 4.2.52 was defined as a constructible submesh of the mesh realization of the cubical compactification (which latter realization was given by an ordinary stratified realization). One would expect the realization of general n-truss maps to also be defined via cubical compactifications; however, cubical compactification is *not* naively functorial and so this approach requires some care, as follows.

Construction 4.2.56 (Cubical compactification for truss bundle maps). Let p and q be n-truss bundles over the posets X and Y respectively, and let $F : p \to q$ be a truss bundle map. Set the cubical compactification of the truss bundle map F to be the truss bundle map $\overline{F}$ defined as the composite

$$\overline{p} \xrightarrow{\mathsf{cr}} p \xrightarrow{F} q \xrightarrow{\mathsf{ci}} \overline{q}$$ ⌟

Remark 4.2.57 (Semifunctoriality of cubical compactification). Though cubical compactification, as given for objects in Definition 4.2.50 and for morphisms in Construction 4.2.56, does not preserve identities and so is not a functor, per se, it does preserve composition of maps, and so is in that sense a semifunctor. ⌟

Convention 4.2.58 (Semifunctors referred to as functors). In a rather pointed abuse of terminiology, we will paper over the aforementioned fact that compactification does not preserve identities, and will willfully use the term 'functor' to refer to semifunctors, particularly for the mesh realization functors constructed using compactification. ⌟

Construction 4.2.59 (Mesh map realizations of truss maps). For an n-truss map $F : T \to S$, the n**-mesh map realization** $\|F\|_{\mathsf{M}} : \|T\|_{\mathsf{M}} \to \|S\|_{\mathsf{M}}$ is the lift of the closed n-mesh map realization $\|\overline{F}\|_{\mathsf{M}} : \|\overline{T}\|_{\mathsf{M}} \to \|\overline{S}\|_{\mathsf{M}}$ (see Construction 4.2.48) of the cubical compactification $\overline{F} : \overline{T} \to \overline{S}$ (see Construction 4.2.56), along the defining constructible substratifications

$\|T\|_{\mathsf{M}} \hookrightarrow \|\overline{T}\|_{\mathsf{M}}$ and $\|S\|_{\mathsf{M}} \hookrightarrow \|\overline{S}\|_{\mathsf{M}}$, i.e. according to the following diagram:

$$\begin{array}{ccc} \|T\|_{\mathsf{M}} & \xrightarrow{\|F\|_{\mathsf{M}}} & \|S\|_{\mathsf{M}} \\ {\scriptstyle \|\mathsf{ci}\|_{\mathsf{M}}}\big\downarrow & & \big\downarrow{\scriptstyle \|\mathsf{ci}\|_{\mathsf{M}}} \\ \|\overline{T}\|_{\mathsf{M}} & \xrightarrow{\|\overline{F}\|_{\mathsf{M}}} & \|\overline{S}\|_{\mathsf{M}} \end{array} .$$

Note that this construction is given just for realizations of truss maps, not truss bundle maps. For bundles (and their maps), the stratified realization of a base fundamental poset (or poset map) need not faithfully encode a base stratification (or map thereof); we address the generalization to bundles later.

Now observe that the association $F \mapsto \|F\|_{\mathsf{M}}$ from a truss map to its mesh map realization is (semi)functorial; note Convention 4.2.58.

Notation 4.2.60 (Mesh realization functor). Constructions 4.2.52 and 4.2.59 together yield the **n-mesh realization functor**

$$\|-\|_{\mathsf{M}} : \mathsf{Trs}_n \to \mathcal{M}esh_n$$

from n-trusses to n-meshes.

Note that whenever a truss map is singular or regular, its mesh map realization is singular or regular, respectively. Thus in particular the mesh realization functor restricts to a functor from closed trusses with singular maps to closed meshes with singular maps, and restricts to a functor from open trusses with regular maps to open meshes with regular maps.

Observation 4.2.61 (Failure of topological enrichment for mesh realization). The preceding construction of mesh realizations does not define an enriched functor $\|-\|_{\mathsf{M}} : \mathcal{T}rs_n \not\to \mathcal{M}esh_n$; the difficulty is already visible in the discontinuity of the stratified realization functor on posets (see Remark B.2.22).

Observation 4.2.62 (Mesh realization is right inverse to the fundamental truss). Note that, after restricting either to closed meshes and trusses and their singular maps, or to open meshes and trusses and their regular maps, the mesh realization of Notation 4.2.60 is right inverse to the fundamental truss functor of Corollary 4.2.20, in the sense that there is unique natural isomorphism $\mathbb{T}_{\mathsf{T}} \circ \|-\|_{\mathsf{M}} \cong \mathrm{id}$.

Of course, forgetting the enrichment of the target of the functor in Notation 4.2.60 provides a realization $\|-\|_{\mathsf{M}} : \mathsf{Trs}_n \to \mathsf{Mesh}_n$, which is, without restriction, right inverse to the fundamental truss functor $\mathbb{T}_{\mathsf{T}} : \mathsf{Mesh}_n \to \mathsf{Trs}_n$.

4.2.5.3. ✻ Constructing cubical compactifications. We now provide the deferred construction of cubical compactifications, first for 1-trusses, then for 1-truss bundles, and finally for n-truss bundles.

Construction 4.2.63 (Cubical compactification of 1-trusses). For a 1-truss T, its cubical compactification 1-truss $\overline{T}$ (in the sense of Definition 4.2.50, and leaving the inclusion and retraction implicit) is obtained from the 1-truss T by adjoining a new upper, and respectively lower, singular endpoint if the upper, respectively lower, endpoint of the given 1-truss T is regular. ⌟

Construction 4.2.64 (Cubical compactification of 1-truss bundles). For a 1-truss bundle $p : T \to X$, its cubical compactification 1-truss bundle $\overline{p} : \overline{T} \to X$ (in the sense of Definition 4.2.50) is obtained from the bundle p by compactifying each fiber of the bundle according to Construction 4.2.63, and extending the 1-truss bordisms to the compactified fibers in the unique endpoint-preserving way. ⌟

Construction 4.2.65 (Cubical compactification of n-truss bundles). Let p be an n-truss bundle over a poset X, consisting of 1-truss bundle maps $p_i : T_i \to T_{i-1}$. Suppose by induction we have constructed the cubical compactification $\mathsf{ci} : p_{<n} \rightleftarrows \overline{p}_{<n} : \mathsf{cr}$ of the $(n-1)$-truncated bundle $p_{<n}$. (The starting case is given by Construction 4.2.64.)

Pull back the top bundle $p_n : T_n \to T_{n-1}$ along the retraction $\mathsf{cr}_{n-1} : \overline{T}_{n-1} \to T_{n-1}$ to obtain the 1-truss bundle $\mathsf{cr}^*_{n-1} p_n$ over the truncated compactification $\overline{T}_{n-1}$. Pulling back again along ci_{n-1} of course recovers the original bundle p_n, i.e. $\mathsf{ci}^*_{n-1}\mathsf{cr}^*_{n-1} p_n = p_n$. Thus we have a 1-truss subbundle map $\mathrm{Tot}(\mathsf{ci}_{n-1}) : p_n \hookrightarrow \mathsf{cr}^*_{n-1} p_n$ and a 1-truss bundle map $\mathrm{Tot}(\mathsf{cr}_{n-1}) : \mathsf{cr}^*_{n-1} p_n \to p_n$; these maps form an inclusion-retraction pair of bundles $p_n \rightleftarrows \mathsf{cr}^*_{n-1} p_n$. Applying Construction 4.2.64 to the bundle $\mathsf{cr}^*_{n-1} p_n$ provides the fiberwise compactification inclusion-retraction pair $\mathsf{cr}^*_{n-1} p_n \rightleftarrows \overline{\mathsf{cr}^*_{n-1} p_n}$. Now set $\overline{p}_n := \overline{\mathsf{cr}^*_{n-1} p_n}$ as the top bundle of the desired cubical compactification, and compose the two given inclusion-retraction pairs to obtain the pair $\mathsf{ci}_n : p_n \rightleftarrows \overline{p}_n : \mathsf{cr}_n$. Altogether, the cubical compactification $\overline{p}$ is the n-truss bundle obtained by augmenting the truncation $\overline{p}_{<n}$ with the bundle $\overline{p}_n$, along with the resulting inclusion-retraction pair $\mathsf{ci} : p \rightleftarrows \overline{p} : \mathsf{cr}$, as required. ⌟

Remark 4.2.66 (Universal property of cubical compactifications). That the previous construction satisfies the universal property indicated in Definition 4.2.50 can be seen inductively as follows. Let $\mathsf{ci} : p \rightleftarrows \overline{p} : \mathsf{cr}$ be the cubical compactification from Construction 4.2.65, and consider another retractable compactification $\iota : p \rightleftarrows q : \rho$. Inductively assume we have constructed the $(n-1)$-truncation $r_{<n}$ of the desired 'factorizing bundle' r over $X \times [1]$.

As an inductive invariant, assume that whenever there is a relation $(0, x) \to (1, y)$ in the total poset of the $(n-1)$-truss bundle $r_{<n}$, then there is a relation $\mathsf{cr}_{n-1}(x) \to \rho_{n-1}(y)$ in the total poset of $p_{<n}$. From this, construct the full n-truss bundle r by augmenting $r_{<n}$ with the 1-truss bundle r_n, defined by

$$r_n|_{(0,x)\to(1,y)} := q_n|_{\iota_{n-1}(\rho_{n-1}(y))\to y} \circ \overline{p}_n|_{\mathsf{cr}_{n-1}(x)\to\rho_{n-1}(y)}$$

To see that this construction again satisfies the inductive invariant assume by contradiction that we have $(0, x) \to (1, y)$ but no arrow $\mathsf{cr}_n(x) \to \rho_n(y)$. By definition of r, there must be some z in the same fiber of p_n as $\rho_n(y)$, such that $\mathsf{cr}_n(x) \to z$ and $\iota(z) \to y$. Since $\iota \circ \rho \to \mathrm{id}_q$, z must be a neighbor of $\rho_n(y)$. If z is regular, there is a fiber arrow $z \to \rho_n(y)$. If z is singular, we must have $\rho_n(y) = z$. Both contradict our assumption. ⌟

CLD did not check that last remark.

4.2.5.4. ✻ Realizations of truss bundles. We previously constructed mesh realizations for n-trusses, but it remains to address the case of n-truss bundles. Naively, at least for closed n-truss bundles, one could just iteratively apply the Construction 4.2.45 of 1-mesh bundle structures on the stratified realizations of closed 1-truss bundles, to obtain a putative n-mesh bundle realization. However, two successive problems arise. (1) Our primary interest in mesh realization is as a weak inverse to a (suitably enriched, suitably restricted) fundamental truss functor $\mathsf{Mesh}_n(B, g) \to \mathsf{Trs}_n(\sqcap g)$; and typically, even for a quite well behaved stratification (B, g), the stratified realization $\|\sqcap g\|$ of the entrance path poset $\sqcap g$ is not even remotely the original stratification (B, g), so we would not have produced anything like a candidate inverse. The natural way to address that problem is to assume the stratification (B, g) is cellulable and to choose a cellular refinement (B, c); at least for such a cellular stratification, the stratified realization $\|\sqcap c\|$ is remotely like the original stratification (B, c). (2) However, in general that stratified realization is still not exactly the original stratification, and so we will need to construct a suitable stratified retraction $(B, c) \to \|\sqcap c\|$ along which we can pull back the naive mesh bundle realization. Finally then we can coarsen that pulled-back mesh bundle according to the base coarsening $(B, c) \to (B, g)$ to obtain the desired mesh realization.

We implement that strategy as follows, begining with a choice of cellular refinement along with a stratified regular cell closure that will be needed for the subsequent retraction construction.

Terminology 4.2.67 (Compactified cellulation). Given a cellulable stratification (B, g), a 'compactified cellulation' (g, c, X) is a refinement of (B, g) by a cellular stratification (B, c), together with a dense constructible substratification $(B, c) \hookrightarrow \|X\|$, where X is a combinatorial regular cell complex. (Here dense refers to the unstratified inclusion of spaces, i.e. $\overline{B} = |X|$.) ⌟

We can proceed directly to building the required retraction $(B, c) \to \|\sqcap c\|$ from the cellular stratification of the base to the stratified realization of its fundamental poset.

Construction 4.2.68 (Cellular inclusion-retractions). Given a compactified cellulation (g, c, X) for a stratification (B, g), abbreviate $Y := \sqcap c$. Note that $Y \hookrightarrow X$ is a dense subposet (i.e. $X^{\geq Y} = X$), and by assumption $B \subset |X|$ is a dense subspace. Observe that the image of the realization map $|Y \hookrightarrow X|$ lands entirely in $B \subset |X|$; let

$$\mathsf{Ci} : |Y| \hookrightarrow B$$

denote that inclusion. Construct a corresponding retraction

$$|Y| \twoheadleftarrow B : \mathsf{Cr}$$

as follows. Consider the simplicial complex NX, i.e. the nerve of the poset X, and decompose each simplex x as a simplicial join $y \star z$, where y is the subsimplex of x spanned by vertices in Y, and z is the subsimplex spanned by vertices in $X \setminus Y$. Note that $|x| \setminus |z| \cong |y| \times [0,1)$; define the retraction on each intersection $|x| \cap B$ by the projection $|y| \times [0,1) \to |y|$. The resulting inclusion-retraction pair $(\mathsf{Ci}, \mathsf{Cr})$ is cell-preserving, and so provides a stratified inclusion and retraction, as required:

$$\mathsf{Ci} : \|Y\| \rightleftarrows (B, c) : \mathsf{Cr}$$ ⌟

Equipped with this retraction from the base cellular stratification, we can describe the mesh realization for closed bundles, and their maps, as follows.

Construction 4.2.69 (Realization of closed n-truss bundles). For a base stratification (B, g), with a compactified cellulation (g, c, X) and the resulting cellular inclusion-retraction pair $\mathsf{Ci} : \|\sqcap c\| \rightleftarrows (B, c) : \mathsf{Cr}$ as in the previous construction, consider a closed n-truss bundle p over the fundamental poset $\sqcap g$.

Pull the bundle back along the coarsening $\sqcap c \to \sqcap g$; denote the result $p^c := (\sqcap c \to \sqcap g)^* p$. That pullback is a tower of closed 1-truss bundles

$$S_n \xrightarrow{p^c_n} S_{n-1} \xrightarrow{p^c_{n-1}} \ldots \xrightarrow{p^c_1} S_0 = \sqcap c\ .$$

The stratified realization of this tower

$$\|S_n\| \xrightarrow{\|p^c_n\|} \|S_{n-1}\| \xrightarrow{\|p^c_{n-1}\|} \ldots \xrightarrow{\|p^c_1\|} \|S_0\| = \|\sqcap c\|$$

obtains the structure of a closed n-mesh bundle by iterated application of Construction 4.2.45.

Next pull back that n-mesh bundle along the retraction $\mathsf{Cr} : (B, c) \to \|\sqcap c\|$, and finally coarsen the mesh bundle along the base coarsening $(B, c) \to (B, g)$ (recall the original truss bundle was constant on strata of g). The result is the **closed n-mesh bundle realization** $\|p\|_{\mathsf{M}}$ of the n-truss bundle p. ⌟

Construction 4.2.70 (Realization of maps of closed n-truss bundles). Consider a stratification (B, g) with compactified cellulation (g, c, X) and the associated cellular inclusion-retraction pair $\mathsf{Ci} : \|\sqcap c\| \rightleftarrows (B, c) : \mathsf{Cr}$. Let $F : p \to q$ be a base-preserving map of closed n-truss bundles over the fundamental poset $\sqcap g$. We construct the **closed n-mesh bundle map realization** $\|F\|_{\mathsf{M}} : \|p\|_{\mathsf{M}} \to \|q\|_{\mathsf{M}}$, a base-preserving n-mesh bundle map over the stratification (B, g).

As before, let p^c and q^c denote the pullbacks of the truss bundles p and q along the coarsening $\sqcap c \to \sqcap g$; note that the resulting maps $p^c \to p$ and $q^c \to q$ are truss bundle coarsenings. Similarly let $F^c : p^c \to q^c$ denote the truss bundle map lifting $F : p \to q$ along those coarsenings. The stratified realization $\|F^c\| : \|p^c\| \to \|q^c\|$ is a mesh bundle map over $\|\sqcap c\|$. Now lift that realization along the pullbacks $\mathsf{Cr}^* \|p^c\| \to \|p^c\|$ and $\mathsf{Cr}^* \|q^c\| \to \|q^c\|$, and denote the resulting mesh bundle map $\mathsf{Cr}^* \|F^c\|$, as in the diagram:

$$\begin{array}{ccc} \mathsf{Cr}^* \|p^c\| & \xrightarrow{\mathsf{Cr}^* \|F^c\|} & \mathsf{Cr}^* \|q^c\| \\ \downarrow & & \downarrow \\ \|p^c\| & \xrightarrow{\|F^c\|} & \|q^c\| \end{array}$$

Finally push forward the mesh bundle map $\mathsf{Cr}^* \|F^c\|$ along the coarsening $(B, c) \to (B, g)$ to obtain the desired mesh bundle realization $\|F\|_{\mathsf{M}} : \|p\|_{\mathsf{M}} \to \|q\|_{\mathsf{M}}$. ⌟

As in the non-bundle case, to define the mesh realization for general (not closed) truss bundles, and their maps, we just need to compactify the bundles, before realizing them and then restricting to a suitable constructible substratification.

Construction 4.2.71 (Realization of general n-truss bundles). Let p be an n-truss bundle over the fundamental poset $\sqcap g$ of a cellulable stratification (B, g), with a compactified cellulation (g, c, X). Form the cubical compactification $\overline{p}$ as in Definition 4.2.50. Construct the closed n-mesh bundle $\|\overline{p}\|_{\mathsf{M}}$ by the preceding construction. Finally, take the constructible substratification $\|p\|_{\mathsf{M}} \hookrightarrow \|\overline{p}\|_{\mathsf{M}}$ whose fundamental poset subtruss bundle is the cubical compactification inclusion $p \hookrightarrow \overline{p}$; the result is the **$n$-mesh bundle realization** $\|p\|_{\mathsf{M}}$ of the given n-truss bundle p. ⌟

We may finally consider the corresponding construction for maps, and the resulting functor.

Construction 4.2.72 (Realization of general n-truss bundle maps). Again consider a stratification (B, g) with compactified cellulation (g, c, X), and let $F : p \to q$ be a base-preserving map of n-truss bundles over the fundamental poset $\sqcap g$. Form the cubical compactification $\overline{F} : \overline{p} \to \overline{q}$ as in (the base-preserving case of) Construction 4.2.56. Construct the mesh bundle map realization $\|\overline{F}\|_{\mathsf{M}} : \|\overline{p}\|_{\mathsf{M}} \to \|\overline{q}\|_{\mathsf{M}}$ according to Construction 4.2.70. Finally, restrict in the source and target along the constructible substratifications $\|p\|_{\mathsf{M}} \hookrightarrow \|\overline{p}\|_{\mathsf{M}}$ and $\|q\|_{\mathsf{M}} \hookrightarrow \|\overline{q}\|_{\mathsf{M}}$ to obtain the desired **n-mesh bundle map realzation** $\|F\|_{\mathsf{M}} : \|p\|_{\mathsf{M}} \to \|q\|_{\mathsf{M}}$. ⌟

The above realization of a truss bundle map is constructed in terms of the realization of the cubical compactification of the map. Thus, the semi-functoriality of the cubical compactification, as in Remark 4.2.57, propagates to this realization; nevertheless, by Convention 4.2.58, we elide the semi-ness of the resulting functoriality.

Construction 4.2.73 (Mesh bundle realization functor). Together, the above constructions yield the **n-mesh bundle realization functor**

$$\|-\|_{\mathsf{M}} : \mathsf{Trs}_n(\sqcap g) \to \mathcal{M}esh_n(B, g)$$

from the category of truss bundles over the poset $\mathbb{\Gamma}g$ to the category of n-mesh bundles over the stratification (B, g). ⌟

Observation 4.2.74 (Mesh bundle realization is right inverse). As in the non-bundle case of Observation 4.2.62, after restricting either to closed mesh and truss bundles and their singular maps, or to open mesh and truss bundles and their regular maps, the mesh bundle realization of Construction 4.2.73 is right inverse to the fundamental truss bundle functor of Corollary 4.2.19, in the sense that there is a unique natural isomorphism $\mathbb{\Gamma}_{\mathsf{T}} \circ \|-\|_{\mathsf{M}} \cong \mathrm{id}$.

Again as in the non-bundle case, forgetting the enrichment of the target of the functor in Construction 4.2.73 provides a realization $\|-\|_{\mathsf{M}} : \mathsf{Trs}_n(\mathbb{\Gamma}g) \to \mathsf{Mesh}_n(B, g)$, which is, without restriction, right inverse to the fundamental truss bundle functor $\mathbb{\Gamma}_{\mathsf{T}} : \mathsf{Mesh}_n(B, g) \to \mathsf{Trs}_n(\mathbb{\Gamma}g)$. ⌟

Note that we have restricted attention to realizations of base-preserving maps of truss bundles. That restriction is partly for simplicity and brevity, though also reflects the fact that, when allowing non-base-preserving maps, the fundamental truss functor $\mathbb{\Gamma}_{\mathsf{T}} : \mathsf{MeshBun}_n \to \mathsf{TrsBun}_n$ destroys even the homotopy class of the base map and so cannot possibly have a weak inverse.

If desired, add remark here on the non-base-preserving case. Need to restrict or lift to cellular base stratifications.

Remark 4.2.75 (Higher homotopical structure of base stratifications). To correctly capture the higher homotopical structure of base stratifications and their maps in our construction of mesh bundle realization functors, we briefly describe two possible avenues of generalization.

First, one may generalize combinatorial base structures from fundamental posets to fundamental ∞-categories: truss and mesh bundles are naturally replaced by their categorical analogs in this case, see Remark 4.2.3. However, our definition of fundamental ∞-categories in terms of quasicategories does not lend itself to a topologically faithful construction of realizations (rather, realizations become 'homotopically faithful', recovering original spaces and maps only up to stratified homotopy equivalence), making it less useful for certain key construction in combinatorial topology.[28]

A second approach requires choices of combinatorial representations for both stratifications *and* their maps at the outset: in essence, in addition to choosing compactified cellulations, we also chose cellular maps between those cellulations to represent our stratified maps. (This is akin to the classical combinatorial-topological situation where all maps are combinatorially represented; the machinery of compactifications developed in this section generalizes the classical setup to work not only with closed cellular structures but also their 'dual' open counterparts.) Truss bundles and their maps carry these choices of combinatorial representatives in the form of appropriate coarsenings, which then allows us to realize bundles and maps in a functorial way, using a straight-forward generalization of the preceding constructions. ⌟

[28] An alternative definition of fundamental ∞-categories, based on 'posets with weak equivalences', is described in Section B.3.3 and remedies this shortcoming.

4.2.5.5. ✻ Realizations of truss coarsenings. The mesh realization $\|-\|_{\mathsf{M}} : \mathsf{Trs}_n \to \mathsf{Mesh}_n$ constructed in the preceding sections provides a right inverse to the fundamental truss $\sqcap_{\mathsf{T}} : \mathsf{Mesh}_n \to \mathsf{Trs}_n$. However, the mesh realization of a truss coarsening is not necessarily a mesh coarsening; thus the mesh realization does not restrict to a functor from the category $\mathsf{Trs}_n^{\mathrm{crs}}$ of n-trusses and their coarsenings to the category $\mathsf{Mesh}_n^{\mathrm{crs}}$ of n-meshes and their coarsenings. We remedy this situation by constructing a distinct *mesh coarsening realization* functor $\|-\|_{\mathsf{M}}^{\mathsf{crs}} : \mathsf{Trs}_n^{\mathrm{crs}} \to \mathsf{Mesh}_n^{\mathrm{crs}}$, with the feature that for any n-truss coarsening $F : T \to S$, the mesh coarsening realization $\|F\|_{\mathsf{M}}^{\mathsf{crs}} : \|T\|_{\mathsf{M}} \to \|S\|_{\mathsf{M}}$ is homotopic, as a map of n-meshes, to the ordinary mesh realization $\|F\|_{\mathsf{M}}$.

Construction 4.2.76 (Mesh coarsening realizations of closed truss coarsenings). Given closed n-trusses $T = (p_n, ..., p_1)$ and $S = (q_n, ..., q_1)$, and a coarsening $F : T \to S$, we construct an n-mesh coarsening $\|F\|_{\mathsf{M}}^{\mathsf{crs}} : \|T\|_{\mathsf{M}} \to \|S\|_{\mathsf{M}}$, with the following properties:

- the fundamental truss $\sqcap_{\mathsf{T}}\|F\|_{\mathsf{M}}^{\mathsf{crs}}$ is the given coarsening F;
- the components $\|F_i\|_{\mathsf{M}}^{\mathsf{crs}} := (\|F\|_{\mathsf{M}}^{\mathsf{crs}})_i : \|T_i\|_{\mathsf{M}} \to \|S_i\|_{\mathsf{M}}$ are *linear* on each simplex of the realization $\|T_i\|_{\mathsf{M}} = \|T_i\|$.

Assume inductively that we have constructed the mesh coarsening realization $\|F_{<n}\|_{\mathsf{M}}^{\mathsf{crs}} : \|T_{<n}\|_{\mathsf{M}} \to \|S_{<n}\|_{\mathsf{M}}$, with the indicated properties. Define the top component $\|F_n\|_{\mathsf{M}}^{\mathsf{crs}} : \|T_n\|_{\mathsf{M}} \to \|S_n\|_{\mathsf{M}}$ on the vertices $x \in T_n \subset \|T_n\| = \|T_n\|_{\mathsf{M}}$ as follows. For each element $y \in T_{n-1}$, for all the elements $x \in T_n$ in the fiber of the 1-truss bundle p_n over y, pick image points $\|F_n\|_{\mathsf{M}}^{\mathsf{crs}}(x)$ in the fiber of the 1-mesh bundle $\|q_n\|_{\mathsf{M}}$ over $\|F_{n-1}\|_{\mathsf{M}}^{\mathsf{crs}}(y)$, subject to the following conditions:

- there is a strict inequality $\|F_n\|_{\mathsf{M}}^{\mathsf{crs}}(x) < \|F_n\|_{\mathsf{M}}^{\mathsf{crs}}(x')$ (in the framed realization order of the 1-mesh fiber) whenever $x \prec x'$ (in the frame order of the 1-truss fiber).
- the image point $\|F_n\|_{\mathsf{M}}^{\mathsf{crs}}(x)$ lies in the stratum of $\|S_n\|_{\mathsf{M}} = \|S_n\|$ corresponding to the poset element $F_n(x) \in S_n$.

Next extend the map $\|F_n\|_{\mathsf{M}}^{\mathsf{crs}}$ linearly to all simplices of the realization $\|T_n\|_{\mathsf{M}} = \|T_n\|$. The resulting map $\|F_n\|_{\mathsf{M}}^{\mathsf{crs}}$ is a coarsening, and thus the realization $\|F\|_{\mathsf{M}}^{\mathsf{crs}}$ is an n-mesh coarsening as desired. ⌟

Recall from Construction 4.2.59 that the mesh map realization of a truss map was constructed as a substratification of the mesh map realization of the cubical compactification of Construction 4.2.56. For coarsenings, the approach is similar, except that we will utilize a distinct compactification map, as follows.

Observation 4.2.77 (Cubically compactified coarsening). Let $F : T \to S$ be a coarsening map of n-trusses. There is a unique coarsening map $\overline{F}^{\mathsf{crs}} : \overline{T} \to \overline{S}$ between the cubical compactifications, such that the following two squares commute:

$$\begin{array}{ccc} \overline{T} & \xrightarrow{\overline{F}^{\mathsf{crs}}} & \overline{S} \\ \mathsf{cr}\big\downarrow \big\uparrow\mathsf{ci} & & \mathsf{cr}\big\downarrow \big\uparrow\mathsf{ci} \\ T & \xrightarrow[F]{} & S \end{array} .$$

We refer to this map $\overline{F}^{\mathsf{crs}} : \overline{T} \to \overline{S}$ as the 'cubically compactified coarsening', and note well that this map is *not* the cubical compactification, in the sense of Construction 4.2.56, of the coarsening F. (The dangerously close terminology is defensible since the cubical compactification of a coarsening is typically not itself a coarsening.)

We omit the explicit construction of cubically compactified coarsenings, which proceeds inductively in a similar spirit to the construction of the ordinary cubical compactifications. (Key in this inductive construction is that, every coarsening of 1-truss bundles factors as a base-preserving coarsening followed by a pullback; as the latter is preserved by cubical compactification of 1-truss bundles in the sense or Construction 4.2.65, this reduces our claim to the case of 1-mesh bundles over a fixed base.)

Note that unlike the semifunctoriality of cubical compactification, from Remark 4.2.57, the cubically compactified coarsening construction preserves identities and so is properly functorial. ⌟

CXD to check the above observation. CLD has not. Need to think through bordisms with different endpoint types, and inductive extensions and such.

Construction 4.2.78 (Mesh coarsening realizations of truss coarsenings). Given n-trusses T and S and a coarsening $F : T \to S$, we construct the **mesh coarsening realization** $\|F\|^{\mathsf{crs}}_{\mathsf{M}} : \|T\|_{\mathsf{M}} \to \|S\|_{\mathsf{M}}$, with similar properties to the closed case:

- the fundamental truss $\sqcap_{\mathsf{T}} \|F\|^{\mathsf{crs}}_{\mathsf{M}}$ is the given coarsening F;
- the components $\|F_i\|^{\mathsf{crs}}_{\mathsf{M}} := (\|F\|^{\mathsf{crs}}_{\mathsf{M}})_i : \|T_i\|_{\mathsf{M}} \to \|S_i\|_{\mathsf{M}}$ are linear on each open simplex of the realization $\|T_i\|_{\mathsf{M}} \hookrightarrow \|\overline{T}_i\|_{\mathsf{M}}$.

Apply Observation 4.2.77 to obtain the cubically compactified coarsening $\overline{F}^{\mathsf{crs}} : \overline{T} \to \overline{S}$, and then apply Construction 4.2.76 to obtain the mesh coarsening $\|\overline{F}^{\mathsf{crs}}\|^{\mathsf{crs}}_{\mathsf{M}} : \|\overline{T}\|_{\mathsf{M}} \to \|\overline{S}\|_{\mathsf{M}}$. Lift that coarsening along the constructible substratifications $\|\mathsf{ci}\|_{\mathsf{M}} : \|T\|_{\mathsf{M}} \hookrightarrow \|\overline{T}\|_{\mathsf{M}}$ and $\|\mathsf{ci}\|_{\mathsf{M}} : \|S\|_{\mathsf{M}} \hookrightarrow \|\overline{S}\|_{\mathsf{M}}$ to yield the desired coarsening $\|F\|^{\mathsf{crs}}_{\mathsf{M}} : \|T\|_{\mathsf{M}} \to \|S\|_{\mathsf{M}}$. ⌟

Because by construction $\|F\|^{\mathsf{crs}}_{\mathsf{M}}$ and $\|F\|_{\mathsf{M}}$ have the same fundamental truss map, i.e. $\sqcap_{\mathsf{T}}\|F\|^{\mathsf{crs}}_{\mathsf{M}} = F = \sqcap_{\mathsf{T}}\|F\|_{\mathsf{M}}$, it follows from Proposition 4.2.40 and Remark 4.2.41 that the mesh coarsening realization $\|F\|^{\mathsf{crs}}_{\mathsf{M}}$ and the mesh map realization $\|F\|_{\mathsf{M}}$ are homotopic mesh maps; in that sense the mesh coarsening realization is a suitable homotopical replacement for the mesh map realization.

Observation 4.2.79 (Mesh realizations of truss degeneracies). Note that the mesh realization exhibits an asymmetry between coarsenings and degeneracies. Though coarsenings require the above special treatment to ensure their realization are again coarsenings, degeneracies require no such care. Indeed, given an n-truss degeneracy map $F : T \to S$, the mesh map realization $\|F\|_{\mathsf{M}} : \|T\|_{\mathsf{M}} \to \|T\|_{\mathsf{M}}$ is a mesh degeneracy, and so the mesh realization restricts to a functor $\|-\|_{\mathsf{M}} : \mathsf{Trs}_n^{\deg} \to \mathsf{Mesh}_n^{\deg}$, providing the expected right inverse to the restricted fundamental truss functor $\sqcap_{\mathsf{T}} : \mathsf{Mesh}_n^{\deg} \to \mathsf{Trs}_n^{\deg}$. ⌟

4.2.6. ⧫Proofs of the equivalences and their applications. We finally assemble the proof of Theorem 4.2.1 and Theorem 4.2.2, that the fundamental truss functor and the mesh realization functor provide weak equivalences between (certain) ∞-categories of meshes and 1-categories of trusses, and similarly for mesh bundles and truss bundles.

Proof of Theorem 4.2.1 and Theorem 4.2.2. It suffices to show that the fundamental truss functors $\sqcap_{\mathsf{T}} : \bar{\mathcal{M}}esh_n(B, g) \to \bar{\mathsf{Trs}}_n(\sqcap g)$ and $\sqcap_{\mathsf{T}} : \mathring{\mathcal{M}}esh_n(B, g) \to \mathring{\mathsf{Trs}}_n(\sqcap g)$ are weak equivalences of ∞-categories. We argue in the first (closed-singular) case; the second (open-regular) case is analogous. We need to check the following (see [**Lur09a**, Def. 1.1.3.6]):

(1) for each closed n-truss bundle q, there exists a closed n-mesh bundle p whose fundamental truss bundle $\sqcap_{\mathsf{T}}(p)$ is equivalent to the given truss bundle q;
(2) for each pair of mesh bundles p and p', the fundamental truss functor hom space map $\sqcap_{\mathsf{T}} : \bar{\mathcal{M}}esh_n(B, g)(p, p') \to \bar{\mathsf{Trs}}_n(\sqcap g)(\sqcap_{\mathsf{T}}(p), \sqcap_{\mathsf{T}}(p'))$ is a weak equivalence of topological spaces.

The first statement is immediate from Construction 4.2.69 of the mesh bundle realization $\|q\|_{\mathsf{M}}$ with fundamental truss bundle $\sqcap_{\mathsf{T}}\|q\|_{\mathsf{M}} = q$; see also Construction 4.2.73 and Observation 4.2.74. The second statement follows from the weak faithfulness of the fundamental truss functor, as in Proposition 4.2.40, together with the observation that the fibers of the fundamental truss hom space map are never empty. That non-emptiness is ensured by the mesh bundle map realization, as in Construction 4.2.70, with again reference to Construction 4.2.73 and Observation 4.2.74. □

In particular, we thus note that the ∞-category $\bar{\mathcal{M}}esh_n$ (and similarly $\mathring{\mathcal{M}}esh_n$) is 1-truncated, and thus can be thought of as an ordinary 1-category (cf. [**Lur09a**, Prop. 2.3.4.18]).

Proof of Theorem 4.2.5. The induced functor $\sqcap_{\mathsf{T}} : \mathcal{M}Bord_n \to N\mathsf{TBord}^n$ sends a $[k]$-simplex of $\mathcal{M}Bord_n$, which by definition is an n-mesh bundle p over the stratified k-simplex $\|[k]\|$, to the classifying functor $\chi_{\sqcap_{\mathsf{T}} p} : [k] \to \mathsf{TBord}^n$ (see Construction 2.3.44) of the fundamental truss $\sqcap_{\mathsf{T}} p$.

To show that this functor $\sqcap_{\mathsf{T}}$ is a trivial fibration (see [**Lur09a**, Ex. 2.0.0.2]) we need to show that it has the right lifting property with respect to the boundary inclusion $\partial\Delta[k] \hookrightarrow \Delta[k]$ (where $\Delta[k]$ is the simplicial set represented by the poset $[k]$).

Suppose we have maps $\alpha : \partial\Delta[k] \to \mathcal{M}Bord_n$ and $\beta : \Delta[k] \to N\mathsf{TBord}^n$ such that $\sqcap_{\mathsf{T}} \circ \alpha = \beta|_{\partial\Delta[k]}$. First, reinterpret $\beta : \Delta[k] \to N\mathsf{TBord}^n$ as a functor $[k] \to \mathsf{TBord}^n$; that functor classifies an n-truss bundle q over $[k]$. By Construction 4.2.73, we have an n-mesh bundle realization $\|q\|_{\mathsf{M}}$ over the stratified simplex $\|[k]\|$. Second, restricting the map α to each facet of $\partial\Delta[k]$ defines a mesh bundle on that facet; glue those mesh bundles together to obtain an n-mesh bundle p over the boundary $\partial \|[k]\|$. (That this gluing is indeed a *posetal* mesh bundle relies on the existence of the filling map β.) By construction, the restricted mesh bundle $\|q\|_{\mathsf{M}}|_{\partial\|[k]\|}$ and the glued mesh bundle p have the same fundamental truss bundles. Apply Proposition 4.2.22 to provide a bundle isomorphism $\kappa : p \cong \|q\|_{\mathsf{M}}|_{\partial\|[k]\|}$.

Identify the stratified simplex with the stratified simplex with an additional boundary collar: $\|[k]\| \cong \|[k]\| \cup_{\partial\|[k]\| \cong \partial\|[k]\| \times \{1\}} (\partial \|[k]\| \times [0,1])$. Finally construct the required filler $\overline{\alpha} : \|[k]\| \to \mathcal{M}Bord_n$ as follows: define the filler on the collar $\partial \|[k]\| \times [0,1]$ to be the mapping cylinder of the bundle isomorphism κ, then extend over the remaining simplex $\|[k]\|$ by the bundle $\|q\|_{\mathsf{M}}$.

Any trivial fibration induces an equivalence of quasicategories as claimed; see [**Rez22**, §23.10]. □

Synopsis. Having completed the proofs of the equivalences between meshes and trusses, we proceed to two applications. First, we introduce mesh blocks and the notion of framed subdivision of framed cells, and then provide a classification of such subdivisions in terms of combinatorial subdivisions of truss blocks. Second, we introduce mesh braces and observe the duality between mesh blocks and braces and more generally the duality equivalence of closed and open meshes.

4.2.6.1. Mesh blocks and subdivisions of framed regular cells. Concatenating the equivalence of meshes and trusses, with the previous equivalence of trusses and collapsible framed cell complexes, provides of course a relationship between meshes and collapsible framed cell complexes, which will in turn allow us to define a notion of framed subdivision of framed cells, and then to classify such subdivisions in terms of combinatorial subdivisions of truss blocks.

From Theorem 4.2.1, we have the equivalence of closed meshes and closed trusses, witnessed by the fundamental truss $\sqcap_{\mathsf{T}}$ and mesh realization $\|-\|_{\mathsf{M}}$. From the earlier Theorem 3.1.2, we have the equivalence of closed trusses and collapsible framed cell complexes, witnessed by the cell gradient ∇_{C} and truss integration $\int_{\mathsf{T}}$. As in Terminology 4.2.7, we compose these functors to define the mesh-to-cell gradient functor $\nabla_{\mathsf{MC}} := \nabla_{\mathsf{C}} \circ \sqcap_{\mathsf{T}}$ and the cell-to-mesh realization functor $\|-\|_{\mathsf{CM}} := \|-\|_{\mathsf{M}} \circ \int_{\mathsf{T}}$, witnessing the following equivalence.

Corollary 4.2.80 (Equivalence of closed meshes and collapsible framed cell complexes). *The mesh-to-cell gradient functor* ∇_{MC} *and the cell-to-mesh* *realization functor* $\|-\|_{\mathsf{CM}}$ *provide a weak equivalence*

$$\bar{\mathcal{M}}\mathit{esh}_n \underset{\|-\|_{\mathsf{CM}}}{\overset{\nabla_{\mathsf{MC}}}{\rightleftarrows}} \mathsf{CollFrCellCplx}_n$$

between the ∞*-category of closed* n*-meshes and the 1-category of collapsible* n*-framed regular cell complexes.*

Recall that the equivalence between trusses and collapsible framed cell complexes restricted to an equivalence between truss blocks and framed cells. We can transport that restriction across the mesh-to-truss equivalence as follows.

Definition 4.2.81 (Mesh blocks). An n**-mesh block** is a closed n-mesh whose total space is the closure of a single stratum. It is more specifically an n**-mesh** m**-block** if that dense stratum is of dimension m. ⌟

Remark 4.2.82 (Mesh blocks and truss blocks). Mesh blocks are exactly those closed meshes whose fundamental truss is a truss block. ⌟

Example 4.2.83 (Mesh blocks). Back in Figure I.6, the first closed mesh is a 3-mesh 3-block. The fundamental truss of that mesh was depicted (on the right) in Figure 3.1, along with (on the left) the corresponding framed regular cell; that cell bears an unmistakable resemblance to the total stratification of the mesh block.

By contrast, in Figure I.6, the second closed mesh is not a mesh block, but has two submeshes, each of which is a mesh block. The front one of those two mesh blocks has fundamental truss block depicted (on the right) in the middle example of Figure C.6, along with (on the left) the corresponding framed regular cell.

The mesh realization of any truss block is of course a mesh block; so the mesh realizations of the truss blocks of Figures 2.51 and 2.52 provide two further examples. ⌟

Corollary 4.2.84 (Equivalence of mesh blocks and framed cells). *The mesh-to-cell functor* ∇_{MC} *and the cell-to-mesh functor* $\|-\|_{\mathsf{CM}}$ *provide a weak equivalence between the* ∞*-category of* n*-mesh blocks (as a full subcategory of* $\bar{\mathcal{M}}\mathit{esh}_n$*) and the 1-category of* n*-framed regular cells.*

Observation 4.2.85 (Stratified realizations and cell-to-mesh realizations). Since mesh realizations of closed trusses are constructed using ordinary stratified realizations, it follows that for any collapsible framed regular cell complex $(X, \mathcal{F})$, there is a canonical stratified homeomorphism $\|(X, \mathcal{F})\|_{\mathsf{CM}} \cong \|X\|$ of the total stratification of cell-to-mesh realization and the stratified realization of the combinatorial regular cell complex X (as seen above in Example 4.2.83). ⌟

The translation of framed regular cells into mesh blocks provides a framed topological realization of framed regular cells, namely via the n-framed realization of the mesh block, as in Construction 4.1.70.[29] We can leverage this framed topological realization to define a notion of *framed subdivisions* of framed regular cells.

For context, first recall the notion of (unframed) subdivision of a combinatorial regular cell.

TERMINOLOGY 4.2.86 (Subdivisions of regular cells). Let X be a combinatorial regular cell. A 'subdivision' of X is a combinatorial regular cell complex Y, together with a stratified coarsening $F : \|Y\| \to \|X\|$ of stratified realizations. ⌟

A subdivision $F : \|Y\| \to \|X\|$ typically restricts, on a closed cell $\left\|Y^{\geq y}\right\| \hookrightarrow \|Y\|$, $y \in Y$, to a non-cellular map $F|_{Y^{\geq y}} : \left\|Y^{\geq y}\right\| \hookrightarrow \|X\|$ (see Definition 1.3.16). In this sense, the notion of subdivision of cells is not immediately combinatorializable. In fact, no *computable* combinatorial description whatsoever exists, in the sense that subdivisions of cells cannot be recognized or enumerated by any algorithmic method.

The framed analog of subdivisions of combinatorial regular cells is given as follows.

DEFINITION 4.2.87 (Framed subdivisions of framed regular cells). Let $(X, \mathcal{F})$ be an n-framed regular cell. A **framed subdivision** of $(X, \mathcal{F})$ is an n-framed regular cell complex $(Y, \mathcal{G})$, together with a stratified coarsening $F : \|Y\| \to \|X\|$ of stratified realizations, such that, for each closed cell $Y^{\geq y}$, $y \in Y$, the restriction $F|_{Y^{\geq y}} : \|(Y^{\geq y}, \mathcal{G}|_{Y^{\geq y}})\|_{\mathsf{CM}} \hookrightarrow \|(X, \mathcal{F})\|_{\mathsf{CM}}$ is a mesh map. ⌟

That these are called 'framed' subdivisions is appropriate because maps of meshes are intrinsically framed; consider Observation 4.1.88, which notes that the top component of a mesh map provides an n-framed map of euclidean subspaces in the sense of Definition 4.1.86. (Note that in this definition of a framed subdivision, the subdividing complex $(Y, \mathcal{G})$ is not assumed to be collapsible; but it will be a consequence of the classification of framed subdivisions that any subdividing complex is in fact collapsible.)

EXAMPLE 4.2.88 (Framed subdivisions). In Figure 4.24, we illustrate various subdivisions of the central square framed cell. The top row shows four framed regular cell complexes and indicates implicit corresponding coarsening maps, providing framed subdivisions of the framed cell. The bottom row shows four framed regular cell complexes, obtained from the top row by small changes to the cell structure or framings; each of these complexes does coarsen to the square cell, and so provides a non-framed subdivision, but those coarsenings are not (and cannot be chosen to be) framed subdivisions. ⌟

[29] Note that the n-framed realization of the cell-to-mesh realization of a framed cell need not be a framed realization of the framed cell in the sense of Terminology 1.3.36 because it is not necessarily linear on each simplex of the underlying framed simplicial complex of the cell. That discrepancy is not serious, though, since the cell-to-mesh realization of a framed cell is always homotopic to a realization that is in fact linear on each simplex.

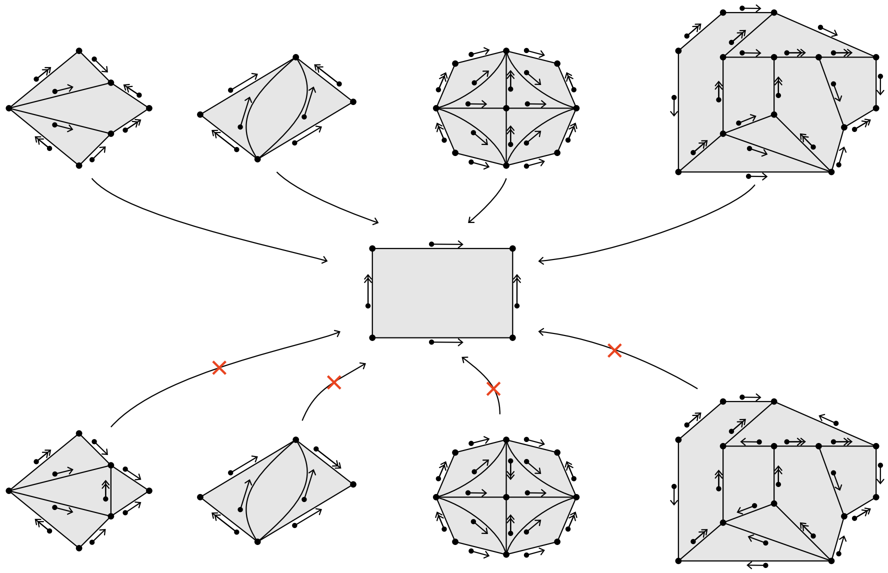


FIGURE 4.24. Framed subdivisions and non-framed subdivisions of the 2-framed square cell.

TERMINOLOGY 4.2.89 (Combinatorial subdivisions of truss blocks). Let B be an n-truss block. A 'combinatorial subdivision' of the block B is a closed n-truss T, together with a truss coarsening $T \to B$. (Given such a combinatorial subdivision, we will also say that the truss combinatorially subdivides the framed cell corresponding to the truss block.) ⌟

By contrast with ordinary subdivisions, framed subdivisions admit a combinatorial and indeed computable classification; specifically, according to Theorem 4.2.8, framed subdivisions of framed cells (in the sense of Definition 4.2.87) correspond to combinatorial subdivisions of truss blocks (in the sense of Terminology 4.2.89), which are themselves algorithmically recognizable and enumerable. We may now give the proof, which proceeds by induction on the cell dimension, using the notions of spacer and section simplices developed in Section 3.3.1.

CXD could review the next proof

PROOF OF THEOREM 4.2.8. Suppose we have a framed regular cell $(X, \mathcal{F})$ and a framed regular cell complex $(Y, \mathcal{G})$, together with a truss block B whose cell gradient $\nabla_{\mathsf{C}} B$ is the framed cell X and a closed truss T whose cell gradient $\nabla_{\mathsf{C}} T$ is the framed cell complex Y; suppose moreover that the truss T combinatorially subdivides the block B via the truss coarsening $F : T \to B$. We need to produce a corresponding framed subdivision of the cell X by the complex Y; that is, by definition, a stratified coarsening $\|Y\| \to \|X\|$ of stratified realizations, which is a mesh map on each closed cell.

Consider the mesh coarsening realization $\|F\|^{\mathsf{crs}}_{\mathsf{M}} : \|T\|_{\mathsf{M}} \to \|B\|_{\mathsf{M}}$, from Construction 4.2.76, and more specifically its total stratified map $(\|F\|^{\mathsf{crs}}_{\mathsf{M}})_n$:

$(\|T\|_{\mathsf{M}})_n \to (\|B\|_{\mathsf{M}})_n$. Recall that the total stratification of the mesh realization is the stratified realization of the total truss poset, which is the stratified realization of the cell gradient (by Construction 3.3.17), which by assumption is the framed cell complex: $(\|T\|_{\mathsf{M}})_n = \|T_n\| = \|\nabla_{\mathsf{C}} T\| = \|(Y, \mathcal{G})\|$. Similarly $(\|B\|_{\mathsf{M}})_n = \|(X, \mathcal{F})\|$. Thus the total stratified map of the mesh coarsening realization provides the required stratified coarsening $\|Y\| \to \|X\|$, and by construction it is a mesh map on cells as required.

Conversely, suppose we have an n-framed regular cell $(X, \mathcal{F})$ and an n-framed regular cell complex $(Y, \mathcal{G})$, together with a framed subdivision, i.e. a stratified coarsening $F : \|Y\| \to \|X\|$, which is a mesh map on closed cells. We need to produce a truss T and truss block B, whose cell gradients are the complex Y and the cell X respectively, together with a truss coarsening $T \to B$.

Assume by induction that we have the desired implication for $(n-1)$-framed cells and cell complexes. Now, consider the case when the n-framed cell $(X, \mathcal{F})$ is a section cell. Necessarily, in this case, all the cells of the complex $(Y, \mathcal{G})$ are also section cells. Recall from Construction 3.3.12 that we may form the integral proframed cell of the n-framed cell $(X, \mathcal{F})$, and since it is a section cell, the first map in that proframe tower has the form $(X, \mathcal{F}) \to (X, \mathcal{F}_{n-1})$; in particular that map induces an isomorphism of stratified realizations. Similarly, the first map of the integral proframe of each closed cell of the subdivision has the form $(Y^{\geq y}, \mathcal{G}|_{Y^{\geq y}}) \to (Y^{\geq y}, (\mathcal{G}|_{Y^{\geq y}})_{n-1})$. Let $(Y, \mathcal{G}_{n-1})$ denote the complex of the projected cells $(Y^{\geq y}, (\mathcal{G}|_{Y^{\geq y}})_{n-1})$, and note that the map of complexes $(Y, \mathcal{G}) \to (Y, \mathcal{G}_{n-1})$ induces an isomorphism of stratified realizations. Of course the subdivision F of $(X, \mathcal{F})$ by $(Y, \mathcal{G})$ induces a subdivision F_{n-1} of $(X, \mathcal{F}_{n-1})$ by $(Y, \mathcal{G}_{n-1})$. By induction, the framed complex $(Y, \mathcal{G}_{n-1})$ is collapsible and the subdivision F_{n-1} has a corresponding combinatorial subdivision, i.e. an $(n-1)$-truss coarsening $T^{n-1} \to B^{n-1}$; that coarsening may be considered (by adding identities at the top) as an n-truss coarsening $T^n \to B^n$, providing the desired combinatorial subdivision for the framed subdivision F.

Next, consider the case when the n-framed cell $(X, \mathcal{F})$ is a spacer cell. Let $\partial_- X$ denote the lower section cell of the spacer $(X, \mathcal{F})$, and let $\partial_- Y$ denote the framed complex determined by the preimage of the section $\partial_- X$ under the subdivision F. Observe that the framed subdivision F restricts to a framed subdivision $F_- : \|\partial_- Y\| \to \|\partial_- X\|$. The top 1-mesh bundle p_n of the n-mesh $\|(X, \mathcal{F})\|_{\mathsf{CM}}$ induces a map $q_n : \|Y\| \to \|\partial_- Y\|$ such that the following diagram commutes:

$$\begin{array}{ccc} \|Y\| & \xrightarrow{F} & \|X\| \\ {\scriptstyle q_n}\big\downarrow & & \big\downarrow{\scriptstyle p_n} \\ \|\partial_- Y\| & \xrightarrow{F_-} & \|\partial_- X\| \end{array}$$

The map q_n is in fact a 1-mesh bundle (and F is a map of 1-mesh bundles); that follows by separate consideration of each cell in the base $\|\partial_- Y\|$, using induction up the section-and-spacer tower of cells over each given cell of the base. By induction, the base $\|\partial_- Y\|$ is the total stratification of an $(n-1)$-mesh (and F_- is a map of $(n-1)$-meshes), and so the map q_n provides the top 1-mesh bundle of an n-mesh; let M denote that n-mesh and abuse notation by referring to the resulting mesh coarsening also simply as F. By construction, the mesh-to-cell gradient $\nabla_{\mathsf{MC}} M$ is the cell complex $(Y, \mathcal{G})$. Thus the complex $(Y, \mathcal{G})$ is collapsible, with the fundamental truss $\sqcap_{\mathsf{T}} M$ having cell gradient $\nabla_{\mathsf{C}} \sqcap_{\mathsf{T}} M \cong (Y, \mathcal{G})$, and the required combinatorial subdivision is given by the fundamental truss map of the mesh map F. □

EXAMPLE 4.2.90 (Combinatorial subdivisions). Figure 4.24 depicted four framed subdivisions of the square framed 2-cell. Such subdivisions are classified, according to Theorem 4.2.8, by truss coarsenings of the 2-truss integrating that framed 2-cell. In Figure 4.25, we illustrate the mesh realizations (via their framed embeddings in $\mathbb{R}^2$) of the four trusses that classify the given four framed subdivisions. ⌟

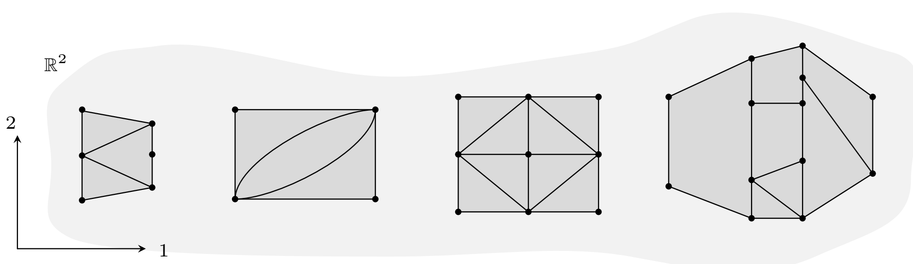


FIGURE 4.25. Mesh realizations of combinatorial subdivisions of the 2-framed square cell.

OBSERVATION 4.2.91 (Space of framed subdivisions). Given a framed cell $(X, \mathcal{F})$ and a fixed collapsible framed cell complex $(Y, \mathcal{G})$, the space $\mathcal{S}ub\mathcal{D}iv((Y, \mathcal{G}); (X, \mathcal{F}))$ of framed subdivisions of that cell by that complex (i.e. the space of stratified coarsenings that are cell-wise mesh maps) can be identified with the space of mesh coarsenings $\mathcal{M}esh_n^{\mathrm{crs}}(\|(Y, \mathcal{G})\|_{\mathsf{CM}}, \|(X, \mathcal{F})\|_{\mathsf{CM}})$. The fundamental truss map $\sqcap_{\mathsf{T}} : \mathcal{S}ub\mathcal{D}iv((Y, \mathcal{G}); (X, \mathcal{F})) \to \mathsf{Trs}_n^{\mathrm{crs}}(\textstyle\int_{\mathsf{T}}(Y, \mathcal{G}), \textstyle\int_{\mathsf{T}}(X, \mathcal{F}))$, from such subdivisions to truss coarsenings, is a weak homotopy equivalence. ⌟

REMARK 4.2.92 (Framed subdivisions can be made piecewise linear). Of course a priori a framed subdivision may be a highly nonlinear stratified map from the cell complex $(Y, \mathcal{G})$ to the cell $(X, \mathcal{F})$. But recall, from Construction 4.2.78, that we constructed piecewise linear mesh coarsening realizations of truss coarsenings, and recall, from the comments following that construction, that mesh coarsening realization is right inverse to the fundamental truss. It follows then from the preceding observation that any framed subdivision is homotopic to a piecewise linear framed subdivision. ⌟

REMARK 4.2.93 (Concrete computability of framed versus simplicial subdivision). Since framed subdivisions are classified in terms of truss coarsenings, and truss coarsenings are a finitely determined combinatorial structure which may be enumerated at worst by brute exhaustion, framed subdivisions are, as advertised, decidably enumerable. This feature is in contrast to for instance unframed simplicial subdivisions (or the theory of subdivision for various other combinatorial shapes), which are not specified or classified by the morphisms in any finitary combinatorial category. Indeed, the most effective combinatorial specification of a simplicial subdivision of a simplex is as a path, of a priori unbounded length, of Pachner moves (see [**Pac91**]). ⌟

4.2.6.2. Dualization of meshes. As a immediate collorary of the dualization of trusses and the equivalence of meshes and trusses, we now construct the dualization functors between the category of closed meshes with singular maps and the category of open meshes with regular maps.

PROOF OF COROLLARY 4.2.9. The 'mesh dualization' functors

$$\dagger : \bar{\mathcal{M}}esh_n \simeq \mathring{\mathcal{M}}esh_n : \dagger$$

are defined as the following composites:

$$\bar{\mathcal{M}}esh_n \underset{\|-\|_{\mathsf{M}}}{\overset{\sqcap_{\mathsf{T}}}{\rightleftarrows}} \bar{\mathsf{T}}\mathsf{rs}_n \underset{\dagger}{\overset{\dagger}{\rightleftarrows}} \mathring{\mathsf{T}}\mathsf{rs}_n \underset{\sqcap_{\mathsf{T}}}{\overset{\|-\|_{\mathsf{M}}}{\rightleftarrows}} \mathring{\mathcal{M}}esh_n$$

The central arrows (labeled by '†') are the dualization functors of trusses, as described in Remark 2.3.59. Since each component functor in the composites is an equivalence, so are the mesh dualization functors. □

Recall from Section 2.3.3.4 that the duals of truss blocks are truss braces; similarly, the duals of mesh blocks are mesh braces, as follows.

DEFINITION 4.2.94 (Mesh braces). An n**-mesh brace** is an open n-mesh whose total space has a single stratum that is in the closure of every other stratum. It is more specifically an n**-mesh** m**-brace** if that single 'codense' stratum is of dimension m. ⌟

EXAMPLE 4.2.95 (Mesh blocks and their dual mesh braces). In Figure 4.26, we depict a 2-mesh 2-block and its dual 2-mesh 0-brace. In the introductory Figure I.6(a), we depicted a 3-mesh 3-block and its dual 3-mesh 0-brace. That 3-mesh block corresponds to a 3-truss block and in turn to a 3-framed regular cell; that truss block and that regular cell are depicted at the top of the appendix Figure C.6. ⌟

EXAMPLE 4.2.96 (Closed meshes and their dual open meshes). In Figure 4.27, we depict a closed 2-mesh and its dual open 2-mesh. Similarly, in the introductory Figure I.6(b), we depicted a closed 3-mesh and its dual open

3-mesh. That closed 3-mesh contains two 3-mesh blocks, which correspond to two 3-mesh braces in the dual open 3-mesh. Each of those two 3-mesh blocks corresponds to a 3-framed regular cell; one of them is depicted in the middle of appendix Figure C.6, along with its corresponding 3-truss and its dual 3-mesh brace (and the inclusion of that brace in the larger dual 3-mesh).

Altogether then in Figure I.6 there are three mesh blocks, or equivalently three regular cells; combining those and a fourth cell, which is a vertical reflection of the first cell, gives a regular cell complex illustrated previously in Figure 1.56. ⌟

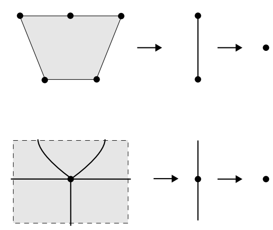


Figure 4.26. A mesh block and its dual mesh brace.

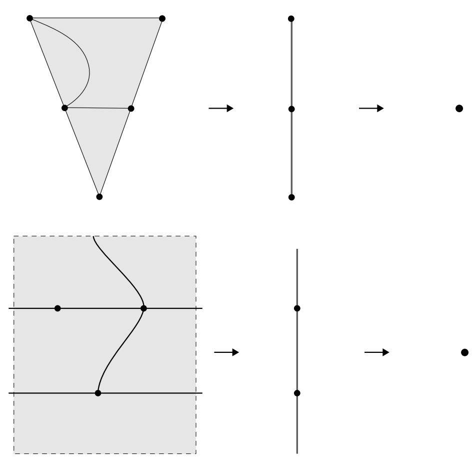
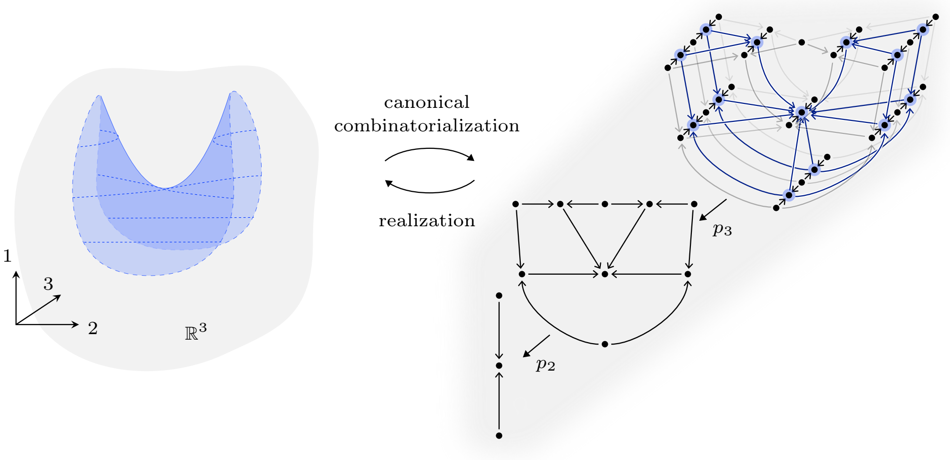


Figure 4.27. A closed mesh and its dual open mesh.

Remark 4.2.97 (Dual cell complexes via compactification). We cannot naively transport the self-duality of meshes to a self-duality of regular cell complexes; given a collapsible regular cell complex, there is a corresponding closed mesh, which dualizes to an open mesh, which does not a priori correspond back to a cell complex. However, we may nevertheless provide a duality of framed complexes 'up to compactification' as follows. Given a collapsible framed regular cell complex $(X, \mathcal{F})$, define its 'dual complex' to be the collapsible framed regular cell complex $\nabla_{\mathsf{C}}\,\overline{(\textstyle\int_{\mathsf{T}}(X,\mathcal{F}))^{\dagger}}$, that is the cell gradient of the compactification (see Definition 4.2.50) of the dual of the integral truss of the given complex. ⌟

CHAPTER 5

Tame stratifications and their combinatorializability



In this final chapter, we begin our study of framed combinatorial *stratified* topology, or more precisely, of its trivially framed model stratifications which we will refer to as *tame stratifications*. Tame stratifications provide a general class of stratifications defined by the property of being *meshable*, i.e., admitting a framed refinement by some n-mesh. Their trivial n-framing is induced by a framed embedding into standard n-framed $\mathbb{R}^n$. Importantly, tame stratifications admit a canonical decomposition into local combinatorial building blocks, called tame *cells* and tame *singularities*, which derive from the earlier notions of mesh blocks and braces. This decomposition relies on fundamental results concerning the combinatorial classification of framed stratified homeomorphism classes of tame stratifications, that we will establish in this chapter.

Tame stratifications may themselves be regarded as the building blocks, or 'universal probing objects', of more general global framed stratified spaces, much as disks and spheres are the building blocks of topological spaces. While we will not develop the resulting global theory here, the machinery of tame stratifications presented in this chapter provides the necessary foundation for such applications. The chapter concludes with an overview of these and other forthcoming directions.

4 : Note above: not true that every tame stratification can be built up from singularities, since the central point stratum of the singularity might not be a separate stratum of the tame stratification

5 : Paragraph here that overviews

5.1. ♦Tame stratifications and tame embeddings

Recall that we have been working toward a definition of a class of stratifications that is sufficiently general as to encode all reasonable finitary topological phenomena, while still admitting a combinatorial classification and, crucially, for which the problem of equivalence is algorithmically decidable. Recall that meshes provide, via their realizations, stratified cellulations of subspaces of euclidean space, which are exceptionally well behaved, in the sense that they descend along the standard projections, constructibly and inductively, to mesh cellulations of all lower-dimensional euclidean spaces. Equipped with this foundational class of mesh stratifications, we may consider all stratifications of euclidean subspaces that admit a mesh refinement. Implicitly and terminologically asserting that such stratifications satisfy the desiderata (of finitary phenomenological generality, combinatorializability, and decidability), we call these *tame stratifications*. An example of a tame stratification is illustrated on the right in **??**; this is a stratification of an open ball in $\mathbb{R}^3$ with two bulk strata, two surface strata, four line strata, and four point strata. Because coarsening preserves tameness, another tame stratification is obtained by merging all the surface, line, and point strata into a single 2-spherical stratum.

Of course, we care about stratifications outside of euclidean space, for instance about stratifications of abstract compact manifolds. We may expand our attention to that wider context, simply by considering stratified spaces with an embedding into euclidean space, whose stratified image is a constructible substratification of a tame stratification of an open euclidean subspace. We call these *tame embeddings*, and will find that they inherit the combinatorializability and decidability properties of tame stratifications. An example of a tame stratification is illustrated by the whole of **??**; on the left is a stratification of the 2-sphere, with two surface strata, four line strata, and four point strata, and the indicated stratified embedding onto a constructible substratification of the right tame stratification. Another example is obtained by merging all the strata of the source (and as before the non-bulk strata of the target); that provides a tame embedding of the unstratified 2-sphere. As it happens this embedding is the first of eleven nontrivial stages of the Morin eversion of the sphere.

OUTLINE. In Section 5.1.1, we define tame stratifications as stratifications admitting a mesh refinement, and tame embeddings as embeddings whose stratified image is a constructible substratification of an open tame stratification. In Section 5.1.2, we provide an overview of the combinatorial classification of tame stratifications by normalized stratified trusses. Finally in Section 5.1.3, we preview applications of the classification to polyhedrality and computability: we highlight the fact that framed stratified homeomorphisms of tame stratifications are homotopic to piecewise-linear homeomorphisms, and the fact that framed stratified homeomorphism of tame stratifications is algorithmically decidable.

5.1.1. ⧫The definitions. Recall that a *coarsening* is a stratified map that is a homeomorphism of underlying spaces. Also recall that the source of a coasening is called a *refinement* of the target. Given a coarsening, each stratum of the target is decomposed into a disjoint union of the images of strata of the source.

Given a class of stratifications, we may consider its 'coarsening closure', which contains all those stratifications obtained by coarsening a stratification in the given class. Applying that coarsening closure to the class of mesh stratifications provides the notion of tame stratifications, as follows. Recall that an n-mesh M comes equipped with an n-realization $\gamma : M_n \hookrightarrow \mathbb{R}^n$, and we refer to the image $\gamma(M_n) \subset \mathbb{R}^n$ as the support of the mesh.

Definition 5.1.1 (Tame stratifications). An **n-tame stratification** is a stratification (Z, f) of a space $Z \subset \mathbb{R}^n$, that admits a refinement by an n-mesh. (That is, there is an n-mesh M with support $\gamma(M) = Z$, whose realization $\gamma : (M_n, f_n) \to (Z, f)$ is a coarsening.) ⌟

Terminology 5.1.2 (Support of a tame stratification). Given a tame stratification (Z, f), we refer to the space $Z \subset \mathbb{R}^n$ as its 'support'. ⌟

Note that the support of a tame stratification is a bounded subset of euclidean space, by our standing assumption that realizations of meshes are bounded (see the comments after Convention 4.1.13). Abusing notation, we may write $M \to f$ as shorthand for the mesh refinement $(M_n, f_n) \to (Z, f)$ of a tame stratification.

It is often convenient to consider, more generally, stratifications that, though they don't per se admit a mesh refinement, have a stratified open neighborhood that admits a mesh refinement. That generalization allows us to focus attention on the most geometrically relevant aspects of the stratification, and provides the notion of ambiently tame stratifications, as follows.

Definition 5.1.3 (Ambiently tame stratifications). An **ambiently n-tame stratification** is a stratification (Y, e) of a space $Y \subset \mathbb{R}^n$, that is a constructible substratification of a tame stratification (Z, f), with $Z \subset \mathbb{R}^n$ an open subspace. ⌟

Though we have thusfar overwhelmingly focused on the nature of stratified structures within euclidean space, we care of course about general spaces and stratified spaces, in particular for instance about all manifolds. We may finally make the leap into that wider context, by considering embeddings of general stratified spaces into euclidean space, with ambiently tame image, as follows.

3 : Would remove / rewrite the "thus far" sentence: (1) we are just at the beginning of chapter 5 (2) if referring chapter 4, meshes are strats with a realization map; which is exactly what tame embeddings are. Could rewrite the "finally make the leap" part as well (less fanfare = better?). Ultimately, we just want ι (realization) to be a *choice of structure*. (In contrast, for mesh-support-like Z in prev def this makes no sense, bc it's already a *mesh* support which comes with built-in realization!)

Definition 5.1.4 (Tame embeddings). An **n-tame embedding** of a stratified space (W, g) is an embedding $\iota : W \hookrightarrow \mathbb{R}^n$, whose stratified image $\iota(W, g)$ is an ambiently tame stratification. ⌟

We think of an n-tame embedding of a stratified space (W, g) as being a choice of expressive and informative structure, analogous to a choice of n-framing on a simplicial or regular cell complex, or more classically to a choice of Morse function on a manifold.

Terminology 5.1.5 (Tame open neighborhoods). Given a tame embedding ι of the stratified space (W, g), a 'tame open neighborhood' of the tame embedding is a choice of open neighborhood Z of the image $\iota(W)$, for which there exists a tame stratification (Z, f) containing the stratified image $\iota(W, g)$ as a constructible substratification. (When context forestalls ambiguity, we may refer to a tame open neighborhood simply as an 'open neighborhood'.) ⌟

Example 5.1.6 (2-Tame stratifications). In Figure 5.1, we depict three 2-tame stratifications. In each case the underlying space Z is the union of the colored strata (where each connected colored region represents a single stratum). The first example is a polytope Z, regarded as a subspace with trivial stratification. The second example has the same subspace Z but with a non-trivial stratification. In the third example, the subspace is neither compact nor open (dashed lines indicate open boundaries). ⌟

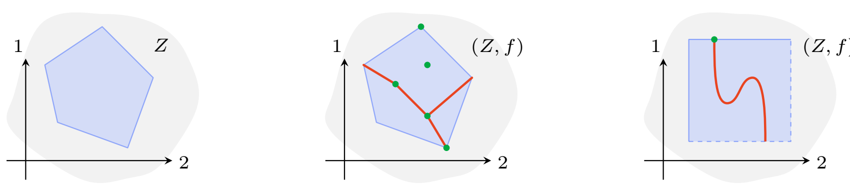


Figure 5.1. 2-Tame stratifications.

Example 5.1.7 (2-Tame embeddings). In Figure 5.2, we depict three examples of 2-tame embeddings. The first example is a 2-tame embedding of the (trivially stratified) circle. The tame open neighborhood can be chosen to be the open dashed square $Z = I^2$; there is a tame stratification of that neighborhood having strata the circle and the two components of its complement. Note that we have only drawn the image of the tame embedding, and so this may also be considered simply as an example of an ambiently tame stratification. We almost always let the source of the tame embedding be implicit in our illustrations; however one should keep the source in mind as the abstract space or stratification that is being framed by the specific embedding. The second example is a 2-tame embedding of a stratification (by manifolds) of the wedge of two circles. The third example is a tame embedding of a different stratification of the same space, namely the stratification with a single non-manifold stratum. ⌟

Example 5.1.8 (3-Tame stratifications). In Figure 5.3, we depict three 3-tame stratifications. The first example is a 3-polytope (the associahedron as it happens), regarded as a subspace with trivial stratification. The second

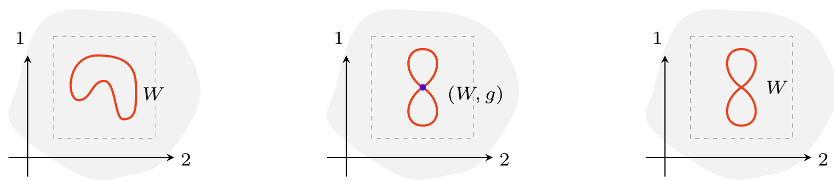


Figure 5.2. 2-Tame embeddings.

example is a cylinder with closed sides and open ends, stratified by a winding line stratum and its complement. The third example is a partially-closed, partially-open prism, stratified by a cuspidal surface stratum, a snaked line stratum, a point stratum, and two bulk strata of the remaining complement. ⌟

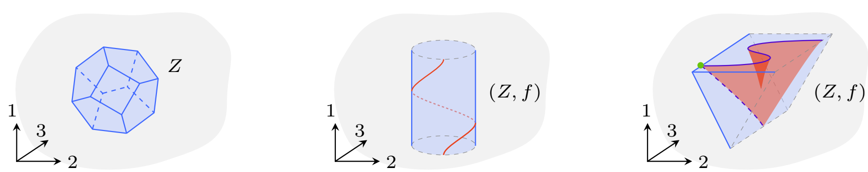


Figure 5.3. 3-Tame stratifications.

Example 5.1.9 (3-Tame embeddings). In Figure 5.4, we depict three examples of 3-tame embeddings. As before we depict only the image of the embedding. The first example represents a braid structure as a tame embedding of two intervals. The second example is a 3-tame embedding of a circle, with target an unknot with nontrivial writhe. The third example is a tame embedding of a Möbius band, stratified by its interior and its boundary. (In all three cases, the tame open neighborhood can be taken to be the open dashed 3-cube.) ⌟

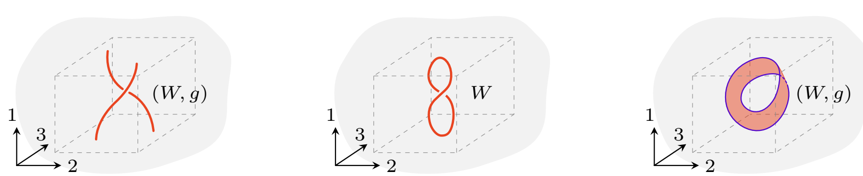


Figure 5.4. 3-Tame embeddings.

Example 5.1.10 (4-Tame stratifications and embeddings). In Figure I.7, we depicted two 4-tame stratifications. In both cases, the subspace $Z \subset \mathbb{R}^4$ is the open 4-cube. The first stratification has three 2-dimensional substrata and a single open bulk stratum; this tame stratification encodes the classical third Reidemeister move. The second stratification has a single 3-dimensional substratum and two open bulk complement strata; this tame stratification encodes the swallowtail singularity. Discarding the bulk strata, both cases may instead be considered as ambiently tame stratifications, and indeed the given coloration more directly suggests that interpretation. Furthermore, conceiving of these stratifications as depicting the images of embeddings, the first example may be considered as a 4-tame embedding of the disjoint union of three open 2-cubes, and the second example may be considered as a 4-tame embedding of an open 3-cube. ⌟

Example 5.1.11 (Untame embedding behavior). To clarify the nature of the tameness condition, in Figure 5.5 we depict a less-straightforwardly tame embedding along with an embedding that fails to be tame. The first example on the left has an infinite oscillation but nevertheless is a tame embedding of the stratified closed interval. The second example on the right also has an infinite oscillation but is not a tame embedding: the 1-dimensional stratum has infinitely many critical points with respect to the ambient standard projection $\mathbb{R}^2 \to \mathbb{R}^1$, and so no neighborhood could be refined by a mesh (as meshes have finitely many strata).

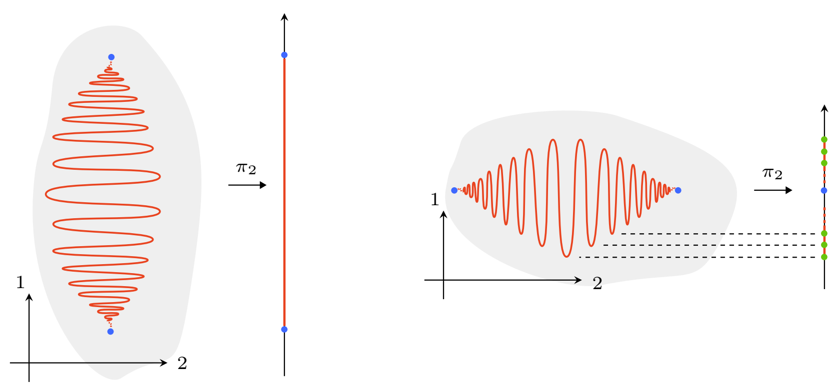


Figure 5.5. Tame and untame oscillations.

Embeddings can be untame without any overt infinitary behavior. For instance, consider the surface depicted on the left in Figure 4.11, as an embedding of the closed square into $\mathbb{R}^3$, with the third coordinate axis aligned with the right vertical closed interval. That is an apparently reasonable embedding of a manifold with boundary, but it is not tame: the ambient standard projection $\mathbb{R}^3 \to \mathbb{R}^2$, when restricted to the image of the embedding, exhibits an intrinstically unconstructible entrance path convergence structure, which no amount of refinement can eliminate.

Another subtle failure of tameness can occur when two embeddings, each of which is itself tame, exhibit a non-local interaction that is globally untame. In Figure 5.6, we depict such an embedding of two disconnected stratified intervals. Each of the two intervals is tamely embedded, a bit like the left case in Figure 5.5; however, the projection of the joint embedding to $\mathbb{R}^2$ has infinitely many intersection points, and so cannot be refined by a mesh. Note that it is possible to slightly perturb the embedding to one that is in generic position with respect to the projection, and is then in fact tame. ⌟

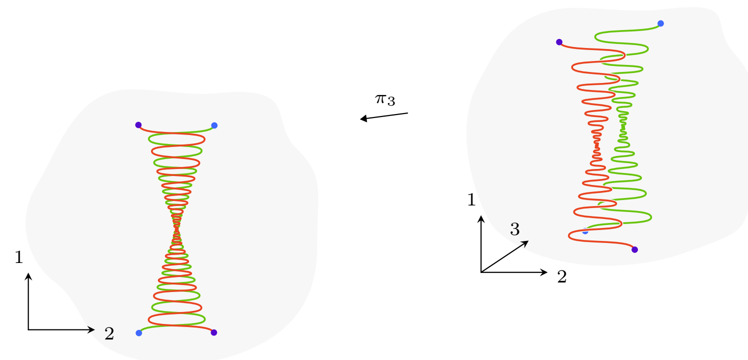


FIGURE 5.6. Nonlocal untameness.

REMARK 5.1.12 (Tameness and triangulability). Because meshes are in particular cellular stratifications, any tame embedding of a compact manifold has triangulable image (and so in particular the manifold is triangulable). Thus an untriangulable compact manifold (for instance the E_8 4-manifold [**Fre82**]) admits no tame embedding into euclidean space whatsoever. ⌟

REMARK 5.1.13 (Tame submersions). Morse functions are a class of especially well-behaved maps from manifolds to 1-dimensional euclidean space; these maps are typically of negative codimension. 'Higher Morse functions' ought to be a class of suitably and especially well-behaved maps from manifolds to n-dimensional euclidean space. Tame embeddings provide a precisely defined such class of maps in the case of *zero or positive codimension maps*. However, we can just as well handle the case of *negative codimension maps*, as follows. An **n-tame submersion** of a stratified space (W, g) is a map $\kappa : W \to \mathbb{R}^n$, that lifts along the projection $\mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ to an $(n+k)$-tame embedding $\iota : W \hookrightarrow \mathbb{R}^{n+k}$. The notion of tame submersion is a robust higher analog of Morse functions for both unstratified and stratified manifolds. ⌟

REMARK 5.1.14 (Tame immersions). Of course, classically, immersions are a natural and rewarding generalization of embeddings. One may informally consider the notion of immersion as obtained from the notion of embedding by allowing discrete (rather than singleton or empty) fibers; more formally, one may characterize an immersion as a local submersion onto its image, again with discrete fibers. In the tame context, a suitable definition is obtained as follows. An **n-tame immersion** of a stratified space (W, g) is an n-tame submersion $\kappa : W \to \mathbb{R}^n$ with discrete fibers. For instance, a 2-tame immersion of the circle is obtained by composing the middle 3-tame embedding from Figure 5.4 with the projection $\pi_3 : \mathbb{R}^3 \to \mathbb{R}^2$; the image is the subspace depicted on the right in Figure 5.2. ⌟

In considering the above notions and terminology of tame submersion and tame immersion, one should keep in mind that the adjectives *framed* and *stratified* are implicit throughout; in particular, tame submersions are actually stratified topological submersions, after a suitable refinement whose existence is ensured by tameness.

Recall from Definition 4.1.86 that, for subspaces $Z \subset \mathbb{R}^n$ and $Z' \subset \mathbb{R}^n$, a map $F : Z \to Z'$ is framed if it descends along the projections $\pi_{>i} = \pi_{i+1} \circ \ldots \pi_{n-1} \circ \pi_n : \mathbb{R}^n \to \mathbb{R}^i$ to maps $F_i : \pi_{>i}(Z) \to \pi_{>i}(Z')$. That notion provides a notion of framed maps of tame stratifications and tame embeddings, as follows.

Terminology 5.1.15 (Framed maps of tame stratifications). Given n-tame stratifications (Z, f) and (Z', f'), a **framed map of tame stratifications** (also simply a 'framed stratified map') $F : (Z, f) \to (Z', f')$ is a stratified map, whose underlying map of spaces $Z \to Z'$ is framed. ⌟

Terminology 5.1.16 (Framed maps of tame embeddings). Given n-tame embeddings $\iota : (W, g) \hookrightarrow \mathbb{R}^n$ and $\iota' : (W', g') \hookrightarrow \mathbb{R}^n$, a **framed map of tame embeddings** (also called again a 'framed stratified map') $F : \iota \to \iota'$ is a framed map $Z \to Z'$ of tame open neighborhoods $Z \supset \iota(W)$ and $Z' \supset \iota'(W')$, which restricts to a stratified map $\iota(W, g) \to \iota'(W', g')$. ⌟

Terminology 5.1.17 (Framed stratified homeomorphisms). A 'framed stratified homeomorphism of tame stratifications' is a framed stratified map that is also a stratified homeomorphism. Similarly a 'framed stratified homeomorphism of tame embeddings' is a framed stratified map $\iota \to \iota'$ whose map of tame open neighborhoods is a homeomorphism $Z \cong Z'$ and whose restriction is a stratified homeomorphism $\iota(W, g) \cong \iota'(W', g')$. ⌟

Example 5.1.18 (Framed stratified homeomorphism). In Figure 5.7, we depict a framed stratified homeomorphism F between two 2-tame embeddings of the circle. The framed map is between the indicated tame open neighborhoods Z and Z', and factors through the projections $\pi_1 : \mathbb{R}^2 \to \mathbb{R}^1$ by a map F_1, as required. (Of course the map of tame open neighborhoods is itself an example of a framed stratified homeomorphism of 2-tame stratifications.) ⌟

Example 5.1.19 (Non-framed stratified homeomorphism). In Figure 5.8, by contrast, we depict a stratified homeomorphism of two 2-tame embeddings of the circle, which though is not a framed map. ⌟

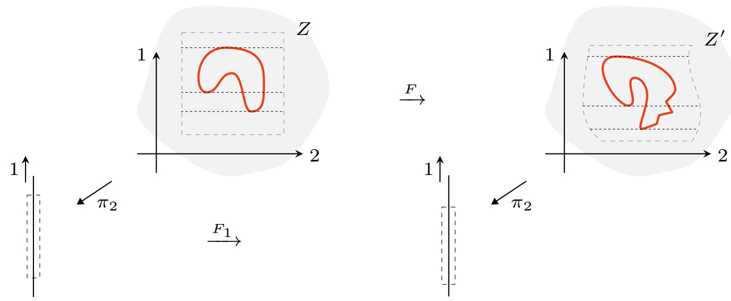


FIGURE 5.7. Framed stratified homeomorphism.

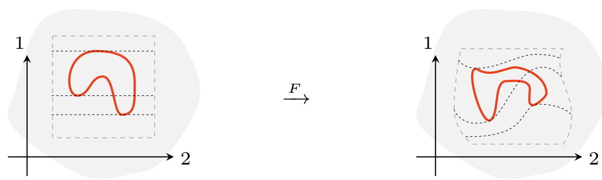


FIGURE 5.8. Non-framed stratified homeomorphism.

EXAMPLE 5.1.20 (Framed stratified map). In Figure 5.9, we depict a framed map of 3-tame embeddings, from an embedding of the 2-disc to an embedding of the thrice-punctured 2-sphere. This framed map is not a stratified homeomorphism, but it is a framed stratified homeomorphism onto its image, and so is in that sense a 'framed substratification'. ⌟

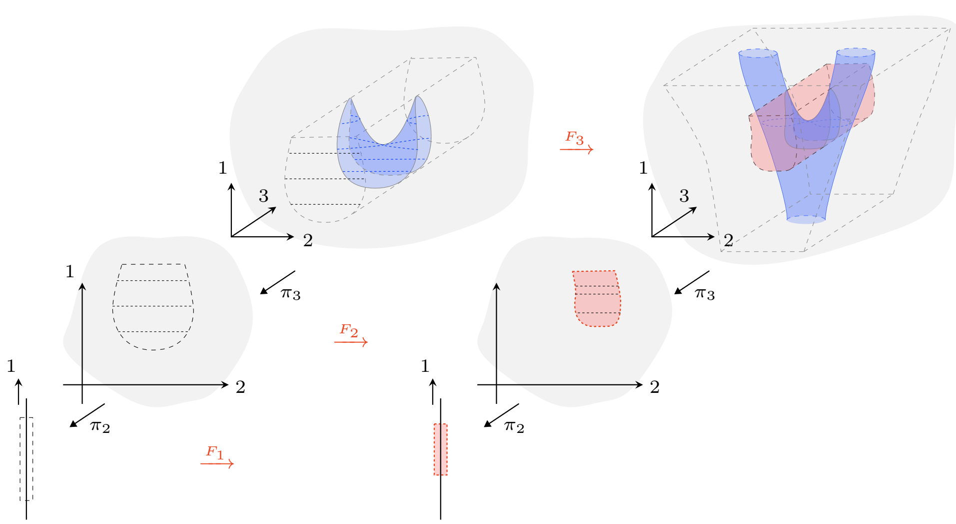


FIGURE 5.9. A framed substratification.

5.1.2. ♦Overview of the classification. Tame stratifications (and similarly tame embeddings) are defined as purely stratified topological structures, satisfying the condition of admitting a mesh refinement. We know that meshes are combinatorially classifiable (by trusses), but a tame stratification does not come equipped with any preferred refining mesh; it is therefore quite unexpected that, as we will prove, tame stratifications nevertheless also permit a combinatorial classification. At root such a classification is possible because there will turn out to be a unique *coarsest* mesh refining a given tame stratification.[30] An intricate discussion of joins of mesh stratifications, and the consequent exhibition of coarsest refining meshes, will occupy all of Section 5.2; the resulting classification of tame stratifications will be detailed and established in Section 5.3.1.

For a given tame stratification, there is a collection of refining meshes, each of which has its combinatorial truss counterpart; to identify the trusses corresponding to these refining meshes, we will need to encode the given stratification as a combinatorial structure on the fundamental trusses of those meshes. That encoding is achieved by the notion of stratified trusses, as follows. Recall from Definition 2.3.6 that a labeled n-truss has a functor from the total poset T_n to a labeling category. Regarding that poset as a topological space via its specialization topology (see Convention B.1.1), the labeling functor may itself be the characteristic map of a topological stratification (see Definition B.1.19 and Remark B.1.27).

Definition 5.1.21 (Stratified n-trusses). A **stratified n-truss** is a labeled n-truss T, whose labeling lbl_T is the characteristic map of a stratification of the total poset T_n of the truss. ⌟

The correspondence between n-meshes and n-trusses extends to a correspondence between n-meshes refining tame stratifications and n-trusses equipped with stratifications. Specifically, given an n-mesh M with a coarsening $\gamma : M \to f$ to a tame stratification f, the corresponding stratified n-truss is the fundamental n-truss $\sqcap_{\mathsf{T}} M$ together with the labeling $\sqcap\gamma : \sqcap M_n \to \sqcap f$ by the fundamental poset map of the coarsening.

Any stratified n-truss so obtained from a refining n-mesh is a combinatorial encoding of a tame stratification. However, to obtain a unique combinatorial representative, we concentrate on those stratified trusses that do not admit any further truss coarsening that is compatible with the given stratification; such stratified trusses will correspond to the aforementioned coarsest refining meshes. These uncoarsenable stratified trusses are called ‘normalized’ and are defined as follows.

[30] Contrast, for instance, the situation of triangulable manifolds, which also have, by assumption, combinatorializable refinements; but there is certainly no unique coarsest triangulation of a manifold, and triangulable manifolds are altogether hopelessly not combinatorially classifiable.

Definition 5.1.22 (Normalized stratified n-trusses). A stratified n-truss T is **normalized** when any label-preserving truss coarsening $T \to S$ is the identity. ⌟

The main result of this chapter is the combinatorial classification of tame stratifications by their correspondence with normalized stratified trusses.

Theorem 5.1.23 (Classification of tame stratifications). *Framed stratified homeomorphism classes of n-tame stratifications correspond bijectively with isomorphism classes of normalized stratified n-trusses.*

The stated correspondence takes a tame stratification to the fundamental stratified truss of its coarsest refining mesh. The proof of this result therefore hinges on the existence of such a coarsest mesh. That existence will follow in turn from the following crucial property of meshes: given any two meshes with the same support, there is a finest mutual coarsening mesh, which we call the *mesh join*.[31] As a tower of stratified bundles, the mesh join is simply the stage-wise join (in the lattice of stratifications and their coarsenings) of the two stratifications; technical attention is required merely to verify that the join is indeed again a mesh.

Remark 5.1.24 (The case of tame embeddings). There is an analogous classification for tame embeddings: framed stratified homeomorphism classes of n-tame embeddings correspond to isomorphism classes of normalized, ambient stratified open n-trusses. This result is also developed in Section 5.3.1 and established as Theorem 5.3.33. ⌟

5.1.3. ⧫Preview of applications. The classification of tame stratifications has several noteworthy consequences, which we group into those concerning *polyhedrality* and those concerning *computability*. The polyhedrality applications will be elaborated and established in Section 5.3.2, and the computability applications will be detailed and derived in Section 5.3.3.

A polyhedral stratification of a subspace of euclidean space is a stratification whose strata are unions of open simplices of a linear realization of a finite simplicial complex. Most immediately, the classification of tame stratifications implies that any tame stratification is framed stratified homeomorphic to a canonical polyhedral stratification (namely a stratified mesh realization of the corresponding normalized stratified truss).

Corollary 5.1.25 (Tame stratifications are polyhedral). *Any tame stratification is framed stratified homeomorphic to a canonical polyhedral stratification.*

Of course, since an ambiently tame stratification is a constructible substratification of a tame stratification, it is also framed stratified homeomorphic to

[31] Again we may contrast the situation of triangulations: two triangulations almost never have a mutual coarsening triangulation, even less so a distinguished finest such.

a polyhedral stratification (in fact, again a canonical one); thus any tame embedding is framed stratified homeomorphic to one with polyhedral stratified image. Conversely, polyhedral stratifications are always tame, as follows.

PROPOSITION 5.1.26 (Polyhedral stratifications are tame). *Any polyhedral stratification in $\mathbb{R}^n$ is the stratified homeomorphic image of an n-tame embedding.*

Not only are tame stratifications framed stratified homeomorphic to polyhedral stratifications, but in this context, the notions of 'framed stratified homeomorphism' and of 'framed stratified piecewise-linear homeomorphism' coincide. That result is another headline consequence of the classification of tame stratifications, as follows.

6 : There is a discrepancy between the terminology in the intro and here, polyhedral vs piecewise linear

7 : Also need to remember to flip Thm 12 and Thm 13 in the intro, and fix their headers. Moral progression is tame in euclidean space, then toward more general complexes ie n-graphs.

THEOREM 5.1.27 (Tame Hauptvermutung). *Any framed stratified homeomorphism between polyhedral stratifications is homotopic to a framed stratified piecewise-linear homeomorphism.*

That homotopy to a PL homeomorphism is in fact unique up to contractible choice. Of course this tame Hauptvermutung is startling because the classical, unframed analog fails: there are bounded polyhedral stratifications that are topologically stratified homeomorphic but not piecewise-linearly stratified homeomorphic.

REMARK 5.1.28 (Framed vs. framed PL vs. PL). We may summarize, as follows, the relationships established between framed, framed piecewise-linear, and purely piecewise-linear phenomena. The functor from tame polyhedral stratifications (i.e. polyhedral stratifications considered with framed stratified piecewise-linear maps) to tame stratifications is surjective on equivalence classes, by Corollary 5.1.25; furthermore, that functor is injective on equivalence classes by Theorem 5.1.27.

The functor from tame polyhedral stratifications to polyhedral stratifications is surjective on equivalence classes, by Proposition 5.1.26. However, that functor is far from being injective on equivalence classes, because framed stratified piecewise-linear homeomorphism is a much finer equivalence relation than (unframed) stratified piecewise-linear homeomorphism. ⌟

Having considered the polyhedrality of tame stratifications, we turn to the computability of tame stratifications.

Recall that any tame stratification has an associated unique coarsest refining mesh, and of course its combinatorial counterpart, a normalized stratified truss. It is by no means clear that there is any systematic way of identifying that coarsest mesh, or equivalently its normalized truss. But in fact we will prove that *stratified truss coarsening is confluent*, in the sense that any maximal chain of coarsenings of a stratified truss ends in its normalized truss. Consequently, at worst, any brute or greedy coarsening algorithm will yield the normalized truss, and therefore the corresponding coarsest mesh.

Corollary 5.1.29 (Canonical coarsest mesh refinements are computable). *Given a tame stratification, one can algorithmically determine its coarsest refining mesh.*

In fact we will sketch an efficient algorithm for stratified n-truss normalization (and thus stratified mesh coarsening), that proceeds by normalizing the 1-truss fibers over the projected $(n-1)$-truss, iteratively in decreasing depth within the $(n-1)$-stage truss poset, and then inductively proceeding down the truss tower using stratifications induced by increasingly expressive classifying functors.

Having established that we can compute the coarsest refining meshes of tame stratifications, and observing that the existence of a stratified isomorphism of corresponding normalized stratified trusses is certainly algorithmically determinable, we will find that equivalence of tame stratifications is decidable, as follows.

Theorem 5.1.30 (Framed stratified homeomorphism of tame stratifications is decidable). *Given two tame stratifications, one can algorithmically decide whether they are framed stratified homeomorphic.*

Of course, the classical, unframed analog of this result, whether stratified or unstratified, fails wildly: homeomorphism of (even combinatorially presented) topological spaces is algorithmically undecidable.

The computable classification of tame stratifications enables the fundamental construction of a *dual stratification* of any tame stratification: from a tame stratification, determine its coarsest refining mesh, take the associated normalized stratified truss, form the dual stratified truss (consisting of the dual truss and the stratification given by the opposite labeling functor), and finally build the stratified mesh realization. The confluence of stratified truss coarsenings dualizes to the confluence of stratified truss degeneracies; consequently any tame stratification has a computable, unique maximally degenerated normal quotient stratification.

8 : If we introduce terminology for the dual, make sure this aligns with it

The preceeding computability results concern n-tame stratifications, so in particular stratifications of subspaces of n-dimensional euclidean space. We may drastically extend the scope of these results by considering n-framed cell complexes, that need not embed in $\mathbb{R}^n$ but that admit a piecewise-linear map to $\mathbb{R}^n$ that is a framed embedding on each individual cell; we refer to such complexes as *realizable framed complexes*. Realizable framed complexes are a precise, computable, convenient, and completely general notion of 'higher-dimensional directed acyclic graphs'.

The join stratification techniques that enabled us to construct a coarsest mesh refining a given tame stratification, generalize to establish the existence and computability of a unique *coarsest cell structure* of any given realizable framed complex.[32] As the computability of the coarsest refining mesh of a

[32] Yet again, note the sharp contrast to the classical case: an unframed regular cell complex typically has numerous distinct and incompatible coarsest cell structures.

tame stratification ensured the decidability of equivalence of tame stratifications, similarly the computability of the coarsest cell structure leads to a corresponding decidability result for realizable complexes, as follows.

Theorem 5.1.31 (Framed homeomorphism of realizable complexes is decidable). *Given two realizable framed regular cell complexes, one can algorithmically decide whether they are framed piecewise-linearly homeomorphic.*

The coarsest cell structure, and the decidability of framed homemorphism, is useful even for complexes that do embed in euclidean space. In particular, tame embeddings inherit framed cell structures from the coarsest refining meshes of their tame neighborhoods, and those cell structures admit a further, often substantial, coarsening simplicifcation to their coarsest cell structures as framed complexes.

5.2. Coarsest meshes

Recall that an n-tame stratification (Z, f) is a stratification of a subspace $Z \subset \mathbb{R}^n$ that can be refined by an n-mesh M (see Definition 5.1.1). In this section, we show that there is a canonical such n-mesh refinement: namely, we construct the *coarsest* refining n-mesh of a given tame stratification.

9 : Here up not done

Outline. In Section 5.2.1, we introduce joins of stratifications as the finest mutual coarsening, define mesh joins as stage-wise stratification joins, and prove that the mesh join is indeed a mesh. In Section 5.2.2, we define the coarsest refining mesh of a tame stratification as a refining mesh that coarsens all others, and the minimal coarsest refining mesh of a tame embedding as one that cannot be coarsened or constructibly shrunk; we establish that both coarsest and minimal coarsest meshes exist and are unique. Finally, in Section 5.2.3, we provide a variety of examples of coarsest and minimal coarsest meshes of tame stratifications and tame embeddings, respectively.

5.2.1. ⧫Mesh joins.

Synopsis. We define the join of two stratifications as their finest mutual coarsening, and introduce joins of meshes and joins of mesh bundles as towers of join stratifications. We then prove the key lemma that mesh joins are themselves meshes, inductively from the join stability of 1-mesh bundles, which itself follows using auxiliary results concerning the projections and bounds of joined strata.

5.2.1.1. The definition of mesh joins. The join of two stratifications of the same space is their finest mutual coarsening. That coarsening may be, however, only a *pre*stratification; recall that prestratifications, unlike stratifications, allow cycles in their formal entrance path relation (see Terminology B.1.8).

Definition 5.2.1 (Joins of stratifications). Given stratifications f and g of a space X, the **join** $f \vee g$ is the prestratification of X, that coarsens both f and g, and such that, for any other prestratification h coarsening both f and g, there is a coarsening from $f \vee g$ to h. ⌟

Construction 5.2.2 (Joins of stratifications). Given a space X and stratifications (X, f) and (X, g), let $\sim$ denote the transitive closure of the relation, on the union of the set of strata of f and the set of strata of g, given by

$$s \sim t \iff (s \cap t \neq \emptyset) \ .$$

The join $f \vee g$ is the prestratification of X whose strata are the nonempty connected subspaces $\bigcup_{s \in \mathsf{s}} s$, where s ranges over the equivalence classes of the relation $\sim$. ⌟

Remark 5.2.3 (Joins as pushouts). The joins of (pre)stratifications (X, f) and (X, g) may also be characterized as the following pushout in the category of prestratifications:

$$\begin{array}{ccc} \mathsf{discr}(X) & \longrightarrow & (X, f) \\ \big\downarrow & & \big\downarrow \\ (X, g) & \longrightarrow & (X, f \vee g) \end{array}$$

Here $\mathsf{discr}(X)$ is the discrete stratification (see Terminology B.1.17).

Note that this pushout is preserved when taking fundamental preorders. Thus in particular, the pushout of a span of posets (in the category of preorders) need not be a poset (but merely a preorder). ⌟

EXAMPLE 5.2.4 (Joins of stratifications). In Figure 5.10, we depict two examples of a join of stratifications; the first case joins two stratifications of the closed pentagon, while the second case joins two stratifications of the closed rectangle. Note that in the second case, the join is only a prestratification, and its fundamental poset merely a preorder. ⌟

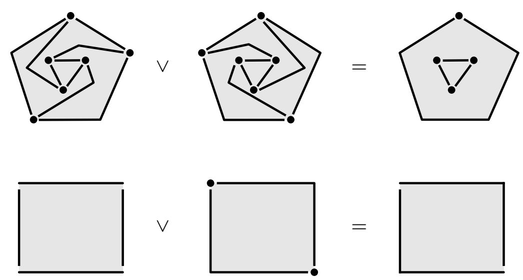


FIGURE 5.10. Joins of stratifications.

NOTATION 5.2.5 (Equivalence classes and strata in joins). Given stratifications (X, f) and (X, g), we will abuse notation and denote the strata of their join $f \vee g$ by, for instance, $\mathsf{s} := \bigcup_{s \in \mathsf{s}} s$; that is, we denote by s both an equivalence class of strata of f and g, and a stratum of the join $f \vee g$. Thus we may write $r \in \mathsf{s}$, where r is a stratum, to mean a member of the *equivalence class* s. And we may write $x \in \mathsf{s}$, where x is a point, to mean a point in the *stratum* s.

Further, we will denote by s_f the subset of the equivalence class s consisting of strata of f, and by s_g the subset of the equivalence class consisting of strata of g. ⌟

In constructing the joins of meshes and of mesh bundles, we will want and need to restrict attention to sufficiently nice base stratifications; we codify that presumption as follows.

TERMINOLOGY 5.2.6 (Sufficiently nice stratifications). A stratification is 'sufficiently nice' when it is finite, frontier-constructible, and cellulable. ⌟

Proposition 5.2.7 (Joins of sufficiently nice stratifications). *Given sufficiently nice stratifications (X, f) and (X, f'), their join $(X, f \vee f')$ is itself a sufficiently nice stratification.*

Proof. Finiteness and cellulability are both preserved under coarsening, so those properties immediately propagate to the join. To see that the join is also frontier-constructible, argue as follows. Suppose the closure of a stratum s intersects the stratum r, but does not contain it. Then there is a stratum $r \in \mathsf{r}$ which itself intersects the closure of s but is not contained in it. Assume $r \in \mathsf{r}_f$; then since s is the union of the strata in s_f, there is an $s \in \mathsf{s}_f$ whose closure intersects r but does not contain it, contradicting the frontier constructibility of f. □

Observation 5.2.8 (Joins of stratified maps). Given stratified maps $F : (X, f) \to (X', f')$ and $G : (X, g) \to (X', g')$ that are identical as maps of underlying spaces $X \to X'$, there is another stratified map

$$F \vee G : (X, f \vee g) \to (X', f' \vee g')$$

with the same underlying map of spaces. We call it the join of the maps F and G. ⌟

We will be primarily interested in the joins of meshes and their bundles. In the construction of those joins, it will be convenient to consider meshes as subspaces of euclidean space, rather than as abstract spaces equipped with an n-realization into euclidean space.

Convention 5.2.9 (Keep n-realizations implicit). Given an n-mesh M with n-realization γ (as described in Construction 4.1.70), we will usually identify the i-th stage space M_i with its image under the map $\gamma_i : M_i \hookrightarrow \mathbb{R}^i$, and so elide the maps γ_i entirely. Similarly given an n-mesh bundle p over a base (B, g), we identify the space M_i with its image in $B \times \mathbb{R}^i$ and suppress the realization maps $\gamma_i : M_i \hookrightarrow B \times \mathbb{R}^i$ (from Construction 4.1.77). ⌟

Definition 5.2.10 (Mesh joins). Consider two n-meshes M and M' consisting, respectively, of the 1-mesh bundles $p_i : (M_i, f_i) \to (M_{i-1}, f_{i-1})$ and $p'_i : (M'_i, f'_i) \to (M'_{i-1}, f'_{i-1})$; assume these meshes have identical support $M_n = M'_n \subset \mathbb{R}^n$, and so identical projected support $M_i = M'_i \subset \mathbb{R}^i$. The **mesh join** $M \vee M'$ is the tower of stratified maps $p_i \vee p'_i : (M_i, f_i \vee f'_i) \to (M_{i-1}, f_{i-1} \vee f'_{i-1})$. ⌟

Definition 5.2.11 (Join of mesh bundles). Similarly, consider two n-mesh bundles p and p', consisting of the 1-mesh bundles $p_i : (M_i, f_i) \to (M_{i-1}, f_{i-1})$ and $p'_i : (M'_i, f'_i) \to (M'_{i-1}, f'_{i-1})$; assume these have the same base and identical support. The **join of mesh bundles** $p \vee p'$ is again the tower $p_i \vee p'_i$. ⌟

When the join of mesh bundles is in fact a mesh bundle, we refer to it also as the ‘mesh bundle join’.

EXAMPLE 5.2.12 (Mesh joins). In Figure 5.11, we depict the mesh join of two open 2-meshes (represented via their realizations in $\mathbb{R}^2$). Note that the mesh join is again a 2-mesh. ⌟

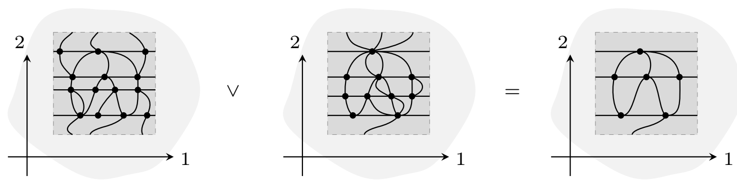


FIGURE 5.11. The join of two open meshes.

5.2.1.2. ✻ The mesh join lemma. The crucial claim, as noted in the preceding example, is that joins of meshes are themselves meshes; the proof of that claim will occupy this section.

KEY LEMMA 5.2.13 (Join stability of meshes). *Given n-meshes M and M' with identical support, their mesh join $M \vee M'$ is itself an n-mesh.*

PROOF. By induction in n, we may assume the mesh join $M_{<n} \vee M'_{<n}$ of the $(n-1)$-truncations $M_{<n}$ and $M'_{<n}$ is an $(n-1)$-mesh. For the inductive step, we need to show that the stratified map $p_n \vee p'_n : f_n \vee f'_n \to f_{n-1} \vee f'_{n-1}$ is a 1-mesh bundle. The next Lemma 5.2.14 establishes this is the case, provided that f_{n-1} and f'_{n-1} are sufficiently nice stratifications, in the sense of Terminology 5.2.6, and that $f_{n-1} \vee f'_{n-1}$ is a cellular stratification. That sufficient niceness is ensured inductively using Observation 4.1.66, Observation 4.1.67, and Proposition 4.1.63. The cellularity of $f_{n-1} \vee f'_{n-1}$ is given by Observation 4.1.75, since the join $M_{<n} \vee M'_{<n}$ is an $(n-1)$-mesh by induction. □

LEMMA 5.2.14 (Join stability of 1-mesh bundles). *Let (B, g) and (B, g') be sufficiently nice stratifications. Consider 1-mesh bundles $p : (M, f) \to (B, g)$ and $p' : (M, f') \to (B, g')$ with the same underlying map of spaces, and their join $p \vee p' : (M, f \vee f') \to (B, g \vee g')$.*

(1) The join $p \vee p'$ is a categorical 1-mesh bundle.
(2) When $g \vee g'$ is cellular, the join $p \vee p'$ is a 1-mesh bundle.

EXAMPLE 5.2.15 (A categorical bundle join). As described in the previous lemma, the join of 1-mesh bundles need only be a categorical 1-mesh bundle, if the base stratification join is not cellular. In Figure 5.12, we illustrate this situation, of a categorical mesh bundle join over a non-cellular base. ⌟

Working toward the proof of this lemma, we first establish the following two auxiliary results. (Note again that the stratified maps p and p' and $p \vee p'$ all have the same underlying map of spaces; for brevity, we typically and abusively refer to that map of spaces simply as p, no matter the stratification under consideration.)

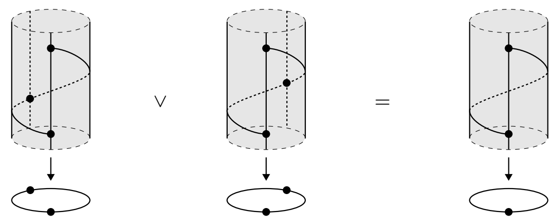


FIGURE 5.12. A categorical 1-mesh bundle as a join.

- In Lemma 5.2.16, we show that for any stratum s in $f \vee f'$, the image $p(\mathsf{s})$ is exactly a stratum r in $g \vee g'$.
- In Lemma 5.2.18, we show that for each stratum s in $f \vee f'$, there are continuous sections $\hat{\gamma}^{\pm}_{\mathsf{s}} : p(\mathsf{s}) \to p(\mathsf{s}) \times \mathbb{R}$ that fiberwise bound s from above and below.

LEMMA 5.2.16 (Joined strata project onto joined strata). *For 1-mesh bundles $p : (M, f) \to (B, g)$ and $p' : (M, f') \to (B, g')$ with identical underlying maps of spaces, the image under $p \vee p'$ of any stratum s of the join $f \vee f'$ is exactly a stratum in the join $g \vee g'$.*

PROOF. Since the join $p \vee p'$ is a stratified map, the image $p(\mathsf{s})$ is certainly contained in some stratum r of $g \vee g'$. Since p is a 1-mesh bundle, the images of strata of f are exactly strata of g. Recall from Observation 5.2.8 the class r_g of strata of g inside the stratum r of $g \vee g'$. Consider the subclass $\mathsf{r}^{\mathsf{s}}_p$ of r_g consisting of the images of strata in s_f. The union of strata in $\mathsf{r}^{\mathsf{s}}_p$ is exactly $p(\mathsf{s})$, and of course the union of strata in r_g is exactly r. It suffices then to show that $\mathsf{r}^{\mathsf{s}}_p = \mathsf{r}_g$.

Suppose, for contradiction, that $\mathsf{r}^{\mathsf{s}}_p \subsetneq \mathsf{r}_g$. Then we can find a stratum r' in the class $\mathsf{r}_{g'}$ that intersects both $\mathsf{r}^{\mathsf{s}}_p$ and $\mathsf{r}_g \setminus \mathsf{r}^{\mathsf{s}}_p$. Pick a stratum r in $\mathsf{r}^{\mathsf{s}}_p$ that intersects r'. By the definition of $\mathsf{r}^{\mathsf{s}}_p$, there is a stratum s in the class s_f projecting to r. Since r intersects r', there is at least one stratum s' in f' that projects to r' and intersects s. But r' intersects $\mathsf{r}_g \setminus \mathsf{r}^{\mathsf{s}}_p$ and so some point of s' projects into a stratum of $\mathsf{r}_g \setminus \mathsf{r}^{\mathsf{s}}_p$; the stratum of s_f containing that point projects onto that stratum of $\mathsf{r}_g \setminus \mathsf{r}^{\mathsf{s}}_p$, a contradiction. □

Recall from Notation 4.1.15, Notation 4.1.25, and Definition 4.1.28 that 1-mesh bundles must have continuous upper and lower realization bounds. In the process of showing that joins of 1-mesh bundles are 1-mesh bundles, we will need to know that every stratum similarly has continuous upper and lower bounds, in the following sense.

NOTATION 5.2.17 (Lower and upper fiber bounds). For a subspace $s \subset B \times \mathbb{R}$ and a point $x \in B$ in the image of s under the projection $\pi : B \times \mathbb{R} \to B$, the 'lower and upper fiber bounds' $\hat{\gamma}^{-}_{s}(x)$ and $\hat{\gamma}^{+}_{s}(x)$ are the lower and upper bounds of the fiber $s_x := s \cap \pi^{-1}(x) \subset \mathbb{R}$. ⌟

By Lemma 5.2.16, the image of a stratum s of the join $f \vee f'$ is a stratum of the base join $g \vee g'$. Therefore the fiber bounds are well defined over that whole base stratum, and in fact they are continuous, as follows.

LEMMA 5.2.18 (Joined strata are bounded by continuous sections). *Consider 1-mesh bundles $p : (M, f) \to (B, g)$ and $p' : (M, f') \to (B, g')$ over sufficiently nice base stratifications. For any stratum s of the join $f \vee f'$, the functions $\hat{\gamma}^{\pm}_{\mathsf{s}} : p(\mathsf{s}) \to p(\mathsf{s}) \times \mathbb{R}$, mapping the base point x to the lower and upper fiber bounds $\hat{\gamma}^{\pm}_{\mathsf{s}}(x)$, are continuous.*

PROOF. Abbreviate $\mathsf{r} = p(\mathsf{s})$. We begin with the following two observations.

(1) For each base point $x \in \mathsf{r}$, the fiber bound $\hat{\gamma}^{\pm}_{\mathsf{s}}(x) \in \mathsf{r} \times \mathbb{R}$ is either the realization bound $\gamma^{\pm}(x)$ of the bundle p (or equivalently of p') at x, or it is a point in singular strata of both f and f'. Indeed, it cannot be a point in a regular stratum of either f or f', since regular strata intersect fibers of the projection $B \times \mathbb{R} \to B$ in open intervals.

(2) For each stratum v in the equivalence class r, the map $\hat{\gamma}^{\pm}_{\mathsf{s}}$ restricts to a continuous map on v. Indeed, assume $v \in \mathsf{r}_g$ (or similarly $v \in \mathsf{r}_{g'}$). Then the intersection $\mathsf{s} \cap p^{-1}(v)$ is exactly a union of strata in f, namely those strata $\mathsf{s}_v \subset \mathsf{s}$ that lie over v. By the previous observation, the image $\hat{\gamma}^{\pm}_{\mathsf{s}}(v)$ of the fiber bound $\hat{\gamma}^{\pm}_{\mathsf{s}} : v \to v \times \mathbb{R}$ is therefore either equal to the realization bound $\gamma^{\pm}(v)$ or to some singular stratum in f lying over the stratum v. In either case, the function $\hat{\gamma}^{\pm}_{\mathsf{s}}$ is continuous on that stratum v.

If a stratum $u \in \mathsf{r}$ contains a point x at which $\hat{\gamma}^{\pm}_{\mathsf{s}}$ is not continuous (that is, $\hat{\gamma}^{\pm}_{\mathsf{s}}$ is not continuous in any neighborhood of x inside the stratum r), we will say that u is 'bad'; otherwise, we say that u is 'good'. Note that by the second observation above, discontinuities cannot occur within a stratum, and so bad strata cannot be *minimal* elements in r_g (or $\mathsf{r}_{g'}$), where r_g (or $\mathsf{r}_{g'}$) is considered as a full subposet of $\sqcap(g)$ (or $\sqcap(g')$). In particular, there exist at least some good strata, namely the minimal elements in r_g and $\mathsf{r}_{g'}$.

We now show that, given a bad stratum u, the map $\hat{\gamma}^{\pm}_{\mathsf{s}}$ is, in fact, discontinuous at *all* points of u. Assume $u \in \mathsf{r}_g$ is bad with a discontinuity at $x \in u$ (the argument is the same when $u \in \mathsf{r}_{g'}$). By the finiteness of the base stratification g, we can pick a stratum $\tilde{u}$ adjacent to u, such that $\hat{\gamma}^{\pm}_{\mathsf{s}}$ is discontinuous at x when restricted to the union $u \cup \tilde{u}$. As mentioned above, the subspace $\hat{\gamma}^{\pm}_{\mathsf{s}}(\tilde{u})$ is either a singular stratum in f or is the realization bound $\gamma^{\pm}(\tilde{u})$. It follows, by constructibility of the 1-mesh bundle $p : f \to g$ and by continuity of the bundle bounds $\gamma^{\pm}$, that either the closure of the image $\hat{\gamma}^{\pm}_{\mathsf{s}}(\tilde{u})$ contains *all* or else *none* of the image $\hat{\gamma}^{\pm}_{\mathsf{s}}(u)$. The latter case must hold, since the former would imply continuity at x (within $u \cup \tilde{u}$). By frontier-constructibility of g, the stratum $\tilde{u}$ contains all of u in its closure, and thus $\hat{\gamma}^{\pm}_{\mathsf{s}}$ is discontinuous at all points in u, as claimed.

To see finally that $\hat{\gamma}^{\pm}_{\mathsf{s}}$ is continuous on all of the base stratum r, we argue by contradiction as follows. Assume a bad stratum u exists, and suppose $u \in \mathsf{r}_g$ (again the argument is the same when $u \in \mathsf{r}_{g'}$). Denote by $\mathsf{r}^{u}_{g',1}$ the subclass of $\mathsf{r}_{g'}$ consisting of strata that intersect u. Note all strata in $\mathsf{r}^u_{g',1}$ are bad, since they each intersect u in at least one point. Moreover, the union of strata in $\mathsf{r}^u_{g',1}$ *strictly* includes u and, since good strata exist, does not cover r. (The inclusion is strict since u is not already all of r, and the stratum r of the join $g \vee g'$ is by definition the transitive closure of the stratum intersection relation.) Denote by $\mathsf{r}^u_{g,1}$ the subclass of r_g consisting of strata that intersect $\mathsf{r}^u_{g',1}$. Again, all strata in $\mathsf{r}^u_{g,1}$ are bad and their union strictly includes the union of strata in $\mathsf{r}^u_{g',1}$, but does not cover r. Denote by $\mathsf{r}^u_{g',2}$ the subclass of $\mathsf{r}_{g'}$ intersecting $\mathsf{r}^u_{g,1}$. Repeating the argument in this way, we obtain a strictly increasing infinite sequence $\mathsf{r}^u_{g',1} \subset \mathsf{r}^u_{g',2} \subset \mathsf{r}^u_{g',3} \subset \cdots \subset \mathsf{r}_{g'}$; the existence of such a sequence contradicts the finiteness of the stratification g'. □

We can now prove that the join of 1-mesh bundles is again a 1-mesh bundle.

Proof of Lemma 5.2.14. We first verify that the join $p \vee p' : f \vee f' \to g \vee g'$ is a stratified bundle whose fibers are 1-meshes. Consider a stratum s of $f \vee f'$ lying over a stratum $\mathsf{r} = p(\mathsf{s})$ of $g \vee g'$. In the preceding proof, we observed that for each stratum $v \in \mathsf{r}_g$ (or $v \in \mathsf{r}_{g'}$), either $\hat\gamma^\pm_{\mathsf{s}}(v) = \gamma^\pm(v)$ or else $\hat\gamma^\pm_{\mathsf{s}}(v)$ is a singular stratum of f (or f' respectively). Because that image $\hat\gamma^\pm_{\mathsf{s}}(v)$ being the realization bound (or being a singular stratum) propagates across strata intersections in the base, in fact either $\hat\gamma^\pm_{\mathsf{s}}(\mathsf{r}) = \gamma^\pm(\mathsf{r})$ or else $\hat\gamma^\pm_{\mathsf{s}}(\mathsf{r})$ is both a union of singular strata of f and a union of singular strata of f'; in the latter case, $\hat\gamma^\pm_{\mathsf{s}}(\mathsf{r})$ is a stratum of $f \vee f'$.

It follows that either (1) $\hat\gamma^-_{\mathsf{s}}(\mathsf{r})$ and $\hat\gamma^+_{\mathsf{s}}(\mathsf{r})$ are disjoint, or (2) $\hat\gamma^-_{\mathsf{s}}(\mathsf{r}) = \hat\gamma^+_{\mathsf{s}}(\mathsf{r})$. If (1) holds, then the stratum s is isomorphic to a product of the base stratum r and an open interval. If (2) holds, then the stratum s is a section of the bundle p over the stratum r. It follows, using the fiber bound continuity established in Lemma 5.2.18, that $p \vee p' : f \vee f' \to g \vee g'$ is stratified-locally trivial, and has 1-mesh fibers, as required.

The bundle certainly inherits its continuous realization bounds $\gamma^\pm$ from p. It remains only to verify that the join $p \vee p' : f \vee f' \to g \vee g'$ is constructible. We first verifty path-dependent constructibility, in the sense of Remark 4.1.41, or more precisely in the sense mentioned immediately after that remark. Consider an entrance path $\alpha : \mathsf{r} \to \mathsf{u}$ and a singular stratum s with $p(\mathsf{s}) = \mathsf{r}$. Define the lift entrance path $\beta : \mathsf{s} \to \mathsf{v}$ as follows: take $\beta|_{[0,1)}$ to lift $\alpha|_{[0,1)}$ along the homeomorphism $p : \mathsf{s} \to \mathsf{r}$, and set $\beta(1)$ to be the limit $\lim_{t\to 1} \beta(t)$. That this limit, and thus entrance path, exists, and that the target stratum v is singular, follows from the constructibility of f and f'. (Though the entrance path α need not be an entrance path in either g or g', there is a sequence $\{t_i \in [0,1)\}$ converging to 1, and an entrance path $\tilde\alpha : \tilde{\mathsf{s}} \to \tilde{\mathsf{r}}$ of either g or g' with $\tilde\alpha(t_i) = \alpha(t_i)$; the lift of $\tilde\alpha$ exists, ensuring the lift of α exists.) That entrance path is uniquely determined by the bare topology of the fibers. Altogether then $p \vee p'$ is a categorical 1-mesh bundle.

Finally, consider the case when the base stratification join $g \vee g'$ is cellular, i.e. by definition a constructible substratification of a locally finite regular cell complex. Recall from Proposition 1.3.13 that regular cell complexes are stratified realizations of cellular posets. Thus Proposition 4.1.43 ensures that the join $p \vee p'$ is, in fact, a 1-mesh bundle. □

That establishes the join stability of 1-mesh bundles; the join stability of n-meshes, as in Key Lemma 5.2.13, follows as previously discussed.

We briefly mention two further forms of join stability.

Lemma 5.2.19 (Join stability for mesh bundles). *Let p and p' be n-mesh bundles over the same cellular base (B, g) and with the same support in $B \times \mathbb{R}^n$. The join $p \vee p'$ is itself an n-mesh bundle.*

The proof of Key Lemma 5.2.13 applies here, verbatim after replacing 'meshes' by 'mesh bundles'.

It will be useful to consider joins in the situation where two meshes do not have identical support, but merely one support is contained in the other.

Lemma 5.2.20 (Stability for relative mesh joins). *Let M and M' be n-meshes, such that the support Z of M is a subspace of the support of M'. Denote by $M'|_Z$ the tower of stratified maps obtained by restricting the tower M' to Z. The stage-wise join $M \vee (M'|_Z)$ is itself an n-mesh.*

We omit a detailed verification; the proof follows the same structure and ideas as that of Key Lemma 5.2.13, but requires additional care regarding strata of M' that only partially intersect the support Z of M.

5.2.2. ♦The coarsest mesh constructions.

Synopsis. We define the coarsest refining mesh of a tame stratification, as a mesh refinement that coarsens all other mesh refinements; using mesh joins, we prove that coarsest refining meshes always exist. We then define minimal coarsest refining meshes of tame embeddings, as refining meshes that cannot be coarsened and also cannot be shrunk to a constructible substratification; we show that minimal coarsest refining meshes also exist and are unique.

5.2.2.1. Coarsest refining meshes of tame stratifications. Equipped with mesh joins, we may barrel directly into a discussion of coarsest refining meshes.

Terminology 5.2.21 (Meshes refining meshes). Given meshes M and N with the same support, we say that 'N refines M' or equivalently 'M coarsens N' if the identity map of the underlying support spaces is a mesh coarsening $N \to M$. ⌟

Definition 5.2.22 (Coarsest refining meshes). A **coarsest refining mesh** of an n-tame stratification (Z, f) is an n-mesh M refining (Z, f), such that for any other n-mesh N refining (Z, f), the mesh N also refines the mesh M. ⌟

Of course, if a tame stratification has a coarsest refining mesh, it has a unique coarsest refining mesh. The fact that coarsest refining meshes always exist is a fundamental and indispensable feature of the theory of tame stratifications.

THEOREM 5.2.23 (Canonical meshes of tame stratifications). *Every n-tame stratification has a coarsest refining n-mesh.*

PROOF. Given any two n-meshes M and N, both refining the n-tame stratification (Z, f), the mesh join $M \vee N$ (provided by Key Lemma 5.2.13) is another n-mesh refining (Z, f), and $M \vee N$ coarsens both M and N. Since meshes are finite stratifications, any chain of mesh coarsenings must terminate. That termination is necessarily a coarsening of every mesh refining the tame stratification, providing a (unique) coarsest refining mesh, as required. □

Illustrative examples of such coarsest refining meshes of tame stratifications will be given later in Figure 5.14, Figure 5.16, and Figure 5.18.

As one may expect, coarsest refining meshes are compatible with framed stratified homeomorphisms of tame stratifications, in the following sense. Recall from Observation 4.1.88 that mesh isomorphisms $F : M \cong N$ determine and are determined by framed stratified homeomorphisms $F_n : (M_n, f_n) \cong (N_n, g_n)$.

OBSERVATION 5.2.24 (Transporting meshes along homeomorphisms). Given a mesh M with support Z, and a framed homeomorphism $F : Z \to W$, there is a 'pushforward mesh' F_*M with support W, such that there is an n-mesh isomorphism $M \cong F_*M$ with top compontent having support map $F : Z \to W$.

Conversely, given a mesh N with support W, there is a 'pullback mesh' F^*N with support Z, such that there is an n-mesh isomorphism $F^*N \cong N$ with top component again having support map $F : Z \to W$. ⌟

PROPOSITION 5.2.25 (Framed stratified homeomorphisms preserve coarsest refining meshes). *Let (Z, f) and (W, g) be n-tame stratifications with coarsest refining meshes M and N, respectively. Any framed stratified homeomorphism $F : f \cong g$ induces an n-mesh isomorphism $F : M \to N$ between the coarsest refining meshes.*

PROOF. Since F is a framed stratified homeomorphism from the stratification f to the stratification g, the pushforward mesh F_*M refines g. Thus the mesh F_*M refines the coarsest refining mesh N. If F_*M were strictly finer than N (i.e. the identity is a nontrivial coarsening $F_*M \to N$), then pulling back along F would yield a nontrivial coarsening $M = F^*(F_*M) \to F^*N$. But M is already the coarsest refining mesh. Thus $F_*M = N$ and so F is a mesh isomorphism $M \cong N$, as required. □

5.2.2.2. ✻ Minimal coarsest refining meshes of tame embeddings. The notion of coarsest refining meshes of tame stratifications has an analog for tame embeddings. However, defining that analog requires a bit more care, as tame embeddings do not come with a predetermined choice of mesh support.

Once we fix a tame open neighborhood of a tame embedding, there is certainly a canonical coarse mesh, as follows.

Remark 5.2.26 (Canonical meshes of tame embeddings with neighborhoods). Given an n-tame embedding $\iota : (W, g) \hookrightarrow \mathbb{R}^n$, with a chosen tame open neighborhood Z, there is a canonical refining open n-mesh with support Z. Specifically, let (Z, g_+) denote the stratification consisting of the strata of $\iota(W, g)$ and the connected components of the complement $Z \setminus \iota(W)$. Note that (Z, g_+) is tame; the desired refining mesh is the coarsest refining mesh of the stratification (Z, g_+). ⌟

The preceding construction depends on a choice of tame neighborhood. We can avoid that dependency by considering meshes that both cannot be coarsened and also are minimal, in the following sense.

Terminology 5.2.27 (Refining embeddings by meshes). Given an n-tame embedding $\iota : (W, g) \hookrightarrow \mathbb{R}^n$, we say an open n-mesh M 'refines the embedding' ι if each stratum in $\iota(W, g)$ is a union of strata of (M_n, f_n). We will write and draw this refinment as $M \to \iota$ or $M \to \iota(W, g)$, as an analog of the coarsening of stratifications. ⌟

Definition 5.2.28 (Minimal coarsest refining meshes). For a tame embedding ι, a **minimal coarsest refining mesh** is an open mesh M refining the embedding ι, such that

(1) the mesh M cannot be strictly coarsened to another mesh refining the embedding, and
(2) the mesh M contains no proper constructible substratification, which is itself an open mesh refining the embedding. ⌟

Example 5.2.29 (Minimal coarsest refining mesh). In Figure 5.13, we depict a tame embedding and a number of meshes refining stratified neighborhoods of that embedding. The embedding is of an X crossing, with closed upper endpoints and open lower endpoints, into $\mathbb{R}^2$, with a single stratum. Mesh (a) refines a stratified neighborhood of the embedding, but is not an open mesh and therefore not a refinement of the embedding per se. Mesh (b) can be coarsened to another refining mesh, though it contains no proper constructible substratification that is an open mesh refining the embedding. Mesh (c) cannot be coarsened to another refining mesh, but it does contain a proper constructible substratification that is an open mesh refining the embedding. Mesh (d) is finally a minimal coarsest refining mesh. ⌟

Further examples of minimal coarsest refining meshes are given later, on the left of Figure 5.15, in Figure 5.17, and in Figure 5.19.

Notice that coarsest refining meshes and minimal coarsest refining meshes are defined rather differently: the former via the universal property of being the coarsest, the latter via the property of being both uncoarsenable and

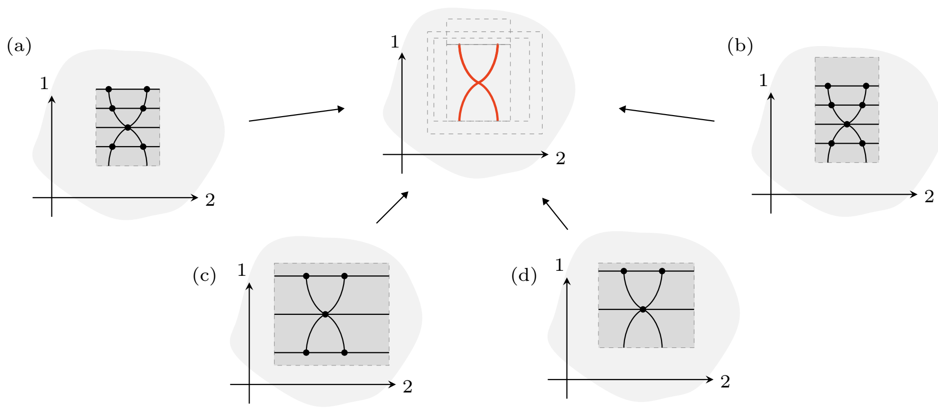


FIGURE 5.13. Refining meshes and a minimal coarsest refining mesh.

unshrinkable. As a result, the proof of the existence of these structures, though based on similar ideas, has a superficially different structure.

THEOREM 5.2.30 (Canonical meshes of tame embeddings). *Every n-tame embedding has a unique (up to n-mesh isomorphism) minimal coarsest refining n-mesh.*

PROOF RECIPE. Existence is straightforward: given any refining mesh, repeatedly either strictly coarsen it to another refining mesh or take a proper constructible substratification that is an open refining mesh; as the stratifications are finite, this process must terminate in a minimal coarsest refining mesh. It remains to show the minimal coarsest refining mesh is unique.

First, given the tame embedding $\iota : (W, g) \hookrightarrow \mathbb{R}^n$, construct a 'projected' $(n-1)$-tame embedding $\iota_{n-1} : (W_{n-1}, g_{n-1}) \hookrightarrow \mathbb{R}^{n-1}$, with $W_{n-1} := \pi_n \circ \iota(W) \subset \mathbb{R}^{n-1}$, as follows. Pick any tame open neighborhood Z of $\iota(W)$, and consider the canonical refining mesh M. (In fact, any other refining mesh would also suffice.) Define a filtration $X_0 \subset X_1 \subset \cdots \subset X_{n-2} \subset X_{n-1} = W_{n-1}$, with $X_i^\circ := X_i \setminus X_{i-1}$ being an open subset of the i-skeleton of (M_{n-1}, f_{n-1}) (seen as a cell complex): inductively in decreasing i, set X_i° to be the maxmal open subset of X_i on which $\pi_n : \iota(W, g) \to W_{n-1}$ restricts to a stratified bundle, when stratifying X_i° by its connected components. Then let g_{n-1} be the stratification of W_{n-1} induced by this filtration (as in Remark B.1.48). (The resulting stratification g_{n-1} depends neither on the choice of tame open neighborhood, nor the choice of refining mesh.)

Now pick two minimal coarsest refining meshes M and M' of the tame embedding ι. Observe that the $(n-1)$-truncations (M_{n-1}, f_{n-1}) and (M'_{n-1}, f'_{n-1}) are minimal coarsest refining meshes of ι_{n-1}. By induction, there is an $(n-1)$-mesh isomorphism $F_{n-1} : (M_{n-1}, f_{n-1}) \cong (M'_{n-1}, f'_{n-1})$, thus in particular an isomorphism of the corresponding fundamental trusses. Further inductively claim and assume that identified strata coincide pointwise, except when one stratum has points outside the support of the mesh containing the other stratum. For the inductive step: note that neither (M_n, f_n) nor (M'_n, f'_n) can have singular strata with points outside the support of the other mesh (since removing all such singular strata would yield a coarser refining mesh); and singular strata in the joint support $M_n \cap M'_n$ must be identical in the two meshes (otherwise construct a coarser refining mesh by taking a mesh bundle join). That much implies the inductive claim about strata coinciding, and shows that M and M' have identical fundamental trusses, therefore are isomorphic as meshes, as required. □

Note that, in fact, the framed stratified homeomorphism constructed in the preceding proof, between two minimal coarsest refining meshes, can (inductively) be chosen to completely fix all strata that pointwise coincide in the two meshes.

Observation 5.2.31 (Minimal coarsest mesh inductive construction). There is a more systematic construction of a minimal coarsest mesh, of a tame embedding $\iota : (W, g) \hookrightarrow \mathbb{R}^n$, as follows. First, by induction, construct a minimal coarsest refining $(n-1)$-mesh (M_{n-1}, f_{n-1}) of the projected tame embedding ι_{n-1} (obtained as in the preceding proof recipe). Next, refine g (only as much as necessary) to obtain a stratified bundle $\tilde{g} \to f_{n-1}$. Finally, construct a minimal coarsest refining 1-mesh bundle $\tilde{\tilde{g}} \to f_{n-1}$, that refines the stratified bundle $\tilde{g}$; this extends the $(n-1)$-mesh to an n-mesh and provides the required minimal coarsest refining n-mesh of ι. ⌟

Lemma 5.2.32 (Framed stratified homeomorphisms preserve minimal coarsest refining meshes). *Let $\iota : (W, g) \hookrightarrow \mathbb{R}^n$ and $\iota' : (W', g') \hookrightarrow \mathbb{R}^n$ be n-tame embeddings with minimal coarsest refining meshes M and N, respectively. If there is a framed stratified homeomorphism $\iota \cong \iota'$, then there exists an n-mesh isomorphism $M \cong N$ between the minimal coarsest refining meshes.*

Proof. Observe that given a tame embedding, for any tame open neighborhood Z, there is a minimal coarsest refining mesh contained in Z. (By an inductive argument, any minimal coarsest refining mesh can have its support shrunk sufficiently, while fixing all strata entirely contained in Z.) By definition, the framed stratified homeomorphism of tame embeddings is a homeomorphism of tame open neighborhoods (that restricts to a stratified homeomorphism of the embedding images); a minimal coarsest refining mesh may be transported across that homeomorphism. The result follows from the uniqueness assurance of Theorem 5.2.30. □

5.2.3. ⧫Examples of coarsest meshes. We illustrate a range of examples of coarsest refining meshes of tame stratifications and minimal coarsest refining meshes of tame embeddings, in dimensions 2, 3, and 4. We see in practice how these coarsest meshes record changes in the stratified homeomorphism type of the fibers of the standard tower of projections, wholistically encoding all the singularities of all strata under those projections along with the interactions local and nonlocal of those singularities and strata.

Example 5.2.33 (Coarsest and non-coarsest refining meshes). In Figure 5.14, we depict a tame stratification, in the middle, along with two refinements on the left and right. The right refinement coarsens to the left refinement and is therefore not itself coarsest; the left refinment is in fact the coarsest refining mesh.

In Figure 5.15, we similarly depict a tame embedding, in the middle, along with two refinments on the left and right. Again the right refinment coarsens to the left, and is therefore not minimal coarsest; the left refinment is the minimal coarsest refining mesh. ⌟

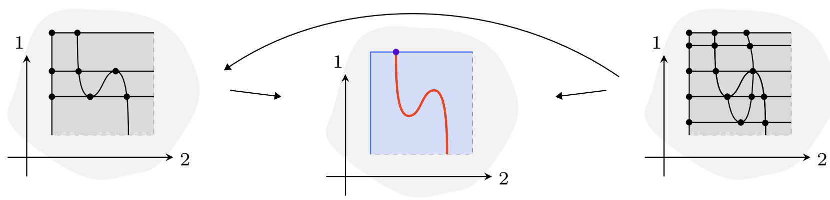


Figure 5.14. Refining meshes of a tame stratification.

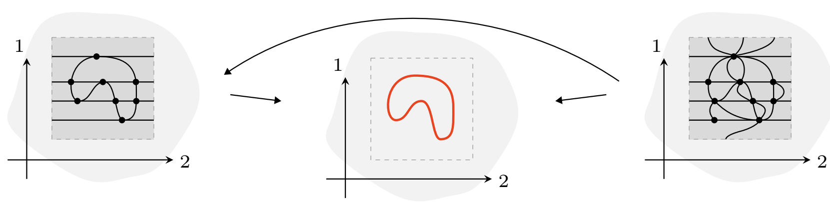


Figure 5.15. Refining meshes of a tame embedding.

Example 5.2.34 (Coarsest and minimal coarsest meshes in dimension 2). In Figure 5.16, we depict two 2-tame stratifications, namely a stratified polytope (on the left) and an unstratified polytope (on the right), along with their shared coarsest refining mesh.

In Figure 5.17, we depict two 2-tame embeddings, namely the figure eight with a basepoint stratum (on the left) and the figure eight as a single stratum (on the right), along with their shared minimal coarsest refining mesh. ⌟

Example 5.2.35 (Coarsest meshes in dimension 3). In Figure 5.18, we depict the coarsest refining meshes of two 3-tame stratifications. The lower left stratification is a cylinder with a single 3-dimensional bulk stratum and a single 1-dimensional line stratum. Notice that the line stratum has no singularities with respect to either the projection to $\mathbb{R}^2$ or to $\mathbb{R}^1$. Nevertheless in the coarsest refining mesh it is split into three segments, because the left and right edges of the cylinder are singular for the projection to $\mathbb{R}^2$, and the line stratum intersects those singular loci at two points.

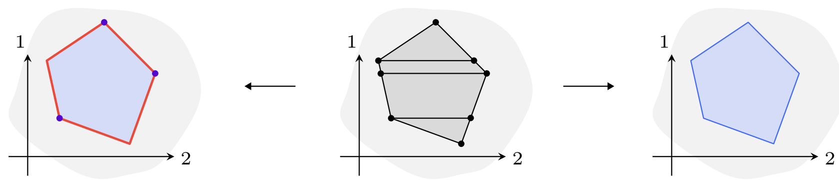


FIGURE 5.16. The coarsest refining mesh of two 2-tame stratifications.

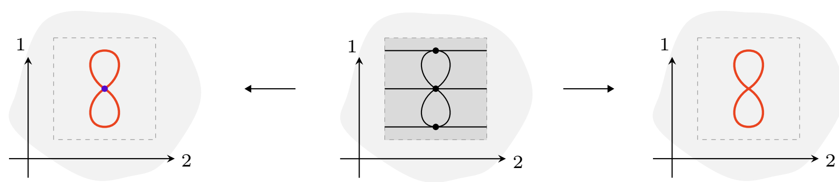


FIGURE 5.17. The minimal coarsest refining mesh of two 2-tame embeddings.

The lower right stratification is a half-closed half-open prism with two 3-dimensional bulk strata and a single 2-dimensional surface stratum. The surface stratum has a smooth arc of singlularities of the projection to $\mathbb{R}^2$, and that arc itself has a cuspidal point singularity for that projection and also at the same point an ordinary Morse singularity for the projection to $\mathbb{R}^1$. The coarsest mesh isolates the singular arc and splits it at the cusp point. ⌟

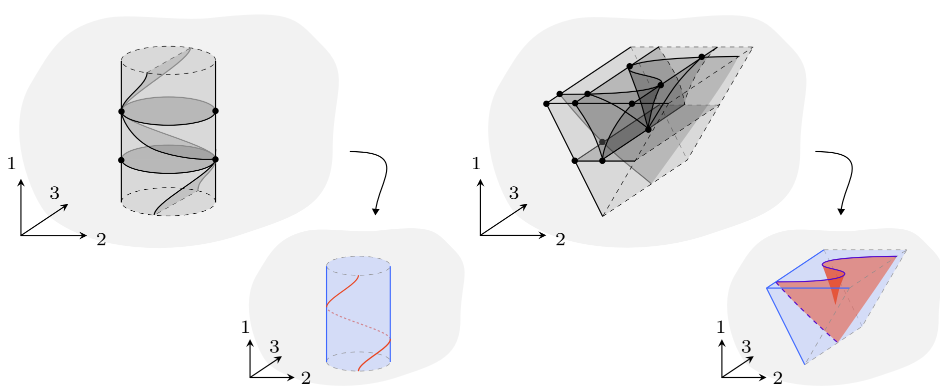


FIGURE 5.18. Coarsest meshes of 3-tame stratifications.

EXAMPLE 5.2.36 (Minimal coarsest meshes in dimension 3). In Figure 5.19, we depict the minimal coarsest refining meshes of two 3-tame embeddings (both trivially stratified). The lower left embedding is a pair of pants surface. Its minimal coarsest mesh records the seams of those pants (singular for projection to $\mathbb{R}^2$) and the split of the inner seam at its central point (singular for projection from the seam to $\mathbb{R}^1$).

On the lower right is the Hopf embedding of the circle, that is, an embedding ι such that the projection $\pi_3 \circ \iota : S^1 \to \mathbb{R}^2$ is an immersion with a single double point. The minimal coarsest mesh records the preimages of that double point, along with the two Morse points of the projection to $\mathbb{R}^1$. This open mesh is dual to a closed mesh, which in turn corresponds to a regular cell complex; in this sense, that regular cell complex is dual to the Hopf circle, as illustrated and informally observed all the way back in Figure 1.56. ⌟

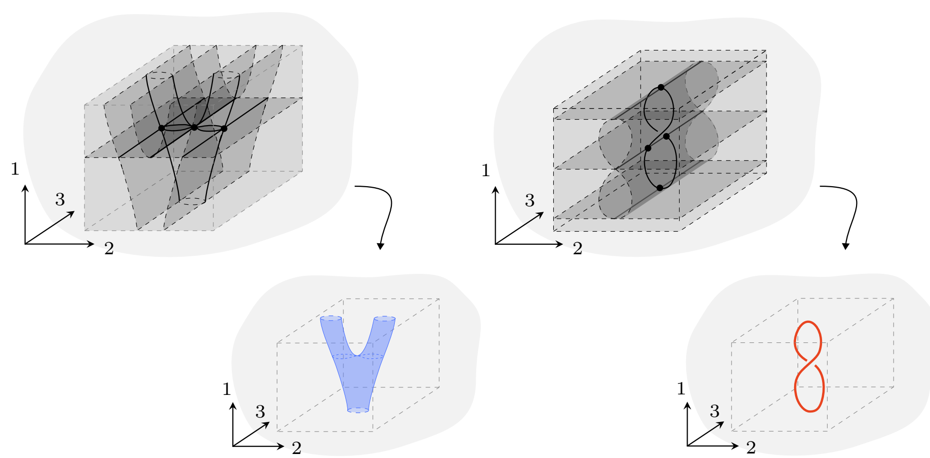


FIGURE 5.19. Minimal coarsest meshes of 3-tame embeddings.

EXAMPLE 5.2.37 (Coarsest meshes in dimension 4). Recall from Example 5.1.10 the description of the two 4-tame stratifications illustrated in Figure I.7, encoding respectively the third Reidemeister move and the swallowtail singularity. In Figure I.8 and Figure I.9, we depicted the coarsest refining meshes of these two stratifications. In the Reidemeister case, the second stage of the mesh provides a concise portrait of the geometry of the braid crossings. Similarly in the swallowtail case, the second stage of the mesh displays the interaction of the two cusp points with the self braiding of the fold locus. ⌟

5.3. Tractability of tame stratifications

In this section we discuss a first set of key properties of tame stratifications.

OUTLINE. We prove the central theorem about the combinatorializability of tame stratifications in Section 5.3.1, establishing that tame stratifications are classified by a special class of labeled trusses. We then generalize these results to tame stratified *bundles* in Section 5.3.1.5. In Section 5.3.2 we discuss a set of corollaries relating to the piecewise linear structure of tame stratifications and their maps, showing in particular that the notions of topological and PL framed stratified homeomorphism coincide, and that any PL stratification is tame. In Section 5.3.3 we record corollaries regarding the computability of tame stratitications, showing that framed stratified homeomorphism is decidable (with an extension to a class of 'realizable' framed regular cell complexes). Finally, in Section 5.3.3.2 we introduce the key notions of tame singularities and tame cells, and discuss the geometric dualization construction of tame stratifications that relates them.

5.3.1. ⧫Combinatorializability.

10 : Here up not done

11 : Definitely will need some clear motivation, to pave the way for stratified trusses and stratified meshes

SYNOPSIS. We recall the notion of stratified trusses, as labeled trusses whose labeling is the characteristic map of a stratification, and the notion of normalized stratified trusses as those admitting no label-preserving truss coarsening. We then define stratified meshes, as tame stratifications equipped with a choice of mesh refinement. We provide a correspondence between stratified trusses and stratified meshes, via a fundamental stratified truss construction and a stratified mesh realization construction. We observe that a stratified mesh has no mesh coarsening exactly when the corresponding stratified truss is normalized, and thereby complete the proof of the classification of tame stratifications by normalized stratified trusses. Finally, we discuss bundles of tame stratifications and their classification by normalized stratified truss bundles.

5.3.1.1. ⧫Stratified trusses. Recall from Definition 5.1.21 that a stratified n-truss is a poset-labeled n-truss T whose labeling lbl_T is the characteristic map of a stratification on the total poset T_n.

NOTATION 5.3.1 (Fundamental posets and strata of stratified trusses). To highlight that a labeled truss $T = (\underline{T}, \mathrm{lbl}_T)$ is a stratified truss, we will usually denote the poset of labels by $\mathbb{\Pi}(T)$, and refer to it as the 'fundamental poset' of the stratified truss. The 'strata' of the stratified truss T are, by definition, the connected subposets of T_n given by the preimages $\mathrm{lbl}_T^{-1}(x)$, for $x \in \mathbb{\Pi}(T)$. ⌟

EXAMPLE 5.3.2 (Stratified trusses). In Figure 5.20, we depict a stratified 2-truss with four strata. We indicate the labeling map lbl_T by coloring preimages in the same color as their target object in the labeling poset. In later examples, we often leave the labeling map implicit, and simply provide the coloring of the total poset of the stratified truss. ⌟

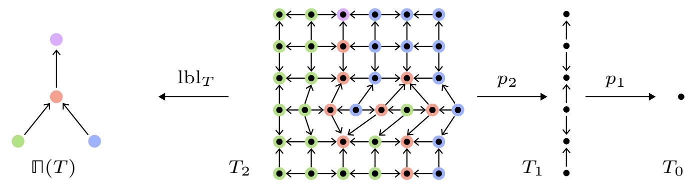


Figure 5.20. A stratified 2-truss.

The condition on a stratified truss, that the labeling map is a characteristic map, can be rephrased combinatorially as follows.

Terminology 5.3.3 (Quotient and connected-quotient maps). A 'quotient map' of posets is a surjective poset map for which a subposet of the codomain is open if and only if its preimage is open in the domain. (Recall a subposet is open when it is downward closed.) A 'connected-quotient map' of posets is a quotient map of posets whose preimages are connected. ⌟

Observation 5.3.4 (Characteristic maps are connected-quotient maps). A labeling $\mathrm{lbl}_T : T_n \to P$ of a truss T in a poset P is the characteristic map of a stratification if and only if it is a connected-quotient map. This is show (in a slightly more general form) in Lemma B.1.39. ⌟

Remark 5.3.5 (Stratification from subposet decomposition). Given an n-truss T, any decomposition of T_n into connected subposets determines a stratified n-truss, with underlying truss T and with strata being the given subposets. ⌟

As a consequence of the preceding remark, we can construct stratified trusses from arbitrary poset-labelings, as follows.

Construction 5.3.6 (Stratifications from poset labelings). Given any poset-labeled truss $T = (\underline{T}, \mathrm{lbl}_T)$, there is an associated stratified truss $\tilde{T}$, with strata being the connected components of the non-empty preimages of the labeling lbl_T. (This is an example of the 'connected component splitting' construction, formalized in Construction B.1.44 in the broader context of general stratifications.) ⌟

Example 5.3.7 (Truss stratifications via poset-labelings). The preceding construction is convenient when illustrating stratified trusses: we may replace a given characteristic map with a labeling in a smaller poset, whose connected component splitting recovers the characteristic map; this reduces the number of labeling colors, without any sacrifice in clarity. For instance, Figure 5.21 depicts a poset labeling of a truss, whose associated stratification is the one previously given in Figure 5.20. ⌟

Terminology 5.3.8 (Ambient stratified trusses). An 'ambient stratified n-truss' is a stratified truss T with a chosen subset of strata $A \subset \mathbb{P}(T)$ called

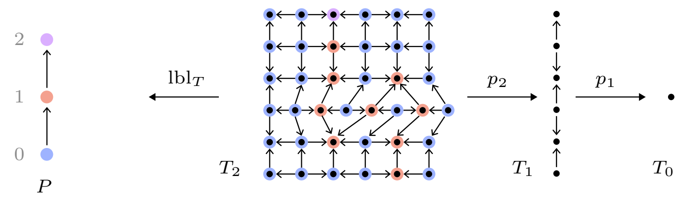


FIGURE 5.21. A stratified truss represented by a poset labeling.

'ambient strata'. In illustrations of ambient stratified trusses, we typically leave the ambient strata uncolored and indicate the corresponding poset element by a white circle.

Note that in practice, we will be concerned exclusively with the case when each ambient stratum is open (and the terminology is meant to suggest this), but we do not insist on this condition.

EXAMPLE 5.3.9 (An ambient stratified truss). In Figure 5.22, we depict an ambient stratified 2-truss, utilizing both our convention for uncolored ambient strata and for poset-labeled stratifications. There is a single '0-dimensional' stratum, colored pastel purple, two '1-dimensional' strata, colored pastel red, and three '2-dimensional' strata, uncolored.

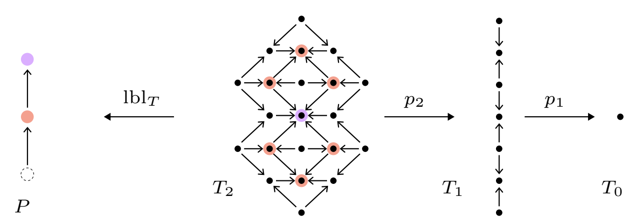


FIGURE 5.22. An ambient stratified 2-truss.

The notion of maps of stratified trusses is directly inherited from the notion of maps of labeled trusses, see Terminology 2.3.34, as follows.

DEFINITION 5.3.10 (Maps of stratified trusses). A **map of stratified n-trusses** $F : T \to S$ is simply a map of labeled n-trusses; i.e. there is an underlying map of n-trusses $\underline{F} : \underline{T} \to \underline{S}$ and a map of labelings $\mathrm{lbl}_F : \sqcap(T) \to \sqcap(S)$ such that $\mathrm{lbl}_F \circ \mathrm{lbl}_T = \mathrm{lbl}_S \circ F_n$.

Note that a map of stratified trusses provides an actual map of stratifications of the total posets (see Definition B.2.1).

Recall that a map of labeled n-trusses was called a coarsening (see Terminology 2.3.65) when every constituent 1-truss bundle map is a surjective regular map preserving endpoint types. In the specific case of stratified trusses, we restrict the use of that term, and also distinguish two special cases, as follows.

TERMINOLOGY 5.3.11 (Coarsenings of stratified trusses). Let $F : T \to S$ be a map of stratified n-trusses.

- The map F is a **label coarsening** if the underlying truss map $\underline{F}$ is the identity, and the label map lbl_F is a connected-quotient map (see Terminology 5.3.3).
- The map F is a **truss coarsening** if the underlying truss map $\underline{F}$ is a coarsening of n-trusses, and the label map lbl_F is the identity.
- The map F is a **coarsening** if the underlying truss map $\underline{F}$ is a coarsening of n-trusses, and the label map lbl_F is a connected-quotient map. ⌟

With this terminology at hand, recall from Definition 5.1.22 that a stratified truss is normalized when it has no non-identity (label-preserving) truss coarsening.

REMARK 5.3.12 (Label coarsenings are stratified coarsenings). Note that, when the stratified truss map $(\underline{F}, \mathrm{lbl}_F) : (\underline{T}, \mathrm{lbl}_T) \to (\underline{S}, \mathrm{lbl}_S)$ is a label coarsening, the top component $\underline{F}_n = \mathrm{id}_{T_n}$ of the identity truss map $\underline{F} = \mathrm{id}_T$ induces a coarsening of stratified spaces $(T_n, \mathrm{lbl}_T) \to (T_n, \mathrm{lbl}_S)$ (see Lemma B.2.12). ⌟

EXAMPLE 5.3.13 (A truss coarsening). In Figure 5.23, we depict a truss coarsening F of stratified 2-trusses. By Terminology 5.3.11, the label map is the identity, and is not drawn. Note that the target of this coarsening is still not normalized. ⌟

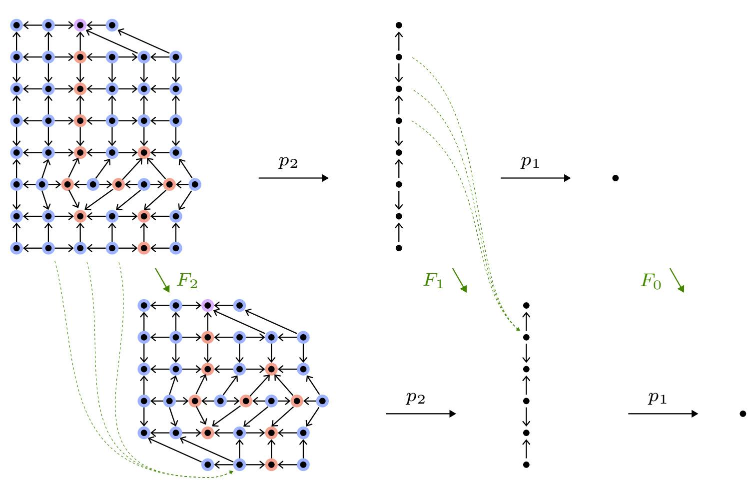


FIGURE 5.23. A truss coarsening of stratified trusses.

Note that every coarsening can be written both as a unique composite of a truss coarsening followed by a label coarsening, and as a unique composite of a label coarsening followed by a truss coarsening.

Recall that truss blocks and truss braces (see Definition 2.3.73 and Definition 2.3.100) are the local building components of closed trusses and of open trusses, respectively. Anticipating our development of tame cells and tame singularities as local components of tame stratifications, we introduce the following combinatorial components of stratified trusses.

Definition 5.3.14 (Truss cells and truss singularities). An **n-truss m-cell** T is a stratified n-truss m-block whose initial element $\bot \in T_n$ is label isolated, in the sense that $\mathrm{lbl}_T^{-1}(\mathrm{lbl}_T(\bot)) = \{\bot\}$.

Dually, an **n-truss m-singularity** T is a stratified n-truss m-brace whose terminal element $\top \in T_n$ is label isolated. ⌟

Example 5.3.15 (Truss singularities). In Figure 5.24, we depict a few truss singularities. For each n-truss m-singularity, the terminal element will later correspond to a stratum of a tame singularity of dimension m.

Dual truss cells, of each truss singularity, can be obtained by stagewise dualizing the given posets. For each resulting n-truss $(n-m)$-cell, the initial element will later correspond to a stratum of a tame cell of dimension $n-m$. ⌟

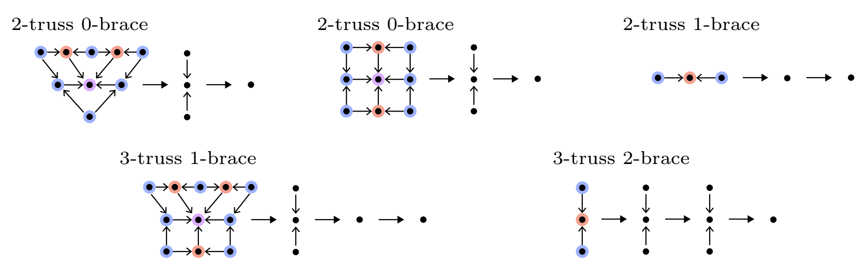


Figure 5.24. Truss singularities.

5.3.1.2. ♦Stratified meshes. As a stratified truss is a truss together with groupings of its elements into strata, a stratified mesh is a mesh together with groupings of its cells into strata; in fact we have already encountered that structure (viewed from the opposite perspective) as a tame stratification together with a mesh refinement.

Definition 5.3.16 (Stratified meshes). A **stratified n-mesh** is an n-tame stratification (Z, f) together with a choice of refining mesh $M \to f$. ⌟

Recall that the notation $M \to f$, for a refining mesh of a tame stratification, is shorthand for a refinement $(M_n, f_n) \to (Z, f)$ by the top stage of the mesh.

We will typically denote a stratified mesh by the pair (M, f), leaving the refinement implicit; this notation also suggests the interpretation that the stratification f is a stratification 'of the mesh M' in the sense that it encodes a merging of the mesh cells into larger strata.

Definition 5.3.17 (Maps of stratified meshes). A **map of stratified n-meshes** $F : (M, f) \to (N, g)$ is an n-mesh map $F : M \to N$ whose top component $F_n : (M_n, f_n) \to (N_n, g_n)$ induces a map of stratifications $f \to g$. ⌟

Recall that a map of n-meshes was called a coarsening (see Terminology 4.1.91) when every constituent 1-mesh bundle map is a coarsening on every fiber. In the specific case of stratified meshes, we restrict the use of that term, and distinguish two special cases.

Terminology 5.3.18 (Coarsenings of stratified meshes). Let $F : (M, f) \to (N, g)$ be a map of stratified n-meshes.

- The map F is a **stratification coarsening** if $M = N$, and $F_n : f \to g$ is a coarsening of stratifications.
- The map F is a **mesh coarsening** if $F : M \to N$ is a coarsening of n-meshes, and $f = g$.
- The map F is a **coarsening** if $F : M \to N$ is a coarsening of n-meshes and $F_n : f \to g$ is a coarsening of stratifications. ⌟

Example 5.3.19 (A mesh coarsening). In Figure 5.25, we depict a mesh coarsening of stratified 2-meshes. ⌟

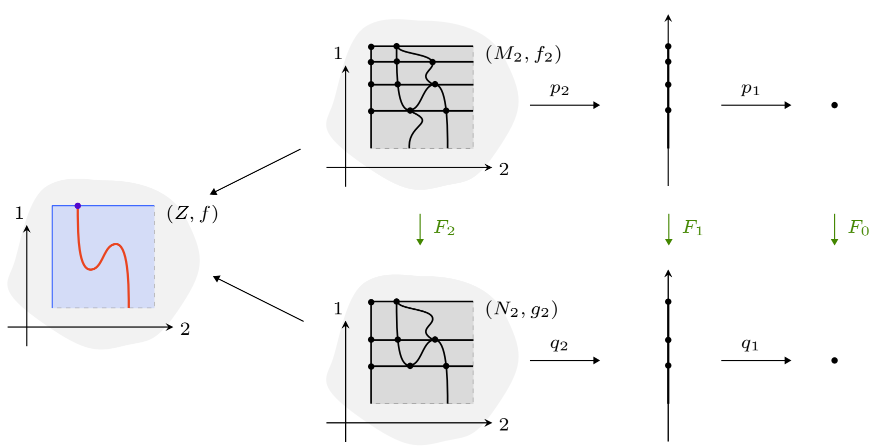


Figure 5.25. A mesh coarsening of stratified meshes.

Recall from Definition 5.3.14 that a truss cell is a stratified truss block whose initial element is label isolated, and similarly a truss singularity is a stratified truss brace whose terminal element is label isolated. The precisely corresponding notions for stratified meshes are as follows.

Definition 5.3.20 (Mesh cells and singularities). An **n-mesh m-cell** (M, f) is a stratified n-mesh m-block, for which the refinement $M \to f$ maps the (dense) m-dimensional stratum of M_n onto a stratum of f.

Dually, an **n-mesh m-singularity** (M, f) is a stratified n-mesh m-brace, for which the refinement $M \to f$ maps the (codense) m-dimensional stratum of M_n onto a stratum of f. ⌟

Example 5.3.21 (Mesh cells and singularities). In Figure 5.26, we depict three stratified 2-mesh singularities (M, f). In each case, the refining mesh is shown on top, and the tame stratification is shown on the bottom. The first two are 2-mesh 0-singularities, while the third is a 2-mesh 1-singularity. For each truss m-singularity, the codense stratum is of dimension m.

In Figure 5.27 we depict the dual stratified 2-mesh cells. The first two are 2-mesh 2-cells, and the third is a 2-mesh 1-cell. Naturally, the dense cell of each mesh m-cell is of dimension m. ⌟

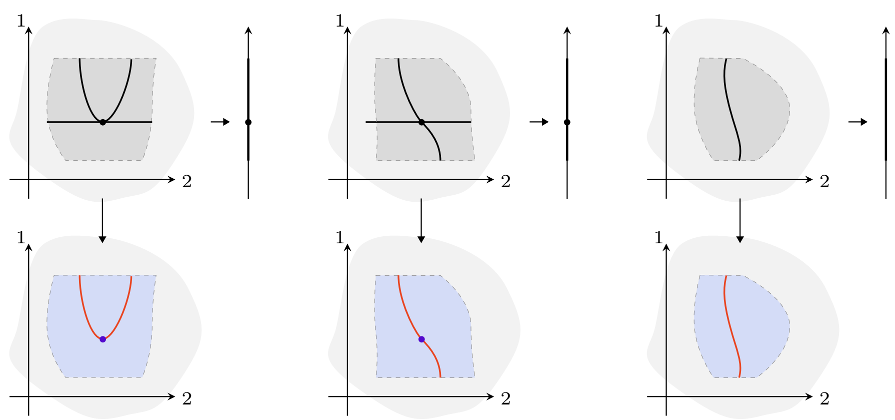


Figure 5.26. Mesh singularities.

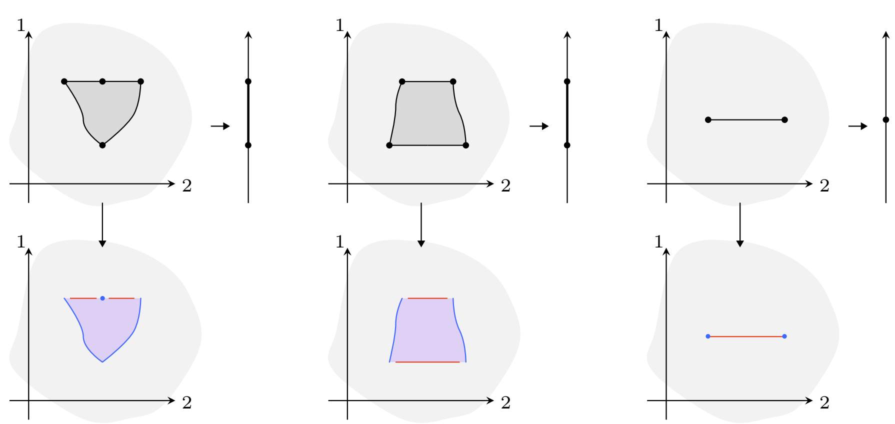


Figure 5.27. Mesh cells.

5.3.1.3. ♦Correspondence of stratified trusses and stratified meshes. Based of course on the equivalence between bare trusses and meshes, we now describe the correspondence between stratified trusses and stratified meshes.

In one direction, to pass from stratified meshes to stratified trusses, we may take the fundamental stratified truss, as follows.

Definition 5.3.22 (Fundamental stratified trusses). Given a stratified n-mesh (M, f), the **fundamental stratified truss** $\sqcap_{\mathsf{T}}(M, f)$ is the stratified n-truss $(\sqcap_{\mathsf{T}} M, \sqcap_{\mathsf{lbl}}(M, f))$, whose underlying truss is the fundamental truss $\sqcap_{\mathsf{T}} M$ of the mesh M, and whose labeling $\sqcap_{\mathsf{lbl}}(M, f)$ is the fundamental poset map $\sqcap(f_n \to f)$ of the coarsening of stratifications $f_n \to f$. ⌟

Example 5.3.23 (A stratified mesh and its fundamental stratified truss). Recall the stratified mesh in the top row of Figure 5.25. The fundamental stratified truss of that stratified mesh is the stratified truss in the top row of Figure 5.23. ⌟

Definition 5.3.24 (Fundamental stratified truss maps). Given a map of stratified meshes $F : (M, f) \to (N, g)$, the **fundamental stratified truss map** $\sqcap_{\mathsf{T}} F : \sqcap_{\mathsf{T}}(M, f) \to \sqcap_{\mathsf{T}}(N, g)$ is the map of stratified trusses whose underlying map of trusses is the fundamental truss map $\sqcap_{\mathsf{T}}(F : M \to N)$, and whose labeling map is the fundamental poset map $\sqcap(F_n : f \to g)$. ⌟

Example 5.3.25 (A mesh coarsening and its fundamental stratified truss coarsening). The fundamental stratified truss map of the mesh coarsening in Figure 5.25 is the truss coarsening in Figure 5.23. ⌟

In the other direction, to pass from stratified trusses to stratified meshes, we may take the stratified mesh realization, as follows. We will make use of the fact that stratified coarsenings of a given stratification are determined by connected-quotient maps of the fundamental poset of the stratification (see Lemma B.2.12).

Definition 5.3.26 (Stratified mesh realizations). Given a stratified n-truss $T = (\underline{T}, \mathrm{lbl}_T)$, the **stratified mesh realization** $\|T\|_{\mathsf{M}}$ is the stratified n-mesh $(\|\underline{T}\|_{\mathsf{M}}, \|T\|_{\mathsf{str}})$, whose underlying mesh is the mesh realization $\|\underline{T}\|_{\mathsf{M}}$ of the truss $\underline{T}$, and whose stratification $\|T\|_{\mathsf{str}}$ is determined by coarsening the stratification $(\|\underline{T}\|_{\mathsf{M}})_n$ according to the fundamental poset map $\sqcap(\mathrm{lbl}_T)$ of the labeling of the stratified truss. ⌟

Definition 5.3.27 (Stratified mesh map realizations). Given a map $F = (\underline{F}, \mathrm{lbl}_F) : T \to S$ of stratified trusses $T = (\underline{T}, \mathrm{lbl}_T)$ and $S = (\underline{S}, \mathrm{lbl}_S)$, the **stratified mesh map realization** $\|F\|_{\mathsf{M}} : \|T\|_{\mathsf{M}} \to \|S\|_{\mathsf{M}}$ is the map of stratified meshes given by the mesh map realization $\|\underline{F}\|_{\mathsf{M}} : \|\underline{T}\|_{\mathsf{M}} \to \|\underline{S}\|_{\mathsf{M}}$. ⌟

Note that the top component of the stratified mesh map realization indeed induces a stratified map $(\|F\|_{\mathsf{M}})_n : \|T\|_{\mathsf{str}} \to \|S\|_{\mathsf{str}}$, as required.

The equivalence of meshes and trusses (see Section 4.2) implies that the fundamental stratified truss and stratified mesh realization provide an equivalence in the stratified case, in the following sense.

PROPOSITION 5.3.28 (Correspondence of stratified meshes and trusses). *Let T be a stratified truss, and let (M, f) be a stratified mesh.*

(1) There is a unique isomorphism of stratified trusses $T \cong \mathbb{\Pi}_{\mathsf{T}}(\|T\|_{\mathsf{M}})$.
(2) There is an isomorphism of stratified meshes $(M, f) \cong \|\mathbb{\Pi}_{\mathsf{T}}(M, f)\|_{\mathsf{M}}$, which is unique up to contractible choice of homotopy.

PROOF. The first claim follows since there is a unique isomorphism of trusses $\underline{T} \cong \mathbb{\Pi}_{\mathsf{T}}\|\underline{T}\|_{\mathsf{M}}$, and since (suppressing that isomorphism) an equality of labelings $\mathbb{\Pi}_{\mathsf{lbl}}(\|T\|_{\mathsf{str}}) = \mathrm{lbl}_T$. The second claim follows since there is a mesh isomorphism $M \cong \|\mathbb{\Pi}_{\mathsf{T}} M\|_{\mathsf{M}}$, unique up to contractible choice of homotopy by the balanced case of weak faithfulness of the fundamental truss (see Proposition 4.2.40 and Remark 4.2.41), and that isomorphism induces a stratified homeomorphism $f \cong \|\mathbb{\Pi}_{\mathsf{T}}(M, f)\|_{\mathsf{str}}$. □

REMARK 5.3.29 (Correspondence of stratified mesh and truss maps). The fundamental stratified truss and stratified mesh realization also provide a mutual inverse correspondence in the case of maps of stratified meshes and stratified trusses (up to contractible choice of homotopy on the stratified mesh map side). ⌟

Recall that the mesh realization of a truss coarsening need not be a coarsening, and that necessitated Construction 4.2.78 of a special mesh coarsening realization for truss coarsenings. The realization of stratified truss coarsenings requires corresponding care, as follows.

DEFINITION 5.3.30 (Stratified mesh coarsening realization). Given a coarsening of stratified trusses $F : T \to S$, the **stratified mesh coarsening realization** $\|F\|^{\mathsf{crs}}_{\mathsf{M}} : \|T\|_{\mathsf{M}} \to \|S\|_{\mathsf{M}}$ is the map of stratified meshes given by the mesh coarsening realization $\|\underline{F}\|^{\mathsf{crs}}_{\mathsf{M}} : \|\underline{T}\|_{\mathsf{M}} \to \|\underline{S}\|_{\mathsf{M}}$. ⌟

Note that the top component of the stratified mesh coarsening realization does induce a stratified map $(\|F\|^{\mathsf{crs}}_{\mathsf{M}})_n : \|T\|_{\mathsf{str}} \to \|S\|_{\mathsf{str}}$, as required.

OBSERVATION 5.3.31 (Correspondence of coarsening notions). Given a coarsening, or a mesh coarsening, or a stratification coarsening $F : (M, f) \to (N, g)$ of stratified meshes, then the fundamental stratified truss map $\mathbb{\Pi}_{\mathsf{T}}(F) : \mathbb{\Pi}_{\mathsf{T}}(M, f) \to \mathbb{\Pi}_{\mathsf{T}}(N, g)$ is, respectively, a coarsening, or a truss coarsening, or a label coarsening of stratified trusses.

Conversely, given a coarsening, or a truss coarsening, or a label coarsening $F : T \to S$ of stratified trusses, then the stratified mesh coarsening realization $\|F\|^{\mathsf{crs}}_{\mathsf{M}} : \|T\|_{\mathsf{M}} \to \|S\|_{\mathsf{M}}$ is, respectively, a coarsening, or a mesh coarsening, or a stratification coarsening of stratified meshes. ⌟

5.3.1.4. ♦Normalization and coarsest refinements. Recall that by Definition 5.2.22, a coarsest refining mesh of a tame stratification is one that is coarser than any other refining mesh, and of course in practice any refining mesh that cannot be coarsened is a coarsest refining mesh. We can rephrase this notion in terms of stratified meshes as follows: a coarsest refining mesh M of a tame stratification f is a stratified mesh (M, f) that admits no non-identity (stratification-preserving) mesh coarsening. That rephrasing has an immediate combinatorial correlate, already presented in Definition 5.1.22: a normalized stratified truss is a stratified truss $(\underline{T}, \mathrm{lbl}_T)$ that admits no non-identity (label-preserving) truss coarsening.

The correspondence of stratified meshes and trusses thus specializes as follows.

Lemma 5.3.32 (Relating coarsest refinements and normalization). *Consider a stratifed mesh (M, f) and a stratified truss T, such that T is the stratified fundamental truss of (M, f), or equivalently (M, f) is the stratified mesh realization of T. The mesh M is the coarsest refining mesh of the stratification f if and only if the stratified truss T is normalized.*

Proof. By Observation 5.3.31, any mesh coarsening provides (on the fundamental stratified truss) a truss coarsening, and any truss coarsening provides (on the stratified mesh realization) a mesh coarsening; thus the stratified mesh cannot be mesh coarsened exactly when the stratified truss cannot be truss coarsened. □

This lemma provides the last ingredient for the proof of the classification of tame stratifications by normalized stratified trusses.

Proof of Theorem 5.1.23. Given a tame stratification (Z, f), we may take its coarsest refining mesh M (by Theorem 5.2.23). Changing the tame stratification by a framed stratified homeomorphism only changes the coarsest refining mesh by a mesh isomorphism (by Proposition 5.2.25). Next form the fundamental stratified truss $\sqcap_{\mathsf{T}}(M, f)$; that truss is normalized, by Lemma 5.3.32.

Conversely, given a normalized stratified truss T, we take its stratified mesh realization $\|T\|_{\mathsf{M}} = (\|\underline{T}\|_{\mathsf{M}}, \|T\|_{\mathsf{str}})$, and so have in particular the corresponding tame stratification $\|T\|_{\mathsf{str}}$. Changing the stratified truss by a balanced isomorphism only changes the tame stratification by a framed stratified homeomorphism. Note that the mesh $\|\underline{T}\|_{\mathsf{M}}$ is a coarsest refining mesh of $\|T\|_{\mathsf{str}}$, again by Lemma 5.3.32.

These two associations are mutually inverse, as required, by Proposition 5.3.28. □

We may now state and prove the analogous classification for tame embeddings.

Theorem 5.3.33 (Classification of tame embeddings). *Framed stratified homeomorphism classes of n-tame embeddings are in correspondence with isomorphism classes of normalized ambient stratified open n-trusses.*

Proof. The proof is anlogous to that of Theorem 5.1.23, but uses the minimal coarsest refining mesh instead of the coarsest refining mesh; the strata of the mesh that are not in the image of the embedding are considered ambient, and become the ambient strata of the fundamental stratified truss. □

Example 5.3.34 (Classifying 2-tame stratifications and embeddings). In Figure 5.28, we illustrate the classification of tame stratifications and embeddings in dimension 2. The first case depicts a 2-tame stratification and its corresponding normalized stratified truss. The second and third cases depict 2-tame embeddings and their corresponding normalized ambient stratified open trusses. ⌟

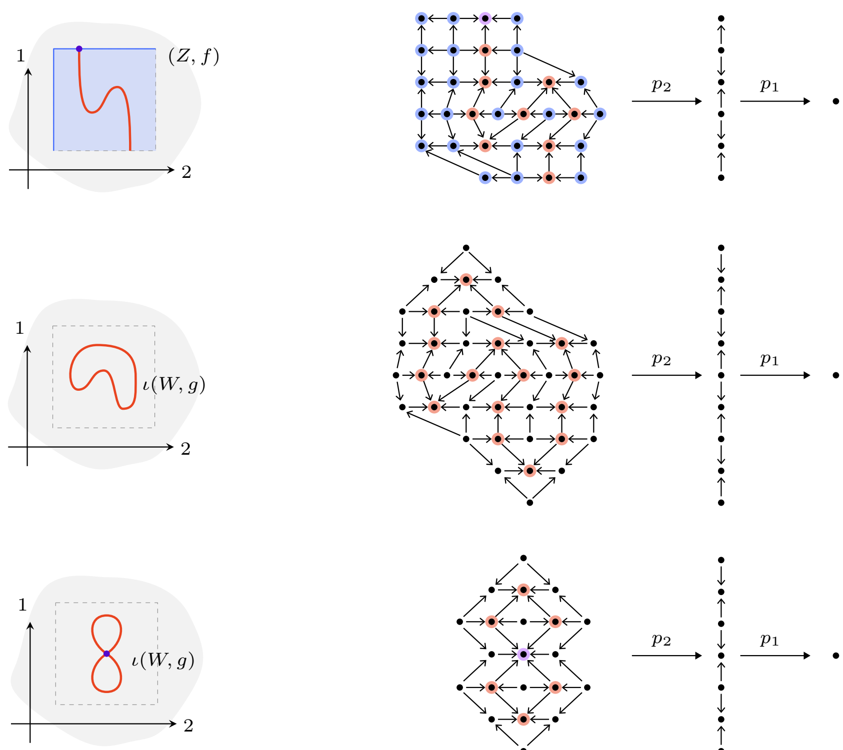


Figure 5.28. Normalized stratified trusses classifying 2-tame stratifications and embeddings.

Example 5.3.35 (Classifying 3-tame stratifications). In Figure 5.29, we depict a 3-tame stratification and its classifying normalized stratified 3-truss. The 3-mesh corresponding to the underlying unstratified 3-truss was shown in Figure 4.1. ⌟

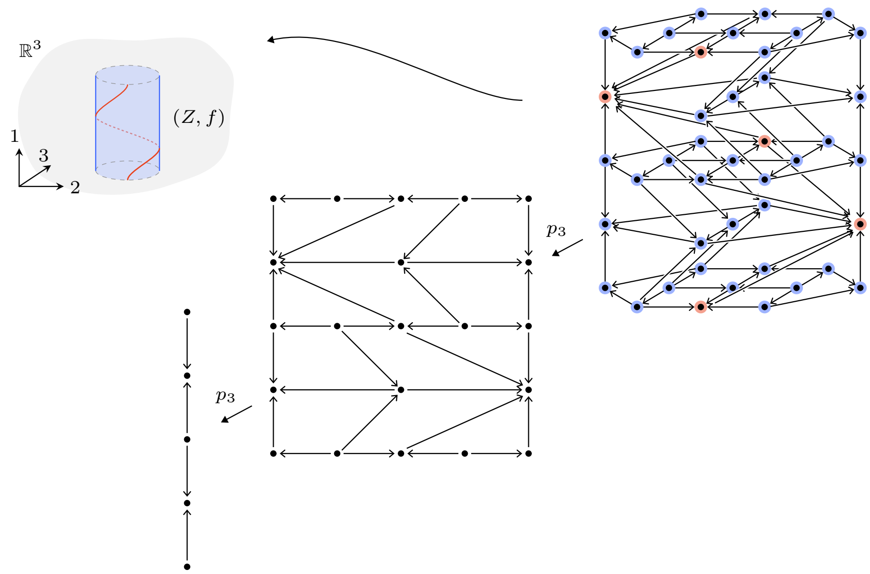


FIGURE 5.29. The normalized stratified truss of a 3-tame stratification.

EXAMPLE 5.3.36 (Classifying 3-tame embeddings). In Figure 5.30, we depict a 3-tame embedding (namely the braid isotopy) and its classifying normalized ambient stratified truss. The 3-mesh corresponding to the underlying unstratified 3-truss was shown on the left in Figure 4.16.

Similarly, in Figure 5.31, we depict another 3-tame embedding (namely the cusp singularity) and its classifying normalized ambient stratified 3-truss. The 3-mesh corresponding to the underlying unstratified 3-truss was shown earlier in Figure 4.15. ⌟

5.3.1.5. ⧫Tame stratified bundles. We next discuss the combinatorial classification of *bundles* of tame stratifications.

Recall that a stratified bundle is a stratified map that is a locally trivial bundle within each base stratum. We will need the following fiberwise framed version of that notion.

TERMINOLOGY 5.3.37 (Framed stratified bundles). Consider a stratified map $q : (Z, f) \to (B, g)$ with a realization (i.e. a base-preserving embedding) into the trivial bundle $B \times \mathbb{R}^n \to B$. The map q is a 'framed stratified bundle' if for each stratum s of the base g and for each point $x \in s$ there is an open neighborhood $x \in U \subset s$, and a stratification $(F \subset \mathbb{R}^n, h)$, such that the restriction $(q^{-1}(U), f) \to U$ is framed bundle isomorphic (i.e. the bundle isomorphism is a fiberwise framed map) to the trivial bundle with fiber (F, h). ⌟

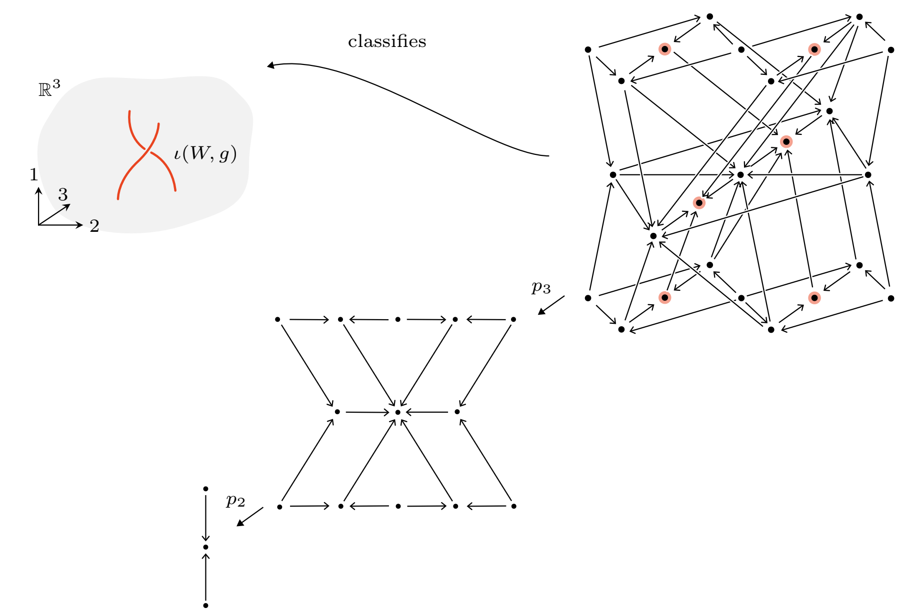


FIGURE 5.30. The normalized ambient stratified truss of the braid.

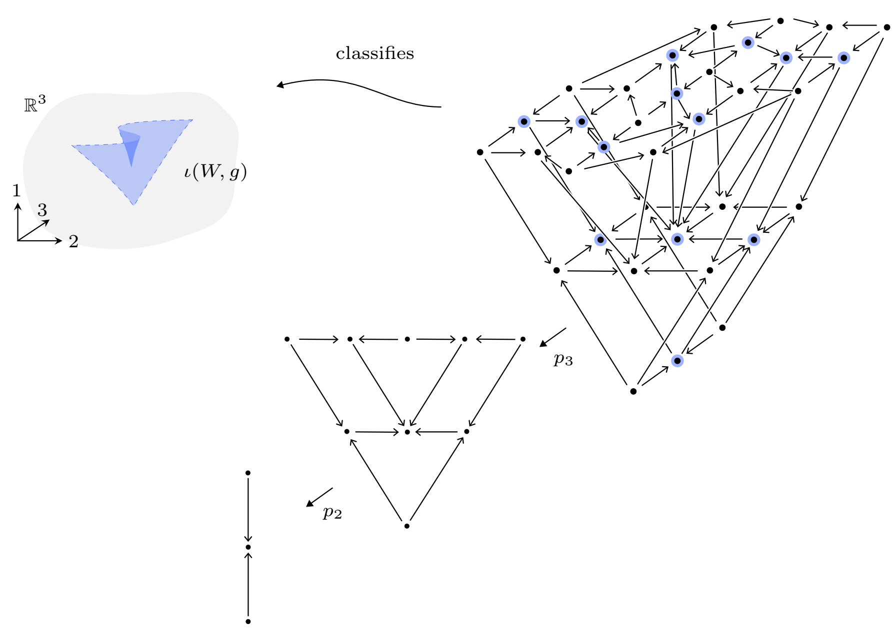


FIGURE 5.31. The normalized ambient stratified truss of the cusp.

Of course any framed stratified bundle is, in particular, a stratified bundle. It will be convenient, and without conceptual consequence, to restrict attention to the case where the realizing map is a subspace inclusion, i.e. to assume $Z \subset B \times \mathbb{R}^n$.

DEFINITION 5.3.38 (Tame stratified bundles). Let (B, g) be a stratification, together with a cellulation (B, c). An **n-tame stratified bundle** over the base cellulation $c \to g$ is a framed stratified bundle $q : (Z, f) \to (B, g)$, for which there exists an n-mesh bundle over the cellulated base (B, c), which refines the total stratification of the bundle. (That is, there is an n-mesh bundle M with support $\gamma(M) = Z$, whose realization $\gamma : (M_n, f_n) \to (Z, f)$ is a coarsening.) ⌟

Assuming the base (B, g) is sufficiently nice (so in particular cellulable), the proofs of Theorem 5.2.23 and Proposition 5.2.25 concerning coarsest meshes carry over (using Lemma 5.2.19 in place of Key Lemma 5.2.13) to the case of coarsest mesh bundles. That generalization yields the following result.

THEOREM 5.3.39 (Coarsest refining mesh bundles). *For a sufficiently nice base stratification (B, g) with a fixed cellulation (B, c), every tame stratified bundle (Z, f) over the cellulation $c \to g$ has a unique coarsest refining mesh bundle over the cellulated base (B, c). Moreover, every stratified bundle homeomorphism of tame stratified bundles preserves this coarsest refining mesh bundle.* □

We now consider the combinatorial counterpart of tame stratified bundles with their coarsest refining mesh bundles. Recall that we defined a stratified truss to be simply a labeled truss, and a normalized stratified truss to be one that admits no label-preserving truss coarsening. In the bundle case, we must adopt a different formulation, to account for the combinatorial analog of the base cellulation, as follows.

DEFINITION 5.3.40 (Normalized labeled truss bundles). A labeled n-truss bundle p is **normalized** if any label-preserving and base-preserving truss bundle coarsening (of p) is the identity. ⌟

DEFINITION 5.3.41 (Stratified n-truss bundles). Let P be a poset, together with a connected quotient map of posets $\phi : Q \to P$. A **stratified n-truss bundle** over the poset quotient $Q \to P$ is a labeled n-truss bundle p over the poset Q, whose labeling lbl_p is a connected-quotient map, such that for each preimage $U := \phi^{-1}(x \in P) \subset Q$, the restricted bundle $p|_U$ normalizes to a constant labeled truss bundle. ⌟

For a sufficiently nice base stratification (B, g), together with a cellulation (B, c), we can thus consider stratified truss bundles over the fundamental poset quotient $\sqcap c \to \sqcap g$.

The classification of tame bundles can now proceed as follows.

THEOREM 5.3.42 (Classification of tame stratified bundles). *Let (B, g) be a sufficiently nice stratification, together with a cellulation (B, c). Framed stratified bundle homeomorphism classes of n-tame stratified bundles over the cellulation $c \to g$ are in correspondence with base-preserving isomorphism classes of normalized stratified n-truss bundles over the fundamental poset quotient $\sqcap c \to \sqcap g$.*

PROOF. The proof is analogous to that of Theorem 5.1.23, using now the fundamental truss bundle construction and the mesh bundle realization construction: from a tame stratified bundle, take the coarsest refining mesh bundle, and form its fundamental truss bundle together with the evident stratification; from a normalized stratified truss bundle, take its mesh bundle realization together with again the evident stratification. □

REMARK 5.3.43 (Tame bundle embeddings). As we have generalized tame stratifications to the bundle case, we may generalize tame embeddings to the bundle case, as follows. A 'tame bundle embedding' of a stratified bundle $q : (W, h) \to (B, g)$ is an embedding $\iota : W \hookrightarrow B \times \mathbb{R}^n$, whose stratified image extends (constructibly) to an open neighborhood stratification that is a tame stratified bundle.

The classification of tame embeddings generalizes accordingly: minimal coarsest refining mesh bundles always exist, and as a consequence tame bundle embeddings (over cellulated bases) are classified (up to framed stratified bundle isomorphism) by normalized ambient stratified truss bundles (over the corresponding fundamental poset quotient) (up to bundle isomorphism). ⌟

EXAMPLE 5.3.44 (Classification of tame stratified bundles and tame bundle embeddings). In Figure 5.32, on the left we depict a 1-tame stratified bundle (Z, f) over a stratified circle (B, g) with cellulation (B, c), and a 1-tame bundle embedding ι of a stratified bundle $(W, h) \to (B, g)$. On the right, we depict the corresponding normalized stratified 1-truss bundle and normalized ambient stratified 1-truss bundle, respectively. ⌟

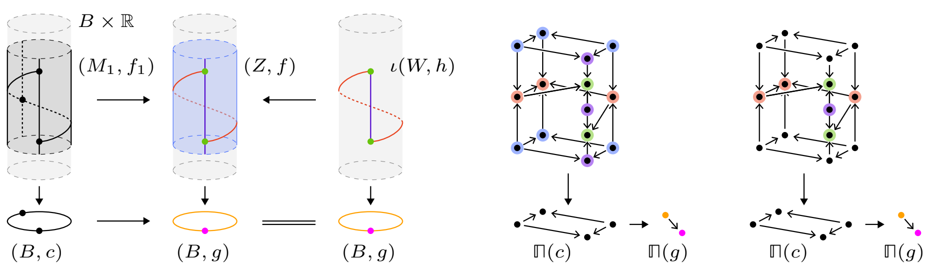


FIGURE 5.32. Tame stratified bundles and tame bundle embeddings with their corresponding normalized stratified truss bundles.

5.3.2. Polyhedrality.

Synopsis. We introduce polyhedral stratifications as those refinable by constructible substratifications of simplicial complexes, and show that any tame stratification is canonically framed stratified homeomorphic to a polyhedral stratification. We prove conversely that any polyhedral stratification is tame embedded, by inductively constructing refining meshes of simplicial complexes. We then establish the tame Hauptvermutung, showing that framed stratified homeomorphisms between polyhedral stratifications are homotopic to framed PL homeomorphisms, in contrast to the classical Hauptvermutung which fails in the non-framed setting. Finally, we discuss the relationship to o-minimal structures.

12 : Here up not done

5.3.2.1. ⧫Tameness and polyhedral stratifications. As a first application of the combinatorializability of tame stratifications, we will now see that tame stratifications are polyhedral, in the sense of being refinable by constructible substratifications of piecewise linear realizations of simplicial complexes, and conversely polyhedral stratifications are tame embeddings.

Terminology 5.3.45 (Linear realizations and polyhedra). A 'linear realization' of a finite simplicial complex K is an embedding $\iota : |K| \hookrightarrow \mathbb{R}^n$ that is linear on each simplex. The image $\iota(|K|) \subset \mathbb{R}^n$ of a linear realization is called a 'polyhedron' (see [**RS72**, Defn. 1.1]). ⌟

Henceforth we assume all simplicial complexes are finite, without comment. Note that given a linear realization of a complex, its image has a stratification by the open simplices of the complex. For convenience, we often suppress the embedding ι of a linear realization, and informally consider the complex as being a subspace of euclidean space.

Definition 5.3.46 (Polyhedral stratifications). A **polyhedral stratification** is a stratification (Z, f) of a euclidean subspace $Z \subset \mathbb{R}^n$, that can be refined by a constructible substratification of the linear realization of a simplicial complex. ⌟

Note that, though a polyhedron is always compact, the support Z of a polyhedral stratification may certainly be non-compact.

As a corollary of the combinatorializability of tame stratifications, we may now prove that any tame stratification is framed stratified homeomorphic to a polyhedral stratification.

Proof of Corollary 5.1.25. Given a tame stratification (Z, f), consider its coarsest refining mesh M. By the proof of Theorem 5.1.23, this tame stratification is framed stratified homemorphic to the stratified mesh realization $\|\mathbb{\Pi}_{\mathsf{T}}(M, f)\|_{\mathsf{str}}$ of the stratified fundamental truss $\mathbb{\Pi}_{\mathsf{T}}(M, f)$. That realization is a polyhedral stratification by construction (see Construction 4.2.47 and Construction 4.2.52). □

We now show conversely that every polyhedral stratification is the image of a tame embedding. We will use the following observation and construction.

Observation 5.3.47 (Image refinements). Let K be a finite simplicial complex, and $F : |K| \to \mathbb{R}^n$ a (not-necessarily injective) simplex-wise linear map. There exists a simplicial complex L (considered as a subspace of $\mathbb{R}^n$) such that $\mathrm{im}(F) = |L|$, and, for each simplex $x \in K$, the image $F(|x|)$ is a union of simplices $|y|$, with $y \in L$ (see [**RS72**, Thm. 2.15]). We call the simplicial complex L an 'image refinement' of the map F. ⌟

Construction 5.3.48 (Refining meshes of linearly realized complexes). Let K be a simplicial complex with a linear realization $\iota : |K| \hookrightarrow \mathbb{R}^n$. We will construct an open n-mesh M that refines the realization in the sense of Terminology 5.2.27. For brevity, we will let the realization embedding ι be implicit, consider the complex as a subspace of euclidean space, suppress the geometric realization from our notation, and refer to the mesh simply as 'refining the complex' K.

Using Observation 5.3.47, take an image refinement L for the projection $\pi_n : K \to \mathbb{R}^{n-1}$. By induction, construct an $(n-1)$-mesh $M_{<n}$ that refines L. Next, refine the complex K to a stratification $\tilde{K}$ such that $\pi_n : \tilde{K} \to (M_{n-1}, f_{n-1})$ is a stratified bundle. (Specifically, take $\tilde{K}$ to be the refinement of K whose strata are the connected components of the spaces $\pi_n^{-1}(s) \cap r$, for s and r being the strata of f_{n-1} and K respectively.) Extend the stratification $\tilde{K}$ to an open 1-mesh bundle $p_n : (M_n, f_n) \to (M_{n-1}, f_{n-1})$ such that $\tilde{K} \hookrightarrow (M_n, f_n)$ is a constructible substratification. Augmenting $M_{<n}$ with the bundle p_n provides the required mesh M refining the complex K. ⌟

Example 5.3.49 (Mesh refinements of linear complexes). In Figure 5.33, we depict the inductive procedure, given in the previous construction, for producing a mesh refinement of a linearly realized simplicial complex. Specifically, on the top left is a realized simplicial complex K, which projects to a complex $\pi_3 K$ refined by the complex L in the lower left. That complex in turn has the mesh refinement shown in the lower right. In the upper right, we do not draw the whole stratification (M_3, f_3) but just the fibers of the 1-mesh bundle $(M_3, f_3) \to (M_2, f_2)$ over the point strata of f_2, along with the constructible substratification $\tilde{K}$. ⌟

We can now record the proof that every polyhedral stratification is the image of a tame embedding.

Proof of Proposition 5.1.26. Let (Z, f) be a polyhedral stratification, and let K be a linearly realized simplicial complex that has a constructible substratification refining (Z, f). By Construction 5.3.48, there is an open mesh M refining the realized complex, in the sense that the mesh has a constructible substratification that literally refines the complex. Transitively, the polyhedral stratification is refined by a constructible substratification

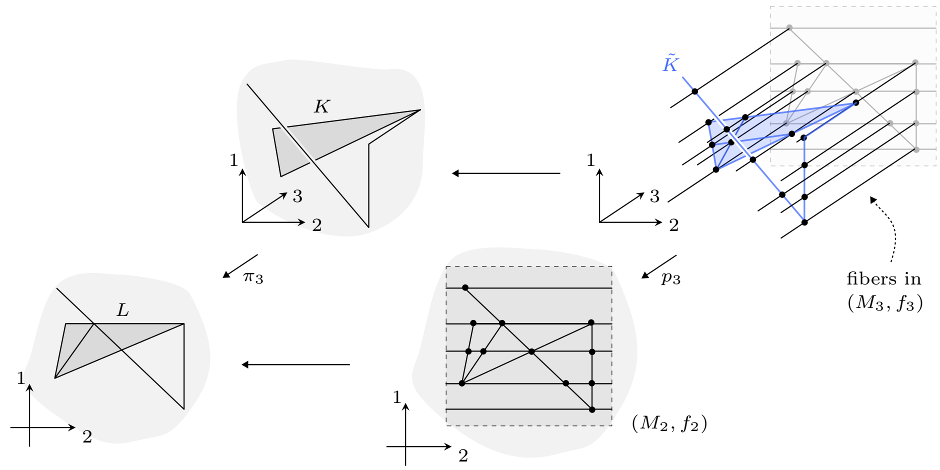


FIGURE 5.33. Inductive construction of mesh refinements for polyhedral stratifications.

of an open mesh. Coarsening the mesh according to that coarsening of its constructible substratification provides a tame stratification with the polyhedral stratification as a constructible substratification. Thus the polyhedral stratification is a tame embedding, as desired. □

The above relationship between tame and polyhedral stratifications makes, of course, crucial use of the linear structure of mesh realizations of trusses. For later use we record the following terminology and observation concerning that linear structure.

TERMINOLOGY 5.3.50 (Linear meshes). A mesh M is called 'linear' if there is a mesh isomorphism $\|\mathbb{\Pi}_{\mathsf{T}} M\|_{\mathsf{M}} \cong M$ that is linear on each open simplex of the mesh realization $\|\mathbb{\Pi}_{\mathsf{T}} M\|_{\mathsf{M}}$ of the fundamental truss $\mathbb{\Pi}_{\mathsf{T}} M$. ⌟

OBSERVATION 5.3.51 (Linear refining meshes of linearly realized complexes). Every linearly realized simplicial complex (and thus every polyhedral stratification) has a linear refining n-mesh (again in the sense of Terminology 5.2.27). This follows by the method of Construction 5.3.48, by inductively choosing the $(n-1)$-mesh $M_{<n}$ to be linear, and then choosing the 1-mesh bundle p_n such that the augmented n-mesh M is again linear. ⌟

5.3.2.2. The tame Hauptvermutung. As a second application of the combinatorializability of tame stratifications, we will now prove that every framed stratified homeomorphism is homotopic to a piecewise linear framed stratified homeomorphism, which is to say we prove a framed stratified Hauptvermutung.

We begin with a brief recollection of the classical Hauptvermutung (see, for example, [**RCS⁺96**]).

Disproven Conjecture 5.3.52 (Hauptvermutung). *Every homeomorphism $P \cong Q$ between polyhedra P, Q is homotopic to a PL homeomorphism.*

The Hauptvermutung, and its related conjectures, have been famously disproven in various ways: not only are there polyhedra for which mutliple different PL structures exist (where 'PL structure' means 'PL homeomorphism class'), but there are also topological manifolds with different PL structures, and even topological manifolds that admit no PL structure at all (that is, the inclusion of PL manifolds into topological manifolds is not surjective on homeomorphism classes). The failure of the Hauptvermutung may be understood as a symptom of the 'wildness' of topological homeomorphism, which finds no counterpart in combinatorial topology; 'taming' this wildness is possible, for example, through the classical techniques of o-minimal structures (cf. [**Shi14**], [**Shi13**]), though often in technically cumbersome ways. We provide a brief comparison between the axioms of o-minimality and framed tame structures later in this section.

Our goal now is to show that a 'tame' variant of the Hauptvermutung holds naturally in the setting of tame stratifications. An adequate classical analog to this statement (for the case of stratifications embedded in $\mathbb{R}^n$) may be phrased as follows. Note that we take a 'stratified homotopy' to mean a homotopy of stratified maps that is constant on fundamental posets.

Disproven Conjecture 5.3.53 (Ambient stratified Hauptvermutung). *Every stratified homeomorphism between (compact) polyhedral stratifications is stratified homotopic to a PL stratified homeomorphism.*

Like its non-ambient counterpart, the ambient Hauptvermutung fails to hold in general.[33]

Disproof of the ambient stratified Hauptvermutung. We construct homeomorphic polyhedral stratifications that are not PL homeomorphic, as follows.

Step 1: Take simplicial complexes K and L that are both PL homotopy 5-tori but not PL homeomorphic. Then both K and L are homeomorphic to the 5-torus T^5. (*References*: Hsiang and Shaneson showed that there are non-standard PL homotopy 5-tori [**HS70**]. Later, Hsiang and Wall showed that all PL homotopy 5-tori are homeomorphic to the 5-torus [**HW69**].)

Step 2: PL embed both K and L in $\mathbb{R}^{10}$; we may assume these embeddings are locally flat. (*References*: Rourke and Sanderson construct the required embeddings in [**RS72**, Theorem 5.5]; local flatness follows from the remark following that theorem.)

Step 3: Enlarge to $n = 12$. Since the embeddings have high codimension and locally flat images, they become ambient isotopic by a compactly supported isotopy. (*References*: This follows from [**DV09**, Thm. 4.4.2]; see also Corollary 4.4.3 loc. cit., which states that homotopic topological embeddings

[33] We are indebted to Mark Powell for outlining the given counter-example to us.

of high codimension into a PL manifold with 1-LCC images are compactly supported ambient isotopic. Here, 1-LCC stands for '1-local-co-connected' and is implied by local flatness; see Prop. 1.3.1 loc. cit.)

Step 4: The final time slice of the isotopy provides a topological homeomorphism. However, a PL homeomorphism cannot exist by the initial choice of PL structures. □

In contrast, the 'tame' variation of the Hauptvermutung holds. Recall Theorem 5.1.27, which states that any framed stratified homeomorphism between polyhedral stratifications is stratified homotopic to some framed PL homeomorphism. In fact, we will see that the homotopy is unique up to contractible choice (among stratified homotopies of framed stratified homeomorphisms).

Proof of Theorem 5.1.27. Let (Z, f) and (W, g) be polyhedral stratifications that are framed stratified homeomorphic by a framed stratified homeomorphism $F : (Z, f) \cong (W, g)$. By definition, F is defined on a tame open neighborhood U of Z and maps U homeomorphically to a tame open neighborhood V of W. Shrinking the domain of F if necessary, chose Q to be a minimal coarsest refining mesh of (Z, f) with support U. Note that FQ is then a minimal coarsest refining mesh of (W, g).

Using Observation 5.3.51, construct *linear* open n-meshes M and N that refine (Z, f) and (W, g) (see Terminology 5.3.50), and such that M and N are supported on U and V, respectively. Set $T = \mathbb{\sqcap}_{\mathsf{T}} M$ and $S = \mathbb{\sqcap}_{\mathsf{T}} N$, and pick isomorphisms $\|T\|_{\mathsf{M}} \cong M$ and $\|S\|_{\mathsf{M}} \cong N$ that are linear on each simplex. Using that Q is minimal coarsest, we obtain truss coarsenings $T \to R$ and $S \to R$ for the truss $R = \mathbb{\sqcap}_{\mathsf{T}} Q$. Using Construction 4.2.78, we realize these coarsenings as maps that are linear on each simplex. We obtain the PL maps in the upper row of the following diagram:

$$\begin{array}{ccccccccc}
(M_n, f_n) & \longleftarrow & \|T_n\|_{\mathsf{M}} & \longrightarrow & \|R_n\|_{\mathsf{M}} & \longleftarrow & \|S_n\|_{\mathsf{M}} & \longrightarrow & (N_n, g_n) \\
\uparrow & & & & \sim & & & & \uparrow \\
(U, f) & & & & \xrightarrow[F]{\qquad\qquad\qquad\qquad} & & & & (V, g)
\end{array} .$$

Define $G : U \to V$ via the zig-zag of PL maps complementing F in this diagram (vertical maps in the diagram are simply identities on underlying spaces). By construction, this map induces the claimed framed stratified homeomorphism $G : (Z, f) \to (W, g)$.

Observe that F and G induce n-mesh maps $F, G : Q \to FQ$ on the respective minimal meshes of (Z, f) and (W, g) which are identical on fundamental trusses (namely, both yield the identity map). By weak faithfulness of the fundamental truss functor, we deduce that F and G must be homotopic as n-mesh maps, and uniquely so up to contractible choice. It follows that they are homotopic as framed stratified maps $F, G : (Z, f) \to (W, g)$ as well (again, considered on appropriate open neighborhoods of Z and W). □

REMARK 5.3.54 (Equivalence of all triangulations). Given a polyhedral stratification (Z, f), consider two different triangulations K and L. Running the preceding proof, setting $(W, g) = (Z, f)$ and $F = \mathrm{id}$ and choosing M to refine K and N to refine L, shows that the two triangulations are piecewise linearly equivalent. ⌟

REMARK 5.3.55 (Contractibility of framed structure groups). The fact that the Hauptvermutung fails in the nonframed setting but holds in the framed setting is underpinned by the following intuition: while the automorphism groups $\mathrm{Aut}_{\mathrm{TOP}}(\mathbb{R}^n)$ and $\mathrm{Aut}_{\mathrm{PL}}(\mathbb{R}^n)$ differ, the groups of framed automorphisms $\mathrm{Aut}^{\mathrm{fr}}_{\mathrm{TOP}}(\mathbb{R}^n)$ and $\mathrm{Aut}^{\mathrm{fr}}_{\mathrm{PL}}(\mathbb{R}^n)$ (see Definition 4.1.86) are both *contractible* and thus trivially equivalent. Indeed, one would therefore expect that the same result holds when working with classical notions of framings [**MW97, War83**], that is, the notions of 'framed topological' and 'framed PL' homeomorphisms (up to details of their definitions) should be equivalent. ⌟

5.3.2.3. Relation to o-minimality. Tame stratifications are related to o-minimal structures in $\mathbb{R}^n$ [**VdD98, Cos00**]: o-minimal structures, in some sense, *axiomatize* the property of spaces being *projectable* along the standard projections $\pi_n : \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}^{n-1}$. In fact, many of the arguments in the framed topology of meshes regarding the inductive construction of section and spacer cells are reminiscent of the cylindrical algebraic decomposition arguments found in o-minimal topology [**BCR13, Col76**]. However, as we now discuss, a foundational difference in the design of these approaches stems from the constructibility conditions imposed by framed topology.

TERMINOLOGY 5.3.56 ($\mathbb{R}^n$-supported meshes). Identify $\mathbb{R}^n$ with the open n-cube $\mathbb{I}^n = (-1, 1)^n \subset \mathbb{R}^n$ by a fixed framed homeomorphism. An '$\mathbb{R}^n$-supported mesh' simply means a stratification of $\mathbb{R}^n$ that induces an $\mathbb{I}^n$-supported mesh under this identification. ⌟

REMARK 5.3.57 (Tame subsets). Any tame subset $U \subset \mathbb{R}^n$ (where 'tame subset' means the inclusion of U into $\mathbb{R}^n$ is a tame embedding) can be refined by an $\mathbb{R}^n$-supported mesh: indeed, any mesh M with bounded support refining U can be trivially extended to an $\mathbb{R}^n$-supported mesh (formally, this can be achieved by compactifying, then dualizing, then compactifying, and then dualizing the mesh). The class of tame subsets in $\mathbb{R}^n$ is the class of subsets that can be refined by an $\mathbb{R}^n$-supported open n-mesh. ⌟

OBSERVATION 5.3.58 (Tame subsets are not o-minimal). The class of tame subsets satisfies some but not all of the axioms of o-minimality. Below we summarize which o-minimality axioms are satisfied and which are not.

(1) *Closure under finite unions*. Tame subsets are *not* closed under unions, and not even under disjoint unions. Consider, for example, the red and green subsets in $\mathbb{R}^3$ depicted in Figure 5.6: while both are tame individually, their union is not.

(2) *Closure under finite intersections*. A similar argument shows that tame subsets are not closed under finite intersections.
(3) *Closure under taking complements*. Tame subsets are closed under taking complements: indeed, a refining mesh for a tame subset is also a refining mesh for its complement.
(4) *Closure under taking products*. Tame subsets are closed under taking products. Indeed, products for framed constructible structures have been defined in Construction 2.3.55 in the case of trusses, and the case of meshes is fully analogous. Given tame subsets U and U' refined by meshes M and M', their product $U \times U'$ is refined by the mesh product $M \times M'$.
(5) *Closure under projections*. Tame subsets are also closed under taking projections π_n, since any n-mesh may be truncated to an $(n-1)$-mesh.
(6) *Unions of intervals and points determine dimension 1*. Finally, the only tame subsets of $\mathbb{R}^1$ are finite unions of points and open intervals, as required by the axioms of o-minimality. ⌟

Tame subsets contain the o-minimal class of polyhedral subsets. However, the *prototypical* o-minimal class of semi-algebraic sets [**BM88, RG95, RG06**] is not included in the class of tame subsets.

Observation 5.3.59 (Semi-algebraic sets need not be tame). Consider the semi-algebraic subset S of $\mathbb{R}^3$ determined by the equation $x_3(x_1+x_2)-x_2=0$. This subset is not tame: restricting the projection $\pi_3 : S \to \mathbb{R}^2$ to the subset $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ recovers a bundle framed homeomorphic to that shown in Figure 4.11, which fails constructibility, and thus no mesh refining S can exist.

Conversely, note that the individual line subsets $\mathbb{I} \hookrightarrow \mathbb{R}^3$ shown in Figure 5.6 are tame (and even smooth, up to artistic freedom), but they are *not* semi-algebraic. Thus, the classes of tame subsets and semi-algebraic sets are generally incomparable. ⌟

In summary, while both o-minimal tame topology and framed tame topology build on projectability conditions, the latter does away with point-wise axioms requiring the existence of unions and intersections, and instead requires a category-theoretically grounded bundle constructibility condition. The result is a new flavor of tame topology, not directly comparable to classical o-minimal structures such as semi-algebraic sets.

5.3.3. Computability.

Synopsis. We show that stratified truss coarsenings define a confluent rewriting system, with all maximal chains ending in the same *normal form*, which is algorithmically computable. From this, we deduce that framed stratified homeomorphism of tame stratifications is algorithmically decidable by comparing their normalized stratified trusses. We briefly consider how these results translate under geometric duality, with a particular focus on so-called tame cells and tame singularities. Finally, we introduce the notion of realizable framed regular cell complexes as those admitting cell-wise framed realizations in $\mathbb{R}^n$. We prove that such complexes have unique coarsest cell structures akin to our construction of coarsest refining meshes, and that their framed homeomorphism type is also algorithmically decidable.

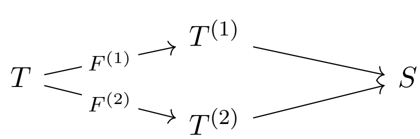
5.3.3.1. Normal forms and framed stratified homeomorphism. We discuss the (efficient) algorithmic construction of coarsest refining meshes of tame stratifications. As a corollary, we deduce that it is decidable whether two tame stratifications are framed stratified homeomorphic. Both problems will be solved by re-phrasing them in combinatorial terms after passing to fundamental trusses. We begin by showing that chaining stratified truss coarsenings is *confluent*, that is, any maximal chain of truss coarsenings eventually terminates in the same normalized stratified truss.

Proposition 5.3.60 (Stratified truss coarsenings are confluent). *Let T be a stratified n-truss. Any maximal chain of non-identity truss coarsenings of T ends in the same normalized n-truss, denoted $[\![T]\!]$, and the truss coarsening $T \to [\![T]\!]$ is unique.*

Terminology 5.3.61 (Normal forms of stratified trusses). We call $[\![T]\!]$ the **normal form** of T. ⌟

Proof of Proposition 5.3.60. We first show confluency of coarsenings. Let $F^{(1)} : T \to T^{(1)}$ and $F^{(2)} : T \to T^{(2)}$ be non-identity truss coarsenings of some stratified truss T. The underlying truss maps $\underline{F}^{(i)} : \underline{T} \to \underline{T}^{(i)}$ are coarsenings of $\underline{T}$. Construct the mesh coarsening realizations $\|\underline{F}^{(i)}\|^{\mathsf{crs}}_{\mathsf{M}} : \|\underline{T}\|_{\mathsf{M}} \to \|\underline{T}^{(i)}\|_{\mathsf{M}}$ using Construction 4.2.78. Up to pulling back $\|\underline{T}^{(i)}\|_{\mathsf{M}}$ along $\|\underline{F}^{(i)}\|^{\mathsf{crs}}_{\mathsf{M}}$, we may assume that $\|\underline{T}^{(i)}\|_{\mathsf{M}}$ and $\|\underline{T}\|_{\mathsf{M}}$ all have the same support in $\mathbb{R}^n$, and that the constructed mesh coarsenings are identities on underlying spaces.

Construct the stratified mesh realization $(\|\underline{T}\|_{\mathsf{M}}, \|T\|_{\mathsf{str}})$ of T. Since $F^{(1)}$ and $F^{(2)}$ are truss coarsenings, both $\|\underline{T}^{(1)}\|_{\mathsf{M}}$ and $\|\underline{T}^{(2)}\|_{\mathsf{M}}$ refine $\|T\|_{\mathsf{str}}$. By Key Lemma 5.2.13, we may take the join of $\|\underline{T}^{(1)}\|_{\mathsf{M}}$ and $\|\underline{T}^{(2)}\|_{\mathsf{M}}$ to obtain the mesh $\|\underline{T}^{(1)}\|_{\mathsf{M}} \vee \|\underline{T}^{(2)}\|_{\mathsf{M}}$, which again refines $\|T\|_{\mathsf{str}}$ and which we abbreviate by M. Construct the fundamental stratified truss $S := \mathbb{\Pi}_{\mathsf{T}}(M, \|T\|_{\mathsf{str}})$, which fits into the following diagram of truss coarsenings



making our initial choices of coarsenings confluent.

Any chain of non-identity truss coarsenings must eventually terminate in some normalized truss. By confluency, this implies all chains must end in the same normalized n-truss $[\![T]\!]$. Note also that the truss coarsening $T \to [\![T]\!]$ is necessarily unique (otherwise, if there were two distinct truss coarsenings

$F^{(1)}$ and $F^{(2)}$ from T to $[\![T]\!]$, we could repeat the above steps to contradict that $[\![T]\!]$ is normalized). □

EXAMPLE 5.3.62 (Confluence of coarsenings). In Figure 5.34 we illustrate two truss coarsenings F and G of the same stratified 2-truss (we leave the projections of the truss implicit, as they appeared earlier in **??**). The coarsenings can be made confluent by the additional truss coarsenings F' and G' as indicated. ⌟

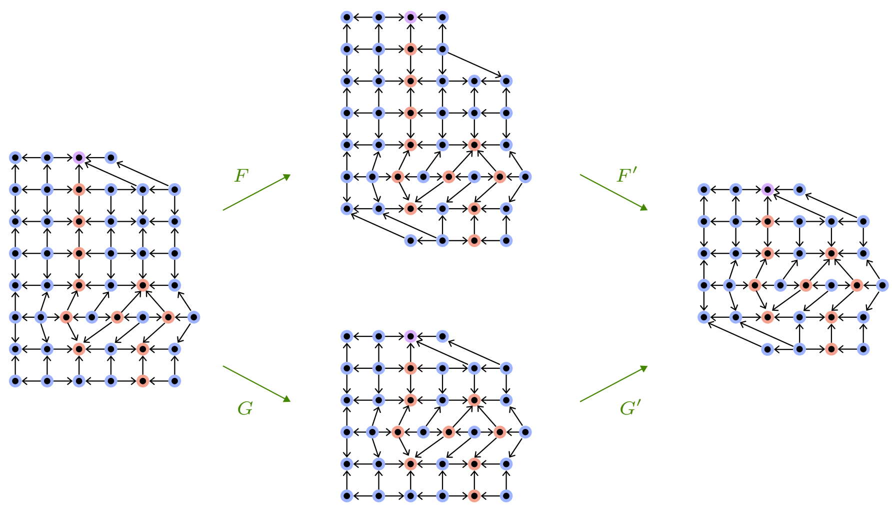


FIGURE 5.34. Confluence of coarsenings.

REMARK 5.3.63 (Mesh joins as truss pushouts). While the preceding proof of Proposition 5.3.60 crucially relies on our earlier construction of mesh joins, one may also prove the statement in *purely combinatorial terms*.

Passing to underlying trusses of the coarsening diagram constructed in the proof of Proposition 5.3.60 yields a pushout diagram in the category $\mathsf{Trs}^{\mathrm{crs}}_n$ of n-trusses and their coarsenings. That pushout is preserved by the 'total poset' functor $(-)_n : \mathsf{Trs}_n \to \mathsf{Pos}$ mapping n-trusses T to their total poset T_n (and maps $F : S \to T$ to poset maps $F_n : S_n \to T_n$). It is also preserved by the truncation functor $(-)_{<n} : \mathsf{Trs}_n \to \mathsf{Trs}_{n-1}$.

The claimed pushouts of truss coarsenings can be constructed by purely combinatorial arguments. The construction is closely related to a *relative* version of our earlier construction of mesh joins, in which the supports of the two meshes to-be-joined are identified via a third mesh that they jointly coarsen (the 'limiting case' of this perspective recovers the characterization of joins via pushouts in Remark 5.2.3, though the required discrete stratification therein has infinitely many strata which and finds no combinatorial counterpart in the theory of trusses).[34] ⌟

[34]Fully analogous to the case of mesh joins, truss coarsening pushouts can be constructed inductively. We leave this construction as a fruitful exercise; the diligent reader may

The preceding arguments generalize to the case of stratified truss bundles (see the discussion of Theorem 5.3.39 and Theorem 5.3.42), which yields the following result.

Observation 5.3.64 (Normal forms of stratified truss bundles). Given a stratified truss bundle p over a base poset B, any chain of base- and label-preserving non-identity truss bundle coarsenings of p ends in the same normalized n-truss bundle over B, denoted $[\![p]\!]$, and the truss coarsening $p \to [\![p]\!]$ is unique. The bundle $[\![p]\!]$ is called the 'normal form' of p. ⌟

Normal forms of stratified trusses (and truss bundles) are evidently computable, though devising an efficient such algorithm requires a little more effort.

Observation 5.3.65 (Normal forms are computable). Given a stratified truss T, there is a finite set of label-preserving surjective truss maps $F : T \to S$. Searching this set for truss coarsenings and selecting the coarsening with smallest codomain provides an algorithm to compute the normal form coarsening $T \to [\![T]\!]$. ⌟

Remark 5.3.66 (Normal forms are efficiently computable). There are better algorithms to compute normal forms than the preceding brute-force search approach. For example, we can apply Observation 5.2.31 to the case of stratified trusses; we give the key steps of the resulting algorithm here, but leave implementation details and correctness to the interested reader.

Given a stratified n-truss T, we will first normalize the stratified 1-truss bundle $(p_n : T_n \to T_{n-1}, \mathrm{lbl}_T)$ of T (see Observation 5.3.64), yielding its normal form $\tilde{p}_n := [\![(p_n, \mathrm{lbl}_T)]\!]$. For this, consider the trivially stratified open 1-truss subbundles of (p_n, lbl_T), i.e. subbundles o whose induced labeling is constant:

$$\begin{array}{ccccc} O_{n-1} & \hookrightarrow & T_n & \xrightarrow{\mathrm{lbl}_T} & \sqcap(T) \\ {\scriptstyle o}\downarrow & & \downarrow{\scriptstyle p_n} & & \\ O_n & \hookrightarrow & T_{n-1} & & \end{array} \qquad (O_{n-1} \xrightarrow{\mathrm{const}} \sqcap(T))$$

Here, $O_i \hookrightarrow T_i$ are open subposets, and o is an open 1-truss bundle. The normal form $\tilde{p}_n$ is obtained by coarsening any such subbundle o to the 'trivial open fiber' bundle $B \times \mathring{\mathbb{T}}_0 \to B$ (where $\mathring{\mathbb{T}}_0$ is the trivial open 1-truss, see Terminology 2.1.23). This can be achieved efficiently by applying these coarsenings to individual fibers, working iteratively through all base points $x \in T_{n-1}$ by *decreasing depth* in the poset T_{n-1} (i.e., by decreasing cell dimension in the corresponding mesh): concretely, in the fiber $p_n^{-1}(x) \to \{x\}$ we coarsen any trivially stratified open neighborhood $O_n \subset p_n^{-1}(x)$ whose downward closure in T_n is also trivially stratified.

encounter that the inductive step involves the iterative construction of certain 1-truss bundle pushouts over a fixed base, a process that must reach a fixpoint after finitely many steps, analogous to the finiteness arguments used in the proof of Lemma 5.2.18.

Next, construct the 'projected' stratified $(n-1)$-truss $T_{<n}$, whose underlying truss is the $(n-1)$-truncation of T, and whose stratification is defined such that an arrow runs within the same stratum if and only if the labeled 1-bordism fiber over that arrow is an identity bordism in $\tilde{p}_n$. We can now inductively apply our algorithm to normalize $T_{<n}$ yielding $\tilde{T}_{<n} := [\![T_{n-1}]\!]$. By construction, there is a unique stratified 1-truss bundle $\tilde{p}'_n$ over $\tilde{T}_{n-1}$, whose pullback along $T_{n-1} \to \tilde{T}_{n-1}$ recovers the normalized bundle $\tilde{p}_n$. Extending $\tilde{T}_{<n}$ with $\tilde{p}'_n$ yields the normalized truss $[\![T]\!]$ as desired. We note that the algorithm equally applies in the case of stratified n-truss bundles.[35] ⌟

We next prove Corollary 5.1.29: recall that this states that coarsest refining meshes of n-tame stratifications (Z, f) can be algorithmically computed.

Proof of Corollary 5.1.29. Let (Z, f) be a tame stratification. By definition, it has some refining mesh M. Compute the fundamental stratified truss $\sqcap_{\mathsf{T}}(M, f)$ and its normal form $[\![\sqcap_{\mathsf{T}}(M, f)]\!]$. The truss coarsening $\sqcap_{\mathsf{T}}(M, f) \to [\![\sqcap_{\mathsf{T}}(M, f)]\!]$ determines a mesh coarsening that coarsens M to the coarsest refining mesh of (Z, f). □

We immediately deduce that framed stratified homeomorphism between tame stratifications is decidable, as stated in Theorem 5.1.30.

Proof of Theorem 5.1.30. Given tame stratifications (Z, f) and (W, g), construct their coarsest refining meshes M and N. Then (Z, f) and (W, g) are framed stratified homeomorphic if and only if the corresponding stratified trusses $\sqcap_{\mathsf{T}}(M, f)$ and $\sqcap_{\mathsf{T}}(N, g)$ are related by a balanced isomorphism (which must be unique if it exists). Since the latter isomorphism problem can be algorithmically solved, so can the former homeomorphism problem. □

The results in this section establish the computational tractability of tame stratifications. We emphasize that the existence of *coarsest* refining meshes plays a fundamental role in this story; conceptually, this antipodal to the classical quest for constructing mutually *refining* triangulations in the classical Hauptvermutung.

5.3.3.2. Tame singularities, tame cells, and their duality. As an application of our combinatorialization results, we discuss two special classes of tame stratifications, called *singularities* and *cells*. These classes are related by a dualization operation that applies to general tame stratifications as we explain. The latter operation also applies to the process of normalization itself, yielding an interesting sibling process of *quotienting degeneracies*.

Building on our earlier notions of truss cells and singularities (see Definition 5.3.14) and mesh cells and singularities (see Definition 5.3.20), we now introduce the notions of tame cells and tame singularities.

[35] In fact, it also applies (verbatim) to the general case of labeled truss bundles. With unsurprising modifications, it can be applied to case truss bundles labeled in quasicategories, see Remark 2.2.75.

DEFINITION 5.3.67 (Tame k-cells). A **tame k-cell** is an n-tame stratification whose corresponding normalized stratified truss is an n-truss k-cell. ⌟

DEFINITION 5.3.68 (Tame k-singularities). A **tame k-singularity** is an n-tame stratification whose corresponding normalized stratified truss is an n-truss k-singularity. ⌟

EXAMPLE 5.3.69 (Cells and singularities). In Figure 5.35 we illustrate tame cells and singularities, and the duality that relates them (as made precise in the next construction). In each case, the singularity is shown on the left, while its dual cell is shown to its right. Note that excepting the top right tame singularity, the other five tame singularities correspond to the five truss singularities in Figure 5.24. ⌟

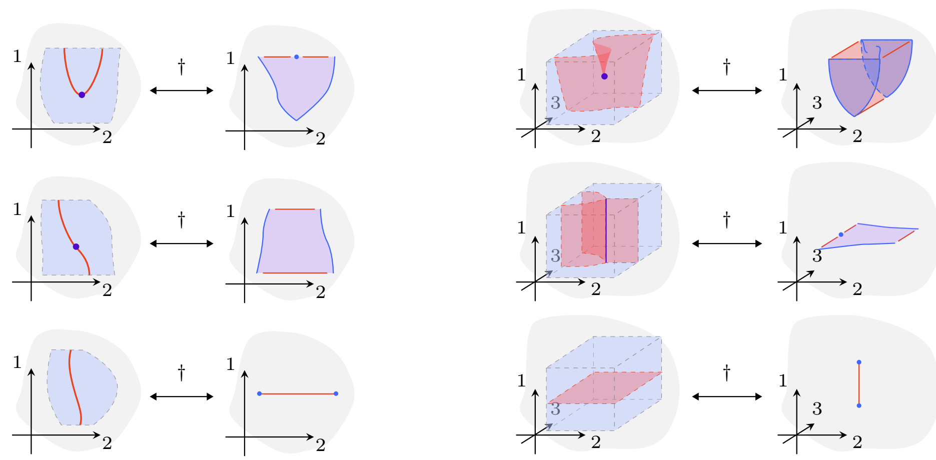


FIGURE 5.35. Tame cells and their dual tame singularities.

The duality of cells and singularities illustrated in the preceding example is formalized by the following construction.

CONSTRUCTION 5.3.70 (Dual stratifications). Let (Z, f) be an n-tame stratification. Its **dual stratification** $(Z^\dagger, f^\dagger)$ is the n-tame stratification obtained as follows. First, construct the corresponding normalized stratified truss T of f as in the proof of Theorem 5.1.23. Define the 'dual stratified truss' $T^\dagger$ to be the stratified truss whose underlying truss is $\underline{T}^\dagger$ and whose labeling is $\mathrm{lbl}_T^{\mathrm{op}} : T_n^{\mathrm{op}} \to \sqcap(T)^{\mathrm{op}}$. Now construct the stratified mesh realization $(M, f^\dagger) := \|T^\dagger\|_{\mathsf{M}}$, and define $Z^\dagger$ to be the underlying space $M_n \subset \mathbb{R}^n$. ⌟

EXAMPLE 5.3.71 (Dual stratifications). In Figure 5.36 we depict two pairs of dual tame stratifications. ⌟

REMARK 5.3.72 (Dual stratified maps). The duality can be applied in the case the *maps* of tame stratifications (to a less strict extent, since map realizations should be considered up to homotopy). Indeed, given a map

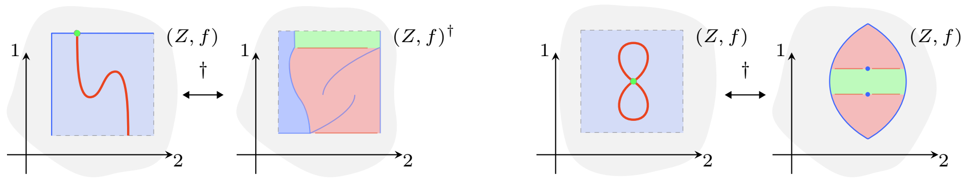


FIGURE 5.36. Duality of tame stratifications.

$F : T \to S$ of stratified trusses, we obtain a 'dual' map $F^\dagger := F^{\mathrm{op}} : T^\dagger \to S^\dagger$ of the respective dual stratified trusses. Analogously, we say maps of tame stratifications $F : (Z, f) \to (Y, g)$ and $G : (Z^\dagger, f^\dagger) \to (Y^\dagger, g^\dagger)$ 'are dual' if their fundamental stratified truss maps are. ⌟

In Proposition 5.3.60 we proved that any chain of non-identity truss coarsenings eventually ends in the same normalized truss. The same holds after dualization. Recall that a degeneracy of stratified trusses is a map $F : T \to S$ whose dual $F^\dagger$ is a coarsening.

COROLLARY 5.3.73 (Stratified truss degeneracies are confluent). *Let T be a stratified n-truss. Any maximal chain of non-identity stratified truss degeneracies of T ends in the same n-truss, and the truss degeneracy from T to it is unique.*

This result is interesting because, unlike coarsenings, degeneracies of stratified trusses may change the topology of stratifications when passing to mesh map realizations. Their behavior generalizes that of ordinary (framed) simplicial degeneracy maps, as the next example illustrates.

EXAMPLE 5.3.74 (Degeneracies of tame stratifications). In Figure 5.37 we depict three framed stratified surjections which are degeneracies. After passing to coarsest refining meshes, all three maps induce degeneracies of stratified meshes; in fact, each of the maps shown is the maximal such degeneracy and the resulting stratification is in 'degeneracy normal form'. ⌟

5.3.3.3. Minimal cell structures and framed homeomorphism. Not all cellulable topological spaces admit a unique (and, in an appropriate sense, canonical) cell structure, and, in general, the same observation holds in the case of framed topology. However, canonical 'minimal' cell structures do exist in the important special case of so-called *realizable* framed regular cell complexes. We sketch a proof of this result, which brings us into the realm of global framed topology. The underlying ideas involved in the proof mirror those in the construction of mesh joins from Key Lemma 5.2.13 and in the comparison of framed and proframed structures from Chapter 3.

TERMINOLOGY 5.3.75 (Realizable framed regular cell complexes). Given an n-framed regular cell complex $(X, \mathcal{F})$, note that each closed cell in the complex is canonically homeomorphic to its cell-to-mesh realization (see

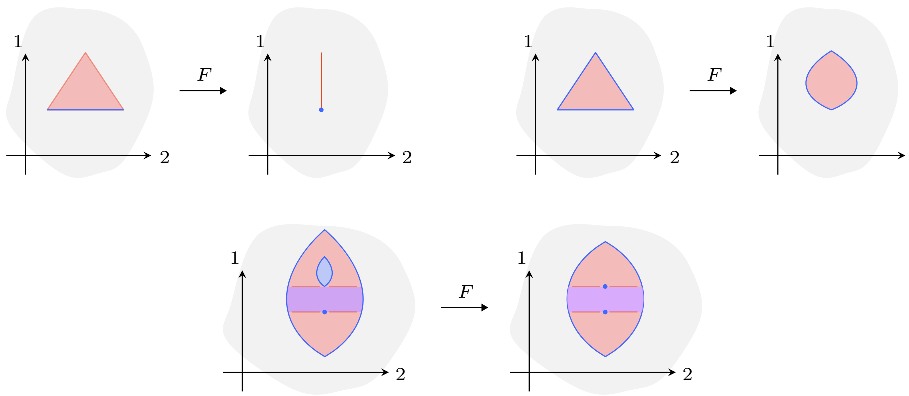


FIGURE 5.37. Degeneracies of tame stratifications.

Observation 4.2.85). We say $(X, \mathcal{F})$ is 'framed realizable' if it admits a 'cell-wise framed realization' $i : |X| \to \mathbb{R}^n$ that is piecewise linear, and restricts to a framed embedding on each individual cell: that is, for each $x \in X$, the map i restricts to an injective n-framed map $\left|X^{\geq x}\right| \hookrightarrow \mathbb{R}^n$ (in the sense of Definition 4.1.86). We refer to framed regular cell complexes that are framed realizable simply as 'realizable framed regular cell complexes'. ⌟

Note that requiring the realization to be PL is a convenient way to ensure tameness of its image (and the images of cells). Note also that any framed realization as defined in Terminology 1.3.36 defines a cell-wise framed realization in the above sense.

EXAMPLE 5.3.76 (Realizable and non-realizable cell complexes). In Figure 5.38 we depict two 1-framed regular cell complexes. The first complex is framed realizable, i.e., admits a cell-wise framed realization in $\mathbb{R}^1$, but the second complex fails to be framed realizable. For better visualization, we may consider both as 2-framed complexes and ask for cell-wise framed realizations in $\mathbb{R}^2$. We can lift the cell-wise framed realization of the first complex along the projection $\pi_2 : \mathbb{R}^2 \to \mathbb{R}^1$, as shown in the figure. But the second complex, necessarily, still fails to have a realization in $\mathbb{R}^2$. ⌟

REMARK 5.3.77 (Directed acyclic graphs). 1-Framed regular cell complexes that are framed realizable are precisely directed acyclic multigraphs (DAMGs): 0-cells represent vertices of these graphs, and framed 1-cells represent their directed edges. In this sense, realizable n-framed regular cell complexes can be regarded as a natural higher-dimensional analog of DAMGs.

This generalization similarly applies to directed acyclic graphs (DAGs): namely, 'n-dimensional directed acyclic graphs' may simply be conceived of as n-framed regular cell complexes that are framed realizable and in which each k-cell, $k > 0$, is uniquely determined by its boundary. ⌟

◇ New fig: a 2-DAG (actually 2-cell complex) and a 2-framed 2d rcc that isn't a 2-DAG [I personally think this is scope creep, graphs are a different paper, tbd]

◇ New fig: a 3-DAG (actually 3-cell complex) and a 3-framed 3d rcc that isn't a 3-DAG [I personally think this is scope creep, graphs are a different paper, tbd]

◇ It would be nice to have another remark saying that n-DAGs are determined by a framed realization. If that's true. Ie you can just have the complex and the realization. [n-DAGs defined above in simple terms, also ... scope

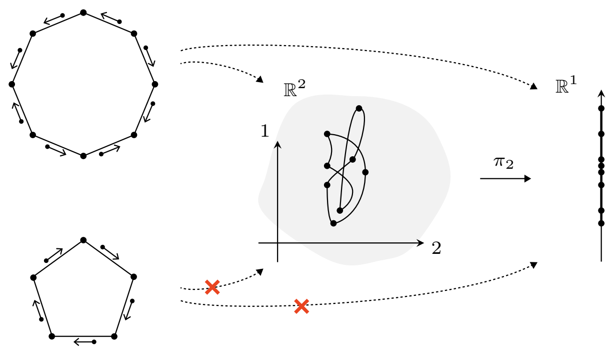


FIGURE 5.38. Realizable and non-realizable cell complexes.

Given an n-framed regular cell complex $(X, \mathcal{F})$, recall that $\|X\|$ denotes its stratified realization: this yields the cell stratification of X, which stratifies the geometric realization $|X|$ by open cell strata.

TERMINOLOGY 5.3.78 (Framed homeomorphisms and coarsenings). Given two realizable n-framed regular cell complexes X and Y, a 'framed PL homeomorphism' $F : X \to Y$ is a PL homeomorphism $F : |X| \to |Y|$ such that, given a cell-wise framed realization i for Y, the composite $i \circ F$ yields a cell-wise framed realization for X. If, in addition, F induces a stratified coarsening $F : \|X\| \to \|Y\|$ we say $F : X \to Y$ is a 'framed coarsening'. ⌟

Implicit in the preceding remark is that the notion of 'framed (PL) homeomorphism' does not depend on the cell structures of the complexes at all; indeed, it merely preserves an induced topological frame structure on the underlying space.

Our main result states the following observation about the cell structures of realizable framed regular cell complexes.

THEOREM 5.3.79 (Coarsest cell structures). *The framed homeomorphism class $[X]$ of a realizable n-framed regular cell complex X has a unique coarsest representative X_{min}, i.e., any $Y \in [X]$ admits a framed coarsening to X_{min}.*

PROOF RECIPE. The proof uses the familiar technique of join stratifications. Pick a framed homeomorphism $F : X \xrightarrow{\simeq} Y$ of realizable framed regular cell complexes, and pick a cell-wise framed realization i for Y. Denote the two given cell stratifications of the underlying space $|Y|$ by $f := F\,\|X\|$ and $g := \|Y\|$. We show that $f \vee g$ is again a framed regular cell complex by showing that closed strata $\overline{s}$ of $f \vee g$ yield framed regular cells $i(\overline{s})$.

The key ingredient is to inductively consider the 'n-framed boundary complexes' $\partial_n X$ and $\partial_n Y$, defined as follows: construct $\partial_n Y$ (and similarly $\partial_n X$) as the subcomplex of Y containing cells c that are mapped homeomorphically by $\pi_n \circ i$ into $\mathbb{R}^{n-1}$, such that at most one cell c' with $c \subset \partial c'$ and $\pi_n(i(c)) = \pi_n(i(c'))$ exists. One checks that F restricts to a framed homeomorphism $F : \partial_n X \xrightarrow{\simeq} \partial_n Y$ and that i restricts to a cell-wise framed realization of $\partial_n Y$. Denote by $\partial_n f$ and $\partial_n g$ the corresponding stratifications of the underlying space of $\partial_n Y$. Crucially, we may consider $\partial_n X$ and $\partial_n Y$ as $(n-1)$-framed regular cell complexes, with $\pi_n \circ i$ providing a cell-wise realization for $\partial_n Y$. Thus, by induction, $\partial_n f \vee \partial_n g$ is a framed regular cell complex. The remaining proof, showing that $f \vee g$ is also a framed regular cell complex, uses the alternating fiber structures of *section* and *spacer* cells (that we have encountered in many parts of framed combinatorial topology) for the appropriately constructed bundle $f \vee g \to \partial_n f \vee \partial_n g$. □

As a consequence of Theorem 5.3.79, we can often vastly simplify the framed cell structures of embedded manifolds and stratifications induced by coarsest refining meshes.

EXAMPLE 5.3.80 (Minimal cell structures of tame embeddings). In Figure 5.39 we depict a tame embedding of the circle in $\mathbb{R}^2$. On the right, we depict a framed regular cell complex obtained by restricting the coarsest refining mesh of the tame embedding to the embedding's image. This is not the coarsest framed cell structure for the underlying framed space; indeed, the coarsest framed cell structure is shown to the left of the former complex. ⌟

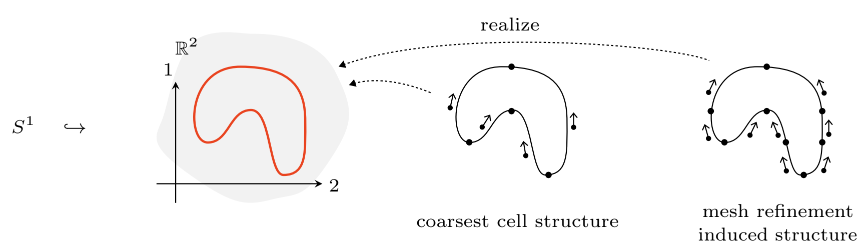


FIGURE 5.39. Coarsest cell structure of a framed embedded circle.

Coarsest cell structures may now be used to algorithmically decide framed homeomorphism of realizable n-framed regular cell complexes.

PROOF OF THEOREM 5.1.31. Given realizable n-framed regular cell complexes X and Y, we first compute their coarsest cell structures $X_{\min}$ and $Y_{\min}$ (e.g., by searching through all possible coarsenings). To check whether X and Y are framed homeomorphic, we check whether $X_{\min}$ and $Y_{\min}$ are identical as framed regular cell complexes. □

EXAMPLE 5.3.81 (Deciding framed homeomorphism of 1-complexes). In Figure 5.40 we illustrate how to decide whether two framed regular cell complexes are homeomorphic in the simple case of 1-complexes (with 1-frame vectors globally pointing from left to right): by the previous theorems, we first compute the coarsest cell structures, and then verify that these coincide for our chosen complexes. ⌟

4 : The result in this section have a further generalization to stratified realizable cell complexes... could remark on that. or not!

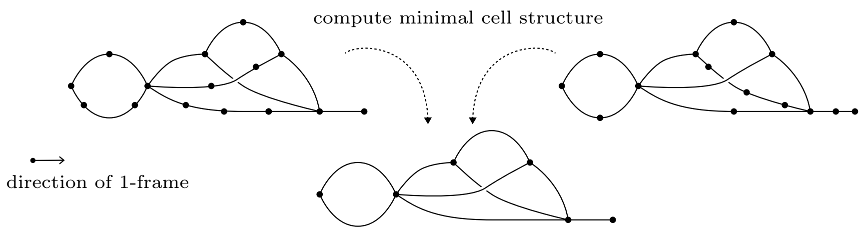


FIGURE 5.40. Decidability of framed homeomorphism for framed 1-complexes.

5.4. Diagrammatic algebra and diagrammatic geometry

In this final section, among a large pool of further topics that deserve discussion, we select and discuss two important research directions that build on framed combinatorial topology. Both are rooted in the interpretation of tame stratifications as certain *diagrams*: namely, as pasting diagrams of morphisms in higher categories (yielding a combinatorial foundation for *diagrammatic higher algebra*), or as gluing diagrams of manifolds embedded in higher-dimensional space (yielding a combinatorial foundation for *diagrammatic geometry*).

OUTLINE. In Section 5.4.1 we first study tame stratifications that are, in an appropriate framed sense, conical stratifications. We then describe the subclass of conical tame stratifications whose normal cones of strata are transversal to the ambient framing: we will call such stratifications **manifold diagrams**. Manifold diagrams solve the long-standing problem of generalizing string diagrams and surface diagrams to higher dimensions, and provide directed-topological semantics to compositions and coherences of higher-categorical morphisms.

In Section 5.4.2 we describe a class of tame stratifications, called **tame tangles**, that model embedded manifolds with corners. We discuss how these tangles admit perturbations in combinatorial terms, and how, applied locally, this yields a combinatorial theory of *singularities*. Crucially, the similarity of combinatorial and differential singularity theory will motivate a set of conjectures, relating tame tangles and differential manifolds. Together these conjectures constitute a first faithful *combinatorialization of smooth structures*.

5.4.1. Manifold diagrams. String diagrams are a ubiquitous tool for describing compositional structures in low dimensions, such as tensor networks [**BCJ11**], spin networks [**BC98, RS95**], proof nets [**Mel06**], Feynman diagrams [**BL11**], quantum circuits and Frobenius algebras [**AC04, CP07**], and, more generally, (symmetric) monoidal categories [**Hot65, JS91**]. String diagrams provide a powerful representation of such structures since they naturally model certain equivalences of compositional constellations via geometric transformations (namely, via stratified isotopies).

Manifold diagrams generalize the idea of string diagrams to higher dimensions. The theory of manifold diagrams builds directly on that of tame stratifications. Concretely, manifold diagrams are tame stratifications with two simple additional constraints: a local triviality condition which is a framed variant of classical conicality, and a transversality condition that ensures strata 'flow in generic directions'.

5.4.1.1. Conicality. One way in which stratified spaces naturally arise is by geometric dualization of cell complexes: the resulting strata intersect cells of the original complex *transversally* and are of dual dimensions. Stratifications obtained in this way exhibit a fundamental property called *conicality*: recall, conicality of a stratification requires the existence of neighborhoods that decompose into a product of a 'tangential' trivially stratified space and a 'normal' stratified cone (see Terminology B.3.1). Conicality is a fundamental ingredient in the theory of stratified spaces. For example, it allows the elegant construction of higher fundamental categories of stratifications (see Section B.3.2), including a formulation of a stratified analog of the classical homotopy hypothesis [**AFR19**].

Example 5.4.1 (Conical stratifications dual to a cell complex). In the center row of Figure 5.41 we depict a neighborhood of the dual stratum of the central 2-simplex in the simplicial complex on the left (obtained by gluing two 3-simplices). This neighborhood decomposes as the product of a tangential $\mathbb{R}^1$ and a transversal cone stratification as shown on the right. ⌟

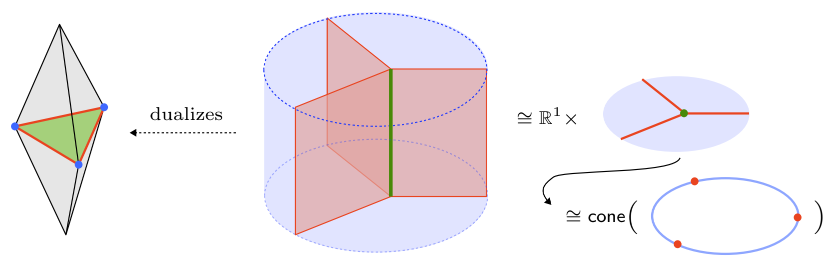


Figure 5.41. A neighborhood as a product of its tangential and its conical transversal parts.

In general, strata in conical stratifications need not be manifolds;[36] however, for our purposes, we assume tangential spaces are always euclidean, which ensures strata are manifolds.

For tame stratifications, there is a natural analog of conicality that admits equivalent formulations both on the topological and the combinatorial side of the theory. In fact, the condition becomes simpler to state since the theory of meshes comes with built-in notions of $\mathbb{R}$-products (as mesh bundles with trivial fibers) and geometric duality; in particular, closed meshes, which are regular cell complex, are dual to open meshes, which are conical by the above logic. (Of course, we know, and self-duality suggests, that all meshes are conical; while this allows strata that are manifolds with corners we focus on the open case here.) Using local building blocks of stratified open meshes as our models for conical neighborhoods we arrive at the following condition.

[36]Extending the complex illustrated in Figure 5.41, consider, for example, the dual of the complex consisting of three 3-simplices attached along a single mutual 2-simplex; the stratum dual to the central 2-simplex would now comprise three lines glued at a point, which is no longer a 1-manifold. Note that this complex no longer embeds in $\mathbb{R}^3$ either. Since we work with cell complexes that can be *realized* in $\mathbb{R}^n$, in an appropriate sense, such situations will never arise for us, and we can assume all strata to be manifolds.

Definition 5.4.2 (Framed conicality). A tame stratification is **framed conical** if all points have neighborhoods that are framed stratified homeomorphic to tame singularities. ⌟

The preceding definition can be unpacked to resemble the classical conicality condition more directly by expressing tame singularities as certain product stratifications: the 'trivial trangential' space $\mathbb{R}^m$ is spanned by all 1-mesh bundles in the coarsest refining mesh of the singularity which have trivial fiber; the remaining 1-mesh bundles span the coordinates of the 'normal cone' stratification. Note that the definition also carries over to the case of tame embeddings, by applying it to the image of an embedding.

Example 5.4.3 (Framed conical and non-conical statifications). Every open mesh, as a cell-wise stratified space, is trivially a framed conical tame stratification. Less trivially, we illustrate a framed conical 2-tame embedding of a stratified circle in Figure 5.42 on the left, while on the right we depict a tame embedding of a stratified circle that fails to be framed conical: indeed, at the two indicated points, no neighborhood exists that is stratified homeomorphic to a tame singularity. ⌟

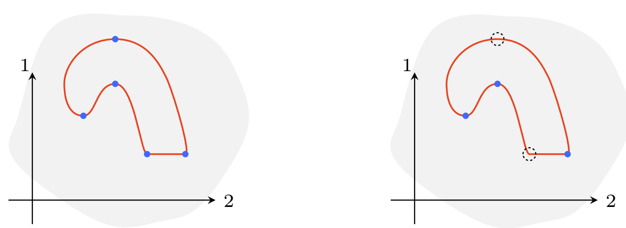


Figure 5.42. Framed conical and non-conical stratifications.

Remark 5.4.4 (Framed conicality, combinatorially). As a consequence of the combinatorialization results in this chapter, framed conicality has the following equivalent combinatorial phrasing: a stratified n-truss (T, lbl_T) is 'conical' if, for each $x \in T$, the stratified subtruss induced by restricting lbl_T to the upper closure $T^{\geq x} \hookrightarrow T$ normalizes to a truss singularity. ⌟

5.4.1.2. Transversality. Strata in framed conical n-tame stratifications inherit a well-defined framing type by restricting the ambient standard framing of $\mathbb{R}^n$. We now turn our attention to those framing types that are, in an appropriate sense, *generic* with respect to that ambient framing. To understand what we mean by genericity, consider the family of tame linear embeddings of the open interval I in $\mathbb{R}^2$ shown in Figure 5.43. Post-compose the embedding with the standard projection $\pi_2 : \mathbb{R}^2 \to \mathbb{R}^1$, which yields a homeomorphism for all but one member of the family. This precisely singles out the green class of embeddings and with them a framing type (indeed, thinking of the embeddings as framed realizations, the green class corresponds exactly to one of two possible framing types of 2-embedded framed 1-simplices, while the blue embedding corresponds to the other framing type). We say that the green embeddings are 'transversal', and in this sense generic, as their image is transversal to the fibers of the projection π_2.

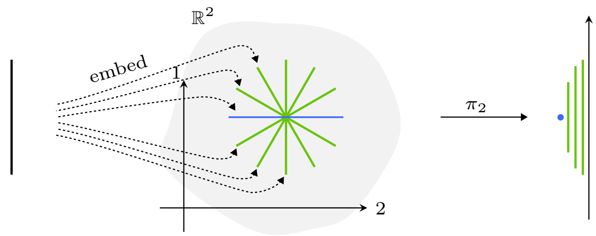


FIGURE 5.43. Genericity of induced framings on the embedded interval.

Transversality generalizes to manifolds of general dimension: a tame embedded k-manifold $\iota : W \hookrightarrow \mathbb{R}^n$ is 'transversal' if post-composition with the projection $\pi_{\leq k} : \mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^k$ yields a local homeomorphism $W \to \mathbb{R}^k$. Similarly, we say that a tame k-singularity is 'transversal' if its central k-stratum is transversal. Transversality can also be conveniently understood at the level of underlying coarsest refining meshes (and has an analogous combinatorial formulation in terms of trusses): namely, a tame k-singularity is transversal if and only if the lowest k bundles $p_k, \ldots, p_1$ in its coarsest refining mesh are 1-mesh bundles with trivial open fiber.

Imposing transversality on all singularities in our earlier definition of framed conicality, we arrive at the following central notion.

DEFINITION 5.4.5 (Manifold diagrams). A **manifold n-diagram** is an n-tame stratification in which all points have neighborhoods that are framed stratified homeomorphic to a transversal tame singularity. ⌟

EXAMPLE 5.4.6 (Manifolds diagrams and non-diagrams). In Figure 5.44, on the left we illustrate a manifold 2-diagram; in contrast, the framed conical tame stratification on the right fails to be a manifold diagram as its central horizontal line stratum is not transversal (cf. Figure 5.43).

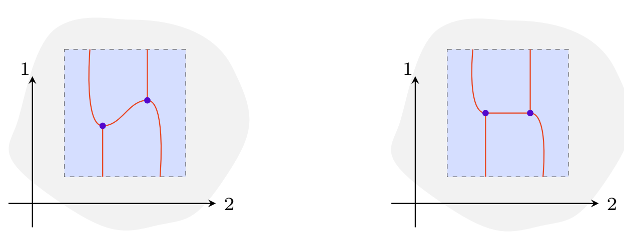


FIGURE 5.44. A manifold diagram and a non-diagram.

⌟

REMARK 5.4.7 (Manifold diagrams, combinatorially). As a consequence of the combinatorialization results in this chapter, manifold diagrams (up to framed stratified homeomorphism) are fully and faithfully captured by the following purely combinatorial notion: a 'combinatorial manifold n-diagram' (T, lbl_T) is a conical stratified n-truss in which neighborhoods $T^{\geq x} \hookrightarrow T$, $x \in T_n$, normalize to n-truss k-singularities whose lowest k truss bundles have trivial open 1-truss fiber. ⌟

The formalization of manifold n-diagrams in all dimensions resolves a long-standing open problem. Previously, stratified topological definitions to capture the notion of manifold diagrams (or versions thereof) had only been given in low dimensions: initially in dimension 2 [**JS91**], later in dimension 3 [**Hum12, BMS12**] (see also [**Tri99**]), and in restricted form in dimension 4 [**CKS96, CS98**]. As mentioned in the beginning of this section, the usefulness of manifold diagrams (and, historically, string diagrams in particular) derives from a correspondence of algebraic compositional equivalences and certain geometric deformations [**Sel10, Had24**]. We can now make the latter precise.

DEFINITION 5.4.8 (Manifold diagram isotopies). A **manifold n-diagram k-isotopy** (or, when $k = 1$, simply a manifold n-diagram isotopy) is a manifold $(n+k)$-diagrams that does not contain l-singularities for $l < k$. ⌟

In particular, we say that two manifold n-diagrams are related by an isotopy (and thus a geometric deformation in the preceding sense) if there exists a manifold diagram isotopy *between them*; more precisely, this means that when restricted the appropriate 'end fibers' of $\pi_{\leq 1} : \mathbb{R}^{n+1} \to \mathbb{R}$ (namely, fibers over points $x \in \mathbb{R}$ in the end strata of the 1-mesh truncation of the isotopy's coarsest refining $(n+1)$-mesh), the isotopy recovers the former diagrams up to framed stratified homeomorphism.

EXAMPLE 5.4.9 (The simplest isotopy). The simplest manifold diagram isotopy is the braid, see Figure 5.4, which runs between two manifold 2-diagrams that each comprise two points embedded in the plane, and whose positions are switched by the isotopy. In algebraic terms, the braid expresses a commutativity relation in the composition of those point strata. ⌟

The dualization constructions of Section 5.3.3.2 apply to our notion of manifold diagrams and yield a geometrically dual counterpart to such diagrams, which may reasonably be called 'cellular diagrams'. These cellular diagrams generalize many kinds of pasting diagrams of morphisms found in the study of higher categories, which leads us to the following somewhat more speculative remark on manifold diagrams as a foundation for higher categories and higher types.

REMARK 5.4.10 (Manifold-diagrammatic higher category theory). The *dual cellular complexes* of manifold diagrams can be directly understood as so-called categorical *pasting diagrams*, i.e., as diagrams that describe the composition of morphisms in higher categories. This interpretation turns manifold diagrams into suitable local models of higher categories [**Cis19**]; the study of the resulting theory of higher categories has been forward-thinkingly termed *manifold-diagrammatic higher category theory*, but is firmly beyond this book's scope.

We nonetheless mention one key observation: the manifold-diagrammatic approach to defining higher categories exhibits a striking difference from existing approaches (including both algebraic and presheaf models [**Lei04, CL04, Ber20**]). This difference concerns the treatment of so-called categorical *coherences*, which determine structural equivalences between pastings (cf. Example 5.4.9). Coherences are both a central feature and a central source of complexity in higher category theory. The novelty of the manifold-diagrammatic approach may be succinctly put as follows:

› In classical approaches, given pastings of morphisms that have some property X, we *require* some coherence to *exist*.
› In the manifold-diagrammatic approach, given pastings of morphisms that have some property X, these pastings *are* coherences.[37]

The resulting 'generatable' perspective on higher-categorical coherences, in which coherences are built from the same combinatorial building blocks as pastings, is connected to several deeper principles at play.[38] It also provides a novel mechanism for expressing finitely specified higher structures (such as *higher inductive types*) without the need for auxiliary machinery that creates new terms to satisfy coherence existence conditions. ⌟

5.4.2. Tame tangles. A tame tangle is a tamely embedded manifold whose image admits a decomposition into traversal 'critical strata'; we make this precise as follows:

Definition 5.4.11 (Tame tangles). An **n-tame m-tangle** is an n-tame embedding $W \hookrightarrow \mathbb{R}^n$ of an m-manifold W whose image can be refined by an n-manifold diagram. ⌟

We remark (but do not prove) that there always exists a unique minimal coarsest n-manifold diagram refining a given tame tangle. This allows us to define the following.

Terminology 5.4.12 (Critical strata of a tame tangle). The 'k-critical strata' of a tame m-tangle are the k-strata of its minimal coarsest refining manifold diagram, for $k \leq m$. ⌟

[37]The property X in question is simply 'being an isotopy', see Definition 5.4.8.

[38]First is the cobordism hypothesis [**BD95**]: our duality of cells and manifold diagrams leads to a combinatorial analog of the classical Thom-Pontryagin construction and its generalization to stratified bordisms [**BRS76, Pon76**]. Second is the principle of semi-strictification: the coherences that are expressed in manifold diagrams are exactly the 'minimal necessary' ones in the sense of semi-strictifiability of higher categories.

Example 5.4.13 (Tangles and their refining manifold diagams). In Figure 5.45 we illustrate three tame tangles with their respective minimal coarsest refining manifold diagrams; note that critical strata precisely run along 'folds' of the manifold with respect to the ambient framing. ⌟

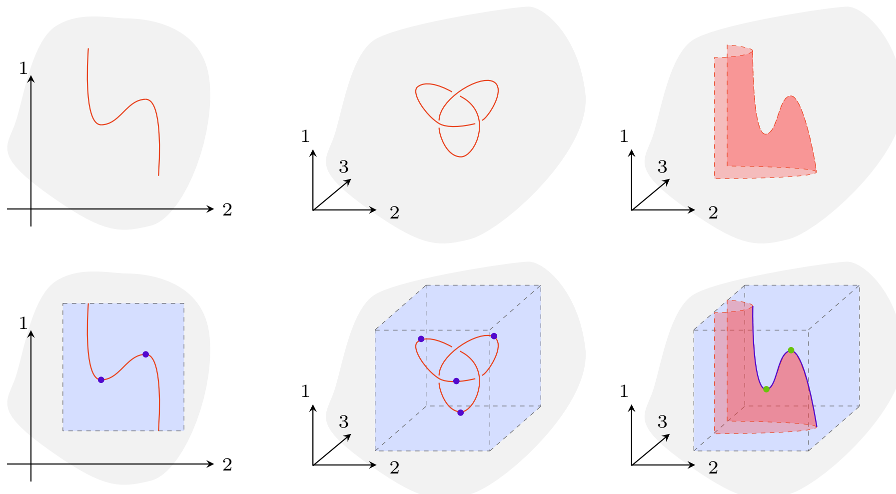


Figure 5.45. Tangles with their minimal coarsest refining manifold diagrams.

Terminology 5.4.14 (Tangle singularities). An m**-tangle** k**-singularity** (or, when $k = 0$, simply an 'm-tangle singularity') is a tame m-tangle whose minimal coarsest refining manifold diagram is a tame k-singularity. ⌟

Example 5.4.15 (Tangle singularities). In Figure 5.46 we illustrate tangle singularities arising from the 3-tame 2-tangle illustrated in the previous figure. ⌟

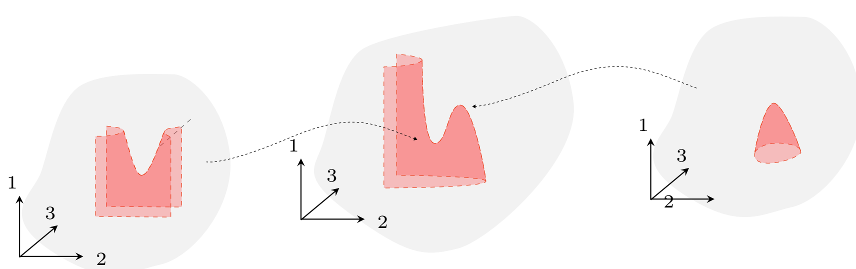


Figure 5.46. Two tangle singularities in a tame tangle.

As before, the results of this chapter allow us to formulate an analogous combinatorial theory of tame tangles. Moreover the notion of tame tangles extends straightforwardly to cases with boundary.[39]

We next turn our attention to the study and classification of tangle singularities; indeed, not all such singularities are created equal, as some are 'more elementary' than others. Intuitively, as a comparative measure of elementarity, one can try define a notion of perturbations, which we sketch as follows.

Terminology 5.4.16 (Tame tangle perturbation). A 'tangle perturbation' is tame bundle over the stratified interval $\|[1]\|$ whose special and generic fibers are n-tame tangle, such that the underlying stratified topological bundle is a trivial bundle over the unstratified interval, i.e., of the form $[0,1] \times g \to [0,1]$. We say that the special fiber tangle 'can be perturbed' to the generic fiber tangle. ⌟

We say that a tangle (0-)singularity W is 'adjacent' to another singularity W' if W can be perturbed to a tame tangle V one of whose singularities is W' such that *all* of V's singularities (including W') are in an appropriate sense, *combinatorially simpler* than W—there are several ways of making precise what 'combinatorially simpler' means, though a basic such measure of complexity is the cardinality of the corresponding normalized stratified truss. We leave a more precise exploration of this design space to future work, and focus on outlining key intuitions of the story for now.

Terminology 5.4.17 (Elementary tangle singularities). We say a singularity is **elementary** if no other singularity is adjacent to it. ⌟

Example 5.4.18 (Elementary and non-elementary singularities). We illustrate an adjacency of singularities in Figure 5.47, where we perturb the 'monkey saddle' (see [**Mil63**]) to a tangle containing two ordinary saddles; both of these singularities which are combinatorially simpler than the monkey saddle and, thus, the monkey saddle is adjacent to the ordinary saddle. In fact, the ordinary saddle can be shown to be an elementary singularity. ⌟

From the definitions outlined above follows a theory of elementary tame tangle singularities that is rather reminiscent of classical singularity theory. Indeed, elementary 2-tangle singularities exactly recover a collection of classical 2-dimensional singularities: the Morse-type singularities (*saddles*, *minima*, and *maxima*) and Cerf-type singularities (*cusps* of 1-parameter families of 1-variable functions).

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[39] And, it is worth emphasizing again that the combinatorial-topological theory of tame stratifications deals elegantly with 'manifolds with corners', which otherwise often require particular technical attention [**Joy09**]; in tame stratified topology, these can be modeled simply using half-open n-trusses and n-meshes.

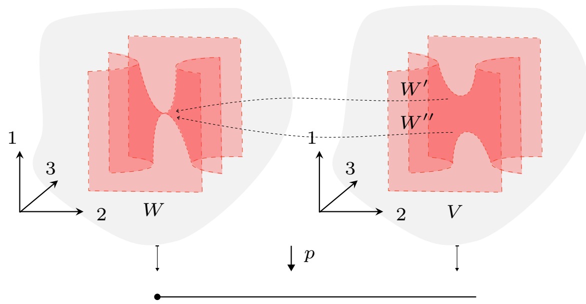


FIGURE 5.47. A perturbation of the monkey saddle into two ordinary saddle singularities.

For codimension-1 m-tangle singularities with $m > 2$ the parallel with classical singularities initially continues into higher dimensions, at least for the so-called *simple* singularities as described by Arnold [**Arn75**] (which, in particular, include the *elementary singularities* that were described in the work of Thom [**TF18**]). These singularities include, for example, the *swallowtail* singularities (succeeding the cusp in Arnold's list of A_k singularities) whose corresponding elementary tangle singularity was depicted already in Figure I.7, but also higher-dimensional singularities such as Arnold's D_4 *singularity*.

As the number of parameters increases, the classical machinery of smooth singularity theory eventually breaks down and non-simple singularities arise (for parameter dimensions above 5, see [**PS14**, §7.6]): the "dimensions of jet spaces outgrow the dimensions of the structure groups", which causes the smooth equivalence relation of singularities to become too fine, leading to uncountably many equivalence classes of singularities. By stark contrast, the combinatorial machinery of elementary tangle singularities 'must work' in all dimensions in that it can, by design, only generate countably many classes of singularities, leading to a profound discrepancy between the two approaches in higher dimensions. The precise nature of elementary tangle singularities and their relation to classical singularity theory remains an open question; we formulate the following problems.

PROBLEM 5.4.19 (Classify elementary singularities in codimension-1). Can we list all elementary tame tangle singularities in codimension 1? Can we make precise a correspondence with simple differential singularities? ⌟

More ambitiously, we could also ask the following.[40]

PROBLEM 5.4.20 (Classify elementary singularities in higher codimension). Can we enumerate all elementary tame tangle singularities in codimension $k > 1$? ⌟

[40] However, higher codimensions bring the complexity of knottings and even exotic smooth sphere structures (known in dimension 7 and upwards).

Remark 5.4.21 (Classifying elementary tangle isotopies). An 'm-tangle k-isotopy' (or, when $k = 1$, simply an m-tangle isotopy) is a tame m-tangle that does not contain any tangle l-singularities for $l < k$. Complementary to the classification of singularities, we could ask for a classification of the *elementary* isotopies, which should be those isotopies that cannot be perturbed into simpler ones. Elementary isotopies include, e.g., the braid or the Reidemeister III. The nature of such elementary isotopies is much less well-studied than that of elementary singularities (and, to our knowledge, no classical definitional counterpart exists), which can be partially attributed to the absence of a framework in which to uniformly represent such isotopies. We leave crafting good questions about elementary tangle isotopies to the reader's imagination. ⌟

In formulating the above problems, we have been agnostic about the structure of the embedding $\iota : W \hookrightarrow \mathbb{R}^n$; in general, it makes sense to at least require ι to be a PL embedding of a PL manifold. Indeed, without this condition, our notion of tangles would include triangulated manifolds that are not PL manifolds, such as those obtained from double suspensions of homology spheres; see [**Bry02**, Thm. 9.1]. However, these triangulated manifolds do not linearly embed in codimension 1, so codimension-1 tangles are not affected by such pathologies.

Beyond PL structures, our illustrations of the correspondence of combinatorial tangle singularities and differential singularities suggest a deeper connection to *smooth* manifolds, which we now outline. Our guiding observation is that the inductive structure of the canonical combinatorial decomposition of a tame tangle (via its minimal coarsest refining mesh) is maximally rigid: it decomposes the tangle as a tower of iterated stratified 1-mesh bundles, whose fibers are 1-dimensional and admit no 'exotic' behavior when glued together. One may therefore expect exotic smooth behavior to arise purely from the *global* compositional interplay of tangle singularities and tangle isotopies. Motivated by the hypothesis that tangle singularities faithfully encode smooth singularities, we conjecture that the *combinatorial structure* of a tangle, which encodes its singularities, isotopies, and their composition, is a faithful invariant for the tangle's smooth structure.

Conjecture 5.4.22 (Framed stratified homeomorphism implies diffeomorphism). *From any two smooth embeddings $e : M \hookrightarrow \mathbb{R}^n$ and $e' : M' \hookrightarrow \mathbb{R}^n$ that correspond to the same normalized stratified truss, we can produce a diffeomorphism $M \cong M'$.*

Note: earlier text restricted attention to codim-1 singularities. But this are only sufficient to described framed bordisms/manifolds. In general, e.g., the Serre automorphism may be trivialized!

Conversely, one may ask whether any smooth manifold can be represented as a tangle. A positive answer seems to be suggested by standard transversality arguments appropriately adapted to the framed stratified setting; we record this possibility in the following conjecture.

Conjecture 5.4.23 (Smooth embedded manifolds are generically tame tangles). *Given a compact smooth k-manifold M, any smooth embedding*

$e : M \hookrightarrow \mathbb{R}^n$ *has an arbitrarily small perturbation such that the image of the perturbed embedding* $e' : M \hookrightarrow \mathbb{R}^n$ *defines a tame* k*-tangle in* $\mathbb{R}^n$*.*

If these conjectures hold, the resulting correspondence of tame tangle representations of manifolds M and smooth structures on M would realize a goal similar to that of MacPherson's program of *combinatorial differential manifolds* [**Mac91**]: the faithful combinatorial representation of smooth structures and the ability to work *smoothly* without direct reference to the continuum $\mathbb{R}$.

APPENDIX A

◊Linear and affine frames

This appendix provides some motivational context concerning various notions of frames, not on combinatorial objects but on classical linear and affine spaces.

In Section A.1.1, we observe that linear orthonormal frames may be reformulated either in terms of sequences of linear subspaces, yielding a notion of *indframes*, or in terms of sequences of linear projections, yielding a notion of *proframes*. Though the notion of orthonormal frame depends on a euclidean structure on the vector space, neither the notion of indframe nor proframe does. We then define not-necessarily-orthonormal frames to be *orthoequivalent* when they induce the same indframe or equvialently proframe; orthoequivalence classes of frames thus provide an effective generalization of orthonormal frames to non-euclidean vector spaces.

In Section A.1.2, we expand attention to three generalized notions of linear frames: an ordinary frame on a vector space corresponds to a trivialization $V \xrightarrow{\simeq} \mathbb{R}^m$, but we may also consider projections $V \twoheadrightarrow \mathbb{R}^k$, leading to a notions of *partial* trivializations and frames, or consider injections $V \hookrightarrow \mathbb{R}^n$, leading to notions of *embedded* trivializations and frames, or altogether consider arbitrary linear maps $V \to \mathbb{R}^n$, leading to notions of *embedded partial* trivializations and frames. Pushing out the reformulations and the generalizations of frames provides further notions of partial, embedded, and embedded partial indframes and proframes, and eventually a notion of orthoequivalence classes of generalized trivializations that consistutes an effective substitute for partial, embedded, or embedded partial orthonormal frames even in the absense of a euclidean structure.

Finally in Section A.2, we briefly discuss the *affine* space analogs of the preceding assortment of linear space concepts. We highlight the crucial asymmetry between affine linear projections and affine linear injections, that leads to a fundamental preferencing for affine proframes over affine indframes. We conclude by emphasizing the conceptual throughline from orthonormal embedded linear frames, to orthoequivalence classes of embedded linear trivializations, to embedded linear proframes, to embedded affine proframes, to embedded simplicial proframes, to embedded simplicial frames, thus back to the starting point of this book.

A.1. Linear frames

A.1.1. Linear trivializations, frames, indframes, and proframes. We recall linear trivializations and linear frames of vector spaces, and introduce the related notions of *linear indframes* and *linear proframes*. For euclidean vector spaces, oriented indframes and oriented proframes provide the same information as orthonormal frames, but indframes and proframes provide a suitable generalization of orthonormal frames to arbitrary vector spaces. The combinatorial analog of proframes in particular plays a pervasive inspirational and technical role in our development of framed combinatorial structures.

We begin with classical linear trivializations and frames.

Definition A.1.1 (Linear trivializations). A **linear trivialization** of an m-dimensional vector space V is a linear isomorphism $V \xrightarrow{\sim} \mathbb{R}^m$. ⌟

Preimages of the standard basis vectors $e_i \in \mathbb{R}^m$ under the linear trivialization map define an ordered list of 'frame vectors' $v_i \in V$. Every linear trivialization therefore determines and is determined by a linear frame in the following sense.

Definition A.1.2 (Linear frames). A **linear frame** of an m-dimensional vector space V is an ordered list $(v_1, v_2, \ldots, v_m) \subset V$ of linearly independent vectors. ⌟

We now want to compare the structure of linear trivializations (and equivalently of linear frames) on vector spaces to the following two structures.

Definition A.1.3 (Linear indframes). A **linear indframe** on an m-dimensional vector space V is a sequence of inclusions of vector spaces V_i, with $\dim(V_i) = i$:

$$0 = V_0 \hookrightarrow V_1 \hookrightarrow V_2 \hookrightarrow \cdots \hookrightarrow V_{m-1} \hookrightarrow V_m = V.$$ ⌟

Definition A.1.4 (Linear proframes). A **linear proframe** on an m-dimensional vector space V is a sequence of projections of vector spaces V^i, with $\dim(V^i) = i$:

$$V = V^m \twoheadrightarrow V^{m-1} \twoheadrightarrow V^{m-2} \twoheadrightarrow \cdots \twoheadrightarrow V^1 \twoheadrightarrow V^0 = 0.$$ ⌟

Observation A.1.5 (Equivalence of indframes and proframes). Note that linear indframes and proframes define the same structure on a vector space. For a linear indframe $\{V_i \hookrightarrow V_{i+1}\}_{0 \leq i < m}$ on V, the corresponding proframe is determined by the cokernels of the sequence of inclusions into the total vector space: $(V \twoheadrightarrow V^{m-i}) := \mathrm{coker}(V_i \hookrightarrow V)$. Conversely, for a linear proframe $\{V^i \twoheadrightarrow V^{i-1}\}_{0 < i \leq m}$ on V, the corresponding indframe is determined by the kernels of the sequence of projections from the total vector space: $(V_i \hookrightarrow V) := \ker(V \twoheadrightarrow V^{m-i})$. An illustration is given in Figure A.1. ⌟

There are two important standard instances of indframes and proframes.

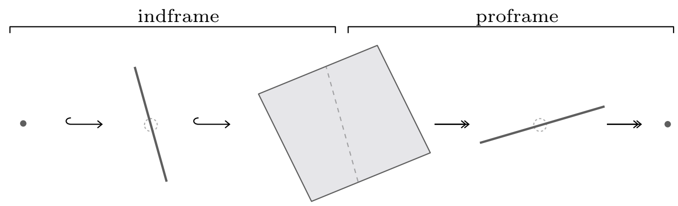


FIGURE A.1. A corresponding indframe and proframe.

TERMINOLOGY A.1.6 (The standard euclidean indframe). The 'standard euclidean indframe' of $\mathbb{R}^n$ is the sequence of subspace inclusions

$$0 \hookrightarrow \mathbb{R} \hookrightarrow \mathbb{R}^2 \hookrightarrow \cdots \hookrightarrow \mathbb{R}^{n-1} \hookrightarrow \mathbb{R}^n$$

where $\mathbb{R}^{i-1} \hookrightarrow \mathbb{R}^i$ is the inclusion as the subspace with first coordinate being zero. ⌟

TERMINOLOGY A.1.7 (The standard euclidean proframe). The 'standard euclidean proframe' of $\mathbb{R}^n$ is the sequence of projections

$$\mathbb{R}^n \twoheadrightarrow \mathbb{R}^{n-1} \twoheadrightarrow \mathbb{R}^{n-2} \twoheadrightarrow \cdots \twoheadrightarrow \mathbb{R}^1 \twoheadrightarrow \mathbb{R}^0$$

where $\mathbb{R}^i \twoheadrightarrow \mathbb{R}^{i-1}$ forgets the last component of vectors in $\mathbb{R}^i$. ⌟

The standard 3-dimensional indframe and proframe are illustrated in Figure A.2.

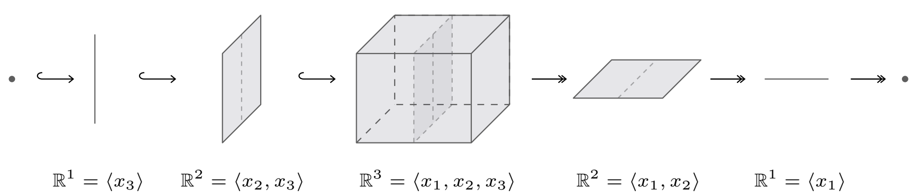


FIGURE A.2. The standard indframe and proframe.

Note that the complement of the image of each standard indframe inclusion $\mathbb{R}^{n-i} = \langle x_{i+1}, \ldots, x_n \rangle \hookrightarrow \mathbb{R}^{n-i+1} = \langle x_i, \ldots, x_n \rangle$ has two components $\mathbb{R}^{n-i+1} \setminus \mathbb{R}^{n-i} = \epsilon_i^- \sqcup \epsilon_i^+$, where the 'negative' component ϵ_i^- and 'positive' component ϵ_i^+ consist of points with i-th numeral coordinate x_i negative and positive, respectively; we let ϵ_i^- and ϵ_i^+ also refer to the images (under the standard indframe inclusions) of these components in the total euclidean space $\mathbb{R}^n$. This assignment of signs to those components gives the standard indframe its standard 'orientation structure'; an orientation of an indframe more generally is such an assignment of signs to the complementary components, as follows.

TERMINOLOGY A.1.8 (Oriented indframe). An 'oriented indframe' on a vector space V is an indframe $\{V_{i-1} \hookrightarrow V_i\}$ together with an association $\nu_i^\pm$ of signs to the connected components of the complement of the image of each inclusion: $V_{m-i+1} \setminus V_{m-i} = \nu_i^- \sqcup \nu_i^+$. (We let ν_i^- and ν_i^+ also refer to the images, under the indframe inclusions, of these components in the total vector space V.) ⌟

An orientation structure on an indframe is equivalent to having an oriented vector space structure on each subspace V_i.

Note that the fiber $\pi_i^{-1}(0)$ over $0 \in \mathbb{R}^{i-1}$ of each standard proframe projection $\pi_i : \mathbb{R}^i \to \mathbb{R}^{i-1}$ is $\mathbb{R}$, and so $\pi_i^{-1}(0) \setminus 0$ has again a 'negative' component $\epsilon_-^i = \mathbb{R}_{<0} \subset \pi_i^{-1}(0)$ and a 'positive' component $\epsilon_+^i = \mathbb{R}_{>0} \subset \pi_i^{-1}(0)$. That assignment of signs gives the standard proframe its standard 'orientation structure'; for a general proframe the corresponding notion is as follows.

TERMINOLOGY A.1.9 (Oriented proframe). An 'oriented proframe' on a vector space V is a proframe $\{p_i : V^i \twoheadrightarrow V^{i-1}\}$ together with an association $\nu_\pm^i$ of signs to the connected components of the complements $p_i^{-1}(0) \setminus 0 = \nu_-^i \sqcup \nu_+^i$. ⌟

An orientation structure on a proframe is equivalent to an oriented vector space structure on each quotient V^i.

Our earlier correspondence of indframes $\{V_i \hookrightarrow V_{i+1}\}$ and proframes $\{p_i : V^i \twoheadrightarrow V^{i-1}\}$ on an m-dimensional vector space V extends to the oriented case. We will write $p_{>i}$ for the composite projection $p_{i+1} \circ ... \circ p_{m-1} \circ p_m : V \twoheadrightarrow V^i$. An orientation structure on the indframe determines an orientation structure on the corresponding proframe by setting $\nu_\pm^i := p_{>i}(\nu_i^\pm)$, and conversely by setting $\nu_i^\pm := p_{>i}^{-1}(\nu_\pm^i)$. (This correspondence is illustrated later in Figure A.3.)

The standard indframe of $\mathbb{R}^m$ can be transported across a trivialization $V \xrightarrow{\simeq} \mathbb{R}^m$, by pulling back the standard subspaces, to give an indframe on the vector space V, and indeed any indframe on V can be obtained this way. Similarly, the standard proframe of $\mathbb{R}^m$ can be transported across a trivialization $V \xrightarrow{\simeq} \mathbb{R}^m$, by composing with the standard projections. These transports are special cases of the following more general constructions, which will be useful later when we consider partial and embedded indframes and proframes.

TERMINOLOGY A.1.10 (Pullback sequence). Given a sequence of vector space inclusions $\{W_i \hookrightarrow W_{i+1}\}$ and a map $F : V \to W_j$, as shown below, we obtain a 'pullback sequence' of inclusions $\{V_i \hookrightarrow V_{i+1}\}$ by iterated pullback:

$$\begin{array}{ccccccccccccc}
0 & \dashrightarrow & V_0 & \dashrightarrow & V_1 & \dashrightarrow & V_2 & \dashrightarrow & \cdots & \dashrightarrow & V_{j-1} & \dashrightarrow & V_j = V \\
 & \searrow & \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow & \lrcorner & \cdots & & \downarrow & \lrcorner & \downarrow F \\
 & & 0 & \hookrightarrow & W_1 & \hookrightarrow & W_2 & \hookrightarrow & \cdots & \hookrightarrow & W_{j-1} & \longrightarrow & W_j
\end{array} .$$

TERMINOLOGY A.1.11 (Restriction sequence). Given a sequence of vector space projections $\{W^i \twoheadrightarrow W^{i-1}\}$ and a map $F : V \to W^j$, as shown below, we obtain a 'restriction sequence' of projections $\{V^{i+1} \twoheadrightarrow V^i\}$ by iterated image factorization:

$$\begin{array}{ccccccccccccc}
V & \dashrightarrow & V^j & \dashrightarrow & V^{j-1} & \dashrightarrow & V^{j-2} & \dashrightarrow & \cdots & \dashrightarrow & V^1 & \dashrightarrow & V^0 = 0 \\
 & \searrow^{F} & \downarrow & & \downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow \\
 & & W^j & \twoheadrightarrow & W^{j-1} & \twoheadrightarrow & W^{j-2} & \twoheadrightarrow & \cdots & \twoheadrightarrow & W^1 & \twoheadrightarrow & 0
\end{array} .$$

OBSERVATION A.1.12 (Trivializations induce oriented indframes and proframes). A trivialization $F : V \xrightarrow{\simeq} \mathbb{R}^m$ induces an oriented indframe on V by taking the pullback sequence of the standard indframe of $\mathbb{R}^m$ along the map F. Similarly a trivialization $F : V \xrightarrow{\simeq} \mathbb{R}^m$ induces an oriented proframe on V by taking the restriction sequence of the standard proframe of $\mathbb{R}^m$ along the map F. Note that when you take the proframe induced by a trivialization, and then take the corresponding indframe in the sense of Observation A.1.5, you obtain exactly the indframe induced by the trivialization. This indframe inherits an orientation $\nu_i^{\pm}$ from the standard orientation of $\mathbb{R}^m$ by requiring $F(\nu_i^{\pm}) = \epsilon_i^{\pm}$.

Note well that distinct linear trivializations may induce the same oriented indframe (or equivalently the same oriented proframe). Considering when trivializations induce the same oriented indframe provides the following equivalence relation.

DEFINITION A.1.13 (Orthoequivalence). Two linear trivializations of the same vector space are **orthoequivalent** if they induce the same oriented indframe, or equivalently the same oriented proframe.

When the vector space has a euclidean structure, each orthoequivalence class of linear trivializations corresponds to a unique orthonormal frame, and so we have the following correspondence.

OBSERVATION A.1.14 (Orthonormal frames, oriented indframes, and oriented proframes are equivalent). Recall that when V is an m-dimensional euclidean vector space, an orthonormal frame is a linear frame whose vectors are orthogonal and of unit length. A frame is orthonormal exactly when its corresponding trivialization is an isometry. Any oriented indframe on V is induced by exactly one isometry $F : V \xrightarrow{\simeq} \mathbb{R}^m$, namely the one such that $F(\nu_i^{\pm}) = \epsilon_i^{\pm}$. Similarly, any oriented proframe is induced by exactly one isometry. Thus, isometric trivializations and orthonormal frames both correspond precisely to oriented indframes and oriented proframes. This correspondence is illustrated in Figure A.3. ⌟

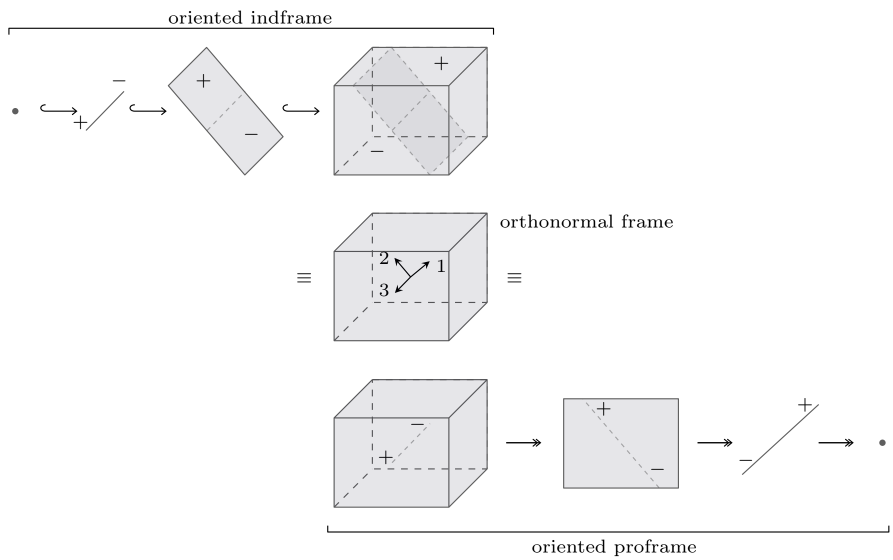


FIGURE A.3. An orthonormal frame and its corresponding oriented indframe and oriented proframe.

REMARK A.1.15 (Orthoequivalence generalizes orthonormality). As discussed, for a euclidean vector space, orthoequivalence classes of trivializations correspond to orthonormal frames, but the former notion is well defined in the absence of a euclidean structure and so provides an effective generalization of orthonormal frames to arbitrary vector spaces. ⌟

A.1.2. Partial and embedded trivializations and frames. Instead of considering linear trivializations $V \xrightarrow{\sim} \mathbb{R}^m$, we may relax our attention to projections $V \twoheadrightarrow \mathbb{R}^k$ or injections $V \hookrightarrow \mathbb{R}^n$ or indeed general linear maps $V \to \mathbb{R}^n$; these provide notions of 'partial', 'embedded', and 'partial embedded' trivializations. We next describe the related generalized notions of frames, indframes, and proframes. The combinatorial analogs of these generalizations, especially of the notion of embedded proframes, will be essential to our development of framed combinatorial complexes that involve gluing structures of different dimensions.

We start with the case of partial trivializations $V \twoheadrightarrow \mathbb{R}^k$ and the related partial frames.

DEFINITION A.1.16 (Linear partial trivializations). A **linear k-partial trivialization** of an m-dimensional vector space V is a linear projection $V \twoheadrightarrow \mathbb{R}^k$. ⌟

DEFINITION A.1.17 (Linear partial frames). A **linear k-partial frame** of an m-dimensional vector space V is an ordered list $(v_1, v_2, \ldots, v_k)$ of k linearly independent vectors in V. ⌟

OBSERVATION A.1.18 (Comparison of partial trivializations and partial frames). If the vector space has a euclidean structure, there is a bijective correspondence between *isometric* partial trivializations and *orthonormal* partial frames, as follows.

Given an isometric partial trivialization $V \twoheadrightarrow \mathbb{R}^k$ (i.e. one that is isometric on the orthogonal complement of the kernel), consider the unique isometric section $\mathbb{R}^k \hookrightarrow V$ (i.e. whose image is the orthogonal complement of the kernel of the trivialization); the image of the standard frame of $\mathbb{R}^k$ under this section is an orthonormal partial frame of V.

Conversely, an orthonormal partial frame $(v_1, v_2, \ldots, v_k)$ of V determines an isometry $\mathbb{R}^k \hookrightarrow V$ sending the standard basis to the partial frame; the resulting partial isometry $V \twoheadrightarrow \mathbb{R}^k$, that splits $\mathbb{R}^k \hookrightarrow V$ and whose kernel is the orthogonal complement of the image of $\mathbb{R}^k \hookrightarrow V$, is an isometric partial trivialization. (This correspondence is illustrated later in Figure A.4.)

In the absence of a euclidean structure, there is no longer a precise correspondence of partial trivializations and partial frames. We may nevertheless think of a partial frame $(v_1, v_2, \ldots, v_k)$ as 'compatible' with a partial trivialization $V \twoheadrightarrow \mathbb{R}^k$ if the map $\mathbb{R}^k \hookrightarrow V$ associated to the partial frame is a section of the trivialization. This compatibility is, though, by no means a bijective correspondence. ⌟

Next we consider the case of embedded trivializations $V \hookrightarrow \mathbb{R}^n$ and the related embedded frames.

DEFINITION A.1.19 (Linear embedded trivialization). A **linear n-embedded trivialization** of an m-dimensional vector space V is a linear inclusion $V \hookrightarrow \mathbb{R}^n$. ⌟

DEFINITION A.1.20 (Linear embedded frame). A **linear n-embedded frame** of an m-dimensional vector space V is an ordered list $(v_1, v_2, \ldots, v_n)$ of n vectors in V, exactly m of which are nonzero, and such that the nonzero vectors are linearly independent. ⌟

OBSERVATION A.1.21 (Comparison of embedded trivializations and embedded frames). If the vector space has a euclidean structure, then there is a many-to-one correspondence of *isometric* embedded trivializations and *orthonormal* embedded frames, as follows.

Given an isometric embedded trivialization $V \hookrightarrow \mathbb{R}^n$, consider the partial isometry $\phi : \mathbb{R}^n \twoheadrightarrow V$, whose kernel is the orthogonal complement of the image of the trivialization. Take the vectors $(\phi(e_1), \phi(e_2), \ldots, \phi(e_n)) \subset V$ and set to zero those vectors that are in the span of the preceding vectors; call the resulting embedded frame $(v_1, v_2, \ldots, v_n) \subset V$. From this, construct an orthonormal embedded frame $(w_1, w_2, \ldots, w_n)$ as follows: if v_i is zero, let w_i be zero, otherwise set w_i to be the unique (suitably signed) unit vector orthogonal to $\langle v_{i+1}, \ldots, v_n \rangle$ inside $\langle v_i, \ldots, v_n \rangle$. Note well, this association, from an isometric embedded trivialization to an orthonormal embedded frame, is far from injective. (The association is illustrated later in Figure A.5.)

Conversely, given an embedded frame $(v_1, v_2, \ldots, v_n) \subset V$ all of whose nonzero vectors are orthonormal, there is an associated isometric embedded trivialization $V \hookrightarrow \mathbb{R}^n$ sending the nonzero $v_i \in V$ to the standard basis vectors $e_i \in \mathbb{R}^n$. Note well that this embedded trivialization is 'axial' in the sense that its image is the span of a subset of the standard basis of euclidean space. Needless to say, this association therefore only hits a very special subset of all embedded trivializations.

In the absence of a euclidean structure, there is no longer such a definite correspondence between embedded trivializations and embedded frames. However, there remains a notion of 'compatibility' between embedded trivializations and embedded frames; that notion will be described later in Terminology A.1.28 using induced proframes. ⌟

Finally, there is a common generalization of partial trivializations and embedded trivializations, and an analogous notion of frames, as follows.

Definition A.1.22 (Linear embedded partial trivializations). A **linear n-embedded k-partial trivialization** of an m-dimensional vector space V is a linear map $V \to \mathbb{R}^n$ with k-dimensional image. ⌟

Definition A.1.23 (Linear embedded partial frames). A **linear n-embedded k-partial frame** of an m-dimensional vector space V is an ordered list of n vectors $(v_1, v_2, \ldots, v_n) \subset V$, exactly k of which are nonzero, and such that the nonzero vectors are linearly independent. ⌟

As in the previous cases, these generalized trivializations do not correspond in a faithful way to these generalized frames, even when there is a euclidean structure. (Nevertheless, there will be a correspondence, mentioned below, of orthoequivalence classes of embedded partial trivializations and orthonormal embedded partial frames; this is illustrated later in Figure A.6.)

The failure of the correspondence of embedded partial trivializations and embedded partial frames can be somewhat remedied by working with suitably orthonormal frames and considering the trivializations up to a suitable notion of orthoequivalence. As in the case of ordinary frames, the notion of orthoequivalence will be based on (now generalized) notions of indframes and proframes, which we introduce presently.

Given a partial trivialization $V \twoheadrightarrow \mathbb{R}^k$, an n-embedded trivialization $V \hookrightarrow \mathbb{R}^n$, or an n-embedded partial trivialization $V \to \mathbb{R}^n$, we may form the pullback sequence, along the trivialization, of the standard euclidean indframe. Sequences of inclusions obtained in this way respectively give notions of partial, embedded, and embedded partial indframes, as follows.

Definition A.1.24 (Linear partial, embedded, and embedded partial indframes). A **linear k-partial indframe** on an m-dimensional vector space V is a sequence of the following form (where $\dim(V_i) = i$):

$$0 \hookrightarrow V_{m-k} \hookrightarrow V_{m-k+1} \hookrightarrow \cdots \hookrightarrow V_{m-1} \hookrightarrow V_m = V.$$

A **linear n-embedded indframe** on an m-dimensional vector space V is a sequence of the following form (where $\dim(V_{m_i}) = m_i$, and, for each i, either $m_{i+1} = m_i + 1$ or $m_{i+1} = m_i$):

$$0 = V_0 = V_{m_0} \hookrightarrow V_{m_1} \hookrightarrow V_{m_2} \hookrightarrow \cdots \hookrightarrow V_{m_{n-1}} \hookrightarrow V_{m_n} = V_m = V.$$

A **linear n-embedded k-partial indframe** on an m-dimensional vector space V is a sequence of the following form (where $\dim(V_{k_i}) = k_i$, and for each i, either $k_{i+1} = k_i + 1$ or $k_{i+1} = k_i$):

$$0 \hookrightarrow V_{m-k} = V_{k_0} \hookrightarrow V_{k_1} \hookrightarrow V_{k_2} \hookrightarrow \cdots \hookrightarrow V_{k_{n-1}} \hookrightarrow V_{k_n} = V_m = V.$$

An n-embedded k-partial indframe on an m-dimensional vector space is simply n-embedded if $k = m$, or simply k-partial if $n = k$.

One defines 'orientations' as before by associating signs to the connected components of the complements $V_{k_i} \setminus V_{k_{i-1}}$ (when those complements are non-empty). ⌟

Similarly, given a partial trivialization $V \twoheadrightarrow \mathbb{R}^k$, an n-embedded trivialization $V \hookrightarrow \mathbb{R}^n$, or an n-embedded partial trivialization $V \to \mathbb{R}^n$, we may form the restriction sequence, along the trivialization, of the standard euclidean proframe. Sequences of projections obtained in this way respectively give notions of partial, embedded, and embedded partial proframes, as follows.

Definition A.1.25 (Linear partial, embedded, and embedded partial proframes). A **linear k-partial proframe** on an m-dimensional vector space V is a sequence of the following form (where $\dim(V^i) = i$):

$$V = V^m \twoheadrightarrow V^k \twoheadrightarrow V^{k-1} \twoheadrightarrow V^{k-2} \twoheadrightarrow \cdots \twoheadrightarrow V^0 = 0.$$

A **linear n-embedded proframe** on an m-dimensional vector space V is a sequence of the following form (where $\dim(V^{m_i}) = m_i$, and, for each i, either $m_{i-1} = m_i - 1$ or $m_{i-1} = m_i$):

$$V = V^m = V^{m_n} \twoheadrightarrow V^{m_{n-1}} \twoheadrightarrow V^{m_{n-2}} \twoheadrightarrow \cdots \twoheadrightarrow V^{m_1} \twoheadrightarrow V^{m_0} = 0.$$

A **linear n-embedded k-partial proframe** on an m-dimensional vector space V is a sequence of the following form (where $\dim(V^{k_i}) = k_i$, and, for each i, either $k_{i-1} = k_i - 1$ or $k_{i-1} = k_i$):

$$V = V^m \twoheadrightarrow V^k = V^{k_n} \twoheadrightarrow V^{k_{n-1}} \twoheadrightarrow V^{k_{n-2}} \twoheadrightarrow \cdots \twoheadrightarrow V^{k_1} \twoheadrightarrow V^{k_0} = 0.$$

An n-embedded k-partial proframe on an m-dimensional vector space is simply n-embedded if $k = m$, or simply k-partial if $n = k$.

One defines 'orientations' as before by associating signs to the connected components of the complements $p_{k_i}^{-1}(0) \setminus 0$, for the projections $p_{k_i} : V^{k_i} \twoheadrightarrow V^{k_{i-1}}$ (when those complements are non-empty). ⌟

Observation A.1.26 (Equivalence of generalized indframes and proframes). In each of the above three generalized cases (namely with the adjectives 'partial', 'embedded', or 'embedded partial'), the notions of indframe and proframe define equivalent structures on a vector space V; one can be constructed from the other as before by taking cokernels and conversely kernels. ⌟

We can now associate generalized indframes and proframes to generalized trivializations and thereby consider the resulting orthoequivalence relation on generalized trivializations.

Observation A.1.27 (Generalized trivializations induce corresponding indframes and proframes). By taking the pullback sequence of the standard indframe of euclidean space, a partial, embedded, or embedded partial trivialization of a vector space V induces a corresponding partial, embedded, or embedded partial indframe of V, referred to as the **induced indframe**. Similarly taking the restriction sequence of the standard proframe of euclidean space produces a corresponding partial, embedded, or embedded partial proframe of V, referred to as the **induced proframe**. Given a generalized trivialization, its induced indframe corresponds to its induced proframe. ⌟

Terminology A.1.28 (Compatibility of generalized frames and trivializations). Given a generalized frame $(v_1, ..., v_n)$ of V and a generalized trivialization $V \to \mathbb{R}^n$, we say the frame and trivialization are **compatible** when each v_i spans the kernel of the projection $V^{k_i} \twoheadrightarrow V^{k_{i-1}}$ in the induced proframe of the trivialization. (The condition can alternatively be phrased in terms of the induced indframe.) ⌟

Definition A.1.29 (Generalized orthoequivalence of trivializations). Two partial, embedded, or embedded partial trivializations of the same vector space are **orthoequivalent** if they induce the same oriented indframe, or equivalently, the same oriented proframe. ⌟

Equipped with the notion of orthoequivalence of trivializations, we can now identify a tighter relationship, in the presence of a euclidean structure, between trivializations and orthonormal frames, as follows. For simplicity, we let orientations and the preservation of orientation structures be mostly implicit.

Observation A.1.30 (Orthoequivalence classes of partial trivializations correspond to orthonormal partial frames). Given a euclidean m-dimensional vector space V and a (not necessarily partial isometry) partial trivialization $V \twoheadrightarrow \mathbb{R}^k$, consider the induced partial indframe $0 \hookrightarrow V_{m-k} \hookrightarrow V_{m-k+1} \hookrightarrow \cdots \hookrightarrow V_{m-1} \hookrightarrow V_m = V$. There is a unique (suitably oriented) unit vector $v_1 \in V_m$ orthogonal to V_{m-1}, and then a unique (suitably oriented) unit vector $v_2 \in V_{m-1}$ orthogonal to V_{m-2}, and so forth. The resulting partial frame $(v_1, v_2, \ldots, v_k)$ is orthonormal.

Conversely, starting with the orthonormal frame, we construct an associated isometric partial trivialization, by the procedure given in Observation A.1.18. Both that constructed isometric partial trivialization and the original partial trivialization have the same induced indframe, and are therefore orthoequivalent. ⌟

The relationship of partial trivializations, partial frames, partial indframes, and partial proframes is illustrated in Figure A.4.

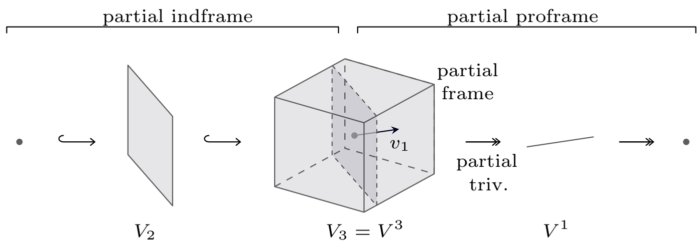


FIGURE A.4. A 1-partial trivialization, frame, indframe, and proframe.

OBSERVATION A.1.31 (Orthoequivalence classes of embedded trivializations correspond to orthonormal embedded frames). Given a euclidean m-dimensional vector space V and a (not necessarily isometric) embedded trivialization $V \hookrightarrow \mathbb{R}^n$, recall from Observation A.1.21 the construction of an associated orthonormal embedded frame: consider the orthogonal projection $\phi : \mathbb{R}^n \twoheadrightarrow V$, and then take a suitable orthnormalization $(w_1, w_2, \ldots, w_n) \subset V$ of the vector sequence $(\phi(e_1), \phi(e_2), \ldots, \phi(e_n)) \subset V$.

Conversely, starting with the orthonormal frame $(w_1, w_2, \ldots, w_n) \subset V$, define an isometric embedded trivialization $V \hookrightarrow \mathbb{R}^n$ by sending each nonzero w_i to $e_i \in \mathbb{R}^n$, again as in Observation A.1.21. The resulting trivialization is orthoequivalent to the original one. ⌟

The relationship of embedded trivializations, embedded frames, embedded indframes, and embedded proframes is illustrated in Figure A.5.

We leave the conceptual pushout of the two previous observations to the exhaustive reader. The correspondence is illustrated in Figure A.6.

REMARK A.1.32 (Orthoequivalence of generalized trivializations generalizes orthonormality). As observed, given a euclidean structure, orthoequivalence classes of embedded partial trivializations have unique embedded partial orthonormal frame representatives; that notion of orthoequivalence classes makes sense, though, in the absence of euclidean structure and so provides an effective substitute for orthonormality of generalized frames. ⌟

A.2. Affine frames

In the previous section we discussed trivializations, frames, indframes, and proframes in the setting of linear vector spaces. We now briefly describe how

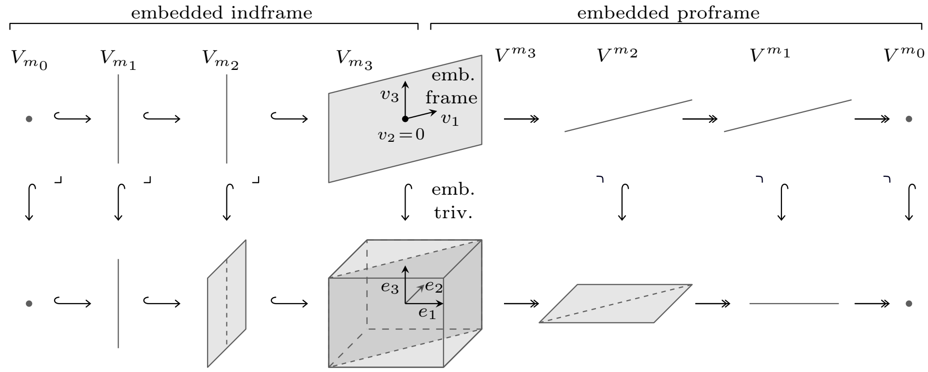


FIGURE A.5. A 3-embedded trivialization, frame, indframe, and proframe.

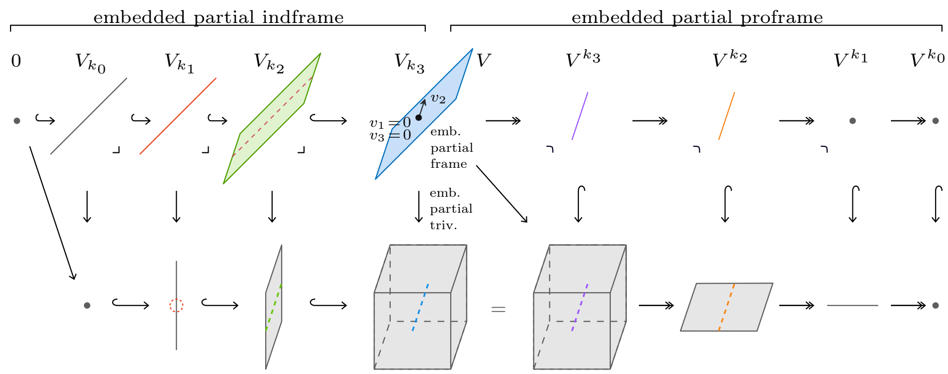


FIGURE A.6. A 3-embedded 1-partial trivialization, frame, indframe, and proframe.

these notions carry over to the case of affine linear spaces. The resulting affine linear structures constitute an instructive analog for a number of the core affine combinatorial structures developed in the book proper; we leave the detailing of that analogy, though, almost entirely to the reader's imagination.

TERMINOLOGY A.2.1 (Affine spaces and their maps). An **affine space** $\mathcal{V}$ is a space freely and transitively acted upon by a vector space $\vec{\mathcal{V}}$, called the 'associated vector space'. Vectors in the associated vector space $\vec{\mathcal{V}}$ are called 'translations'. An **affine map** $\mathcal{F} : \mathcal{V} \to \mathcal{W}$ of affine spaces is a continuous map such that for a necessarily unique linear map $\vec{\mathcal{F}} : \vec{\mathcal{V}} \to \vec{\mathcal{W}}$ (the 'associated vector space map') we have $\mathcal{F}(v) - \mathcal{F}(v') = \vec{\mathcal{F}}(v - v')$. Denote the category of affine spaces and affine maps by Aff. The associated vector space and associated vector space map provide a functor

$$\vec{-} : \mathsf{Aff} \to \mathsf{Vect}.$$

Of course this associated vector space functor has a canonical section, namely the functor

$$\vec{-} : \mathsf{Vect} \to \mathsf{Aff}$$

that simply forgets the origin of the vector space. ⌟

TERMINOLOGY A.2.2 (Geometric realizations of simplices). Given an unordered m-simplex S, its 'geometric realization' $|S|$ (also called the associated 'geometric simplex') is the subspace, of the free vector space $\mathbb{R}\langle S\rangle$ (on the set of vertices of S), consisting of convex combinations of the standard basis. ⌟

The realization $|S|$ is contained in an affine hyperplane of $\mathbb{R}\langle S\rangle$; we denote that hyperplane $\langle S\rangle$. The affine structure of $\langle S\rangle$ restricts to a partial affine structure on $|S|$ ('partial' in the sense that the action by translations of the associated vector space $\langle\vec{S}\rangle$ is partial).

TERMINOLOGY A.2.3 (Affine maps of simplices). Given a simplex S and an affine space $\mathcal{W}$, an **affine map** $|S| \to \mathcal{W}$ is a map that is the restriction of an affine map $\langle S\rangle \to \mathcal{W}$ defined on the associated affine hyperplane.[41] ⌟

NOTATION A.2.4 (Standard geometric simplices). We denote the geometric realization of the standard m-simplex, with vertices $\{0, 1, \ldots, m\}$, by Δ^m, and refer to it as the 'standard geometric m-simplex'. ⌟

TERMINOLOGY A.2.5 (Space of affine vectors). Given an affine space $\mathcal{V}$, the 'space of affine vectors' in $\mathcal{V}$, denoted $\hat{\mathcal{V}}$, is the space of affine embeddings $e : \Delta^1 \hookrightarrow \mathcal{V}$, of the standard geometric 1-simplex Δ^1 into $\mathcal{V}$. Note that the space of affine vectors is itself an affine space (it has an action by $\vec{\mathcal{V}} \oplus \vec{\mathcal{V}}$), and note that there is a canonical affine isomorphism $\mathcal{V} \times \mathcal{V} \cong \hat{\mathcal{V}}$. Any affine map $\mathcal{F} : \mathcal{V} \to \mathcal{W}$ induces (by postcomposition) a map of spaces of affine vectors $\hat{\mathcal{F}} : \hat{\mathcal{V}} \to \hat{\mathcal{W}}$. The formation of affine vectors and affine vector maps together provide an 'affine vector functor' $\hat{-} : \mathsf{Aff} \to \mathsf{Aff}$. ⌟

TERMINOLOGY A.2.6 (Basepoint forgetting map). Given an affine space $\mathcal{V}$, there is a canonical 'basepoint forgetting' map $\hat{\mathcal{V}} \to \vec{\mathcal{V}}$, sending an affine vector $e : \Delta^1 \hookrightarrow \mathcal{V}$ to the translation vector $e(1) - e(0)$. ⌟

Equipped with the above affine terminology, we may now crudely transport the notions of trivializations, frames, indframes, and proframes to the affine setting as follows.

TERMINOLOGY A.2.7 (Affine trivializations, frames, indframes, and proframes). By an 'affine trivialization' or 'affine frame' or 'affine indframe' or 'affine proframe' of an affine space $\mathcal{V}$, we will mean respectively a linear

[41] In the main text, abusing terminology, we refer to affine maps $|S| \to \mathcal{W}$, from affine realizations of simplices, as 'linear maps'.

trivialization or linear frame or linear indframe or linear proframe of its associated vector space $\vec{\mathcal{V}}$. ⌟

Though these affine notions may be concisely specified as above in terms of the associated vector spaces, the concepts may also be understood more directly in terms of structures on affine spaces, as follows.

Remark A.2.8 (Affine perspective on affine trivializations and frames). Given an affine space $\mathcal{V}$, an isomorphism $\mathcal{V} \xrightarrow{\simeq} \vec{\mathbb{R}}^m$ of affine spaces would be a purely affine notion of 'affine trivialization'; such an isomorphism provides a linear isomorphism $\vec{\mathcal{V}} \xrightarrow{\simeq} \mathbb{R}^m$, thus an affine trivialization in the previous sense.

For an affine space $\mathcal{V}$, we may ask for a collection of frame vectors $v_i^x : \Delta^1 \to \mathcal{V}$, for all $x \in \mathcal{V}$, with v_i^x based at x in the sense that $v_i^x(0) = x$, and such that the frame vectors are invariant under every translation; such a collection would be a purely affine notion of 'affine frame'. The basepoint forgetting map $\hat{\mathcal{V}} \to \vec{\mathcal{V}}$ sends such a collection of frame vectors $\{v_i^x\}$ to a linear frame of $\vec{\mathcal{V}}$, thus provides an affine frame in the previous sense. Conversely, pulling back a linear frame along the basepoint forgetting map provides a compatible collection of frame vectors based at every point of the affine space. ⌟

We may similarly try to express affine indframes and affine proframes in more native affine terms. However, we encounter the following obstruction.

Observation A.2.9 (Asymmetry of affine projections and injections). Given an affine space $\mathcal{V}$, and a *linear projection* $\vec{\mathcal{V}} \twoheadrightarrow W$ of its associated vector space, there is a canonically induced affine projection $\mathcal{V} \twoheadrightarrow \mathcal{W}$ whose associated vector space map is the given vector space projection; here $\mathcal{W}$ is constructed as the quotient of $\mathcal{V}$ by the action of the kernel $\ker(\vec{\mathcal{V}} \twoheadrightarrow W)$.

By contrast, given a *linear injection* $U \hookrightarrow \vec{\mathcal{V}}$ into the associated vector space, there is no canonical candidate for a corresponding affine injection $\mathcal{U} \hookrightarrow \mathcal{V}$ (whose associated vector space map is the given vector space inclusion). In particular, given an affine projection $\mathcal{V} \twoheadrightarrow \mathcal{W}$, whose associated linear map has the kernel $\ker(\vec{\mathcal{V}} \twoheadrightarrow \vec{\mathcal{W}}) \hookrightarrow \vec{\mathcal{V}}$, there is no canonical choice of 'affine-linear kernel' $\mathcal{U} \hookrightarrow \mathcal{V}$, whose associated vector space map is the given linear kernel. ⌟

Remark A.2.10 (Asymmetry of simplicial degeneracies and affine faces). The asymmetry between affine projections and affine injections has an analog in the affine combinatorics of simplices. Indeed, while all relevant projections of simplicies can be accounted for by honest simplicial degeneracy maps, there are relevant 'affine inclusions' of simplices that simply cannot be expressed as honest simplicial face maps; those inclusions necessitate the introduction of the notion of affine face map of simplices and the related notion of affine kernel of a simplicial degeneracy. ⌟

The mismatch between affine projections and affine inclusions may be marginally ameliorated by working with 'basepoint-wise indframes', as follows.

OBSERVATION A.2.11 (Basepoint-wise affine indframes). Given an affine space $\mathcal{V}$ and an indframe $0 = \vec{\mathcal{V}}^0 \hookrightarrow \vec{\mathcal{V}}^1 \hookrightarrow ... \hookrightarrow \vec{\mathcal{V}}^m = \vec{\mathcal{V}}$ on the associated vector space $\vec{\mathcal{V}}$, we can pull the indframe back along the basepoint forgetting map $\hat{\mathcal{V}} \to \vec{\mathcal{V}}$, to obtain a filtration of the space of affine vectors $\hat{\mathcal{V}}$; this process, roughly speaking, bases a copy of the indframe at every point of $\mathcal{V}$. Still, the structure of an affine indframe cannot be encoded in any faithful and canonical way via honest affine maps into the original affine space $\mathcal{V}$. ⌟

In practice, the aforementioned asymmetry makes affine proframes a *much more convenient tool* than affine indframes. In particular, we can reformulate the notion of affine proframes in natively affine terms as follows.

DEFINITION A.2.12 (Affine proframes). An **affine proframe** on an m-dimensional affine space $\mathcal{V}$ is a sequence of surjective affine maps of the following form (where $\dim(\mathcal{V}^i) = i$):

$$\mathcal{V} = \mathcal{V}^m \twoheadrightarrow \mathcal{V}^{m-1} \twoheadrightarrow \mathcal{V}^{m-2} \twoheadrightarrow ... \twoheadrightarrow \mathcal{V}^1 \twoheadrightarrow \mathcal{V}^0 = 0. \quad ⌟$$

OBSERVATION A.2.13 (Correspondence between linear and affine proframes). Given an affine space $\mathcal{V}$ and a proframe $\vec{\mathcal{V}} = \vec{\mathcal{V}}^m \twoheadrightarrow \vec{\mathcal{V}}^{m-1} \twoheadrightarrow \cdots \twoheadrightarrow \vec{\mathcal{V}}^0$ on its associated vector space $\vec{\mathcal{V}}$, there is a corresponding sequence of affine surjective maps, thus an affine proframe, obtained by applying the construction of Observation A.2.9 to each projection in the linear proframe. Conversely, given any affine proframe on $\mathcal{V}$, in the sense of Definition A.2.12, we obtain a linear proframe of the associated vector space $\vec{\mathcal{V}}$ simply by considering the associated vector space maps. ⌟

The notion of affine proframes (as in Definition A.2.12), and the correspondence of linear and affine proframes (a la Observation A.2.13), generalize straightforwardly to the cases of partial, embedded, and partial embedded proframes.

We may finally, tersely, trace the following motivational thread all the way from classical linear frames to our combinatorial affine frames, as follows. (For definiteness we mention the embedded frame case, though this may be specialized to ordinary frames or generalized to partial ones as desired.) An orthonormal embedded linear frame of a vector space (as in Definition A.1.20) corresponds, by Observation A.1.31, to an orthoequivalence class of embedded linear trivializations, which by Definition A.1.29 corresponds to an embedded linear proframe; such an embedded linear proframe corresponds, by the embedded analog of Observation A.2.13, to an embedded affine proframe. Now, the geometric realization of an embedded proframe on a simplex is evidently and precisely an embedded affine proframe of the affine space geometric realization of that simplex. Finally, that embedded proframe on a simplex corresponds, by Observation 3.2.23, to an embedded frame on the simplex, as in our core Definition 1.1.36. Altogether, frames on simplices provide a faithful affine combinatorial analog of classical linear frames on vector spaces.

APPENDIX B

Stratified topology

The notion of *stratified space* (or *singular space*) refers to a decomposition of a space into *strata* [**Fri20, Ban07, Wei94, Pfl02**], often ordered by some index of dimension or depth. Frequently, such order is enforced by working with filtrations $X_0 \subset X_1 \subset ... \subset X_{k-1} \subset X_k$ of spaces $X = X_k$ where X_{i-1} is required to be closed in X_i. Equivalently, and more concisely, such a filtration may be expressed by a continuous function $f : X \to [k]^{\mathrm{op}}$ (where $[k]^{\mathrm{op}}$ is the poset $(0 \leftarrow 1 \leftarrow ... \leftarrow k)$ topologized such that downward closed subposets are open sets) which allows us to recover X_i as the preimage $f^{-1}[i]^{\mathrm{op}}$, $i \leq k$. This has been generalized by defining stratifications as continuous maps of spaces to any poset, yielding, for instance, definitions of '$\mathscr{S}$-filtered spaces' in [**GM88**, §III.2.2.1] and of 'P-stratifications' in [**Lur12**, Def. A.5.1]. Note, however, posets in the codomain of such continuous maps may contain information that is unrelated to the decomposition of the underlying space, even when the map is surjective.

⋄ Consider whether the last references to the main text (search for creffoot..., crefchapter-4...) can be removed. (So the appendix is self-contained.)

In this appendix, we develop a notion of stratifications which is similarly general, but in which the role of posets faithfully represents topological information about the stratification, yielding a categorically more natural and robust definition. The different definitions coincide in many cases, for instance, in the case of locally finite stratifications.

B.1. Stratified spaces

In our discussion of stratified spaces, we will use the following conventions.

CONVENTION B.1.1 (Specialization topology). Given a preorder $(P, \leq)$ we regard it as a topological space with the *specialization topology*, declaring the open subsets to be those that are downward closed; a subset U is downward closed if $x \leq y$ and $y \in U$ implies that $x \in U$.[42] ⌟

In general, the specialization topology will not be Hausdorff or weakly Hausdorff. However, posets do belong to the category of *compactly generated spaces* [**Str09**] (since all first-countable spaces do): this category is categorically convenient in that it admits cartesian closed structure.

A category convenient for the purposes of homotopy theory is that of compactly generated weakly Hausdorff spaces: the category admits useful categorical constructions, such as closure under certain pushouts, and has a

[42]We frequently write the relation $x \leq y$ as $x \to y$, interpreting preorders and posets as categories.

homotopy theory equivalent to that of categories of 'cell-like' models such as simplicial sets. We fix the following notation.

Notation B.1.2 (Categories of spaces). We denote by TOP the category of *all* topological spaces, by $k\mathsf{Top}$ the subcategory of compactly generated spaces, and by Top the subcategory of compactly generated weakly Hausdorff spaces. The category $k\mathsf{Top}$ is cartesian closed with internal hom denoted by $\mathrm{Map}(-,-)$, and this internal hom is inherited by Top. ⌟

While we will not be concerned with the nuances of these definitions, it will be useful to keep in mind that they represent slightly different conceptions of spaces tailored to specific purposes: for us, all underlying spaces of *stratified realizations*, which are *cell-like* spaces, belong to Top; in contrast, posets as spaces live in $k\mathsf{Top}$. To work in a joint setting of both cell-like and poset-like spaces we therefore assume the following.

Convention B.1.3 (Compactly generated spaces). By default, all spaces will be assumed to be compactly generated. ⌟

Finally, there is also an inverse translation from spaces to posets.

Remark B.1.4 (Specialization order). Given a topological space $X \in k\mathsf{Top}$, we denote by $\mathrm{Spcl}\, X$ its *specialization order*: this is the preorder whose objects are the objects of the underlying set X, and whose morphisms $x \to y$ are given whenever y is contained in the closure of x. ⌟

Note that for a poset P we have $\mathrm{Spcl}\, P = P$. The specialization topology provides a functor, which is an adjoint equivalence between finite preorders and finite topological spaces. The inverse functor is given by the specialization order functor Spcl. The equivalence is *concrete*, meaning that both unit and co-unit of the adjunction are identities on underlying sets.

Before proceeding to our first definitions, we briefly remark on the following key convention that has been, often implicitly, used throughout the book and will play an explicit role in the next sections.

Remark B.1.5 (Entrance versus exit paths). There are categorically dual conventions for the 'fundamental categories $\sqcap_\bullet f$' of stratifications f, based on whether one chooses to work with 'entrance paths' or 'exit paths' in a stratified space. In this book, we chose the former convention over the latter, since it often yields more intuitive (namely, *covariant*) descriptions of constructible bundles [**AFR19**]. An illustration of this observation is given in Figure B.1 for a stratified bundle π over a stratified circle (S^1, f) and its associated covariant functor on the fundamental poset $\sqcap_\bullet f$ defined using entrance paths; all involved notions will be formally defined in this appendix. ⌟

B.1.1. Entrance paths, stratifications, and fundamental posets. In this section we introduce the basic notions of stratified spaces, including their

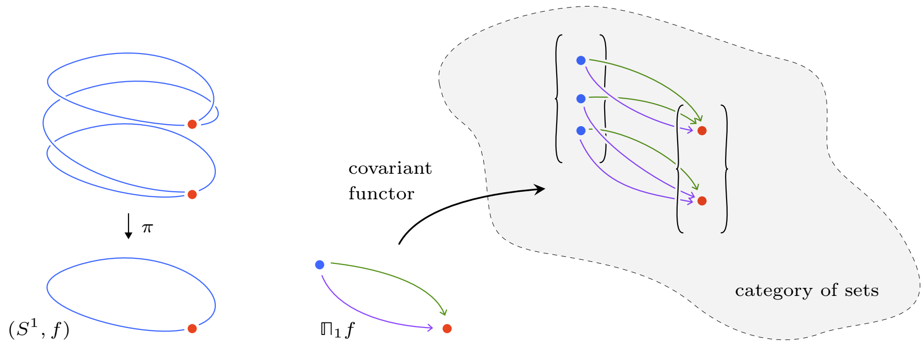


FIGURE B.1. A stratified bundle represented as a covariant functor on fundamental categories.

fundamental posets and their *characteristic functions* which map stratified spaces to their fundamental posets.

A robust definition of stratified spaces is obtained by letting the topological decomposition of a space into strata determine the corresponding poset structure, in terms of the existence of so-called entrance paths between strata, as follows.

DEFINITION B.1.6 (Entrance path). Given a space X and two subspaces X_r and X_s, an **entrance path** from X_r to X_s is a path $\alpha : [0,1] \to X$ with $\alpha([0,1)) \subset X_r$ and $\alpha(1) \in X_s$. ⌟

Here, the path is thought of as *entering* from the former subspace X_r into the latter subspace X_s.

DEFINITION B.1.7 (Formal entrance path). Given a space X and two subspaces X_r and X_s, we say there exists a **formal entrance path** from X_r to X_s, when the closure of X_r has nonempty intersection with X_s. ⌟

In contrast to entrance paths, note that the structure of formal entrance paths is boolean: either there exists a formal entrance path between subspaces or there doesn't. If there is an entrance path from a subspace X_r to a subspace X_s of a space X this implies the existence of a formal entrance path, but the converse need not hold unless additional conditions are imposed (see Lemma B.1.30).

TERMINOLOGY B.1.8 (Formal entrance path relation of a decomposition). Consider a decomposition $\{X_s \subset X\}_{s \in \mathsf{Dec}}$ of a space X into disjoint subspaces X_s (that is, $X = \bigsqcup_{s \in \mathsf{Dec}} X_s$). The 'formal entrance path relation' of the decomposition is the relation on the indexing set Dec of subspaces that has an arrow $r \to s$ exactly when there is a formal entrance path from X_r to X_s. ⌟

Note that the formal entrance path relation of a decomposition is reflexive, but need not be antisymmetric or transitive. Stratifications are exactly those decompositions for which this relation has no cycles, that is for which it is a directed acyclic graph.

DEFINITION B.1.9 (Prestratifications and stratifications). A **prestratification** (X, f) of a space X is a decomposition $f = \{X_s \subset X\}_{s \in \mathsf{Dec}(f)}$ of X into disjoint, nonempty, and connected subspaces indexed by a set $\mathsf{Dec}(f)$, called the 'decomposition set'. The subspaces X_s are called **strata** of (X, f). A **stratification** (X, f) is a prestratification such that the formal entrance path relation on the decomposition set $\mathsf{Dec}(f)$ has no cycles. ⌟

NOTATION B.1.10 (Shorthand for (pre)stratifications). We frequently abbreviate a (pre)stratification (X, f) simply by f, referring to f as a '(pre)stratification on X'. Moreover, we often abbreviate a stratum $X_s \subset X$ simply by its index $s \in \mathsf{Dec}(f)$. ⌟

Observe that, given a stratification (X, f), the transitive closure of the formal entrance path relation on the decomposition set $\mathsf{Dec}(f)$ is a partially ordered set, which has an arrow $r \to s$ exactly when there is a chain of formal entrance paths beginning at r and ending at s. This does not hold true if (X, f) is merely a *pre*stratification, in which case $\mathsf{Dec}(f)$ obtains the structure of a *pre*ordered set.

DEFINITION B.1.11 (Fundamental preorder and poset). For a prestratification (X, f), the **fundamental preorder** $\mathbb{\Pi}(f)$ is the decomposition set of the prestratification together with the transitive closure of the formal entrance path relation. If (X, f) is a stratification, then we refer to $\mathbb{\Pi}(f)$ as the **fundamental poset** of (X, f). ⌟

REMARK B.1.12 (Exit paths and the exit path preorder). Given a prestratification (X, f), the opposite preorder $\mathbb{\Pi}(f)^{\mathrm{op}}$ of the fundamental preorder is also often called the 'exit path preorder'. Its arrows may be understood as 'exit paths': an exit path from X_s to X_r is a path $p : [0, 1] \to X$ with $p(0) \in X_s$ and $p((0, 1]) \subset X_r$; the path is *exiting* from the stratum X_s into the stratum X_r.

Whether to focus on entrance or exit paths is a matter of convention and convenience; in this book, we find that entrance paths have more natural functoriality dependencies and so we work entirely with them, see Remark B.1.5. ⌟

EXAMPLE B.1.13 (Fundamental poset). Figure B.2 shows a stratification of the open 2-disk into five strata, along with its fundamental poset (shown on the right, together with an indication of which poset elements correspond to which strata). ⌟

EXAMPLE B.1.14 (Fundamental poset requiring transitive closure). Figure B.3 depicts a stratification of the open interval, into one open interval and two half-open interval strata, together with its fundamental poset. ⌟

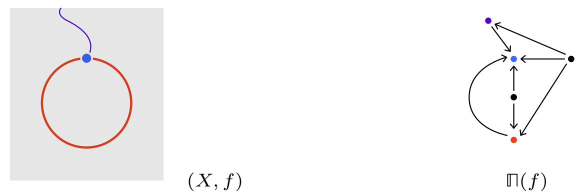


FIGURE B.2. Fundamental poset of a stratification.

FIGURE B.3. A stratification with fundamental poset as the transitive closure of the entrance path relation.

EXAMPLE B.1.15 (Fundamental preorder). In Figure B.4 we depict a decomposition of the circle that is not a stratification but a prestratification, because the formal entrance path relation has a cycle. ⌟

FIGURE B.4. A decomposition that is not a stratification but a prestratification, and its formal entrance path relation.

REMARK B.1.16 (Fundamental posets as generalizations of connected component sets). Fundamental posets play a fundamental role in the theory of stratified spaces: from a perspective of algebraic topology, they are the analog of *connected component sets* $\pi_0 X$ of topological spaces X for stratified spaces.[43] ⌟

TERMINOLOGY B.1.17 (Discrete and indiscrete stratifications). Every space X has an 'indiscrete stratification' whose strata are the connected components of X. The fundamental preorder of the indiscrete stratification is the set of connected components of X. Conversely, every space also has

[43] A yet more abstract analogy can be similarly made from a perspective of higher category theory: fundamental posets are the $(0, 1)$-truncated fundamental categories of stratified spaces, which conversely are the ∞-categorical analogs of posets, a.k.a. $(\infty, 1)$-posets or simply ∞-posets, just as spaces in the form of ∞-groupoids are the ∞-categorical analogs of sets. See later in Remark B.3.19.

a 'discrete prestratification', such that each point becomes its own stratum. The fundamental poset of the discrete stratification of X is the specialization order $\mathrm{Spcl}\,X$ (in particular, the definition of specialization orders can be recovered from the definition of fundamental preorders). ⌟

Unless indicated otherwise, a bare topological space is implicitly given the indiscrete stratification.

TERMINOLOGY B.1.18 (Finite (pre)stratifications). We call a (pre)stratification (X, f) 'finite' if its fundamental preorder $\mathbb{\Pi}(f)$ is finite, and call it 'infinite' otherwise. ⌟

DEFINITION B.1.19 (Characteristic function). Given a prestratification (X, f), we refer to the function $X \to \mathbb{\Pi}(f)$ sending each point $x \in X_r$ to its corresponding stratum $r \in \mathbb{\Pi}(f)$, as the **characteristic function** of the prestratification; we denote the characteristic function of a prestratification (X, f) by $f : X \to \mathbb{\Pi}(f)$. ⌟

A fundamental property of characteristic functions is that they are *finitely continuous*, as follows.

TERMINOLOGY B.1.20 (Finitely continuous maps). A function of topological spaces $F : X \to Y$ is called 'finitely continuous' if for each finite subspace $Q \subset Y$ the function restricts to a continuous map $F : F^{-1}(Q) \to Q$. ⌟

LEMMA B.1.21 (Finite continuity in prestratifications). *Characteristic functions are finitely continuous.*

PROOF. Consider a prestratification (X, f) with characteristic function $f : X \to \mathbb{\Pi}(f)$. Consider a finite subposet $Q \subset \mathbb{\Pi}(f)$, and let $U \subset Q$ be a downward closed subposet of Q (i.e., an open subspace). Arguing by contradiction, assume $f^{-1}(U) \subset f^{-1}(Q)$ is not open. Then there is a point $p \in f^{-1}(U)$ such that each neighborhood of p intersects a preimage $f^{-1}(q)$ of some $q \in Q \setminus U$. Since Q is finite, there must be a $q \in Q \setminus U$ such that $f^{-1}(q)$ intersects *all* neighborhoods of p. This means p lies in the closure of $f^{-1}(q)$. By definition of formal entrance paths there must be an arrow from q to $f(p) \in U$. But this contradicts downward closure of the latter subposet. Thus, $f^{-1}(U) \subset f^{-1}(Q)$ must be open, showing finite continuity of f. □

In the case of *finite* prestratifications, this of course implies that their characteristic functions are continuous in the usual sense. In fact, this also holds for the more general notion of locally finite stratifications. (From now on, we will focus most of our attention on stratifications instead of working in the more general context of prestratifications; nonetheless, several of the following definitions and results still generalize to the case of prestratifications.)

DEFINITION B.1.22 (Locally finite stratifications). A stratification (X, f) is **locally finite** if every stratum s has an open neighborhood $s \subset N(s)$ which is a union of finitely many strata. ⌟

TERMINOLOGY B.1.23 (Locally finite posets). A poset $(P, \leq)$ is 'locally finite' if all downward closures $P^{\leq x} = \{y \mid y \leq x\}$ are finite. ⌟

The definition of local finiteness can also be phrased as a *pointwise local* condition under the further assumption of so-called *frontier-constructibility*.

DEFINITION B.1.24 (Frontier-constructibility). A **frontier-constructible** stratification (X, f) is a stratification in which the closure $\overline{s}$ of each stratum s can be written as a union of strata in f.[44] ⌟

The frontier-constructibility condition is also sometimes simply referred to as the *frontier condition* in the literature. In Lemma B.2.10 we show that this condition holds iff $f : X \to \sqcap(f)$ is an open map.

OBSERVATION B.1.25 (Local finiteness for frontier-constructible stratifications). Given a stratification (X, f) consider the following conditions.

(1) (X, f) is locally finite,
(2) Any point $x \in X$ has an open neighborhood intersecting only finitely many strata,
(3) The fundamental poset $\sqcap(f)$ is locally finite.

In general, only condition (1) and (3) are equivalent. If (X, f) is frontier-constructible, then all three conditions become equivalent. ⌟

LEMMA B.1.26 (Locally finite characteristic functions are continuous). *If (X, f) is a locally finite stratification, then its characteristic function $f : X \to \sqcap(f)$ is continuous.*

PROOF. We show each point $x \in X$ has an open neighborhood restricted to which f is continuous. Let x lie in a stratum $s = f(x) \in \sqcap(f)$. Using local finiteness, pick a neighborhood U of s intersecting only finitely many strata. By definition of formal entrance paths, we can shrink this to a neighborhood U', which is contained in $f^{-1}(\sqcap(f)^{\leq s})$. Continuity of f restricted to U' now follows from the finite stratified case, see Lemma B.1.21, applied to the finite substratification of f obtained from the union of strata in $\sqcap(f)^{\leq s}$. □

REMARK B.1.27 (Characteristic maps). Whenever a characteristic function is continuous we usually refer to it as a characteristic *map*. ⌟

Note, characteristic functions of general infinite (pre)stratifications need not be continuous, as the next example illustrates.

EXAMPLE B.1.28 (Discontinuous characteristic function). In Figure B.5 we depict a stratification of the closed interval with non-continuous characteristic function. Note that the stratification is not locally finite. ⌟

[44] The name 'frontier-constructible' may be thought of as a reference to the observation that in a frontier-constructible stratification the inclusions of closures of strata into the full stratification induce constructible stratified bundles (with empty or singleton fibers). Note also, that the condition equivalently asks for the characteristic map to be open, see Lemma B.2.10.

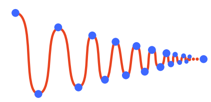


FIGURE B.5. A stratification with non-continuous characteristic function.

As we will see in Lemma B.1.39, there is a precise characterization of those functions $f : X \to P$ from a space to a finite poset which are characteristic maps of stratifications.

Finally, let us revisit the relation of entrance paths and formal entrance paths.

DEFINITION B.1.29 (Pairwise path-connectedness). A stratification (X, f) is called **pairwise locally path-connected** if for each pair of strata $r, s \in \sqcap\!\!\sqcap(f)$ the union $r \cup s \subset X$ is locally path-connected. ⌟

LEMMA B.1.30 (Conditions for dropping formality). *Given a pairwise locally path-connected stratification (X, f), and a formal entrance path from r to s, then there exists an entrance path starting in r and with endpoint some $x \in s$. Moreover, if (X, f) is frontier-constructible, then such a path can be constructed for any $x \in s$.*

PROOF. The proof is straightforward. For the first statement, choose x in the intersection $\overline{r} \cap s$ and use pairwise local path-connectedness. For the second part, note $s \subset \overline{r}$. □

OBSERVATION B.1.31 (Fundamental posets via entrance paths). Note that in a fronter-constructible stratification, the fundamental poset has an arrow from r to s precisely when there is a formal entrance path from r to s (i.e. the transitive closure is not needed). Let us call a stratification 'reasonably regular' when it is pairwise locally path-connected and also frontier constructible. By the preceding lemma, the fundamental poset of a reasonably regular stratification has an arrow from r to s precisely when there is an (actual not formal) entrance path from r to s. ⌟

B.1.2. Poset structures and quotient maps. In this section we relate our definitions of stratifications to the closely related but different notion of *poset structures* on spaces.

DEFINITION B.1.32 (Poset structures). Given a poset P, a P**-structured** space (X, f) is a space X together with a continuous map $f : X \to P$. ⌟

We will first show that characteristic maps of finite stratifications can be understood as a certain class of poset structures. Later we will show that, conversely, every poset structure can be universally *split* into a stratification.

Recall, a surjective continuous map $f : X \to Y$ of spaces is a quotient map if for each subset $U \subset Y$ we have that U is open if and only if $f^{-1}(U)$ is open.

If Y is the specialization topology of a poset, we call f a 'poset quotient'. Poset quotients (to finite posets) admit the following useful characterization.

NOTATION B.1.33 (Covering relation). Given a poset $(P, \leq)$ its covering relation is usually defined as follows: we say $x \in P$ 'covers' $y \in P$, written $y <^{\mathrm{cov}} x$, if $y < x$ is non-refinable (that is, for any $y < z < x$ we have either $y = z$ or $z = x$). ⌟

LEMMA B.1.34 (Quotient maps to finite posets). *For a space X, a finite poset P, and a surjective continuous map $f : X \to P$, the following are equivalent:*

(1) f is a quotient map,
(2) for any cover $p <^{\mathrm{cov}} p'$ in P there is a formal entrance path from $f^{-1}(p)$ to $f^{-1}(p')$.

REMARK B.1.35 (A quotient of posets is a map that is surjective on objects and on covers). In the lemma, if X is itself the specialization topology on a poset Q, then the lemma simplifies to saying that $f : Q \to P$ is a quotient map if and only if f is surjective on objects and on covers. ⌟

PROOF OF LEMMA B.1.34. For $p \in P$, define K^p_0 to be the preimage $f^{-1}(p)$. Set $I^p_0 = \{p\}$. Let I^p_1 be the set of $q \in P$ such that $f^{-1}(q)$ intersects the closure $\overline{K^p_0}$. Note that continuity of f implies that $p \leq q$ for each $q \in I^p_1$. Define K^p_1 to be the union of preimages $f^{-1}(q)$ of $q \in I^p_1$. Set I^p_2 to be the set of $q \in P$ such that $f^{-1}(q)$ intersects the closure $\overline{K^p_1}$, and define K^p_2 to be the union of preimages $f^{-1}(q)$ of $q \in I^p_2$. Repeating this process, since P is finite, we find an index j with $I^p_j = I^p_{j+1}$ and $K^p_j = K^p_{j+1} = \overline{K^p_j}$. Denote these sets by I^p and K^p respectively.

First, assume f is a quotient map. Consider a cover $p < p'$. We claim it is impossible that $p' \notin I^p$: indeed, the complement $X \setminus K^p$ is the preimage of $P \setminus I^p$. Since $X \setminus K^p$ is open and since f is a quotient map, it follows that $P \setminus I^p$ is open which contradicts the assumption that $p < p'$ and $p' \notin I^p$. Thus assume $p' \in I^p$. This implies $f^{-1}(p') \subset K^p$ (and thus intersects K^p). Then there is a sequence $p = p_0 < p_1 < ... < p_k = p'$ with $p_i \in I^p_i$. Since $p < p'$ is a cover we must have $k = 1$, meaning $f^{-1}(p')$ intersects the closure of $f^{-1}(p)$.

Conversely, assume f satisfies that for any cover $p < p'$ in P, the preimage $f^{-1}(p')$ intersects the closure of the preimage $f^{-1}(p)$. Let $Q \subset P$ be a subposet. Let $I^{P\setminus Q}$ and $K^{P\setminus Q}$ be the respective unions of all I^p and K^p for each $p \in P \setminus Q$. If Q is open then $f^{-1}(Q)$ is open by continuity of f. If $f^{-1}(Q)$ is open, then it must be disjoint from $K^{P\setminus Q}$ (by construction of $K^{P\setminus Q}$). Thus $I^{P\setminus Q} = P \setminus Q$. Note that $I^{P\setminus Q}$ is upward closed by our initial assumption. It follows that Q is downward closed, i.e., open as required. □

A central role will be played by poset quotients whose equivalence classes are *connected* in the following sense.

DEFINITION B.1.36 (Connected-quotient maps). For a space X and a finite poset P, a continuous map $f : X \to P$ is called a **connected-quotient map** if it is a poset quotient map whose preimages of points $p \in P$ are connected. (Note, we take 'connected' to also entail 'non-empty'.) ⌟

DEFINITION B.1.37 (Connected-quotient maps between posets). A connected-quotient map $f : Q \to P$ where Q is a poset (endowed with specialization topology) is a poset quotient whose preimages are connected subposets of Q. ⌟

EXAMPLE B.1.38 (Connected-quotient map). In Figure B.6 we depict three maps from the circle to three different posets (color coding images and preimages in the same color). The first map is a connected-quotient map; the second map fails to be a quotient map despite having connected preimages, the third map is a quotient map but fails to have connected preimages. ⌟

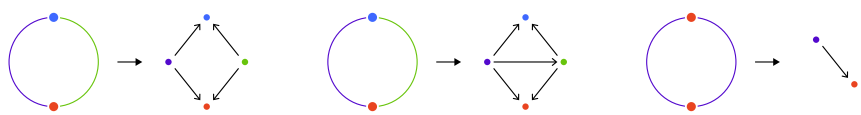


FIGURE B.6. A connected-quotient map and non-examples.

We now characterize stratifications among P-structures.

LEMMA B.1.39 (Characteristic maps are connected-quotient maps). *For a space X, a finite poset P, and a P-structure $f : X \to P$, the following are equivalent:*

(1) f is the characteristic map of a stratification (that is, the decomposition of X into preimages of f is a stratification with characteristic map f and fundamental poset $\sqcap(f) = P$),
(2) f is a connected-quotient map.

PROOF. If f is a characteristic map then, by definition, it has connected preimages and satisfies the second condition in Lemma B.1.34. Thus f is a connected-quotient map.

Conversely, if f is a connected-quotient map, then f defines a stratification by decomposing X into the preimages of f (which are connected by Definition B.1.36). By Lemma B.1.34 the map $f : X \to P$ is exactly the characteristic map of this stratification. □

OBSERVATION B.1.40 (Connected-quotient maps compose). Using the definition of connected-quotient maps one verifies that, given a connected-quotient map $X \to P$ (of some finite stratification on X) and a connected-quotient map $P \to Q$ (of some finite stratification of P), their composite $X \to Q$ yields another connected-quotient map. ⌟

As an immediate corollary of Lemma B.1.39, this observation implies that characteristic maps of finite stratifications compose. As we will see, compositions of characteristic functions describe coarsenings (see Lemma B.2.12).

We end with the remark that similar results hold in the infinite case: the correspondence of characteristic maps and connected-quotient maps from Lemma B.1.39 may be generalized to the context of infinite stratifications by characterizing characteristic functions as *finitely* connected-quotient maps (analogous to the notion of *finite* continuity in Lemma B.1.21), but we forego a discussion of the infinite (and the locally finite) case here.

B.1.3. Factoring poset structures into stratifications and labelings. In this section we show that any poset structure factors into a stratification followed by a *labeling*. A labeling of a stratification in a category C functorially associates data in C to the strata and the formal entrance paths of that stratification.

TERMINOLOGY B.1.41 (Labelings). Let C be a category, and (X, f) a (pre)stratification. A 'C-labeling' (or simply a 'labeling') of (X, f) in C is a functor $L : \mathbb{\Pi}(f) \to \mathsf{C}$. If C is a poset, we also call L a 'poset labeling'. ⌟

We briefly remark on the following higher categorical generalization: instead of considering labelings as functors from the fundamental (preorder or) poset of a (pre)stratification, one may of course also consider functors from the fundamental category or fundamental ∞-category, as defined later in Definition B.3.10.

EXAMPLE B.1.42 (Specialization labelings). Let (X, f) be a finite (pre)stratification. The 'specialization labeling' of X associated to f is the labeling of the discrete prestratification of $X \to \mathrm{Spcl}\,X$ given by the functor $\mathrm{Spcl}\,f : \mathrm{Spcl}\,X \to \mathbb{\Pi}(f)$ (obtained by applying the specialization topology functor to the continuous map $f : X \to \mathbb{\Pi}(f)$). ⌟

We now show that any P-structure canonically factors as a stratification with a discrete labeling on that stratification. This factorization is referred to as the P-structure's *connected component splitting*. Discreteness of the labeling will mean the following.

TERMINOLOGY B.1.43 (Discrete map). A map of posets $F : Q \to P$ is called a 'discrete map' if its preimages are discrete, that is, for each $p \in P$ the preimage $F^{-1}(p)$ contains no non-identity arrows. Note that the condition is equivalent to requiring F to be a conservative functor of categories $Q \to P$. ⌟

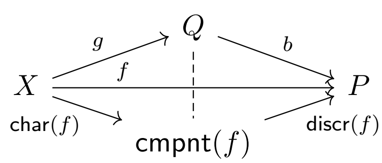
CONSTRUCTION B.1.44 (Connected component splittings). For a P-structure $f : X \to P$, the **connected component splitting** of f is the factorization

$$f = (X \xrightarrow{\mathsf{char}(f)} \mathsf{cmpnt}(f) \xrightarrow{\mathsf{discr}(f)} P)$$

defined as follows. The map $\mathsf{char}(f)$ is the characteristic function of the stratification decomposing X into the connected components of preimages of f; note that the formal entrance path graph cannot have cycles since P is assumed to be a poset and f to be continuous. The discrete map $\mathsf{discr}(f) : \mathsf{cmpnt}(f) \to P$ maps a given connected component of a preimage $f^{-1}(p)$ back to p. ⌟

Note that even if $f : X \to P$ is continuous, the characteristic function $\mathsf{char}(f)$ need not be continuous (see Example B.1.46). We point out three universal properties of connected component splittings: universality among connected-quotient factorizations, universality among discrete map factorizations, and uniqueness among connected-quotient and discrete map factorizations.

Lemma B.1.45 (Universality of connected component splitting). *Let $f : X \to P$ be a P-structure. Assume f factors into maps $g : X \to Q$ and $b : Q \to P$, where g is continuous and b a map of posets. Consider the following diagram:*



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(1) Characteristic map universality*: If g is characteristic, then there is a unique poset map $Q \to \mathsf{cmpnt}(f)$ making the above diagram commute.*
(2) Discrete map universality*: If b is a discrete map, then there is a unique poset map $\mathsf{cmpnt}(f) \to Q$ making the above diagram commute.*
(3) Combined universality*: If g is characteristic and b is a discrete map, then there is a unique poset isomorphism $Q \cong \mathsf{cmpnt}(f)$ making the above diagram commute.*

Proof. We first prove statement (1). Since g is characteristic it has connected preimages. Thus its preimages must lie in the connected components of preimages of f. The map $Q \to \mathsf{cmpnt}(f)$ is the inclusion of strata of g into strata of $\mathsf{char}(f)$.

We next prove statement (2). We first show that preimages of g are unions of strata of $\mathsf{char}(f)$ (i.e., connected components of preimages of f). Let Z be a connected component of a preimage of f. Let $\{q^Z_i\}_{i \in I}$ be the set of objects in Q whose preimages $r^Z_i = g^{-1}(q^Z_i)$ intersect Z. Note that, since b is assumed to be a discrete map, there are no arrows between any q^Z_i in Q. Let Q^Z_i be the downward closure of q^Z_i in Q. Since g is assumed continuous, we have a disjoint open cover $\sqcup_i g^{-1}(Q^Z_i) \cap Z$ of Z. Since Z is connected, the indexing set I must be of cardinality 1. This shows that preimages $g^{-1}(q)$ of g are unions of connected components Z of preimages of f. The map

$\mathsf{cmpnt}(f) \to Q$ can then be defined by mapping the strata $Z \subset g^{-1}(q)$ back to q.

The final statement (3) follows from combining statements (1) and (2). □

Example B.1.46 (Translating P-structures into stratifications). The map on the left in Figure B.6 determines a stratification. This stratification is recovered from the two P-structures shown on the right by connected component splitting. In particular, there are many P-structures with the same underlying stratification. In Figure B.7 we depict another P-structure (again coloring images and preimages in the same color); its connected component splitting recovers the stratification from Figure B.5 with non-continuous characteristic function. ⌟

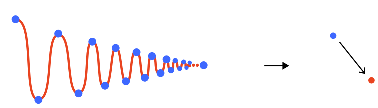

Figure B.7. A non-continuously splitting poset structure.

B.1.4. Relation to other notions of stratifications. Using the constructions from the preceding sections, we now describe the relation of our notion of stratifications to two other frequently used definitions of stratifications, namely to *P-stratifications* and *$\mathscr{S}$-filtered spaces*.

Remark B.1.47 (Relation of stratifications and P-stratifications). Our notion of a P-structure, given by a continuous map from X to P, is also known as a 'P-stratification' (see [**Lur12**, Def. A.5.1]). Given a P-stratification $f : X \to P$ we can construct a stratification with characteristic map $\mathsf{char}(f)$ obtained by connected component splitting from Construction B.1.44 (note that there may be many different P-stratifications $f : X \to P$ that lead to the same stratification in this way). Conversely, every *locally finite* stratification (X, f) arises as the connected component splitting of a P-stratification; indeed, by Lemma B.1.26, we can simply set $P = \sqcap(f)$ and the characteristic map $f : X \to P$ will be continuous. ⌟

Remark B.1.48 (Relation of stratifications and $\mathscr{S}$-filtered spaces). Given a poset $\mathscr{S}$ with unique minimal element $\perp$, an '$\mathscr{S}$-filtration' of a space X is a collection of closed subsets X_s, $s \in \mathscr{S}$, such that $X_\perp = X$ and $X_s \subset X_t$ if $s \geq t$ in $\mathscr{S}$ (see [**GM88**, §III.2.1], up to opposite poset conventions). This defines a continuous map $f_{\mathscr{S}} : X \to \mathscr{S}$, mapping points in the subspace $X_t \setminus \bigcup_{s>t} X_s$ to $t \in \mathscr{S}$. The characteristic function $\mathsf{char}(f_{\mathscr{S}})$ of the connected component splitting of $f_{\mathscr{S}}$ yields a stratification in our sense. Conversely, every stratification (X, f) with continuous characteristic map $f : X \to \sqcap(f)$ yields a $\sqcap(f)^{\lhd}$-filtration of X by setting $X_s = f^{-1}(\sqcap(f)^{\geq s})$ (here, $\sqcap(f)^{\lhd}$ is the poset obtained by adjoining a new bottom element $\bot$ to $\sqcap(f)$, and $\sqcap(f)^{\geq s}$ is the upper closure of an element s in $\sqcap(f)$).[45]

B.1.5. Stratified realizations of posets. In this section we discuss that the geometric realization of any poset P itself carries canonically the structure of a stratification, called the *stratified realization* of P.

Remark B.1.49 (Nerves of posets). Recall the nerve NP of a poset $(P, \leq)$ is the simplicial set whose m-simplices S are the length-m chains of composable arrows in P; in other words, an m-simplex is a map of posets $S : [m] \to P$. The simplex $S : [m] \to P$ is called nondegenerate if it is injective.

Remark B.1.50 (Geometric realizations of posets). Recall the 'geometric realization' $|P|$ of a poset P is obtained by applying the geometric realization of simplicial sets to the nerve of P. Explicitly, $|P|$ is the space of functions $w : \mathrm{obj}(P) \to \mathbb{R}_{\geq 0}$ whose support $\mathrm{supp}(w) \subset \mathrm{obj}(P)$ is the object set of a nondegenerate simplex in P, and whose total weight is fixed, i.e., $\sum_{p \in \mathrm{obj}(P)} w(p) = 1$. We refer to such w as a 'convex combination of objects' of the poset.

Construction B.1.51 (Stratified realizations of posets). The geometric realization $|P|$ of a poset P has a stratification $\|P\|$, called the **stratified realization**, with fundamental poset P itself, constructed as follows: the characteristic function of $\|P\|$ sends a convex combination w of objects of the poset to the minimal object $\min(\mathrm{supp}(w))$ (in P) of the support of that convex combination:

$$\|P\| : |P| \to P \quad , \quad w \mapsto \min(\mathrm{supp}(w)) \in P.$$

The stratum corresponding to the object $p \in P$ is denoted $\mathsf{str}(p) \subset \|P\|$; it consists of all convex combinations w of objects weakly greater than p, whose value at p is nonzero.

As we will see later in Construction B.2.14, stratified realization of posets extends to a functor from the category of posets to the category of stratifications.

Example B.1.52 (Stratified realizations). We illustrate three stratified realizations in Figure B.8: from left to right, we depict the stratified realizations $\|P\|$ of the '1-cell poset' $P = \{-1 \leftarrow 0 \to 1\}$, the 2-simplex $P = \{0 \to 1 \to 2\}$ and the product of two 1-simplices $P = \{0 \to 1\} \times \{0 \to 1\}$.

B.2. Stratified maps

B.2.1. Maps, coarsenings, and substratifications.

13 : Probably some words belong here (cf B.1/B.3). If not need a nobreak.

14 : Probably some words belong here as well (cf B.1.1/B.3.1). If not need a nobreak.

[45] In fact, in the case of finite stratifications, we can always recover (X, f) from an $\mathbb{N}$-filtration of X. Define a depth map depth : $\sqcap(f) \to \mathbb{N}^{\mathrm{op}}$, mapping $s \in \sqcap(f)$ to k if chains in $\sqcap(f)$ starting at s have maximal length $(k+1)$. Define the filtration $X_0 \subset X_1 \subset ... \subset X_{k_{\max}} = X$, where $k_{\max}$ is the maximal depth of elements in $\sqcap(f)$, by setting X_i to be the preimage of $[0, i]$ under the composite depth $\circ f$.

FIGURE B.8. Stratified realizations of posets

DEFINITION B.2.1 (Map of stratifications). Given stratifications (X, f) and (Y, g) with respective characteristic maps $f : X \to \mathbb{\Pi}(f)$ and $g : Y \to \mathbb{\Pi}(g)$, a **map of stratifications** $F : (X, f) \to (Y, g)$ is a continuous map $F : X \to Y$ for which there exists a (necessarily unique) map of posets $\mathbb{\Pi}(F) : \mathbb{\Pi}(f) \to \mathbb{\Pi}(g)$ such that $\mathbb{\Pi}(F) \circ f = g \circ F$. ⌟

TERMINOLOGY B.2.2 (Stratified maps). We frequently refer to maps of stratifications simply as 'stratified maps'. ⌟

EXAMPLE B.2.3 (Map of stratifications). In Figure B.9 we depict a stratified map on the left and a non-stratified map on the right. In both cases, the underlying map of topological spaces is given by vertical projection. ⌟

FIGURE B.9. A stratified map and a non-stratified map.

DEFINITION B.2.4 (Coarsenings and refinements of stratifications). A map of stratifications $F : (X, f) \to (Y, g)$ is a **coarsening** of (X, f) to (Y, g), or, synonymously, a **refinement** of (Y, g) by (X, f), if the underlying map of spaces $F : X \to Y$ is a homeomorphism. ⌟

Note that we use coarsening and refinement as synonyms of dual flavor, i.e., describing dual processes: a coarsening *coarsens* the domain, while a refinement, in opposite direction, *refines* the codomain.

EXAMPLE B.2.5 (Coarsening and refinement). In Figure B.10 we illustrate a coarsening of stratifications on the circle, along with the corresponding refinement indicated by a dashed arrow in opposite direction. ⌟

DEFINITION B.2.6 (Substratification). A stratified map $(X, f) \to (Y, g)$ is a **substratification** if the underlying map $X \subset Y$ is an inclusion and if every stratum s of f is a connected component of $X \cap t$ for some stratum t of g. ⌟

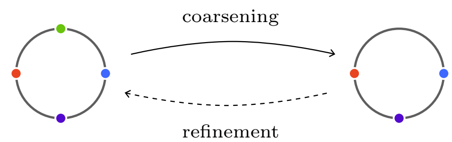

FIGURE B.10. A coarsening and its corresponding refinement of stratifications.

By extension we refer to stratified maps that are not inclusions of underlying sets, but whose underlying map is injective and a stratified homeomorphism onto a substratification, also as 'substratifications'.

EXAMPLE B.2.7 (Substratification). In Figure B.11 we depict two stratified maps: the first is a substratification, which though is not injective on fundamental posets; the second is a substratification which is injective on fundamental posets (and, in fact, it is a *constructible* substratification as we define below). In Figure B.12 we depict a map that is a stratified map whose underlying map is injective, but which fails to be a substratification. ⌟

FIGURE B.11. Two substratifications.

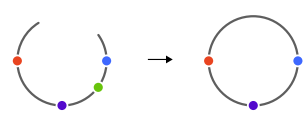

FIGURE B.12. A stratified map that is not a substratification.

DEFINITION B.2.8 (Stratification restrictions). Given a stratification (Y, g) and a subspace $X \subset Y$, the **restriction** $(X, g|_X)$ (also simply written (X, g)) is the substratification of (Y, g) whose strata are the connected components of intersections $X \cap t$ for all strata t of g. ⌟

DEFINITION B.2.9 (Constructible substratifications). A substratification $(X, f) \to (Y, g)$ is **constructible** if every stratum of (X, f) is exactly a stratum of (Y, g). ⌟

B.2.2. Classifying stratifications via stratified maps. In this section we briefly discuss how three classes of stratifications, namely, frontier-constructible stratifications, substratifications, and coarsened stratifications, can be understood in terms of maps.

Recall from Definition B.1.24, a frontier-constructible stratification is a stratification (Y, g) in which the topological closure $\overline{s}$ of each stratum s yields a constructible substratification $(\overline{s}, g|_{\overline{s}})$ of g (by restricting g to $\overline{s}$). Frontier-constructibility has an alternative description purely in terms of characteristic functions as follows.

LEMMA B.2.10 (Frontier-constructible stratifications are those with open characteristic function). *A stratification (X, f) is frontier-constructible if and only if the characteristic function $f : X \to \sqcap(f)$ is an open map.*

PROOF. Assume f is frontier-constructible. Let $U \subset X$ be an open subset. We need to show that $f(U) \subset \sqcap(f)$ is open, which in the specialization topology means that $f(U)$ is downward closed. It suffices (because the fundamental poset is generated by formal entrance paths) to check that given an element $s \in f(U)$ and a formal entrance path $r \to s$, we have $r \in f(U)$. The existence of the formal entrance path $r \to s$ means $s \cap \partial r \neq \emptyset$; frontier-constructibility then implies that $s \subset \partial r$. As $s \in f(U)$, there is some point of the stratum s that is in U, and because U is open, there must be a point of the stratum r that is in U. Thus $r \in f(U)$ as required.

Conversely, assume $f : X \to \sqcap(f)$ is open. It suffices to show that if there is a formal entrance path $r \to s$, i.e., $s \cap \partial r \neq \emptyset$, then $s \subset \partial r$. Suppose there is such an entrance path but by contrast there is a point $p \in s \setminus \partial r = s \setminus \overline{r}$. Then we can choose an open neighborhood $p \in U \subset X$ disjoint from the closure $\overline{r}$. By assumption it follows that the image $f(U)$ is open, which is to say downward closed; thus $s \in f(U)$ and $r \to s$ implies that $r \in f(U)$, contradicting the fact that the neighborhood U does not intersect even the closure of r. □

We can characterize substratifications and coarsenings in terms of fundamental poset maps, as follows.

LEMMA B.2.11 (Substratification from discrete maps). *A map of finite stratified spaces $F : (X, g) \to (Y, f)$ is a substratification if and only if $F : X \to Y$ is a subspace inclusion and $\sqcap(F) : \sqcap(g) \to \sqcap(f)$ is a discrete map.*

PROOF. By definition every substratification is a subspace inclusion of underlying spaces. The fact that $\sqcap(F)$ is a discrete map follows since strata of substratifications are defined as connected components of the intersection of the subspace X with strata of f, and since $\sqcap(f)$ is a poset.

Conversely, assume the stratified map F is a subspace inclusion and that $\sqcap(F)$ is a discrete map. Note that the substratification $(X, f|_X)$ of f is obtained exactly by connected component splitting of $f \circ F : X \to \sqcap(f)$, see Construction B.1.44. Since g is a continuous characteristic map, and since $\sqcap(F)$ is a discrete map, statement (3) of Lemma B.1.45 (applied to $f : Y \to \sqcap(f)$ restricted to $X \subset Y$) shows that F is a substratification as claimed. □

LEMMA B.2.12 (Coarsenings from connected-quotient maps). *Let (X, f) be a finite stratification. Coarsenings of f (up to isomorphism) are canonically in bijection with connected-quotients of $\mathbb{\Pi}(f)$: namely, the bijection takes coarsenings F to their fundamental poset maps $\mathbb{\Pi}(F)$.*

PROOF. Let $F : (X, f) \to (X, g)$ be a coarsening. Then $\mathbb{\Pi}(F)$ is a connected-quotient map since its preimages are connected and it satisfies condition (2) in Lemma B.1.34.

Conversely, let $H : \mathbb{\Pi}(f) \to P$ be a connected-quotient map. Define a stratification (X, g) whose strata are unions of those strata in f that are mapped to the same object in P under H. Since preimages of H are connected, these unions are connected subspaces of X and thus define a prestratification. In fact, since H is a connected-quotient map to a poset P, this defines a stratification with fundamental poset $\mathbb{\Pi}(g) = P$, see Lemma B.1.39. The resulting coarsening $(X, f) \to (X, g)$ is the identity on the underlying space X, and maps fundamental posets by H. □

Again, we forego discussing analogs of the above results for the case of infinite (and locally finite) stratifications.

B.2.3. The category of stratifications. Having defined stratified spaces and maps, we now obtain the category of stratifications.

TERMINOLOGY B.2.13 (The ordinary category of stratifications). Denote by Strat the category of stratifications and their stratified maps. ⌟

Posets faithfully embed into stratifications by the stratified realization functor as follows (recall the construction of stratified realizations $\|P\|$ of posets P from Construction B.1.51).

CONSTRUCTION B.2.14 (Stratified realization functor). Given a map of posets $F : P \to Q$, then the map $\|F\| : \|P\| \to \|Q\|$, mapping $\|F\|(w)(q) = \sum_{p \in F^{-1}(q)} w(p)$, is a stratified map. This yields the 'stratified realization' functor

$$\|-\| : \mathsf{Pos} \to \mathsf{Strat}$$

from the category of posets to the category of stratifications. ⌟

Conversely, the fundamental poset construction previously described yields a functor from the category of stratifications to the category of posets.

CONSTRUCTION B.2.15 (Fundamental poset functor). The association of the fundamental poset $\mathbb{\Pi}(f)$ to the stratification (X, f), and of the map of posets $\mathbb{\Pi}(F)$ to the map of stratifications $F : (X, f) \to (Y, g)$ provides the 'fundamental poset' functor

$$\mathbb{\Pi} : \mathsf{Strat} \to \mathsf{Pos}$$

from the category of stratifications to the category of posets. ⌟

Observation B.2.16 (The fundamental poset inverts stratified realization). The preceding functors form a section-retraction pair: namely, $\sqcap \|-\| = \mathrm{id}$. ⌟

We can further promote the fundamental poset functor to a functor of ∞-categories. We usually model ∞-categories using Top-enriched categories (or else, using quasicategories). For generality, the constructions below will be given in the case of $k\mathsf{Top}$-enriched categories (see Notation B.1.2). Indeed, while our interest usually lies with homotopically well-behaved 'cell-like' spaces in Top, working with $k\mathsf{Top}$ will precisely allow us to include the case of stratified 'poset-like' spaces (see Convention B.1.1).

Notation B.2.17 (The $k\mathsf{Top}$-enriched category of stratifications). Denote by $\mathcal{S}trat$ the $k\mathsf{Top}$-enriched category of stratified spaces and their stratified maps: hom spaces $\mathcal{S}trat((X,f),(Y,g))$ are topologized as subspaces of the internal hom $\mathrm{Map}(X,Y)$ in $k\mathsf{Top}$ (see Notation B.1.2). ⌟

Note that restricting attention only to spaces $X \in \mathsf{Top}$ similarly yields a Top-enriched category of stratifications.

We next define the $k\mathsf{Top}$-enriched category of posets. It will be convenient to assume local finiteness at this point (though it is possible to generalize the discussion below to other cases as well). For the rest of this section we will assume basic familiarity with the categorical theory of (core) compactly generated spaces; see [**ELS04**] for an excellent introduction. We first record the following useful technical facts.

Observation B.2.18 (Properties of locally finite posets). Let P and Q be locally finite posets.

(1) Local finiteness of P implies that P is core compact, and thus exponentiable (see [**ELS04**, Def. 2.8 & Thm. 2.9]).
(2) By probing P with compact Hausdorff probes $\left|P^{\leq x}\right| \to P^{\leq x} \hookrightarrow P$ one verifies that P is compactly generated (see [**ELS04**, Def. 3.1]).
(3) The internal hom $\mathrm{Map}(P,Q)$ in $k\mathsf{Top}$ is the k-ification of the space $C_0(P,Q)$, defined as the set of continuous maps $F : P \to Q$ (which, in our case, are exactly poset maps $P \to Q$) with topology given by subbasic opens $M(U,V) = \{F \mid U \ll F^{-1}(V)\}$ for open $U \subset P$ and open $V \subset Q$ (note, '$\ll$' indicates that U is relatively compact in $F^{-1}(V)$, see [**ELS04**, Def. 4.1, Prop. 5.11, Thm. 5.15]).
(4) Given an open set $V \subset P$ and open set $U \subset P$ that is relatively compact in V then U must be finite. Indeed, consider the open cover $\{V^{\leq x} \mid x \in V\}$. By relative compactness, there exists a finite subcover. By local finiteness of P this subcover only contains finitely many objects. Conversely, any finite open $U \subset V$ is relatively compact. We may thus work with subbasic opens $M(U,V) = \{F \mid F(U) \subset V\}$ for finite open $U \subset P$ and open $V \subset Q$. ⌟

NOTATION B.2.19 (The $k\mathsf{Top}$-enriched category of locally finite posets). The $k\mathsf{Top}$-enriched category $\mathcal{P}os_{\mathrm{lf}}$ of locally finite posets is the $k\mathsf{Top}$-enriched category obtained by topologizing hom sets $\mathsf{Pos}(P,Q)$ using the internal homs $\mathrm{Map}(P,Q)$ in $k\mathsf{Top}$. ⌟

REMARK B.2.20 (The case of finite posets). For finite posets P and Q, the space $\mathrm{Map}(P,Q)$ is exactly the hom poset $\mathsf{Pos}(P,Q)$ of poset functors and natural transformations endowed with specialization topology. ⌟

CONSTRUCTION B.2.21 (Fundamental poset ∞-functor). Let $\mathcal{S}\mathsf{trat}_{\mathrm{lf}}$ denote the full subcategory of $\mathcal{S}trat$ consisting of locally finite stratifications. The fundamental poset functor $\mathsf{Strat} \to \mathsf{Pos}$ induces an ∞-functor of $k\mathsf{Top}$-enriched categories

$$\sqcap : \mathcal{S}\mathsf{trat}_{\mathrm{lf}} \to \mathcal{P}os_{\mathrm{lf}} \quad .$$

The continuity of the functor on hom spaces can be derived using standard arguments as follows. We need to check that $\sqcap : \mathcal{S}\mathsf{trat}_{\mathrm{lf}}((X,f),(Y,g)) \hookrightarrow \mathrm{Map}(X,Y) \to \mathcal{P}os_{\mathrm{lf}}(\sqcap(f),\sqcap(g))$ is continuous. Recall $\mathcal{P}os_{\mathrm{lf}}(\sqcap(f),\sqcap(g)) = kC_0(\sqcap(f),\sqcap(g))$ (see Observation B.2.18). Since k-ification is a right adjoint, it suffices to check continuity of the underlying function mapping into $C_0(\sqcap(f),\sqcap(g))$. Pick a subbasic $M(U,V)$ for the latter space. Since U is finite, we can pick a finite compact set $K = \{p_s \in f^{-1}(s) \mid s \in U\}$, $K \subset X$. Since g is locally finite, the characteristic map $g : Y \to \sqcap(g)$ is continuous. Thus $W = g^{-1}(V)$ is open in Y. Then $M(K,W) = \{F : X \to Y \mid F(K) \subset W\}$ is an open subset of $\mathrm{Map}(X,Y)$ (see [**Str09**, Def. 2.8]), and its intersection with $\mathcal{S}\mathsf{trat}_{\mathrm{lf}}((X,f),(Y,g))$ equals $\sqcap^{-1}M(U,V)$, showing the latter is open as required. ⌟

REMARK B.2.22 (Stratified realization ∞-functor). Converse to the preceding construction, one can try to construct a functor $\|-\| : \mathcal{P}os_{\mathrm{lf}} \to \mathcal{S}\mathsf{trat}_{\mathrm{lf}}$. However, while $\|P\|$ is locally finite if P is, this functor is not continuous on hom spaces. For example, consider $\|-\| : \mathcal{P}os_{\mathrm{lf}}([0],[1]) \to \mathcal{S}\mathsf{trat}_{\mathrm{lf}}(\|[0]\|,\|[1]\|)$. Choose a subbasic $M(\{0\},(1-\epsilon,1])$, $\epsilon > 0$, in the latter space. Neither of the subbasics $M(\{0\},\{0 \to 1\})$ and $M(\{0\},\{0\})$ includes in the preimage of that subbasic.

This failure is symptomatic of an earlier *mistake*, as was detailed in Remark 4.2.17: $\mathcal{P}os_{\mathrm{lf}}$ and $\mathcal{S}trat$ have non-invertible 2-categorical structure that gets inverted by working with topologically enriched categories. ⌟

Finally, we also mention a *tensoredness* property of stratifications. We assume the underlying space of stratifications to be locally compact Hausdorff spaces for the next two constructions (see [**Str09**, Prop. 2.6]).

CONSTRUCTION B.2.23 (Stratified products). Given two stratifications (X,f) and (Y,g), the **product stratification** is simply $(X \times Y, f \times g)$ where $f \times g$ is the characteristic function $X \times Y \to \sqcap(f) \times \sqcap(g)$ obtained by taking the product of characteristic functions $f : X \to \sqcap(f)$ and $g : Y \to \sqcap(g)$. One further defines products of stratified maps by taking products of their underlying continuous maps. This yields the topological 'product' functor

$$(- \times -) : \mathcal{S}trat \times \mathcal{S}trat \to \mathcal{S}trat. \qquad \lrcorner$$

Taking products with topological spaces provides a 'fiberwise Top-tensor' on the category of stratified spaces as follows.

CONSTRUCTION B.2.24 (Fiberwise Top-tensoredness of $\mathcal{S}\mathsf{trat}_{\mathrm{lf}}$). Let (X, f) and (Y, g) be locally finite stratifications and $F : \sqcap(f) \to \sqcap(g)$ a map of their fundamental posets. Denote by $\mathcal{S}\mathsf{trat}_{\mathrm{lf}}(f, g)_F$ the preimage of F of the map $\sqcap : \mathcal{S}\mathsf{trat}_{\mathrm{lf}}(f, g) \to \mathcal{P}os_{\mathrm{lf}}(\sqcap(f), \sqcap(g))$. Using cartesian closedness of Top, identify $\mathrm{Map}(Z, \mathrm{Map}(X, Y)) \cong \mathrm{Map}(Z \times X, Y)$ (for $Z \in \mathsf{Top}$); in particular, we obtain a homeomorphism

$$\mathrm{Map}(Z, \mathcal{S}\mathsf{trat}_{\mathrm{lf}}(f, g)_F) \cong \mathcal{S}\mathsf{trat}_{\mathrm{lf}}(Z \times f, g)_F$$

where the right hand side denotes the space of stratified maps $Z \times f \to (Y, g)$ whose underlying map of fundamental posets is F (noting $\sqcap(Z \times f) \cong \sqcap(f)$)). $\lrcorner$

B.2.4. Stratified bundles and pullbacks. A *stratified bundle* is a stratified map that is locally trivial along each stratum of the base. The notion generalizes the ordinary notion of *fiber bundles* of topological spaces. We assume all spaces in this section to be locally compact Hausdorff spaces.

DEFINITION B.2.25 (Stratified bundles). A stratified map $p : (X, f) \to (Y, g)$ is a **stratified bundle** if for each stratum s of g and each point $x \in s$, there is a neighborhood $U_x \subset s$ inside the stratum s, such that there is a stratification (Z, h) together with an isomorphism of stratifications $\tau : U_x \times h \cong (p^{-1}(U_x), f)$ for which $p \circ \tau : U_x \times h \to U_x$ is the projection. The stratification (Z, h) is called the fiber of p over the stratum s. $\lrcorner$

Note that every fiber bundle is naturally a stratified bundle with indiscrete stratifications on both base and total space. We will usually further assume that all the fibers of a stratified bundle are non-empty, in other words that the underlying map of spaces is surjective. In this case the stratification of the total space determines the stratification of the base space.

OBSERVATION B.2.26 (The base stratification is determined by the total stratification). Suppose $(X, f) \to (Y, g)$ and $(X, f) \to (Y, g')$ are stratified bundles with the same underlying surjective map $F : X \to Y$. Then the stratifications g and g' are equal. $\lrcorner$

Just as fiber bundles can be pulled back along continuous maps, stratified bundles can be pulled back along stratified maps. Here, a *pullback* of stratified maps means the following.

DEFINITION B.2.27 (Pullbacks of stratifications). Given stratifications (X, f), (Y, g), and (Z, h) and maps $F : f \to h$ and $G : g \to h$, the **pullback stratification** $(X, f) \times_{(Z,h)} (Y, g)$ is the stratification $(X \times_Z Y, f \times_h g)$, where $X \times_Z Y$ is the pullback of spaces and $f \times_h g$ is the restriction $f \times g|_{X \times_Z Y}$ of the product stratification $f \times g$ to the pullback space $X \times_Z Y \subset X \times Y$. $\lrcorner$

Example B.2.28 (Pullback stratifications need not be finite or have continuous characteristic function). In Figure B.13 we depict a pullback of finite stratifications that is not finite and does not have continuous characteristic function. ⌟

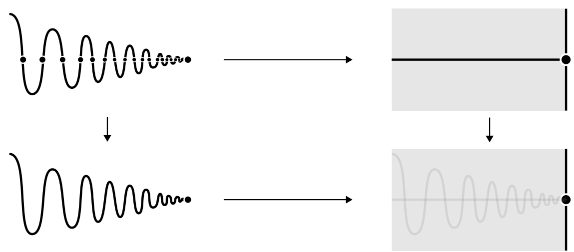

Figure B.13. Pullbacks of stratifications need not preserve finiteness.

Observation B.2.29 (Pullbacks of stratified bundles). Given a stratified bundle $p : (X, f) \to (Y, g)$ and a stratified map $F : (Y', g') \to (Y, g)$, then the pullback map $(X \times_Y Y', f \times_g g') \to (Y', g')$ is a stratified bundle itself, usually denoted by $F^*p : F^*f \to g'$. This follows since stratified maps map strata into strata, which allows us to 'pull back' trivializations of p over neighborhoods in strata of g to trivializations of F^*p over neighborhoods in strata of g'. ⌟

Remark B.2.30 (Constructible bundles). There is a natural strengthening of the notion of stratified bundles, namely to so-called 'constructible bundles' [**GV72, Tre09, MS22**]. The condition of 'constructibility' requires that bundles can be reconstructed up to isomorphism from functorial data associated to the fundamental categorical structure of their base stratifications (here, 'fundamental categorical structure' may refer, for example, to the fundamental poset, fundamental category, fundamental ∞-category, or another variation thereof). ⌟

B.3. Conical and cellulable stratifications

We recall notions of conical and cellulable stratifications. Both are *niceness* conditions on stratifications. Lurie shows in [**Lur12**, App. A] that conical stratifications have *fundamental ∞-categories*, providing a natural generalization of fundamental ∞-groupoids of spaces. We will provide a similar (but simpler) construction in the case of cellulable stratifications.

B.3.1. Conical stratifications. Many of the stratifications in this book satisfy an additional regularity condition called *conicality*. This condition requires neighborhoods around strata that look like the product of a cone *normal* to the stratum and an open set *tangential* in the stratum. Let us first formalize the operation of taking cones on stratifications.

TERMINOLOGY B.3.1 (Stratified cones). Given a stratification (X, f), we can define its **open cone** $\mathsf{cone}(f)$ to be the unique stratification of $\mathsf{cone}(X) = X \times (0,1] \cup_{X\times\{1\}} \top$ containing $f \times (0,1)$ (see Construction B.2.23) as a constructible substratification on $X \times (0,1)$.

Similarly, one defines the **closed cone** $\overline{\mathsf{cone}}(f)$ to be the stratification of $\overline{\mathsf{cone}}(X) = X \times [0,1] \cup_{X\times\{1\}} \top$ which contains f as a constructible substratification on $X \times \{0\}$ and $\mathsf{cone}(f)$ as a constructible substratification on $\mathsf{cone}(X)$. ⌟

Note that the poset $\sqcap(\mathsf{cone}(f))$ is $\sqcap(f)^{\triangleright}$, i.e., obtained from $\sqcap(f)$ by adding a new terminal element $\top$. The map $\mathsf{cone}(f)$ sends the cone point $\top$ to $\top \in \sqcap(f)^{\triangleright}$.

DEFINITION B.3.2 (Conical stratification). A **tubular neighborhood** of a point x of a stratification (X, f) is a neighborhood U_x of x, together with a stratified space $(Y_x, \mathsf{link}(x))$, called a **link** at x, a connected topological space Z_x, called the **tangential neighborhood**, and a stratified homeomorphism

$$Z_x \times \mathsf{cone}(\mathsf{link}(x)) \cong (U_x, f|_{U_x})$$

sending $z \times \top$ to x, for some $z \in Z_x$. (Here $\top$ is the cone point in the cone $\mathsf{cone}(\mathsf{link}(x))$.) A stratification is **conical** if it has a tubular neighborhood at every point. ⌟

EXAMPLE B.3.3 (Conical and non-conical stratifications). In Figure B.14 we illustrate an example of a conical stratification, together with an illustration of a tubular neighborhood as a product of a cone and a space. By contrast, in Figure B.15 we depict two stratifications (the latter of the same space as before, but now decomposed into only two strata) which are not conical stratifications. ⌟

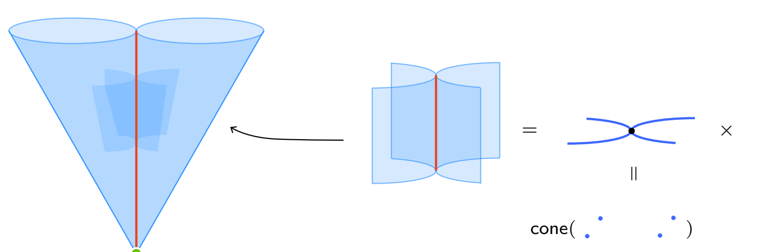

FIGURE B.14. A conical stratification with a tubular neighborhood.

REMARK B.3.4 (Topological stratifications). A conical stratification in which every stratum is a topological manifold is usually (and maybe confusingly) called a 'topological stratification'. In this situation, the tangential

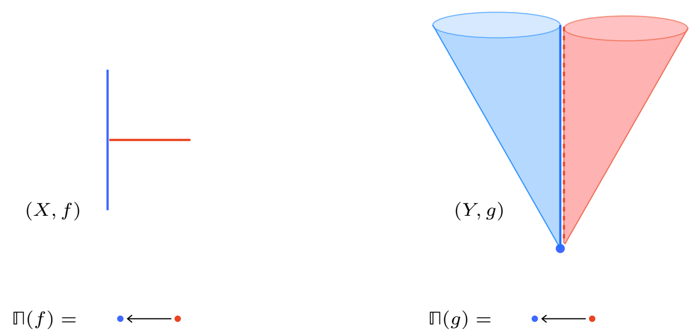

FIGURE B.15. Two non-conical stratifications and their fundamental posets.

spaces Z_x can always be chosen to be euclidean. The conical stratification shown in Figure B.14 is a topological stratification. An instance of a conical stratification (with two strata) that is not a topological stratification is depicted in Figure B.16. ⌟

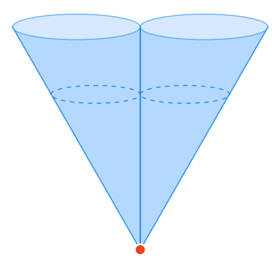

FIGURE B.16. A conical stratification that is not topological.

OBSERVATION B.3.5 (Constructible substratifications inherit conicality). If the stratification (X, f) is conical and $(Y, g) \hookrightarrow (X, f)$ is a constructible substratification (Definition B.2.9), then the stratification (Y, g) is also conical. ⌟

OBSERVATION B.3.6 (Coarsening need not preserve conicality). If the stratification (X, f) is conical and $(X, f) \to (X, g)$ is a coarsening, then the stratification (X, g) need not be conical. ⌟

REMARK B.3.7 (Conical implies frontier-constructible). Every conical stratification is frontier-constructible (see Definition B.1.24). In particular, a conical stratification is locally finite if and only if its fundamental poset is. ⌟

PROPOSITION B.3.8 (Locally finite stratified realizations are conical). *The stratified realization $\|P\|$ (see Construction B.1.51) of any locally finite poset P is conical.*

PROOF. For any poset element $p \in P$, and any point $x \in \mathsf{str}(p)$ of the corresponding stratum, we construct a tubular neighborhood, which is in fact independent of the choice of x within the stratum. Recall, the points of the stratum $\mathsf{str}(p)$ correspond to convex combinations of objects in $P^{\geq p}$ (with non-zero coefficient for p). The *canonical link stratification* at any such point, written $\mathsf{link}(p)$, is defined to be $\|P^{<p}\|$. Note that since $P^{\geq p}$ and $P^{<p}$ include into P we may trivially extend convex combinations of objects in the former to convex combination of objects in P. The resulting inclusion $\mathsf{str}(p) \hookrightarrow \|P\|$ extends to a tubular neighborhood $\mathsf{str}(p) \times \mathsf{cone}(\mathsf{link}(p)) \hookrightarrow \|P\|$ by mapping (cf. [**Lur12**, Prop. A.6.8])

$$\begin{aligned}&\mathsf{str}(p) \times \mathsf{link}(p) \times (0,1] \to \|P\| \\ &\qquad (w_{\mathsf{str}(p)}, w_{\mathsf{link}(p)}, t) \mapsto \big(w(q) := t \cdot w_{\mathsf{str}(p)}(q) + (1-t) \cdot w_{\mathsf{link}(p)}(q)\big)\end{aligned}$$

This verifies conicality of $\|P\|$ as claimed. □

REMARK B.3.9 (Natural non-conical stratifications). Some natural operations lead outside the realm of conical stratifications. For example, while the stratified n-simplex $\|[n]\|$ is conical by the preceding result, the substratification determined by the simplex boundary, denoted by $\partial \|[n]\|$, is not. ⌟

B.3.2. Fundamental higher categories. A central reason for considering conical stratifications is the availability of a good notion of *(higher) entrance path homotopies* between entrance paths, leading to a definition of fundamental ∞-category which we present below. While we usually take ∞-category to mean Top-enriched category, in the context of fundamental ∞-categories of stratifications, we work in terms of quasicategories instead. (In fact, we will later sketch yet another definition given in terms of 'categories with weak equivalences').

DEFINITION B.3.10 (Fundamental ∞-category, [**Lur12**, Rmk. A.6.5]). The **fundamental ∞-category** $\sqcap_\infty(f)$ of a conical stratification (X, f) is the quasicategory with underlying simplicial set having as m-simplices maps of the stratified m-simplex $\|[m]\|$ to f; that is $\sqcap_\infty(f)_m := \mathsf{Strat}(\|[m]\|, f)$. ⌟

Equipped with this notion of fundamental ∞-category, we may recover the fundamental poset and also a notion of *fundamental 1-category* by a process of categorical truncation, as follows.

REMARK B.3.11 (Periodic table and truncations of categories). Recall, an (n, k)-category, where $0 \leq k \leq n+1$, is a category that may have a non-trivial hom sets of i-morphisms for $0 \leq i \leq n$ (non-trivial meaning 'more than one object'), and any i-morphism with $i > k$ is an equivalence (i.e., has an inverse up to higher equivalences). Common cases include $(1,1)$-categories (i.e., ordinary 1-categories), $(\infty,1)$-categories (which we also refer to as ∞-categories), and $(\infty,0)$-categories (also known as ∞-groupoids). But the classification also has interesting low-dimensional edge cases: $(-1,0)$-categories are the booleans, $(0,0)$-categories are sets, and $(0,1)$-categories are preorders (and thus, up to equivalence of preorders, posets).

Keeping the preceding examples in mind, we give a brief description of the construction of (n,k)**-truncations** of (m,l)-categories $\mathcal{C}$ in two cases. First, when $k = n+1$, then the $(n, n+1)$-truncation $\tau_{n,n+1}\mathcal{C}$ is obtained from $\mathcal{C}$ by replacing any non-trivial hom set of $(n+1)$-morphisms by a trivial set; in other words, by a *boolean* (up to the map $\mathsf{Set} \to \mathsf{Bool}$ that detects non-emptiness). Second, if $k = n$, and $l \leq n$, then the (n,n)-truncation $\tau_{n,n}\mathcal{C}$ is obtained from $\mathcal{C}$ by replacing $(n+1)$-morphisms by strict equalities.[46] ⌟

Terminology B.3.12 (0-truncations and 1-truncations). A technically convenient notion of truncations in the preceding sense, that applies to quasicategories as a model of $(\infty,1)$-categories, is described in [**CL18**]. Therein, $(0,1)$-truncations are referred to simply as '0-truncations', and $(1,1)$-truncations are referred to simply as '1-truncations'. We adopt this convention when no confusion arises. ⌟

Observation B.3.13 (Fundamental poset truncation). For a conical stratification (X,f), the 0-truncation of the fundamental ∞-category $\sqcap_\infty(f)$ is the fundamental poset $\sqcap(f)$, i.e., $\tau_{0,1}\sqcap_\infty(f) = \sqcap(f)$. ⌟

Construction B.3.14 (Fundamental 1-categories). The **fundamental 1-category** $\sqcap_1(f)$ of a conical stratification (X,f) is the 1-truncation of the fundamental ∞-category $\sqcap_\infty(f)$. ⌟

A direct definition of the fundamental 1-category, not using the fundamental ∞-category or categorical truncation, proceeds using path components of the space of entrance paths (see [**Woo09**]).

Remark B.3.15 (The fundamental ∞-categories of stratifications). The class of ∞-categories that are (up to equivalence) obtained as the fundamental ∞-categories of conical stratifications can be characterized as precisely the ∞-categories with a conservative functor to a poset (see [**BGH18**, §2.1]). Thus, they could reasonably be called '∞-posets': ∞-posets are to posets, what ∞-groupoids are to sets, and what ∞-categories are to categories (indeed, in each case the left hand term is fully characterized as having a conservative truncation to the right hand term). ⌟

[46] A general definition of (n,k)-truncation can be given in a similar spirit, but would require us to give a precise definition of the structure needed for a general i-morphism to be an equivalence (several, mostly equivalent technical frameworks could be chosen for this purpose): indeed, an (n,k)-truncation, $k < n$, keeps only equivalences as i-morphisms of the original category when $i > k$, and replaces equivalence by strict equality in dimension $(n+1)$.

Recall that an ∞-category is called 0-truncated if its hom spaces are (-1)-types, meaning they are either empty or contractible.

LEMMA B.3.16 (Fundamental ∞-category of stratified realizations). *Given a locally finite poset P, the fundamental ∞-category of its stratified realization $\|P\|$ is equivalent to (the nerve of) P, that is,*

$$\sqcap_\infty \|P\| \simeq NP.$$

In particular, $\sqcap_\infty \|P\|$ is 0-truncated (see [**CL18**]*).*

PROOF. We first check that $\mathcal{C} \equiv \sqcap_\infty \|P\|$ is 0-truncated. It suffices [**CL18**, Prop. 3.12] to show that any sphere $\partial[m] \to \mathcal{C}$, for $m > 1$, has a filler (note, here we think of $[m]$ as a simplicial set, which is sometimes denoted $\Delta[m]$ in the literature). Given such a map $\phi : \partial[m] \to \mathcal{C}$, by the definition of $\sqcap_\infty$, the map ϕ is represented by a continuous map $|\phi| : |\partial[m]| \to |P|$. Pick $x \in P$ such that $|\phi|(0) \in \mathsf{str}(x)$. Then $\operatorname{im}|\phi|$ lies in the closure of $\mathsf{str}(x)$, i.e., in $\left|P^{\geq x}\right| \subset |P|$. Note that $\left|P^{\geq x}\right| = \mathsf{cone}\,|P^{>x}|$. Similarly, identify $|[m]| \cong \mathsf{cone}(|\partial[m]|) = \left(|\partial[m]| \times [0,1]\right)/_{|\partial[m]|\times\{0\}}$. Then define the filler $\psi : |[m]| \to \left|P^{\geq x}\right|$ by mapping $(q,t) \mapsto t \cdot |\phi|(q)$. By construction, ψ sends the interior of $|[m]|$ to the stratum $\mathsf{str}(x)$ and thus is a stratified map as needed.

Since $\mathcal{C}$ is 0-truncated, it is equivalent to $N(\mathrm{ho}(\mathcal{C}))$ [**CL18**, Prop. 3.8]. Furthermore, the homotopy category of any 0-truncated ∞-category has a skeleton that is a poset [**CL18**, Prop. 3.10]. Let Q denote a posetal skeleton of $\mathrm{ho}(\mathcal{C})$. Observe that Q must be isomorphic to P: the map $Q \to P$ sending q to x iff $q \in \mathsf{str}(x)$ witnesses that isomorphism. We thus obtain

$$\sqcap_\infty \|P\| \equiv \mathcal{C} \simeq N(\mathrm{ho}(\mathcal{C})) \simeq NQ \cong NP.$$

as desired. Note that the composed equivalence is the *canonical* one: it takes 0-simplices q of $\sqcap_\infty \|P\|$ to 0-simplices x in NP iff $q \in \mathsf{str}(x)$. □

DEFINITION B.3.17 (0-truncated stratifications). A conical stratification (X, f) is called **0-truncated** if $\sqcap_\infty(f)$ is 0-truncated. ⌟

In particular, the preceding lemma shows that stratified realizations of posets are 0-truncated. This can be aligned with our analogy to sets and spaces.

REMARK B.3.18 (Analogy to sets and spaces). As mentioned earlier, posets are to (sufficiently nice) stratifications what sets are to (sufficiently nice) spaces. The above result provides another facet of this analogy: geometric realizations of sets (thought of as 0-categories) are discrete spaces and their fundamental ∞-groupoids are 0-truncated; similarly, the preceding result shows that fundamental ∞-categories of stratified realizations of posets are 0-truncated. ⌟

REMARK B.3.19 (The higher category of stratifications). Since stratifications have fundamental $(\infty,1)$-categories, the category of stratifications is naturally an $(\infty,2)$-category. This description can be sharpened: we saw that the fundamental category of a stratification is precisely a $(\infty, 1)$-poset (i.e. an (∞, ∞)-category whose truncation functor to its homotopy poset is conservative). Consequently, the category of stratifications is, more precisely, an $(\infty, 2)$-poset (i.e. an (∞, ∞)-category whose truncation functor to its homotopy 2-poset is conservative; here a 2-poset is a poset-enriched category, i.e. a $(1, 2)$-category). The situation is again analogous to the familiar setting of spaces: a space has a fundamental $(\infty, 0)$-category (i.e. an (∞, ∞)-category whose truncation functor to its homotopy set is conservative), and so the category of spaces is an $(\infty, 1)$-category (i.e. an (∞, ∞)-category whose truncation functor to its homotopy 1-category is conservative). $\lrcorner$

B.3.3. Cellulable stratifications. In this final section we discuss cellulable stratifications. Recall, a regular cell complex is a stratification in which strata are open disks (also called its 'open cells') whose closures are closed disks (also called its 'closed cells').

Definition B.3.20 (Cellulable stratifications). The class of **cellulable stratifications** is the smallest class of stratifications containing regular cell complexes with the following closure properties:

(1) If $(X, f) \to (Y, g)$ is a coarsening and (X, f) is cellulable then (Y, g) is cellulable.
(2) If $(Y, g) \subset (X, f)$ is a constructible substratification and (X, f) is cellulable then (Y, g) is cellulable. $\lrcorner$

Remark B.3.21 (Local finiteness assumption). We will assume that our cellulable stratifications have locally finite fundamental posets. One way to ensure this is to require in the above definition that regular cell complexes are locally finite and coarsenings $F : (X, f) \to (Y, g)$ are open finite, i.e., $\sqcap(F)$ is open with finite preimages. $\lrcorner$

Since constructible substratifications and coarsenings commute (i.e., any constructible substratification of a coarsening is a coarsening of a constructible substratification), the definition of cellulable stratifications can be phrased via an intermediate notion of *cellular stratifications* as follows.

Definition B.3.22 (Cellular stratifications). A stratification (Y, f) is called a **cellular stratification** if there exists a regular cell complex X (implicitly taken to be stratified by its cells) and a stratified inclusion $(Y, f) \hookrightarrow X$ making (Y, f) a constructible substratification of X. $\lrcorner$

Terminology B.3.23 (Cellulations). A refinement of a stratification by a cellular stratification is called a 'cellulation'. $\lrcorner$

Observation B.3.24 (Cellulable stratifications). A cellulable stratification is precisely a stratification that can be obtained by coarsening a locally finite cellular stratification. $\lrcorner$

To better understand the definition of cellulable stratifications we briefly discuss two properties of locally finite regular cell complexes. We begin with the property of conicality. Note that the stratification of general (non-regular) cell complexes need not be conical in general, even when the complex is finite (i.e., has only finitely many cells). In contrast, in the regular case, we have the following.

PROPOSITION B.3.25 (Regularity implies conicality). *Locally finite regular cell complexes are conically stratified.*

PROOF. For a regular cell complex X, there is a stratified homeomorphism $X \cong \|\sqcap X\|$ of the complex with the stratified realization of its fundamental poset (see [**Bjö84**, §3]). From Proposition B.3.8 it then follows that locally finite regular cell complexes are conically stratified. □

OBSERVATION B.3.26 (Cellular stratifications are conical). Combining Proposition B.3.25 with Observation B.3.5 it follows that locally finite cellular stratifications are conical. ⌟

OBSERVATION B.3.27 (Cellulable stratifications need not be conical, *but*...). Following Observation B.3.6, cellulable stratifications need not be conical. This means that, a priori, the construction of fundamental ∞-categories introduced in Definition B.3.10 does not apply to cellulable stratifications. However, as we will see shortly, cellulable stratifications have their own natural such definition, and, in fact, they remedy certain shortcomings of conical stratifications: for example, the restriction of the stratification n-simplex $\|[n]\|$ to its boundary is, in general, *not conical* but it is *cellulable*; cellulable stratifications, therefore, will allow us to construct fundamental ∞-categories even for such non-conical spaces. ⌟

REMARK B.3.28 (Cellular links and stars, Top and PL case). Given a regular cell complex X its conicality implies that each cell $x \in X$ has a canonical link stratification. This link is called the **cellular link** $\mathsf{link}(x)$ of x: combinatorially, it is constructed as the stratified realization of the poset $\sqcap X^{<x}$ (see the proof of Proposition B.3.8).

Relatedly, one can define the **cellular star** $\mathsf{star}(x)$ around x as the *simplicial* star around the corresponding vertex x (in the complex $N\sqcap X$) regarded as a subspace of the original regular cell complex, and stratified by the substratifications induced by X *separately* on the *boundary* and the *interior* of the star. Note that this yields stratified inclusions $\mathsf{star}(x) \hookrightarrow X$ and $\overline{x} \hookrightarrow \mathsf{star}(x)$. Removing the boundary of $\mathsf{star}(x)$ yields the **open cellular star**, which contains the open cell x as a stratum, called its **core cell**.

Observe that strata of the cellular link are realizations of half-open intervals $[y, x) := \sqcap X^{<x,\geq y}$. As observed in [**Bjö84**], open intervals (y, x) in cellular posets realize to homology spheres, and thus strata in the cellular link are cones of homology spheres. In general, strata in the cellular link need not be cells.

Observe that strata in the boundary and the interior of the cellular star $\mathsf{star}(x)\setminus\overline{x}$ without its core $\overline{x}$ are *higher* suspensions $(s \times D^m)_{/(a,b)\sim(a',b),b\in\partial D^m}$ of strata s in, respectively, the cellular link of x and the cone of the cellular link of x. By an appropriate generalization of the double suspension theorem [**Edw80, Edw06**] strata of the open cellular star are presumably disks (the suspension of the double cone is the cone of the double suspension).

The failure of cellular link strata to be cells in general can be considered pathological: that link strata need not be cells follows from the fact that links in triangulated manifolds need not be spheres but are, in general, merely homology spheres (the open interval (y, x) appears as a link in the triangulated sphere $|\sqcap X^{>y}|$, cf. [**FP90, Bjö84**]); indeed, the double suspension of a triangulated homology sphere being a sphere serves to illustrate this fact, but relies on infinitary and 'wild' constructions. The situation is remedied precisely by working with PL manifolds: in regular PL cell complexes (i.e., stratified realizations of PL cellular posets) the cellular link is again a regular cell complex. $\lrcorner$

Remark B.3.29 (Cellular links are stratified links). As stated in Observation B.3.26, regular cell complexes are conical stratifications in the sense of Definition B.3.2. We can now manifest this observation in more concrete terms: their open stars provide tubular neighborhoods, with links being their cellular links, and tangential neighborhoods being their core cells. $\lrcorner$

Observation B.3.30 (Cellular stratifications are 0-truncated). Given a locally finite regular cell complex X, we have a canonical equivalence $\sqcap_\infty(X) \simeq N\sqcap(X)$ that takes a point x to the stratum s that it lives in. This follows from Lemma B.3.16 since locally finite regular cell complexes are stratified realizations of locally finite posets. The observation remains true for cellular stratifications (Y, f), by appropriately restricting the equivalence to subcategories resp. subposets determined by constructible substratifications. $\lrcorner$

A common tool for working with cellular stratifications is barycentric subdivisions, whose construction at the level of posets we briefly record here for completeness.

Construction B.3.31 (Barycentric subdivision and conical subdivision). Let X be a combinatorial regular cell complex represented by a cellular poset, and $x \in X$ a cell in X. The **conical subdivision** $\mathsf{CSub}_X(x)$ **of** x in X is the cellular poset obtained by the pushout

$$\begin{array}{ccc} X^{>x} & \hookrightarrow & X^{\geq x} \\ {\scriptstyle \mathrm{id}\times\{0\}}\downarrow & & \downarrow \\ X^{>x} \times [1] & \longrightarrow & \mathsf{CSub}_X(x) \end{array}$$

Note that the 'boundary part' $X^{>x} \times \{1\}$ of $\mathsf{CSub}_X(x)$ is canonically isomorphic to the original boundary $X^{>x}$ of x. This allows us to glue $\mathsf{CSub}_X(x)$ onto $X \setminus x$, thus *replacing* x by the 'interior part' $X^{\geq x}$ of $\mathsf{CSub}_X(x)$.

The **barycentric subdivision** $\mathsf{BSub}(X)$ (cf. [**War96**]) is obtained by, inductively in increasing cell dimensions, replacing all original cells $x \in X$ by their conical subdivisions. There is a 'canonical subdivision map' of posets $\mathsf{BSub}(X) \to X$, obtained as the composite

$$\mathsf{BSub}(X) = X_{(N)} \to X_{(N-1)} \to ...X_{(2)} \to X_{(1)} \to X_{(0)} = X$$

where N is the maximal cell dimension in X, and each $X_{(i)}$ is obtained from $X_{(i-1)}$ by gluing in conical subdivisions of all i-cells, such that the map $X_{(i)} \to X_{(i-1)}$ is defined on newly subdivided i-cells x to map interior parts of $\mathsf{CSub}_{X_{(i)}}(x)$ to $x \in X_{(i-1)}$ (and act as the identity elsewhere). $\lrcorner$

Observation B.3.32 (Barycentric subdivisions are subdivisions). There is a (up to homotopy unique) stratified coarsening $\|\mathsf{BSub}(X)\| \to \|X\|$ whose fundamental poset mapping is the subdivision map $\mathsf{BSub}(X) \to X$. $\lrcorner$

An advantage of working with cellulable stratifications (in comparison to, say, conical stratifications) is that constructions can exploit the 0-truncatedness of regular cell complexes. For example, fundamental ∞-categories of cellulable stratifications are easy to define, by presenting them as *posets with weak equivalences.*[47]

Construction B.3.33 (The fundamental ∞-category for cellulable stratifications). Let (X, f) be a cellulable stratification. We can find a locally finite cellular stratification (Y, g) which refines (X, f). Denote by $F : \mathbb{\Pi}(g) \to \mathbb{\Pi}(f)$ the characteristic map of the refinement. Denote by $W_F = \{\alpha \mid F(\alpha) = \mathrm{id}\}$ the weak equivalences given by those entrance paths in g that become invertible paths in f. The fundamental ∞-category $\mathbb{\Pi}_\infty(f)$ of f is the ∞-category presented by the tuple $(\mathbb{\Pi}(g), W_F)$, i.e., by a category with weak equivalences. $\lrcorner$

Note that an explicit ∞-category $\mathbb{\Pi}_\infty(f)$ can be obtained by localizing $\mathbb{\Pi}(g)[W_F^{-1}]$ (here, we use the idea of 'localization' without reference to a concrete model; but concrete models could be given [**DK80b**] [**DK80a**]). Note that localizing in the traditional setting of 1-categories, yields the homotopy category $\mathrm{ho}(\mathbb{\Pi}_\infty(f))$.

Example B.3.34 (Fundamental categories of the stratified circle). Let us understand the fundamental $\mathbb{\Pi}_\infty(f)$ for three different stratifications of the circle S^1. The stratifications are illustrated in Figure B.17.

First, consider the trivially stratified circle (S^1, f_1). To determine its fundamental ∞-category, choose a regular cell refinement $(Y, g) \to S^1$; this has two 1-cells, glued together at their boundaries. In the presentation of $\mathbb{\Pi}_\infty(f_1)$, all arrows in the fundamental poset of the complex become weak equivalences (indicated by $\sim$ in the figure). The resulting 1-category $\mathrm{ho}(\mathbb{\Pi}_\infty(f_1))$ is equivalent to the category with a single object and a single generating non-identity automorphism (without further relations). Of course,

[47]Categories with weak equivalences (in various variations of the idea) are another well-known model for $(\infty, 1)$-categories. 'Posets with weak equivalences' apply the same structure to posets (regarded as categories). We, however, forego any technical discussion of this notion here, sketching only the key ideas.

in the case of trivial stratifications, the definition of $\sqcap_\infty(f_1)$ coincides with that of the fundamental ∞-groupoid of the underlying space, so our computation is a presentation of $\Pi_\infty S^1$.

Second, consider the circle (S^1, f_2) stratified by a single point and its complement. Note that this stratification is cellulable but not cellular. We can choose a refining regular cell complex as before, and compute $\sqcap_\infty(f_2)$ as before. In this case, the resulting 1-category $\mathrm{ho}(\sqcap_\infty(f_2))$ is equivalent to the category with two objects and two non-identity morphisms running from the first to the second object (without further relations).

Thirdly, consider the circle (S^1, f_3) stratified as the regular cell complex with two 0- and two 1-cells itself. This stratification is certainly cellular. Its fundamental ∞-category $\sqcap_\infty(f_3)$ is 0-truncated; indeed, it can be presented by the poset $\sqcap(f_3)$ (without any weak equivalences) as shown in the figure. ⌟

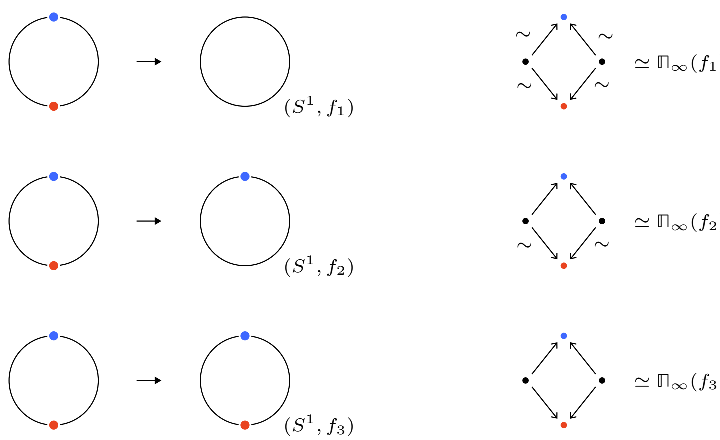

FIGURE B.17. Fundamental categories of the stratified circle.

REMARK B.3.35 (Well-definedness of the fundamental ∞-category). Given a cellulable stratification f, our construction of $\sqcap_\infty(f)$ above depends on a choice of cellular refinement of f. Showing that the resulting ∞-category $\sqcap_\infty(f)$ is independent of this choice is non-trivial in general. However, in the case of *conical* cellulable stratifications one can compare $\sqcap_\infty(f)$ with the construction in Definition B.3.10, and show that the two constructions are indeed equivalent. ⌟

APPENDIX C

◊A menagerie of framed regular cells

This appendix illustrates an assemblage of low-dimensional framed regular cells and their corresponding combinatorial representations as n-truss blocks. That combinatorial classification was the primary content of Chapter 3, and the correspondence is informally visible in the pictures: the total poset of the truss block is the fundamental poset of the regular cell, and successively projecting out the highest frame direction yields a tower of lower-dimensional cells and their associated fundamental posets.

C.1. 2-dimensional cells

There is a unique framed regular 1-dimensional cell, namely the closed framed interval. Already in dimension 2 there are infinitely many framed regular cells; necessarily they all project to the framed interval. In Figure C.1 we illustrate a few of the simplest framed regular 2-cells; the framing is induced by realizing the cells in $\mathbb{R}^2$, and that realization is indicated by the axes in the lower left corner of the figure. The first three cells are familiar shapes, namely the 2-globe, the 2-simplex, and the 2-cube; we refer to the subsequent three 2-cells as the V-cell, the Y-cell, and the X-cell, as they are dual to a V-shaped singularity, a Y-shaped singularity, and an X-shaped singularity, respectively.

C.2. 3-dimensional cells

We next consider framed regular cells in dimension 3. We organize these cells, primarily, by which framed regular 2-cell they project to, and, secondarily, by the cardinality of the total poset of their fundamental truss block.

In Figure C.2 we depict a few of the simplest framed regular 3-cells that project to the 2-globe. The first four cells are the 3-globe, the suspended 2-simplex, the suspended 2-cube, and the suspended Y-cell. The sixth cell is a product of a 2-globe c_2 and a 1-globe c_1; here the product is the usual stratified product (see Construction B.2.23), with framing induced by the product $c_2 \times c_1 \hookrightarrow \mathbb{R}^2 \times \mathbb{R}^1$ of the given realizations $c_2 \hookrightarrow \mathbb{R}^2$ and $c_1 \hookrightarrow \mathbb{R}^1$. The fifth cell is a degeneration of that product, collapsing one of the fiber 1-cells to a point. The last cell already exhibits a more complicated structure, where a central Y-cell collapses to a point on one side, and degenerates asymmetrically to an interval on the other side.

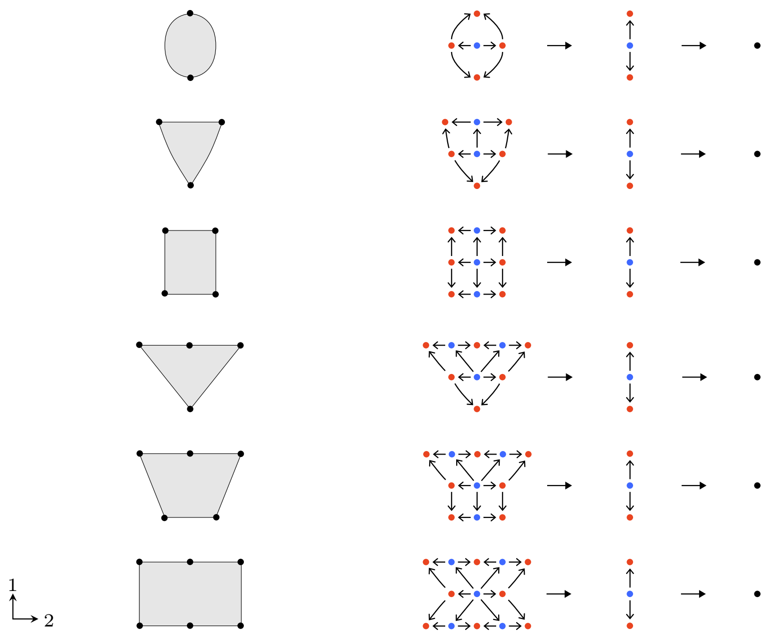

FIGURE C.1. The simplest 2-cells.

In Figure C.3 we similarly depict framed regular 3-cells that project to the 2-simplex. The first such cell is a 'triangoli cell', that is a pillowed 2-simplex having two 2-simplices (glued along their boundaries) as its boundary. The second cell is the cone of the 2-globe c_2; here the cone is the usual stratified closed cone (see Terminology B.3.1), with framing derived by extending the given realization $c_2 \hookrightarrow \mathbb{R}^2$ to a realization $\mathsf{cone}(c_2) \hookrightarrow \mathbb{R} \times \mathbb{R}^2$. The third and fourth cells both involve a central 2-simplex partially degenerating, to an interval or a 2-globe, on one side; the fifth cell has a 2-cube similarly degenerating to an interval. The last two cells both contain a V-cell slice, but along different axes. Altogether, of course, the more elaborate framed 3-cells begin to defy concise description.

In Figure C.4 we depict further 3-cells that project to the 2-simplex. These 3-cells have a somewhat different character than all the previous 3-cells, in that they admit piecewise linear realizations. The first two cells are both 3-simplices, but with distinct framed structures. The third and fourth cells are both square pyramids, but again with distinct framed structures. The last cell is of course a framed product of a 2-simplex and a 1-simplex.

In Figure C.5 we depict framed regular 3-cells that project to the 2-cube. The first such cell is a 'ravioli cell', that is a pillowed 2-cube having two 2-cubes (glued along their boundaries) as its boundary. The second and third cells burst open an edge or two of the ravioli cell into a 2-globe. The fourth cell is a product of a 1-globe c_1 and a 2-globe c_2. We see that the product of framed cells is not commutative: this cell $c_1 \times c_2$ is not framed equivalent to the product $c_2 \times c_1$ shown in Figure C.2. The fifth cell is another, framed distinct from the previous ones, square pyramid. The sixth cell is another product, of a 1-simplex and a 2-simplex, distinct from the product of the 2-simplex and 1-simplex in the previous figure. The last cell is of course the standard framed 3-cube, itself a triple product.

Finally, in Figure C.6, we depict a selection of more complicated 3-cells. To give a deeper understanding of the geometry of these cells, we also now illustrate their dual open meshes. (The framed cell corresponds to the given closed 3-truss, which dualizes to an open 3-truss, which in turn corresponds, by the main equivalence of Chapter 4, to an open 3-mesh.) The first 3-cell projects to the 2-dimensional V-cell; we sometimes refer to this 3-cell as the 3-dimensional 'quadratic cell' because the dual mesh exhibits a quadratic 1-tangle singularity. Note that this cell is the first one in our menagerie that has a boundary slice (in this case the top 2-3-planar slice) that is not in fact a topological cell. Though at first perhaps distressing, that behavior of the boundary slice is still completely combinatorially controlled by the classifying truss block. The second 3-cell projects to the 2-dimensional X-cell; we might refer to this 3-cell as a 'braid cell', since its dual mesh exhibits half of a 1-tangle braid, as illustrated. The third and last 3-cell projects to the 2-globe, and has an X-cell as its 2-3-planar central slice; we refer to it fancifully as the 'treccioni' cell.

C.3. 4-dimensional cells

Though we are reaching the limits of concise visualizability on paper, we also illustrate several framed regular 4-cells. In Figure C.7 we depict the simplest framed regular 4-cell, namely the 4-globe. The green arrow of the 4-frame indicates a fourth dimension (in first coordinate position), corresponding to the direction of the 1-frame vector. In Figures C.8, C.9, and C.10 we similarly depict, respectively, the double cone of the 2-globe, the product of the 2-simplex and the 2-globe, and the (framed distinct) product of the 2-globe and 2-simplex. Finally, in Figure C.11 we illustrate a more complex 4-cell, interpolating from a quadratic 3-cell, through a burst ravioli, to a (circa-2002) reverse astropop.

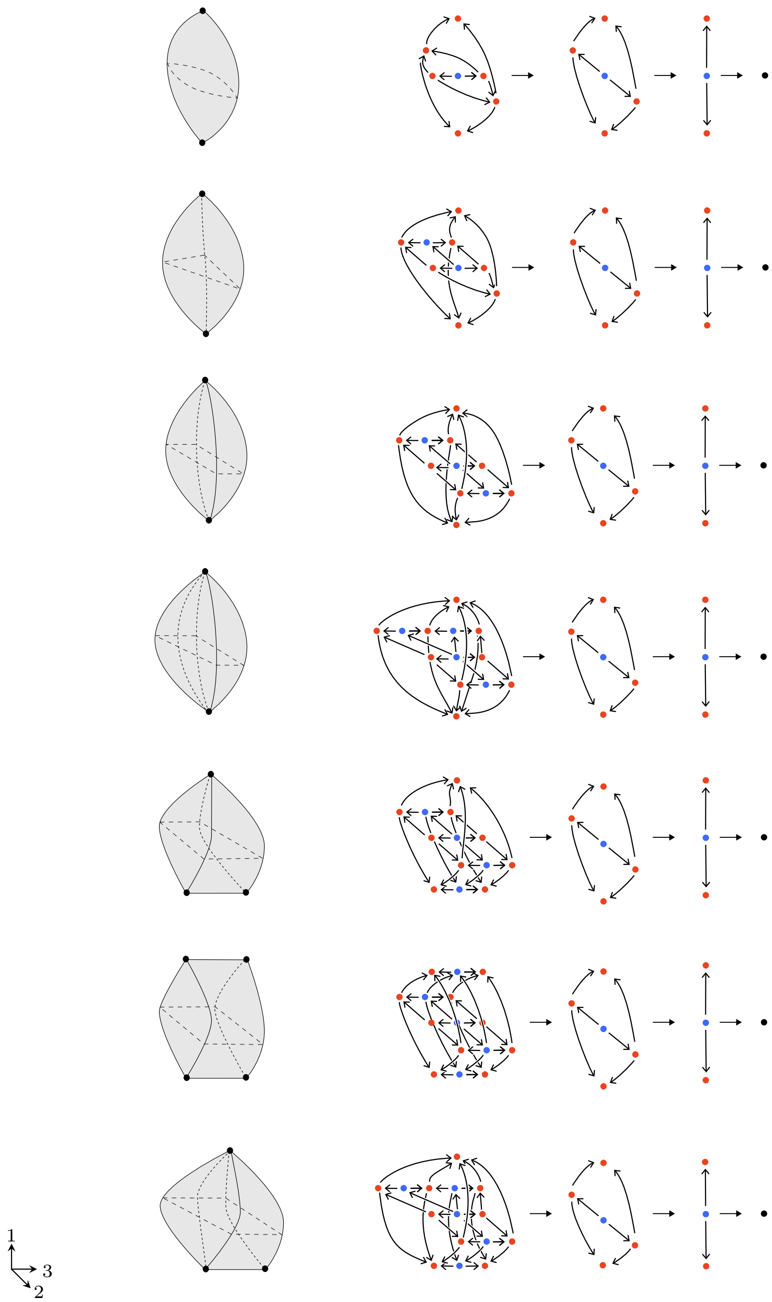

FIGURE C.2. The simplest 3-cells projecting to the 2-globe.

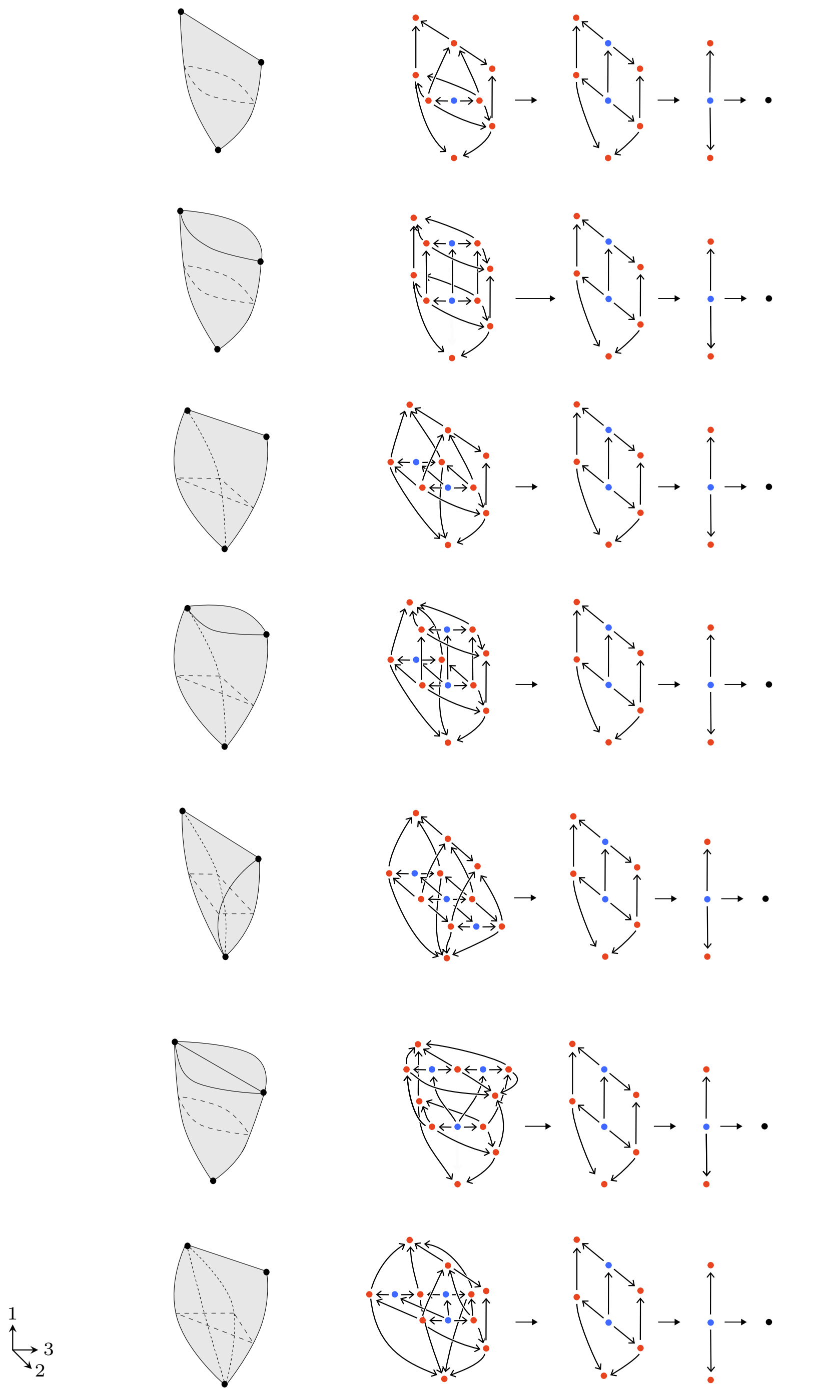

FIGURE C.3. The simplest 3-cells projecting to the 2-simplex.

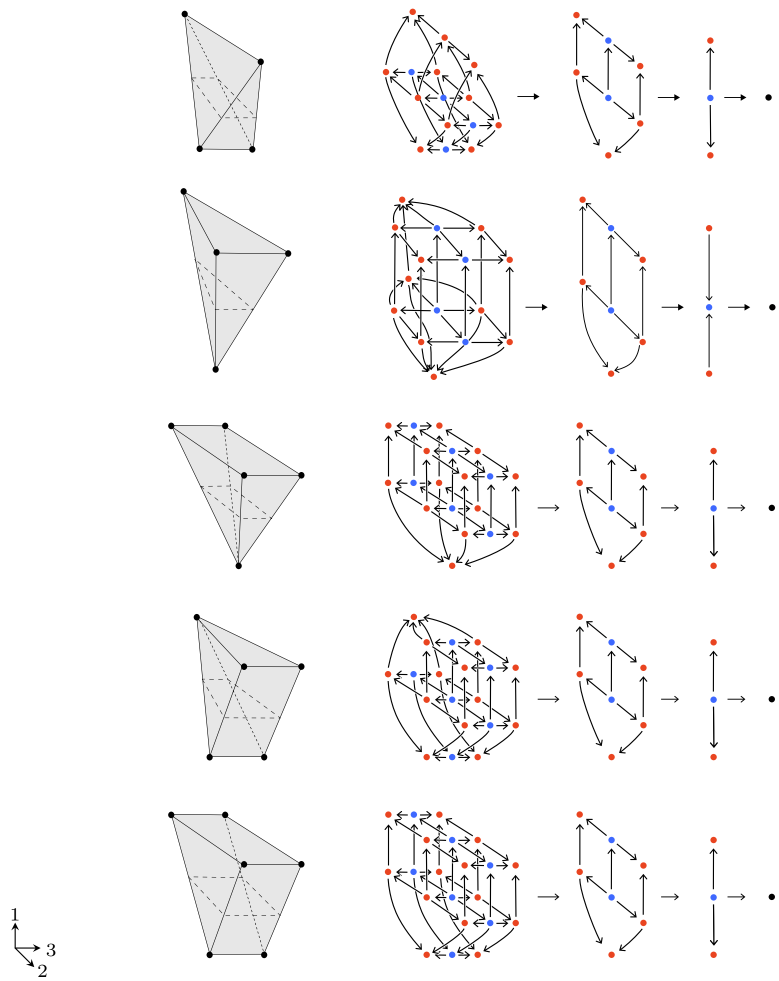

FIGURE C.4. Further 3-cells projecting to the 2-simplex.

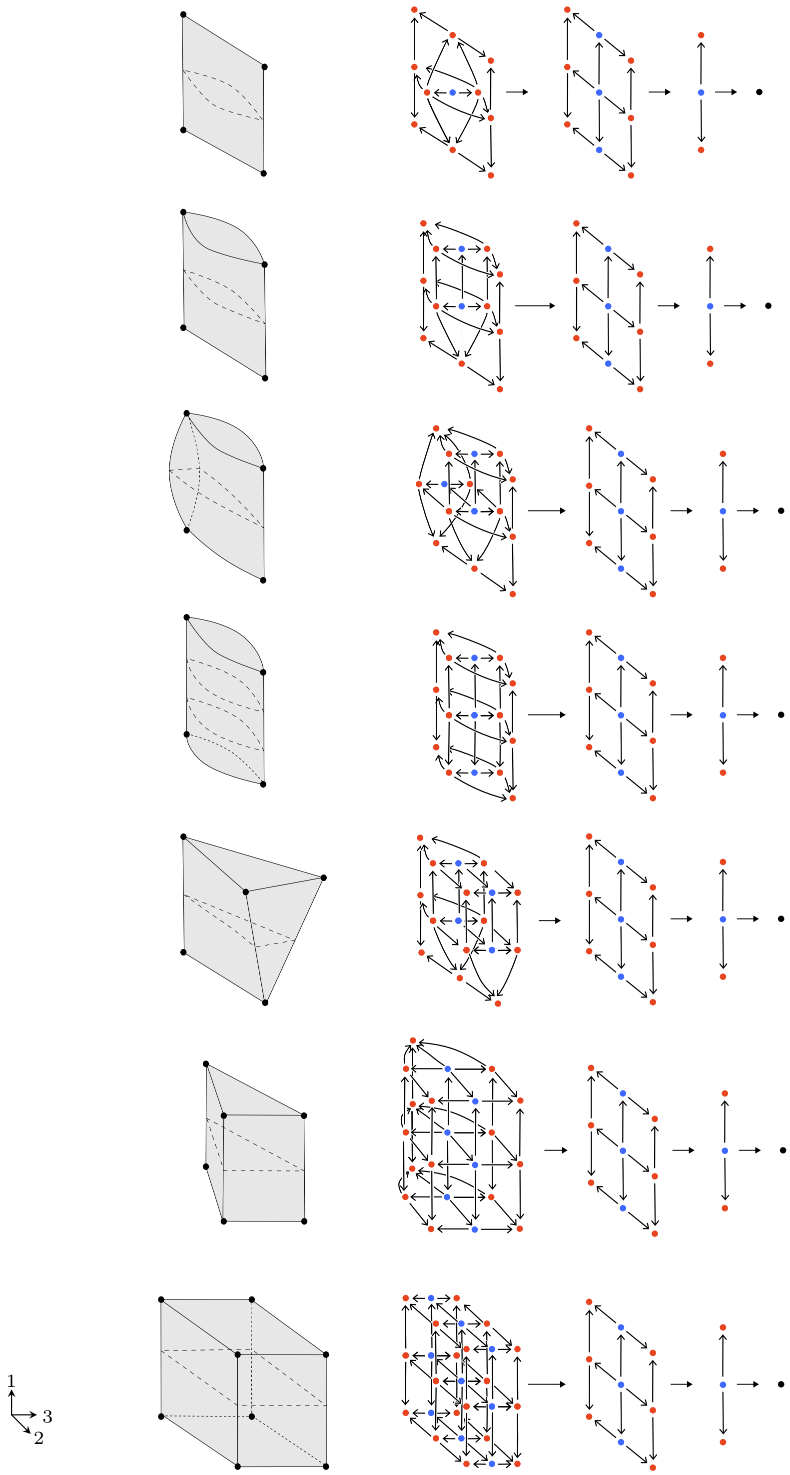

FIGURE C.5. The simplest 3-cells projecting to the 2-cube.

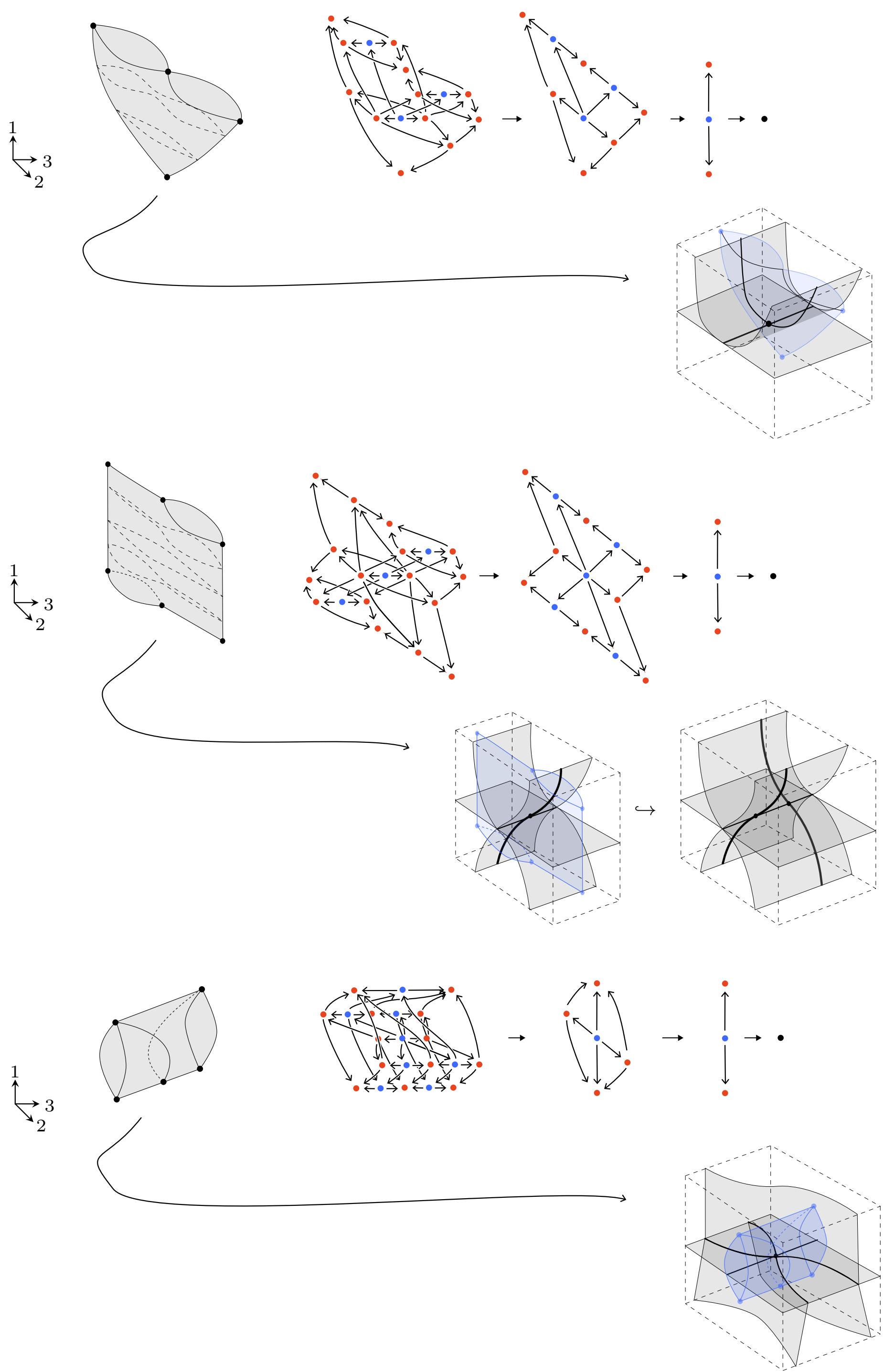

FIGURE C.6. More exotic 3-cells and their dual open meshes.

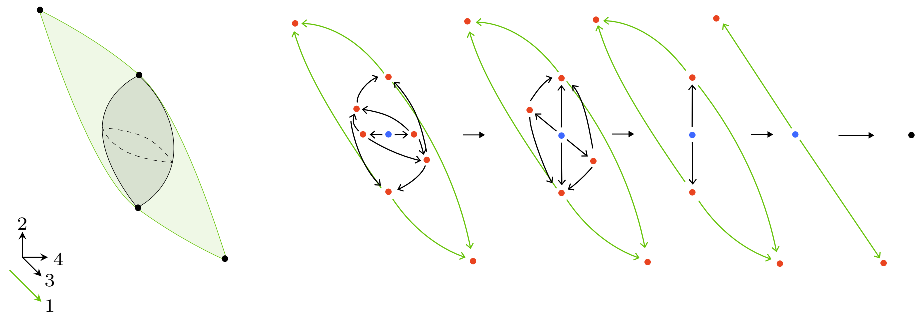

Figure C.7. The 4-globe as a 4-cell.

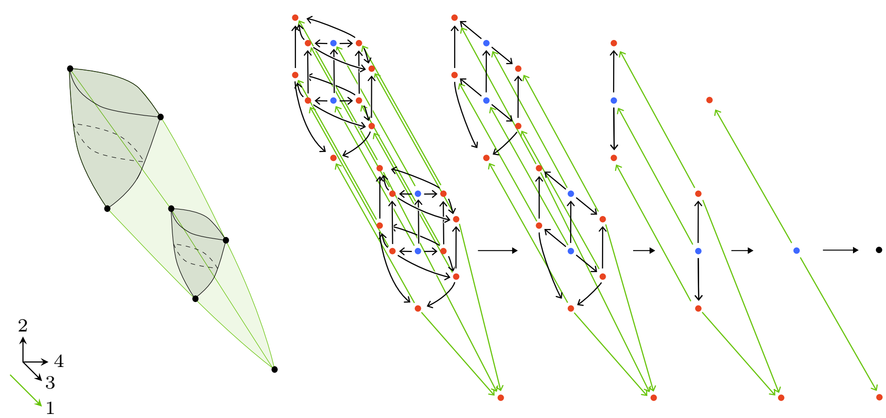

Figure C.8. The double cone of the 2-globe as a 4-cell.

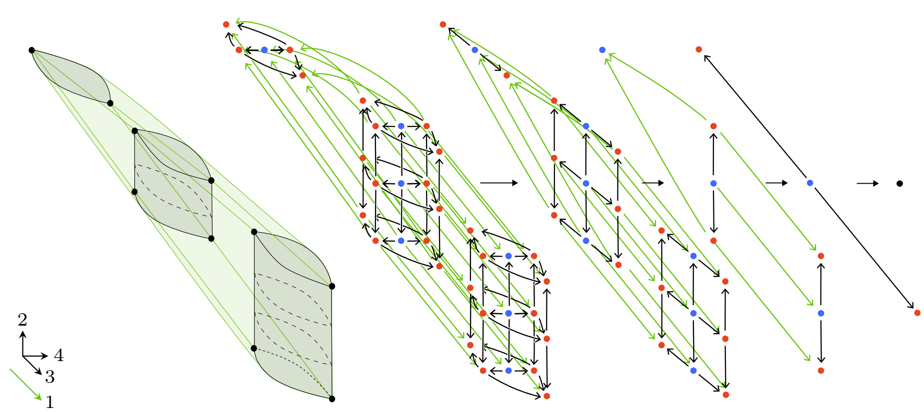

Figure C.9. The product of the 2-simplex and the 2-globe as a 4-cell.

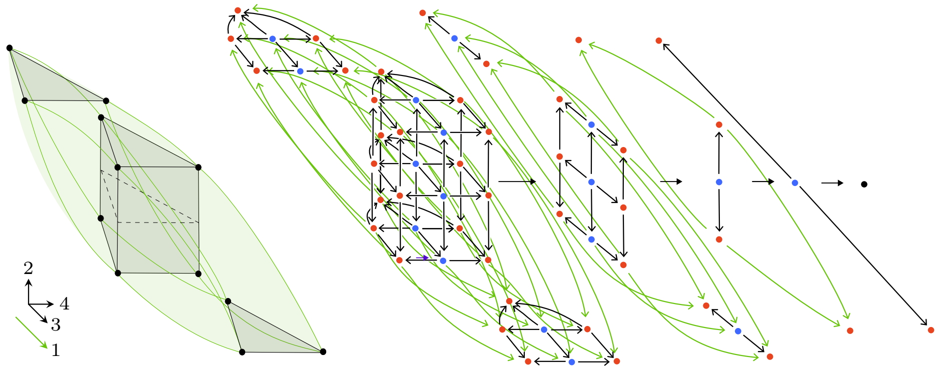

FIGURE C.10. The product of the 2-globe and the 2-simplex as a 4-cell.

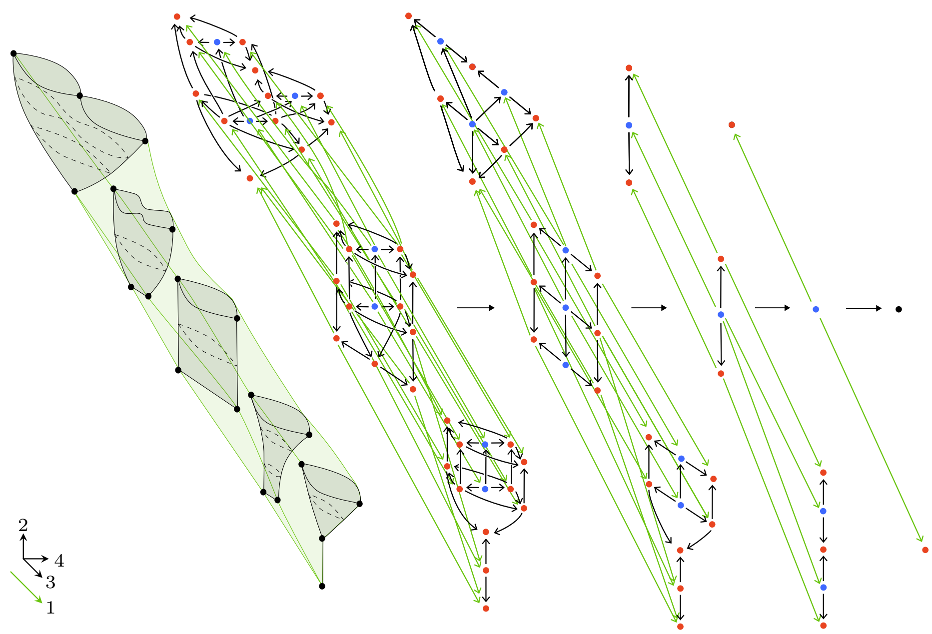

FIGURE C.11. A more involved 4-cell.

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List of Notation

Chapter 1

$\mathcal{F}$	A frame structure. ⇝index: ▷*framed simplex*, ▷*framed simplicial complex*, ▷*framed regular cell complex.*	§1
$\mathcal{F}\|_X$	The restriction of a frame structure. ⇝index: ▷*restricted frame* ▷*restricted framing.*	§1
$r_{\mathcal{F}}$	Realization of a frame structure $\mathcal{F}$. ⇝index.	§1
Δ	The category of ordered simplices.	1.1.2
$\underline{\Delta}$	The category of unordered simplices.	1.1.3
$[m]$	The combinatorial ordered standard m-simplex.	1.1.7
$[m]^{\mathsf{un}}$	The unordered standard simplex.	1.1.9
$\underline{m}$	The ordered set $1 < 2 < ... < m$.	1.1.10
$\|S\|$	The geometric simplex obtained by realizing an unordered simplex	1.1.11
$\|f\|$	The map of geometric simplices obtained by realizing a simplicial map	1.1.11
$\mathsf{spine}\, S$	The ordered collection of spine vectors of an ordered simplex.	1.1.14
$\mathsf{spine}[m]$	The spine of the standard simplex.	1.1.15
$S \twoheadrightarrow T$	A degeneracy, projection, or epimorphism.	1.1.16
$\ker(S \twoheadrightarrow T)$	The kernel of a degeneracy or projection.	1.1.17
$\ker^{\mathrm{aff}}(S \twoheadrightarrow T)$	The affine kernel of a degeneracy between ordered simplices.	1.1.18
$S \hookrightarrow T$	An affine face map between ordered simplices.	1.1.19
$\mathrm{im}^{\mathrm{aff}}(f)$	The affine image of an affine face map.	1.1.20
$\mathrm{coker}^{\mathrm{aff}}(f)$	The affine cokernel of an affine face map.	1.1.21
$\epsilon_i^{\pm}$	The ith positive or negative standard component of $\mathbb{R}^n$.	1.1.28
$\not\perp$	The relation of simplicial vectors being akin.	1.1.49
FrSimp_n	The category of n-embedded framed simplices and their framed maps.	1.1.62
$\mathsf{PartFrSimp}_n$	The category of n-embedded partially framed simplices and their framed maps.	1.1.68
SSet	The category of simplicial sets.	1.2.1
$\mathsf{SimpCplx}$	The category of simplicial complexes and their simplicial maps.	1.2.2
$(-)^{\mathsf{un}}$	The unordering functor, which forgets the order of vertices in each simplex.	1.2.5

$\mathsf{SimpCplx}^{\mathrm{ord}}$	The category of ordered simplicial complexes and their simplicial maps.	1.2.7
X^{ord}	A choice of ordering of an unordered simplicial structure.	1.2.8
$\mathsf{FrSimpCplx}_n$	The category of n-framed simplicial complexes and framed maps.	1.2.21
$\mathsf{Unframe}$	A functor that takes a framed simplicial complex to its underlying simplicial complex.	1.2.22
$h^{\mathcal{F}}$	The highest frame vector of a framed simplex.	1.2.26
q_k	The canonical induced framed k-collapse map, resulting from a sequence of framed elementary k-collapses.	1.2.31
$\mathsf{CollFrSimpCplx}_n$	The full subcategory of n-framed simplicial complexes comprising collapsible framings.	1.2.34
$P^{>x}$	The strict upper closure of an element x in a poset P, consisting of elements $y \in P$ such that $y > x$.	1.3.2
$P^{\geq x}$	The upper closure of an element x in a poset P, consisting of elements $y \in P$ such that $y \geq x$.	1.3.2
NP	The nerve of a poset P.	1.3.3
$\lvert - \rvert$	Geometric realization of a poset.	1.3.3
$\lVert - \rVert$	Stratified realization.	1.3.4
$\mathsf{str}(x)$	The strata of the stratified realization of a poset.	1.3.4
$\sqcap X$	The fundamental poset of a regular cell complex X	1.3.7
$\bot_X$	The initial object of a combinatorial regular m-cell.	1.3.14
$\mathsf{CellCplx}^{\mathcal{S}}$	The category of geometric regular cell complexes.	1.3.17
$\mathsf{CellCplx}$	The category of combinatorial regular cell complexes.	1.3.20
$\mathcal{C}ell\mathcal{C}plx^{\mathcal{S}}$	The topological category of geometric regular cell complexes and their cellular maps.	1.3.23
$\mathsf{CellCplx}^{\mathrm{PL}}$	The category of combinatorial regular PL cell complexes.	1.3.32
$\mathsf{axl}\, x$	The subcomplex spanned by the highest frame vectors of a framed regular cell. ⇝index: ▷ *axel vector.*	1.3.48
FrCell_n	The category of n-framed regular cells.	1.3.54
$\mathsf{FrCellCplx}_n$	The category of n-framed regular cell complexes.	1.3.54
$\mathsf{CollFrCellCplx}_n$	The category of collapsible n-framed regular cell complexes.	1.3.54
$\sqcap^{\mathsf{fr}}$	The framed fundamental poset functor.	1.3.56

Chapter 2

$\trianglelefteq$	The face order of a truss. ⇝index.	§2
$\preceq$	The frame order of a truss. ⇝index.	§2
dim	The dimension map of a truss. ⇝index.	§2
†	The dualization functor. ⇝index: ▷ *dualization functor,* ▷ *dual truss,* ▷ *dual truss bundle,* ▷ *dual truss bordism.*	§2
χ_{-}	The classification functor, taking a (labeled) truss bundle to its classifying functor. ⇝index.	§2
π_{-}	The totalization functor, taking a classifying functor to its total truss bundle. ⇝index.	§2
lbl_X	A labeling structure of X. ⇝index: ▷ *labeling functor,* ▷ C*-labeling,* ▷ *poset labeling.*	§2
$\mathsf{reg}(T)$	The subset of regular elements of a truss. ⇝index: ▷ *regular element set.*	2.1.13

$\mathsf{sing}(T)$	The subset of singular elements of a truss. ⇝index: ▹ *singular element set.*	2.1.13
Trs_1	The category of 1-trusses.	2.1.19
$\mathsf{end}_{\pm}T$	The minimal and maximal element of the frame order of a 1-truss, referred to as the lower and upper endpoint.	2.1.22
$\overset{-\circ}{\mathbb{T}}_k$	The left-closed right-open 1-truss with $2k$ elements.	2.1.23
$\bar{\mathbb{T}}_0$	The trivial closed 1-truss, which has a single, singular element.	2.1.23
$\bar{\mathbb{T}}_k$	The closed 1-truss with $2k+1$ elements.	2.1.23
$\overset{\circ-}{\mathbb{T}}_k$	The left-open right-closed 1-truss with $2k$ elements.	2.1.23
$\mathring{\mathbb{T}}_0$	The trivial open 1-truss, which has a single, regular element.	2.1.23
$\mathring{\mathbb{T}}_k$	The open 1-truss with $2k+1$ elements.	2.1.23
$\mathring{\mathsf{Trs}}_1$	The category of open trusses and regular maps.	2.1.25
$\bar{\mathsf{Trs}}_1$	The category of closed trusses and singular maps.	2.1.25
$\nrightarrow$	A functorial relation between preorders, or a profunctor between categories.	2.1.30
reg^R	The regular function of a 1-truss bordism R, mapping regular elements from the target truss S to regular elements in the source truss T.	2.1.34
sing_R	The singular function of a 1-truss bordism R, mapping singular elements from the source truss T to singular elements in the target truss S.	2.1.34
Bool	The category of boolean values, whose objects are 'true' $\top$ and 'false' $\bot$.	2.1.38
$\mathsf{BoolProf}$	The category of preorders and their boolean profunctors.	2.1.39
rel	The underlying relation functor.	2.1.40
TBord^1	The category of 1-trusses and their bordisms.	2.1.53
$\mathsf{Trs}_1^{\mathsf{r},\partial}$	The category of 1-trusses and their regular maps that preserve regular endpoints.	2.1.67
$\mathsf{Trs}_1^{\mathsf{s},\partial}$	The category of 1-trusses and their singular maps that preserve singular endpoints.	2.1.67
Cyl	The mapping cylinder functor.	2.1.68
coCyl	The mapping cocylinder functor.	2.1.68
R	The associated total poset of a 1-truss bordism R.	2.1.73
$p : T \to B$	A 1-truss bundle with total poset T and base poset B.	2.1.74
$\mathrm{cov}(B)$	The covering relation of the poset B.	2.1.81
TrsBun_1	The category of 1-truss bundles.	2.1.90
$\mathsf{Trs}_1(B)$	The category of 1-truss bundles over a fixed base poset B and their base-preserving maps.	2.1.90
$\mathring{\mathsf{Trs}}_1(B)$	The category of open truss bundles over a fixed base poset B and their regular maps.	2.1.90
$\bar{\mathsf{Trs}}_1(B)$	The category of closed truss bundles over a fixed base poset B and their singular maps.	2.1.90
$\mathrm{Tot}\,F$	The total poset of the bundle classified by F. ⇝index: ▹ *totalization functor.*	2.1.93
π_F	The total 1-truss bundle of a classifying functor F. ⇝index: ▹ *totalization functor.*	2.1.93
$\mathsf{TBord}^1(B)$	The category of 1-truss bundles and their bordisms over a fixed base poset B.	2.1.97

G^*p	The pullback of a truss bundle p along a map G. ⇝index: ▷*pullback truss bundle.*	2.1.103
ΣX	The suspension of a poset X, obtained by adjoining initial and final elements.	2.1.110
Σp	The suspension of a truss bundle $p : T \to B$. ⇝index.	2.1.111
Γ_p	The set of sections of a 1-truss bundle $p : T \to [m]$.	2.2.10
Ψ_p	The set of spacers of a 1-truss bundle $p : T \to [m]$.	2.2.10
$\langle - \rangle$	The scaffold norm of a section or spacer simplex. ⇝index: ▷*scaffold norm of sections*, ⇝index: ▷*scaffold norm of spacers.*	2.2.25
$\mathsf{K}_p^{\pm}$	The bottom and top sections of a 1-truss bundle $p : T \to [m]$.	2.2.28
$\partial_{\pm} L$	The upper and lower boundary sections of a spacer L.	2.2.31
$\Phi_p(z)$	The fiber category in a 1-truss bundle $p : T \to B$ over a nondegenerate simplex $z : [m] \to B$.	2.2.36
$(\underline{-})$	The label-forgetting functor. ⇝index: ▷*label-forgetting functor*, ▷*underlying truss*, ▷*underlying truss bordism*, ▷*underlying truss bundle.*	2.2.49
$\underline{T}$	See $(\underline{-})$.	2.2.49
$\mathsf{TBord}^1_{/\!/-}$	The labeled 1-truss bordism endofunctor on categories. ⇝index: ▷*relabeling functor*, ▷*labeled 1-truss bordism functor*, ▷*category of labeled 1-trusses and their bordisms.*	2.2.50
ι	The bordism-as-profunctor pseudofunctor.	2.2.51
$H_{/\!/\mathsf{C}}$	The vertical comma category of a normal pseudofunctor H from a category T to the bicategory $\mathbb{P}\mathrm{rof}$, over a category C in $\mathbb{P}\mathrm{rof}$.	2.2.53
$p\vert_A$	The restriction of a labeled truss bundle.	2.2.65
$\mathsf{TBord}^1(B)_{/\!/\mathsf{C}}$	The category of C-labeled 1-truss bundles over a base poset B and their bordisms.	2.2.71
$\mathcal{T}\mathit{Bord}^1_{/\!/-}$	The labeled 1-truss bordism endofunctor on quasicategories.	2.2.75
$\mathrm{cod}(R)$	The codomain of a labeled n-truss bordism R.	2.3.13
$\mathrm{dom}(R)$	The domain of a labeled n-truss bordism R.	2.3.13
rel_k^R	The k-stage functorial relation of an n-truss bordism R.	2.3.14
$n\mathsf{TBord}_{/\!/\mathsf{C}}$	The category of C-labeled n-trusses and their bordisms.	2.3.20
$\mathsf{TBord}^n_{/\!/-}$	The n-fold iterated labeled 1-truss bordism functor.	2.3.23
$\mathrm{cov}(T_k)$	The generating arrows of an n-truss bundle.	2.3.30
Trs_n	The category of n-trusses.	2.3.38
LblTrs_n	The category of labeled n-trusses.	2.3.38
TrsBun_n	The category of n-truss bundles.	2.3.38
$\mathsf{LblTrsBun}_n$	The category of labeled n-truss bundles.	2.3.38
$\mathsf{Trs}_n(B)$	The category of n-truss bundles over a poset B and base-preserving maps.	2.3.38
$\mathring{\mathsf{Trs}}_n(B)$	The category of open n-truss bundles and their regular maps.	2.3.38
$\bar{\mathsf{Trs}}_n(B)$	The category of closed n-truss bundles and their singular maps.	2.3.38
$\mathcal{T}\mathit{rs}_n$	The $k\mathsf{Top}$-enriched category of n-trusses.	2.3.39
$\mathcal{T}\mathit{rs}_n(B)$	The $k\mathsf{Top}$-enriched category of n-truss bundles over a poset B and base-preserving maps.	2.3.39
$p_{>k}$	The upper truncation of a labeled n-truss bundle p.	2.3.42

$p_{\leq k}$	The lower truncation of a labeled n-truss bundle p.	2.3.43
$\mathsf{TBord}^n(B)_{//\mathsf{C}}$	The category of C-labeled n-truss bundles over a base poset B and their bordisms.	2.3.50
$q \times T$	The truss product of an unlabeled m-truss bundle q and a C-labeled n-truss T.	2.3.55
$\mathsf{Trs}_n^{\mathrm{crs}}$	The category of n-trusses and their coarsenings.	2.3.66
$\mathsf{Trs}_n^{\mathrm{deg}}$	The category of n-trusses and their degeneracies.	2.3.66
$\perp$	The initial element in the total poset of a block.	2.3.73
Blk_n	The category of n-truss blocks and singular maps.	2.3.78
$\mathbb{X}$	The category of blocks.	2.3.79
$T^{\trianglerighteq x}$	The face block of an element x in a truss T.	2.3.81
BlkSet_n	The category of n-truss block sets.	2.3.85
$\mathbb{X}$	The category of block sets.	2.3.87
$\mathsf{BlkCplx}_n$	The category of block complexes.	2.3.92
$\mathsf{RBlkCplx}_n$	The category of regular block complexes.	2.3.92
$\mathbb{X}$	The category of braces.	2.3.106
BrcSet_n	The category of n-truss brace sets.	2.3.108
$\mathbb{X}$	The category of brace sets.	2.3.110

Chapter 3

∇_{C}	The cell gradient functor, which takes a truss block to its framed regular cell. ⇝index.	§3
$\int_{\mathsf{T}}$	The truss integration functor, which takes a framed regular cell to its truss block. ⇝index.	§3
$\mathcal{P}$	A proframe structure. ⇝index: ▷*proframed simplex* ▷ *proframed simplicial complex*.	3.2.1
$p_\perp$	The initial degeneracy of a k-partial proframe.	3.2.3
$\mathcal{P}^n_{\mathbb{R}}$	The standard euclidean proframe of $\mathbb{R}^n$.	3.2.8
$r_i^{\mathcal{P}}$	A sequence of linear embeddings forming a proframed realization.	3.2.9
$\mathsf{ProFrSimp}_n$	The category of n-embedded proframed simplices and their proframed maps.	3.2.12
$p_{\to i}$	Composite projection $p_{i+1} \cdots p_n : [m] \to [m_i]$ in an n-embedded proframed simplex. This differs from ▷ *upper truncation* of a tower of bundles.	3.2.14
∇	The gradient frame (or framing) functor. ⇝index.	3.2.15
$\int$	An integral proframe (or proframing) construction. ⇝index.	3.2.16
$\mathcal{P}_{\leq i}$	The lower i-truncation of an n-proframing.	3.2.29
K_z	The fiber set over a simplex z for a simplicial map $p : K \to K'$.	3.2.37
$\partial^\pm x$	The upper and lower sections of a spacer simplex x.	3.2.39
$\Phi_{\mathcal{P}}(z)$	The fiber category over a simplex z in an n-proframed simplicial complex.	3.2.40
$\mathsf{CollProFrSimpCplx}_n$	The category of collapsible n-proframed simplicial complexes	3.2.45
$\gamma^\pm$	The lower and upper sections of a spacer cell.	3.3.5
$\mathcal{P}^x$	The integral proframing of the framed regular subcell determined by x.	3.3.8
$\mathsf{axl}\perp$	The highest frame vectors of an framed regular cell.	3.3.16

Chapter 4

$\Vert - \Vert_{\mathsf{M}}$	The mesh realization functor. ⇝index.	§4
$\mathbb{\Pi}_{\mathsf{T}} -$	The fundamental truss functor. ⇝index.	§4
γ	A framed realization of a manifold M or manifold bundle p. ⇝index.	§4
S^1	The standard framed circle.	4.1.1
$\mathbb{R}$	The standard framed euclidean space.	4.1.1
$\gamma^{\pm}$	The upper and lower realization bounds of a 1-mesh or 1-mesh bundle or family. ⇝index: ▷*realization bounds* ▷*mesh family.*	4.1.15
$\mathbb{\Pi} -$	The fundamental poset functor.	4.1.27
$s \to r$	A formal entrance path from a stratum s to a stratum r.	4.1.27
$p : (M, f) \to (B, g)$	A 1-mesh bundle over a base stratification (B, g).	4.1.28
$\mathsf{fib}(s)$	The fiber 1-mesh over a stratum s.	4.1.34
$\mathbb{\Pi}_\infty f$	The fundamental ∞-category of a stratification f.	4.1.38
$\mathbb{\Pi}_1 f$	The fundamental category of a stratification f.	4.1.39
$G^* p$	The pullback of a mesh bundle. ⇝index.	4.1.57
$\overline{p}$	The fiberwise compactification of a 1-mesh bundle p. *This is analogous, but less general, than* ▷cubical compactification *of truss bundles.*	4.1.58
$p_{\leq i}$	The lower i-truncation of an n-mesh bundle p.	4.1.80
$\mathrm{Tot} G$	The pullback of a mesh bundle map G.	4.1.93
$p\vert_X$	The restriction of an n-mesh bundle p to a substratification X in its base.	4.1.94
Mesh_n	The category of n-meshes.	4.1.95
$\mathsf{MeshBun}_n$	The category of n-mesh bundles.	4.1.95
$\mathsf{Mesh}_n(B, g)$	The category of n-mesh bundles over a base stratification (B, g).	4.1.95
$\mathcal{M}esh_n$	The ∞-category of n-meshes.	4.1.97
$\bar{\mathcal{M}}esh_n$	The ∞-category of closed n-meshes and their singular maps.	4.1.97
$\mathring{\mathcal{M}}esh_n$	The ∞-category of open n-meshes and their regular maps.	4.1.97
$\mathcal{M}eshBun_n$	The ∞-category of n-mesh bundles.	4.1.97
$\mathcal{M}esh_n(B, g)$	The ∞-category of n-mesh bundles over a base stratification (B, g).	4.1.97
$\mathcal{M}esh_n^{\mathrm{bal}}(B, g)$	The ∞-category of n-mesh bundles over (B, g) with balanced maps.	4.1.97
$\mathcal{M}Bord_n$	The n-mesh bordism category.	4.1.100
N^{hc}	The homotopy coherent nerve functor.	4.2.6
$\mathcal{C}yl$	The mesh mapping cylinder functor.	4.2.6
$co\mathcal{C}yl$	The mesh mapping cocylinder functor.	4.2.6
$\Vert - \Vert_{\mathsf{CM}}$	The cell-to-mesh realization functor.	4.2.7
∇_{MC}	The mesh-to-cell gradient functor.	4.2.7
$\dagger$	The mesh dualization functor.	4.2.9
c_s	The regular contour of a regular stratum s.	4.2.29
C_b	The catchment area of an open cell b.	4.2.32
π_b	The radial catchment projection from the closed catchment area to the cell b.	4.2.32
ci	The cubical inclusion map in a cubical compactification. ⇝index.	4.2.50

cr	The cubical retraction map in a cubical compactification. ⇝index.	4.2.50
$\overline{p}$, $\overline{T}$	The cubical compactification of a truss bundle p or truss T.	4.2.50
Ci	The inclusion map of a cellular inclusion-retraction pair, from the realization of Y to B.	4.2.68
Cr	The retraction map of a cellular inclusion-retraction pair, from B to the realization of Y.	4.2.68
$\Vert F\Vert^{\mathsf{crs}}_{\mathsf{M}}$	The mesh coarsening realization of a coarsening of n-trusses $F : T \to S$.	4.2.78
$\mathsf{SubDiv}(Y, \mathcal{G}; X, \mathcal{F})$	The space of framed subdivisions between framed regular cell complexes $(Y, \mathcal{G})$ and $(X, \mathcal{F})$.	4.2.91

Chapter 5

(M, f)	Stratified mesh. ⇝index.	§5
$M \to f$	Mesh refinement of a tame stratification f by a mesh M. ⇝index.	§5
$\Vert - \Vert_{\mathsf{M}}$	Stratified mesh realization. ⇝index.	§5
$\sqcap_{\mathsf{T}}(M, f)$	Fundamental stratified truss of a stratified mesh (M, f). ⇝index.	§5
ι	A tame embedding. ⇝index.	5.1.4
$f \vee g$	The join of stratifications f and g.	5.2.1
s	An equivalence class of strata in a join of stratifications.	5.2.5
$M \vee M'$	The join of two n-meshes M and M'.	5.2.10
$p \vee q$	The join of two n-mesh bundles p and q.	5.2.11
FM	The pushforward mesh of a mesh M under a framed homeomorphism F.	5.2.24
$F^{-1}N$	The pullback mesh of a mesh N under a framed homeomorphism F.	5.2.24
$\sqcap(T)$	The fundamental poset of a stratified truss.	5.3.1
$[\![T]\!]$	The normal form of a stratified truss T.	5.3.61
$(Z^\dagger, f^\dagger)$	The dual stratification of an n-tame stratification.	5.3.70
$X_{\min}$	The unique coarsest cell structure representative of the framed homeomorphism class of a realizable n-framed regular cell complex.	5.3.79

Appendix A

V	A vector space.	§A
$\vec{V}$	The associated affine space of a vector space.	A.2.1
$\mathcal{V}$	An affine space	A.2.1
$\vec{\mathcal{V}}$	The associated vector space of an affine space.	A.2.1
$\langle S\rangle$	The affine hyperplane spanned by a simplex.	A.2.2
Δ^m	The standard geometric m-simplex.	A.2.4
$\hat{\mathcal{V}}$	Affine space of affine vectors in an affine space.	A.2.5

Appendix B

TOP	The category of all topological spaces.	B.1.2
$k\mathsf{Top}$	The category of compactly generated spaces.	B.1.2

Top	The category of compactly generated weakly Hausdorff spaces.	B.1.2
$\mathrm{Map}(-,-)$	The internal hom for compactly generated spaces.	B.1.2
$\mathrm{Spcl}\, X$	The specialization order of a topological space X.	B.1.4
$\mathbb{\Pi}(f)$	The fundamental preorder of a prestratification (X, f).	B.1.11
$\mathbb{\Pi}(f)^{\mathrm{op}}$	The exit path preorder of a prestratification (X, f).	B.1.12
$y <^{\mathrm{cov}} x$	The covering relation in a poset.	B.1.33
$\mathsf{cmpnt}(f)$	The connected component set of a P-structure.	B.1.44
$\mathsf{char}(f)$	The characteristic function of the connected component splitting of a P-structure.	B.1.44
$\mathsf{discr}(f)$	The discrete map of the connected component splitting of a P-structure	B.1.44
$\lvert - \rvert$	Geometric realization of a poset or poset map.	B.1.50
$\lVert - \rVert$	The stratified realization of a poset or poset map.	B.1.51
$\mathsf{str}(x)$	The stratum in $\lVert P \rVert$ corresponding to the object $x \in P$.	B.1.51
$g\vert_X$	The restriction of a stratification (Y, g) to a subspace X.	B.2.8
Strat	The category of stratifications and their stratified maps.	B.2.13
$\mathcal{S}\mathit{trat}$	The $k\mathsf{Top}$-enriched category of stratified spaces and their stratified maps.	B.2.17
$\mathcal{P}\mathit{os}_{\mathrm{lf}}$	The $k\mathsf{Top}$-enriched category of locally finite posets.	B.2.19
$\mathcal{S}\mathsf{trat}_{\mathrm{lf}}$	The category of locally finite stratifications.	B.2.21
$(- \times -)$	The topological product functor for stratifications.	B.2.23
$\mathcal{S}\mathsf{trat}_{\mathrm{lf}}(f, g)_F$	The preimage of F in the fundamental poset map $\mathbb{\Pi} : \mathcal{S}\mathsf{trat}_{\mathrm{lf}}(f, g) \to \mathcal{P}\mathit{os}_{\mathrm{lf}}(\mathbb{\Pi}(f), \mathbb{\Pi}(g))$.	B.2.24
$(X, f) \times_{(Z,h)} (Y, g)$	The pullback stratification.	B.2.27
$F^* p$	The pullback of a stratified bundle p along a stratified map F.	B.2.29
$\overline{\mathsf{cone}}(f)$	The closed cone of a stratification (X, f).	B.3.1
$\mathsf{cone}(f)$	The open cone of a stratification (X, f).	B.3.1
$\mathsf{link}(x)$	The link at a point x in a stratification (X, f). *Coincides with the cellular link for regular cell complexes.*	B.3.2
$\mathbb{\Pi}_\infty f$	The fundamental ∞-category of a conical stratification f.	B.3.10
$\mathsf{star}(x)$	The cellular star of a cell x in a regular cell complex.	B.3.28
$\mathsf{link}(x)$	The cellular link of a cell x in a regular cell complex. *Coincides with the link of regular cell complexes as conical stratifications.*	B.3.28
$\mathsf{BSub}(X)$	The barycentric subdivision of a regular cell complex X.	B.3.31
$\mathsf{CSub}_X(x)$	The conical subdivision of a cell x in a regular cell complex X.	B.3.31

Index of Terms

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